Sahlqvist Correspondence Theory for Second-Order Propositional Modal Logic

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Abstract

Modal logic with propositional quantifiers (i.e. second-order propositional modal logic (SOPML)) has been considered since the early time of modal logic. Its expressive power and complexity are high, and its van-Benthem-Rosen theorem and Goldblatt-Thomason theorem have been proved by ten Cate (2006). However, the Sahlqvist theory of SOPML has not been considered in the literature. In the present paper, we fill in this gap. We develop the Sahlqvist correspondence theory for SOPML, which covers and properly extends existing Sahlqvist formulas in basic modal logic. We define the class of Sahlqvist formulas for SOMPL step by step in a hierarchical way, each formula of which is shown to have a first-order correspondent over Kripke frames effectively computable by an algorithm ALBASOMPL. In addition, we show that certain Π_2 -rules correspond to Π_2 -Sahlqvist formulas in SOMPL, which further correspond to first-order conditions, and that even for very simple SOMPL Sahlqvist formulas, they could already be noncanonical.

Keywords: correspondence theory, second-order propositional modal logic, ALBA algorithm, Π_2 -rules, canonicity

1 Introduction

Second-Order Propositional Modal Logic (SOMPL). Modal logic with propositional quantifiers has been considered in the literature since Kripke [29], Bull [10], Fine [18, 19], and Kaplan [17]. This language is of high complexity: its satisfiability problem is not decidable, and indeed not even analytical. In Kaminski and Tiomkin [27], the authors showed that the expressive power for SOMPL whose modalities are S4.2 or weaker is the same as second-order predicate logic. However, not every second-order formula is equivalent to an SOMPL-formula, since

¹For more literature, see [1, 3, 4, 5, 6, 16, 20, 23, 25, 26, 28, 30, 31, 32].

SOMPL-formulas are preserved under generated submodels (see van Benthem [37]). In ten Cate [35], the author proved the analogues of the van Benthem-Rosen theorem (on the model level) and Goldblatt-Thomason theorem (on the frame level) for SOMPL. Therefore, a natural question is: on the frame level, can we find a natural fragment of SOPML-formulas such that each formula in this fragment corresponds to a first-order formula, in the sense of Sahlqvist theory (see [33, 37])? This is what we will answer in the paper.

Correspondence Theory. Typically, modal correspondence theory [37] concerns the correspondence of modal formulas and first-order formulas over Kripke frames, via the tools of standard translation. Syntactic classes (e.g. Sahlqvist formulas [33], inductive formulas [24], etc.) of modal formulas are identified to have first-order correspondents and are canonical, i.e. their validity are closed under taking canonical extensions.

In the present paper, we identify the Sahlqvist formulas of SOMPL, which cover and properly extend the Sahlqvist fragment in basic modal logic. We show that each Sahlqvist SOMPL formula corresponds to a first-order formula by an algorithm ALBA SOPML. In particular, we have the following observations: the SOMPL Sahlqvist formula $\forall p(\Box p \land \forall q(q \rightarrow \Diamond \Diamond q \lor p) \rightarrow p)$ corresponds to $\forall x \forall y (Rxy \land Ryx \rightarrow Rxx)$, which is not modally definable since this property is not preserved under taking bounded morphic image (see Example 7.3); the SOMPL Sahlqvist formula $\forall q (\forall p(p \rightarrow \Diamond p \lor q) \rightarrow q)$ is not canonical (see Example 7.2), which is in contrast to the basic modal logic setting where each Sahlqvist formula is canonical.

Non-standard Rules. Another topic that is related to the present paper is non-standard rules, starting from Gabbay [21] where a non-standard rule for irreflexivity is introduced. These rules have been used in temporal logic [11, 22], region-based theories of space [2, 36] and are used to prove completeness results for modal logic systems with non- ξ -rules [38]. In particular, the so-called Π_2 -rules [7, 8, 34] which generalize both the irreflexivity rule of Gabbay [21] and the non- ξ -rules of Venema [38], have their natural $\forall \exists$ -counterparts, which are essentially $\forall \exists$ -SOMPL formulas, fit naturally into the language of SOMPL. We use the correspondence algorithm to compute the first-order correspondents of a subclass of Π_2 -rules whose $\forall \exists$ -counterparts are SOMPL Π_2 -Sahlqvist formulas.

Our methodology. The present paper use the same methodology as [15, 12]. In the present paper, inspired by the Sahlqvist rules in Santoli [34], we identify the Sahlqvist formulas of SOMPL, which are generalizations of Sahlqvist formulas in basic modal logic and have first-order correspondents. The Sahlqvist fragment of

SOPML is defined in a step-by-step way, and we give an algorithm ALBA SOPML (Ackermann Lemma Based Algorithm) which can successfully reduce Sahlqvist formulas in SOPML to first-order formulas and is sound with respect to Kripke semantics.

Structure of the paper. The structure of the paper is as follows: Section 2 gives the necessary preliminaries. Section 3 gives the definition of Sahlqvist SOPML formulas step by step. Section 4 defines the algorithm ALBA^{SOPML}. Section 5 shows the soundness of the algorithm with respect to Kripke frames. Section 6 shows that the algorithm succeeds on all Sahlqvist SOPML formulas. Section 7 gives some examples and connect them with non-standard rules, and one example shows that even for very simple Sahlqvist SOPML formulas, they can already be non-canonical. Section 8 gives some final remarks and conclusion.

2 Preliminaries

2.1 Language and semantics

In the present paper we consider the unimodal language. Given a set Prop of propositional variables, the second-order propositional modal formulas are defined as follows:

$$\varphi ::= p \mid \bot \mid \top \mid \neg \varphi \mid \varphi \land \varphi \mid \varphi \lor \varphi \mid \varphi \rightarrow \varphi \mid \Box \varphi \mid \Diamond \varphi \mid \forall p \varphi \mid \exists p \varphi$$

where $p \in \mathsf{Prop}$. We use the notation \vec{p} to denote a set of propositional variables and use $\varphi(\vec{p})$ to indicate that the propositional variables occur in φ are all in \vec{p} . We say that an occurrence of a propositional variable p in a formula φ is *positive* (resp. *negative*) if it is in the scope of an even (resp. odd) number of negations (here $\alpha \to \beta$ is regarded as $\neg \alpha \lor \beta$).

The semantics of the second-order propositional modal formulas are defined as follows:

Definition 1. A *Kripke frame* is a pair $\mathbb{F} = (W, R)$ where $W \neq \emptyset$ is the *domain* of \mathbb{F} , the *accessibility relation R* is a binary relation on W. A *Kripke model* is a pair $\mathbb{M} = (\mathbb{F}, V)$ where $V : \mathsf{Prop} \to P(W)$ is a *valuation* on \mathbb{F} . V_X^p denote a valuation which is the same as V except that $V_X^p(p) = X \subseteq W$.

Now the satisfaction relation can be defined as follows: given any Kripke model $\mathbb{M} = (W, R, V)$, any $w \in W$, the basic and Boolean cases are standard, and for modalities and propositional quantifiers,

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\mathbb{M}, w \Vdash \Box \varphi iff for any v such that Rwv, \mathbb{M}, v \Vdash \varphi;

\mathbb{M}, w \Vdash \Diamond \varphi iff there exists v such that Rwv and \mathbb{M}, v \Vdash \varphi;

\mathbb{M}, w \Vdash \exists p\varphi iff for all X \subseteq W, (W, R, V_X^p), w \Vdash \varphi;

\mathbb{M}, w \Vdash \exists p\varphi iff there exists X \subseteq W such that (W, R, V_Y^p), w \Vdash \varphi.
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In order to use the algorithm to compute the first-order correspondents of Sahlqvist SOPML formulas, we will need the following *expanded modal language* which is defined as follows²:

$$\varphi ::= p \mid \mathbf{i} \mid \bot \mid \top \mid \neg \varphi \mid \varphi \land \varphi \mid \varphi \lor \varphi \mid \varphi \to \varphi \mid$$

$$\square \varphi \mid \diamond \varphi \mid \blacksquare \varphi \mid \diamond \varphi \mid \forall p \varphi \mid \exists p \varphi \mid \forall \mathbf{i} \varphi \mid \exists \mathbf{i} \varphi \mid \mathbf{l}(\varphi, \varphi)$$

where $p \in \mathsf{Prop}$, $\mathbf{i} \in \mathsf{Nom}$ is a *nominal*, \blacksquare and \spadesuit are the backward-looking box and diamond respectively, $\forall \mathbf{i}$ and $\exists \mathbf{i}$ are *nominal quantifiers*, and \mathbf{l} is a binary modality. We call a formula *pure* if it does not contain propositional variables or propositional quantifiers (it can contain nominals, nominal quantifiers and the binary modality \mathbf{l}).

The interpretation of the expanded modal language is given as follows: For a valuation V, it is defined as $V : \mathsf{Prop} \cup \mathsf{Nom} \to P(W)$ such that $V(\mathbf{i})$ is a singleton for all $\mathbf{i} \in \mathsf{Nom}$. The additional satisfaction clauses are given as follows (here $V^{\mathbf{i}}_{v}$ denote a valuation which is the same as V except that $V^{\mathbf{i}}_{v}(\mathbf{i}) = \{v\} \subseteq W$.):

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\mathbb{M}, w \Vdash \mathbf{i} iff V(\mathbf{i}) = \{w\};

\mathbb{M}, w \Vdash \blacksquare \varphi iff for any v such that Rvw, \mathbb{M}, v \Vdash \varphi;

\mathbb{M}, w \Vdash \bullet \varphi iff there exists v such that Rvw and \mathbb{M}, v \Vdash \varphi;

\mathbb{M}, w \Vdash \exists \mathbf{i}\varphi iff for all v \in W, (W, R, V_v^{\mathbf{i}}), w \Vdash \varphi;

\mathbb{M}, w \Vdash \exists \mathbf{i}\varphi iff there exists v \in W such that (W, R, V_v^{\mathbf{i}}), w \Vdash \varphi;

\mathbb{M}, w \Vdash \mathbf{l}(\varphi, \psi) iff for all v \in W (if \mathbb{M}, v \Vdash \varphi, then \mathbb{M}, v \Vdash \psi).
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We can extend V to a map from the set of formulas to P(W) in the natural way.

2.2 Inequalities and complex inequalities

We will find it convenient to use the inequality notation $\varphi \leq \psi$ where φ and ψ are formulas. We use lneq to denote the set of all inequalities in the expanded modal language. We define *complex inequalities* as follows:

²Notice that by adding the universal modality A into the language, all of the additional connectives in the expanded modal language can be defined in the language with A. For example, $I(\varphi,\psi)$ can be rewritten as $A(\varphi \to \psi)$, and the backward-looking modality \bullet can be defined by $\bullet \varphi \leftrightarrow \exists p(p \land \forall q(q \to A(p \to q))) \land E(\varphi \land \Diamond p)$ where E is $\neg A \neg$. The expanded modal language is introduced for the convenience of the algorithm, as what is typically done in algorithmic correspondence theory.

Comp ::= Ineq | Comp & Comp | Comp
$$\Rightarrow$$
 Comp | $\forall p$ Comp | $\exists p$ Comp | $\forall i$ Comp | $\exists i$ Comp

Here we assume that the quantifiers have a higher precedence than &, and & is higher than \Rightarrow .

Complex inequalities are interpreted in models $\mathbb{M} = (W, R, V)$ instead of pointed models (\mathbb{M}, w) . The semantics of complex inequalities is defined as follows:

• An inequality is interpreted as follows:

$$(W, R, V) \Vdash \varphi \leq \psi$$
 iff

(for all $w \in W$, if (W, R, V), $w \vdash \varphi$, then (W, R, V), $w \vdash \psi$);

- $(W, R, V) \Vdash \mathsf{Comp}_1 \& \mathsf{Comp}_2 \text{ iff } (W, R, V) \Vdash \mathsf{Comp}_1 \text{ and } (W, R, V) \Vdash \mathsf{Comp}_2;$
- $(W, R, V) \Vdash \mathsf{Comp}_1 \Rightarrow \mathsf{Comp}_2 \text{ iff } ((W, R, V) \Vdash \mathsf{Comp}_1 \text{ implies } (W, R, V) \Vdash \mathsf{Comp}_2);$
- $(W, R, V) \Vdash \forall p \text{Comp iff for all } X \subseteq W, (W, R, V_X^p) \Vdash \text{Comp};$
- $(W, R, V) \Vdash \exists p \text{Comp}$ iff there exists an $X \subseteq W$ such that $(W, R, V_X^p) \Vdash \text{Comp}$;
- $(W, R, V) \Vdash \forall \mathbf{i} \mathsf{Comp} \text{ iff for all } v \in W, (W, R, V_v^{\mathbf{i}}) \Vdash \mathsf{Comp};$
- $(W, R, V) \Vdash \exists i Comp \text{ iff there exists an } v \in W \text{ such that } (W, R, V_v^i) \Vdash Comp.$

2.3 Standard translation

In the correspondence language which is second-order due to the existence of propositional quantifiers in SOPML, we have a binary predicate symbol R corresponding to the binary relation, a set of constant symbols i corresponding to each nominal i, a set of unary predicate symbols P corresponding to each propositional variable p.

Definition 2. The standard translation of the expanded SOPML language is defined as follows:

- $ST_x(p) := Px$;
- $ST_x(i) := x = i;$
- $ST_r(\bot) := \bot$;

- $ST_x(\top) := \top$;
- $ST_x(\neg \varphi) := \neg ST_x(\varphi);$
- $ST_x(\varphi \wedge \psi) := ST_x(\varphi) \wedge ST_x(\psi);$
- $ST_x(\varphi \vee \psi) := ST_x(\varphi) \vee ST_x(\psi);$
- $ST_x(\varphi \to \psi) := ST_x(\varphi) \to ST_x(\psi);$
- $ST_x(\Box \varphi) := \forall y(Rxy \to ST_y(\varphi));$
- $ST_x(\diamond \varphi) := \exists y (Rxy \wedge ST_y(\varphi));$
- $ST_{x}(\blacksquare \varphi) := \forall y(Ryx \to ST_{y}(\varphi));$
- $ST_x(\Phi\varphi) := \exists y(Ryx \land ST_y(\varphi));$
- $ST_x(\forall p\varphi) := \forall PST_x(\varphi);$
- $ST_x(\exists p\varphi) := \exists PST_x(\varphi);$
- $ST_x(\forall \mathbf{i}\varphi) := \forall i ST_x(\varphi);$
- $ST_x(\exists \mathbf{i}\varphi) := \exists i ST_x(\varphi);$
- $ST_{\nu}(\mathbf{I}(\varphi,\psi)) := \forall \nu(ST_{\nu}(\varphi) \to ST_{\nu}(\psi)).$

The following proposition states that this translation is correct:

Proposition 3. For any Kripke model \mathbb{M} , any $w \in W$ and any expanded SOPML formula φ ,

$$\mathbb{M}, w \Vdash \varphi \ \textit{iff} \ \mathbb{M} \models S \, T_x(\varphi)[x := w].$$

For inequalities and complex inequalities, the standard translation is given in a global way:

Definition 4. • $ST(\varphi \le \psi) := \forall x (ST_x(\varphi) \to ST_x(\psi));$

- $ST(Comp_1 \& Comp_2) = ST(Comp_1) \land ST(Comp_2);$
- $ST(Comp_1 \Rightarrow Comp_2) = ST(Comp_1) \rightarrow ST(Comp_2);$
- $ST(\forall p(\mathsf{Comp})) := \forall P(ST(\mathsf{Comp}));$
- $ST(\exists p(\mathsf{Comp})) := \exists P(ST(\mathsf{Comp}));$

- $ST(\forall i(Comp)) := \forall i(ST(Comp));$
- $ST(\exists i(Comp)) := \exists i(ST(Comp)).$

Proposition 5. For any Kripke model \mathbb{M} , any inequality lneq, any complex inequality Comp,

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\mathbb{M} \Vdash \text{Ineq } iff \mathbb{M} \models ST(\text{Ineq});
\mathbb{M} \Vdash \text{Comp } iff \mathbb{M} \models ST(\text{Comp}).
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3 Sahlqvist formulas in second-order propositional modal logic

In this section, we define Sahlqvist formulas of second-order propositional modal logic step by step.

We first define (quantifier-free) positive formulas $POS(\vec{p})$ whose propositional variables are among \vec{p} :

$$\mathsf{POS}(\vec{p}) ::= p \mid \bot \mid \top \mid \mathsf{POS}(\vec{p}) \land \mathsf{POS}(\vec{p}) \mid \mathsf{POS}(\vec{p}) \lor \mathsf{POS}(\vec{p}) \mid \Box \mathsf{POS}(\vec{p}) \mid \Diamond \mathsf{POS}(\vec{p})$$

where p is in \vec{p} . These positive formulas have similar roles to the positive consequent part in Sahlqvist formulas in basic modal logic, which are going to receive minimal valuations. The reason why we do not allow propositional quantifiers in positive formulas is that we want the formula after receiving the minimal valuations to be translated into a first-order formula, while propositional quantifiers will make it second-order.

3.1 The Π_1 -fragment: Sahlqvist formulas in basic modal logic

We define the Π_1 -Sahlqvist antecedent $Sahl_1(\vec{p})$ whose propositional variables are among \vec{p} :

$$Sahl_1(\vec{p}) ::= \Box^n p \mid \bot \mid \top \mid \neg POS(\vec{p}) \mid Sahl_1(\vec{p}) \land Sahl_1(\vec{p}) \mid \diamondsuit Sahl_1(\vec{p})$$

where p is in \vec{p} .

Then the Π_1 -Sahlqvist formulas are defined as $\forall \vec{p}(\mathsf{Sahl}_1(\vec{p}) \to \mathsf{POS}(\vec{p}))$. Indeed, Sahlqvist formulas³ in the basic modal logic setting can be treated as universally quantified by propositional quantifiers which bind all occurrences of propositional variables, so in this sense the Π_1 -Sahlqvist formulas can be taken as the Sahlqvist formulas in basic modal logic.

³In [9, Chapter 3], what we call Sahlqvist formulas are called Sahlqvist implications.

3.2 The Π_2 -fragment

We define the PIA formula PIA(\vec{q} , \vec{p}) as follows:

$$\mathsf{PIA}(\vec{q}, \vec{p}) ::= p \mid \Box \mathsf{PIA}(\vec{q}, \vec{p}) \mid \mathsf{PIA}(\vec{q}, \vec{p}) \land \mathsf{PIA}(\vec{q}, \vec{p}) \mid \mathsf{POS}(\vec{q}) \lor \mathsf{PIA}(\vec{q}, \vec{p})$$

where p is in \vec{p} . Here the PIA formula has two bunches of propositional variables: \vec{q} is to receive minimal valuations for \vec{q} from somewhere else, and \vec{p} is used to compute minimal valuations for \vec{p} . Then it is easy to see that PIA(\vec{q} , \vec{p}) is equivalent to the form $\bigwedge \Box (POS(\vec{q}) \lor \Box (POS(\vec{q}) \lor \ldots p))$, where p is in \vec{p} . Now we can define Π_2 -Sahlqvist antecedents as follows:

$$Sahl_2(\vec{p}) ::= Sahl_1(\vec{p}) \mid \forall \vec{q}(Sahl_1(\vec{q}) \rightarrow PIA(\vec{q}, \vec{p})) \mid Sahl_2(\vec{p}) \land Sahl_2(\vec{p}) \mid \diamondsuit Sahl_2(\vec{p})$$

Then Π_2 -Sahlqvist formulas are defined as $\forall \vec{p}(\mathsf{Sahl}_2(\vec{p}) \to \mathsf{POS}(\vec{p}))$. It is easy to see that formulas of the form $\forall \vec{p}(\mathsf{Sahl}_1(\vec{p}) \land \forall \vec{q}(\mathsf{Sahl}_1(\vec{q}) \to \mathsf{PIA}(\vec{q}, \vec{p})) \to \mathsf{POS}(\vec{p})$ are in the Π_2 -hierarchy.

3.3 The Π_n -fragment

Now for the Π_n -fragment, assume that we have already defined Π_{n-1} -Sahlqvist antecedents $\mathsf{Sahl}_{n-1}(\vec{p})$ and Π_{n-1} -Sahlqvist formulas $\forall \vec{p}(\mathsf{Sahl}_{n-1}(\vec{p}) \to \mathsf{POS}(\vec{p}))$, then we can define Π_n -Sahlqvist antecedents as follows:

$$Sahl_n(\vec{p}) ::= Sahl_{n-1}(\vec{p}) \mid \forall \vec{q}(Sahl_{n-1}(\vec{q}) \rightarrow PIA(\vec{q}, \vec{p})) \mid Sahl_n(\vec{p}) \land Sahl_n(\vec{p}) \mid \diamondsuit Sahl_n(\vec{p$$

Then Π_n -Sahlqvist formulas are defined as $\forall \vec{p}(Sahl_n(\vec{p}) \to POS(\vec{p}))$.

4 The Algorithm ALBA SOMPL

In the present section, we define the correspondence algorithm ALBA^{SOMPL} for second-order propositional modal logic, in the style of [13, 14]. The algorithm receives a Π_n -Sahlqvist formula $\forall \vec{p}(\mathsf{Sahl}_n(\vec{p}) \to \mathsf{POS}(\vec{p}))$ as input and goes in three stages.

1. Preprocessing and first approximation:

The algorithm receives a Π_n -Sahlqvist formula $\forall \vec{p}(\mathsf{Sahl}_n(\vec{p}) \to \mathsf{POS}(\vec{p}))$ as input, and then apply the rewriting rule:

$$\frac{\forall \vec{p}(\mathsf{Sahl}_n(\vec{p}) \to \mathsf{POS}(\vec{p}))}{\forall \vec{p}(\mathsf{Sahl}_n(\vec{p}) \le \mathsf{POS}(\vec{p}))}$$

Then apply the first-approximation rule:

$$\frac{\forall \vec{p}(\mathsf{Sahl}_n(\vec{p}) \leq \mathsf{POS}(\vec{p}))}{\forall \vec{p} \forall \mathbf{i}_0(\mathbf{i}_0 \leq \mathsf{Sahl}_n(\vec{p}) \ \Rightarrow \ \mathbf{i}_0 \leq \mathsf{POS}(\vec{p}))}$$

2. The reduction stage:

In this stage, we aim at reducing $\mathbf{i} \leq \mathsf{Sahl}_n(\vec{p})$ to a complex inequality in which p occurs either in the form $\varphi \leq p$ where φ is pure or in the form $\mathbf{j} \leq \neg \mathsf{POS}(\vec{p})$.

- (a) The commutativity rule and the associativity rule for &;
- (b) The rules for nominals:
 - i. Splitting rule:

$$\frac{\mathbf{i} \le \alpha \land \beta}{\mathbf{i} \le \alpha \& \mathbf{i} \le \beta} (S pl - Nom)$$

ii. Separation rule:

$$\frac{\mathbf{i} \le \alpha \to \beta}{\mathbf{i} \le \alpha \implies \mathbf{i} \le \beta} \ (Sep - Nom)$$

iii. Quantifier rule:

$$\frac{\mathbf{i} \le \forall q\alpha}{\forall q(\mathbf{i} \le \alpha)} (Quant - Nom)$$

iv. Approximation rule:

$$\frac{\mathbf{i} \leq \Diamond \alpha}{\exists \mathbf{j}(\mathbf{j} \leq \alpha \& \mathbf{i} \leq \Diamond \mathbf{j})} (Approx - Nom)$$

The nominals introduced by the approximation rule must not occur in the whole complex inequality before applying the rule.

(c) The residuation rules:

$$\frac{\alpha \leq \Box \beta}{\blacklozenge \alpha \leq \beta} (Res - \Box) \qquad \frac{\alpha \leq \beta \vee \gamma}{\alpha \wedge \neg \beta \leq \gamma} (Res - \vee)$$

(d) The splitting rule:

$$\frac{\alpha \le \beta \land \gamma}{\alpha \le \beta \& \alpha \le \gamma} \ (S \ plitting)$$

(e) The quantifier rules:

$$\frac{\exists \mathbf{j}(\mathsf{Comp}_1) \& \mathsf{Comp}_2}{\exists \mathbf{j}(\mathsf{Comp}_1 \& \mathsf{Comp}_2)} (Scope - \&) \qquad \frac{\exists \mathbf{j}(\mathsf{Comp}_1) \Rightarrow \mathsf{Comp}_2}{\forall \mathbf{j}(\mathsf{Comp}_1 \Rightarrow \mathsf{Comp}_2)} (Scope - \Rightarrow)$$

where $Comp_2$ does not have free occurrences of j.

$$\frac{\forall q \forall p (\mathsf{Comp})}{\forall p \forall q (\mathsf{Comp})} (Ex - pq) \qquad \frac{\forall \mathbf{i} \forall p (\mathsf{Comp})}{\forall p \forall \mathbf{i} (\mathsf{Comp})} (Ex - p\mathbf{i})$$

$$\frac{\forall p \forall \mathbf{i} (\mathsf{Comp})}{\forall \mathbf{i} \forall p (\mathsf{Comp})} (Ex - \mathbf{i}p) \qquad \frac{\forall \mathbf{i} \forall \mathbf{j} (\mathsf{Comp})}{\forall \mathbf{j} \forall \mathbf{i} (\mathsf{Comp})} (Ex - \mathbf{j}\mathbf{i})$$

$$\frac{\forall p(\mathsf{Comp}_1 \Rightarrow (\mathsf{Comp}_2 \& \mathsf{Comp}_3))}{\forall p(\mathsf{Comp}_1 \Rightarrow \mathsf{Comp}_2) \ \& \ \forall p(\mathsf{Comp}_1 \Rightarrow \mathsf{Comp}_3)} \ (\mathit{Spl-Quant-p})$$

$$\frac{\forall \mathbf{i}(\mathsf{Comp}_1 \Rightarrow (\mathsf{Comp}_2 \& \mathsf{Comp}_3))}{\forall \mathbf{i}(\mathsf{Comp}_1 \Rightarrow \mathsf{Comp}_2) \ \& \ \forall \mathbf{i}(\mathsf{Comp}_1 \Rightarrow \mathsf{Comp}_3)} \ (\mathit{Spl} - \mathit{Quant} - \mathbf{i})$$

(f) The Ackermann rule:

In this step, we compute the minimal valuation for propositional variables and use the Ackermann rule to eliminate all the propositional variables.

$$\frac{\forall q(\alpha_1 \leq \beta_1 \& \dots \& \alpha_n \leq \beta_n \& \psi_1 \leq q \& \dots \& \psi_m \leq q \Rightarrow \alpha \leq \beta)}{\alpha_1[\bigvee \psi/q] \leq \beta_1[\bigvee \psi/q] \& \dots \& \alpha_n[\bigvee \psi/q] \leq \beta_n[\bigvee \psi/q] \Rightarrow \alpha[\bigvee \psi/q] \leq \beta[\bigvee \psi/q]}$$

where:

- i. $\varphi[\theta/p]$ means uniformly replace occurrences of p in φ by θ ;
- ii. $\bigvee \psi = \psi_1 \vee \ldots \vee \psi_m$;
- iii. Each α_i is positive, and each β_i negative in q, for $1 \le i \le n$;
- iv. α is negative in q and β is positive in q;
- v. Each ψ_i is pure (therefore q does not occur in ψ_i).
- (g) The packing rule:

$$\frac{\forall \mathbf{i}(\alpha_1 \leq \beta_1 \& \dots \& \alpha_n \leq \beta_n \Rightarrow \alpha \leq \beta)}{\exists \mathbf{i}(\mathbf{l}(\alpha_1, \beta_1) \land \dots \land \mathbf{l}(\alpha_n, \beta_n) \land \alpha) \leq \beta}$$

where β does not contain occurrences of **i**.

3. **Output**: By the execution of the algorithm, we can guarantee (see Theorem 6.1) that given a Π_n -Sahlqvist formula as input, we can rewrite it into a pure complex inequality. Then we use standard translation to translate it into a first-order formula.

From the design of the algorithm, we can see that it is specifically designed for Π_n -Sahlqvist formulas. Therefore, when we try to extend the Π_n -Sahlqvist fragment, we need to revise the rules accordingly.

5 Soundness of ALBA SOPML

In the present section, we will prove the soundness of the algorithm ALBA^{SOPML}. The basic proof structure is similar to [14].

Theorem 5.1 (Soundness). If ALBA^{SOPML} runs successfully on an input Π_n -Sahlqvist formula $\forall \vec{p}(Sahl_n(\vec{p}) \to POS(\vec{p}))$ and outputs a first-order formula $FO(\forall \vec{p}(Sahl_n(\vec{p}) \to POS(\vec{p})))$, then for any Kripke frame $\mathbb{F} = (W, R)$,

$$\mathbb{F} \Vdash \forall \vec{p}(\mathsf{Sahl}_n(\vec{p}) \to \mathsf{POS}(\vec{p})) \ \textit{iff} \ \mathbb{F} \models \mathsf{FO}(\forall \vec{p}(\mathsf{Sahl}_n(\vec{p}) \to \mathsf{POS}(\vec{p}))).$$

Proof. The proof goes similarly to [14, Theorem 8.1]. Let $\forall \vec{p}(Sahl_n(\vec{p}) \leq POS(\vec{p}))$ denote the complex inequality after the first rewrite rule, $\forall \vec{p} \forall \mathbf{i}_0(\mathbf{i}_0 \leq Sahl_n(\vec{p}) \Rightarrow \mathbf{i}_0 \leq POS(\vec{p}))$ denote the complex inequality after the first approximation rule, $Comp(\forall \vec{p} \forall \mathbf{i}_0(\mathbf{i}_0 \leq Sahl_n(\vec{p}) \Rightarrow \mathbf{i}_0 \leq POS(\vec{p})))$ denote the complex inequality after Stage 2, and $FO(\forall \vec{p}(Sahl_n(\vec{p}) \rightarrow POS(\vec{p})))$ denote the standard translation of the complex inequality obtained after Stage 2, then it suffices to show the equivalence from (1) to (5) given below:

- (1) $\mathbb{F} \Vdash \forall \vec{p}(\mathsf{Sahl}_n(\vec{p}) \to \mathsf{POS}(\vec{p}))$
- (2) $\mathbb{F} \Vdash \forall \vec{p}(\mathsf{Sahl}_n(\vec{p}) \leq \mathsf{POS}(\vec{p}))$
- (3) $\mathbb{F} \Vdash \forall \vec{p} \forall \mathbf{i}_0 (\mathbf{i}_0 \le \mathsf{Sahl}_n(\vec{p}) \implies \mathbf{i}_0 \le \mathsf{POS}(\vec{p}))$
- (4) $\mathbb{F} \Vdash \mathsf{Comp}(\forall \vec{p} \forall \mathbf{i}_0 (\mathbf{i}_0 \leq \mathsf{Sahl}_n(\vec{p}) \implies \mathbf{i}_0 \leq \mathsf{POS}(\vec{p})))$
- (5) $\mathbb{F} \models \mathsf{FO}(\forall \vec{p}(\mathsf{Sahl}_n(\vec{p}) \to \mathsf{POS}(\vec{p})))$
 - the equivalence between (1) and (2) follows from Proposition 6;
 - the equivalence between (2) and (3) follows from Proposition 7;
 - the equivalence between (3) and (4) follows from Proposition 8;
 - the equivalence between (4) and (5) follows from Proposition 5.

In the remainder of this section, we prove the soundness of the rules in Stage 1 and 2.

Proposition 6 (Soundness of the first rewrite rule in Stage 1). *The first rewrite rule is sound in both directions in* \mathbb{F} , *i.e. the formula before the rule is valid in* \mathbb{F} *iff the complex inequality after the rule is valid in* \mathbb{F} .

Proof.

$$\begin{split} \mathbb{F} &\Vdash \forall \vec{p}(\mathsf{Sahl}_n(\vec{p}) \to \mathsf{POS}(\vec{p})) \\ \text{iff} \quad \text{for all } V, (\mathbb{F}, V) \Vdash \forall \vec{p}(\mathsf{Sahl}_n(\vec{p}) \to \mathsf{POS}(\vec{p})) \\ \text{iff} \quad \text{for all } V, \text{for all } \vec{X} \subseteq W, (\mathbb{F}, V_{\vec{X}}^{\vec{p}}) \Vdash \mathsf{Sahl}_n(\vec{p}) \to \mathsf{POS}(\vec{p}) \\ \text{iff} \quad \text{for all } V, \text{ for all } \vec{X} \subseteq W, (\mathbb{F}, V_{\vec{X}}^{\vec{p}}) \Vdash \mathsf{Sahl}_n(\vec{p}) \leq \mathsf{POS}(\vec{p}) \\ \text{iff} \quad \text{for all } V, (\mathbb{F}, V) \Vdash \forall \vec{p}(\mathsf{Sahl}_n(\vec{p}) \leq \mathsf{POS}(\vec{p})) \\ \text{iff} \quad \mathbb{F} \Vdash \forall \vec{p}(\mathsf{Sahl}_n(\vec{p}) \leq \mathsf{POS}(\vec{p})). \end{split}$$

Proposition 7 (Soundness of the first approximation rule in Stage 1). *The first approximation rule is sound in both directions in* \mathbb{F} , *i.e. the complex inequality before the rule is valid in* \mathbb{F} *iff the complex inequality after the rule is valid in* \mathbb{F} .

```
\begin{array}{l} \textit{Proof.} \ \ \mathbb{F} \Vdash \forall \vec{p}(\mathsf{Sahl}_n(\vec{p}) \leq \mathsf{POS}(\vec{p})) \\ \text{iff for all } V, (\mathbb{F}, V) \Vdash \forall \vec{p}(\mathsf{Sahl}_n(\vec{p}) \leq \mathsf{POS}(\vec{p})) \\ \text{iff for all } V, \text{ for all } \vec{X} \subseteq W, (\mathbb{F}, V_{\vec{X}}^{\vec{p}}) \Vdash \mathsf{Sahl}_n(\vec{p}) \leq \mathsf{POS}(\vec{p}) \\ \text{iff for all } V, \text{ all } \vec{X} \subseteq W, \text{ all } w \in W, (\mathbb{F}, V_{\vec{X}}^{\vec{p}}), w \Vdash \mathsf{Sahl}_n(\vec{p}) \text{ implies } (\mathbb{F}, V_{\vec{X}}^{\vec{p}}), w \Vdash \mathsf{POS}(\vec{p}) \\ \text{iff for all } V, \text{ all } \vec{X} \subseteq W, \text{ all } w \in W, (\mathbb{F}, V_{\vec{X}, w}^{\vec{p}, \mathbf{i}_0}) \Vdash \mathbf{i}_0 \leq \mathsf{Sahl}_n(\vec{p}) \text{ implies } (\mathbb{F}, V_{\vec{X}, w}^{\vec{p}, \mathbf{i}_0}) \Vdash \mathbf{i}_0 \leq \mathsf{POS}(\vec{p}) \\ \text{iff for all } V, (\mathbb{F}, V) \Vdash \forall \vec{p} \forall \mathbf{i}_0 (\mathbf{i}_0 \leq \mathsf{Sahl}_n(\vec{p}) \Rightarrow \mathbf{i}_0 \leq \mathsf{POS}(\vec{p})) \\ \text{iff } \mathbb{F} \Vdash \forall \vec{p} \forall \mathbf{i}_0 (\mathbf{i}_0 \leq \mathsf{Sahl}_n(\vec{p}) \Rightarrow \mathbf{i}_0 \leq \mathsf{POS}(\vec{p})). \\ \end{array}
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Proposition 8 (Soundness of the rules in Stage 2). The rules in Stage 2 are sound in both directions in \mathbb{F} , i.e. the complex inequality before the rule is valid in \mathbb{F} iff the complex inequality after the rule is valid in \mathbb{F} .

Proof. It suffices to show that each rule in Stage 2 is sound in both directions in \mathbb{F} .

• For the commutativity rule and associativity rule for &, by the validity of $\alpha \land \beta \leftrightarrow \beta \land \alpha$ and $(\alpha \land \beta) \land \gamma \leftrightarrow \alpha \land (\beta \land \gamma)$.

- For the splitting rule for nominals and the splitting rule for arbitrary formulas, it follows from the following equivalence: for all Kripke frame \mathbb{F} and all $V, \mathbb{F}, V \Vdash \alpha \leq \beta \wedge \gamma$ iff $(\mathbb{F}, V \Vdash \alpha \leq \beta)$ and $\mathbb{F}, V \Vdash \alpha \leq \gamma$.
- For the separation rule for nominals, it follows from the following equivalence: for all $\mathbb{F} = (W, R)$ and all V,

```
\mathbb{F}, V \Vdash \mathbf{i} \leq \alpha \to \beta
iff \mathbb{F}, V, V(\mathbf{i}) \Vdash \alpha \to \beta
iff \mathbb{F}, V, V(\mathbf{i}) \Vdash \alpha implies \mathbb{F}, V, V(\mathbf{i}) \Vdash \beta
iff \mathbb{F}, V \Vdash \mathbf{i} \leq \alpha implies \mathbb{F}, V \Vdash \mathbf{i} \leq \beta
iff \mathbb{F}, V \Vdash \mathbf{i} \leq \alpha \Rightarrow \mathbf{i} \leq \beta.
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• For the quantifier rule for nominals, it follows from the following equivalence: for all $\mathbb{F} = (W, R)$ and any V,

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\begin{split} \mathbb{F}, V &\Vdash \mathbf{i} \leq \forall q\alpha \\ \text{iff } \mathbb{F}, V, V(\mathbf{i}) &\Vdash \forall q\alpha \\ \text{iff for all } X \subseteq W, \mathbb{F}, V_X^q, V(\mathbf{i}) &\Vdash \alpha \\ \text{iff for all } X \subseteq W, \mathbb{F}, V_X^q, V_X^q(\mathbf{i}) &\Vdash \alpha \\ \text{iff for all } X \subseteq W, \mathbb{F}, V_X^q &\Vdash \mathbf{i} \leq \alpha \\ \text{iff } \mathbb{F}, V &\Vdash \forall q(\mathbf{i} \leq \alpha). \end{split}
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- For the approximation rule for nominals, it suffices to show that for any $\mathbb{F} = (W, R)$ and any V,
 - 1. if $(\mathbb{F}, V) \Vdash \mathbf{i} \leq \Diamond \alpha$, then there is a valuation $V^{\mathbf{j}}$ such that $V^{\mathbf{j}}$ is the same as V except $V^{\mathbf{j}}(\mathbf{j})$, and $(\mathbb{F}, V^{\mathbf{j}}) \Vdash \mathbf{i} \leq \Diamond \mathbf{j}$ and $(\mathbb{F}, V^{\mathbf{j}}) \Vdash \mathbf{j} \leq \alpha$;
 - 2. if $(\mathbb{F}, V) \Vdash \mathbf{i} \leq \Diamond \mathbf{j}$ and $(\mathbb{F}, V) \Vdash \mathbf{j} \leq \alpha$, then $(\mathbb{F}, V) \Vdash \mathbf{i} \leq \Diamond \alpha$.

For item 1, if $(\mathbb{F}, V) \Vdash \mathbf{i} \leq \Diamond \alpha$, then $(\mathbb{F}, V), V(\mathbf{i}) \Vdash \Diamond \alpha$, therefore there exists a $w \in W$ such that $(V(\mathbf{i}), w) \in R$ and $(\mathbb{F}, V), w \Vdash \alpha$. Now take $V^{\mathbf{j}}$ such that $V^{\mathbf{j}}$ is the same as V except that $V^{\mathbf{j}}(\mathbf{j}) = \{w\}$, then $(V^{\mathbf{j}}(\mathbf{i}), V^{\mathbf{j}}(\mathbf{j})) \in R$, so $(\mathbb{F}, V^{\mathbf{j}}) \Vdash \mathbf{i} \leq \Diamond \mathbf{j}$ and $(\mathbb{F}, V^{\mathbf{j}}) \Vdash \mathbf{j} \leq \alpha$.

For item 2, suppose $(\mathbb{F}, V) \Vdash \mathbf{i} \leq \Diamond \mathbf{j}$ and $(\mathbb{F}, V) \Vdash \mathbf{j} \leq \alpha$. Then $(V(\mathbf{i}), V(\mathbf{j})) \in R$ and $(\mathbb{F}, V), V(\mathbf{j}) \Vdash \alpha$, so $(\mathbb{F}, V), V(\mathbf{i}) \Vdash \Diamond \alpha$, therefore $(\mathbb{F}, V) \Vdash \mathbf{i} \leq \Diamond \alpha$.

• For the residuation rule for \square , it suffices to show that for any $\mathbb{F} = (W, R)$ and any $V, (\mathbb{F}, V) \Vdash \spadesuit \alpha \leq \beta$ iff $(\mathbb{F}, V) \Vdash \alpha \leq \square \beta$.

 \Rightarrow : if $(\mathbb{F}, V) \Vdash \blacklozenge \alpha \leq \beta$, then for all $w \in W$, if (\mathbb{F}, V) , $w \Vdash \blacklozenge \alpha$, then (\mathbb{F}, V) , $w \Vdash \beta$. Our aim is to show that for all $v \in W$, if (\mathbb{F}, V) , $v \Vdash \alpha$, then (\mathbb{F}, V) , $v \Vdash \Box \beta$.

Consider any $v \in W$ such that $(\mathbb{F}, V), v \Vdash \alpha$. Then for any $u \in W$ such that $(v, u) \in R$, $(\mathbb{F}, V), u \Vdash \spadesuit \alpha$. Since $(\mathbb{F}, V) \Vdash \spadesuit \alpha \leq \beta$, we have that $(\mathbb{F}, V), u \Vdash \beta$, so for any $u \in W$ such that $(v, u) \in R$, $(\mathbb{F}, V), u \Vdash \beta$, so $(\mathbb{F}, V), v \Vdash \Box \beta$.

- For the residuation rule for \vee , it follows from the validity of $(\alpha \to (\beta \vee \gamma)) \leftrightarrow ((\alpha \wedge \neg \beta) \to \gamma)$.
- For the quantifier scope rules, it follows from the validity of $\exists x\alpha \land \beta \leftrightarrow \exists x(\alpha \land \beta)$ and $(\exists x\alpha \to \beta) \leftrightarrow \forall x(\alpha \to \beta)$ (where x does not occur in β).
- For the quantifier exchange rules, it follows from the validity of $\forall P \forall Q\alpha \leftrightarrow \forall Q \forall P\alpha, \forall P \forall x\alpha \leftrightarrow \forall x \forall P\alpha$ and $\forall x \forall y\alpha \leftrightarrow \forall y \forall x\alpha$.
- For the quantifier splitting rules, it follows from the validity of $\forall P(\alpha \rightarrow \beta \land \gamma) \leftrightarrow \forall P(\alpha \rightarrow \beta) \land \forall P(\alpha \rightarrow \gamma)$ and $\forall x(\alpha \rightarrow \beta \land \gamma) \leftrightarrow \forall x(\alpha \rightarrow \beta) \land \forall x(\alpha \rightarrow \gamma)$.
- For the Ackermann rule, it suffices to show that for any $\mathbb{F} = (W, R)$ and any V.

 $\mathbb{F}, V \Vdash \forall q (\alpha_1 \leq \beta_1 \& \dots \& \alpha_n \leq \beta_n \& \psi_1 \leq q \& \dots \& \psi_m \leq q \Rightarrow \alpha \leq \beta)$ iff $\mathbb{F}, V \Vdash \alpha_1[\bigvee \psi/q] \leq \beta_1[\bigvee \psi/q] \& \dots \& \alpha_n[\bigvee \psi/q] \leq \beta_n[\bigvee \psi/q] \Rightarrow \alpha[\bigvee \psi/q] \leq \beta[\bigvee \psi/q].$

 \Rightarrow : Easy, by instantiation of the propositional quantifier.

 \Leftarrow : Assmue \mathbb{F} , $V \Vdash \alpha_1[\bigvee \psi/q] \leq \beta_1[\bigvee \psi/q] \& \dots \& \alpha_n[\bigvee \psi/q] \leq \beta_n[\bigvee \psi/q] \Rightarrow \alpha[\bigvee \psi/q] \leq \beta[\bigvee \psi/q]$. Then for any $X \subseteq W$, it suffices to show that if \mathbb{F} , $V_X^q \Vdash \alpha_1 \leq \beta_1 \& \dots \& \alpha_n \leq \beta_n \& \psi_1 \leq q \& \dots \& \psi_m \leq q$, then \mathbb{F} , $V_X^q \Vdash \alpha \leq \beta$. Now assume \mathbb{F} , $V_X^q \Vdash \alpha_1 \leq \beta_1 \& \dots \& \alpha_n \leq \beta_n \& \psi_1 \leq q \& \dots \& \psi_m \leq q$, then $V_X^q(\alpha_i) \subseteq V_X^q(\beta_i)$ for $1 \leq i \leq n$, and $V_X^q(\psi_j) \subseteq X$ for $1 \leq j \leq m$, therefore $V(\bigvee \psi) = V_X^q(\bigvee \psi) \subseteq X$. Since each α_i is positive and each β_i is negative in q, we have that $V(\alpha_i[\bigvee \psi/q]) \subseteq V_X^q(\alpha_i) \subseteq V_X^q(\beta_i) \subseteq V(\beta_i[\bigvee \psi/q])$, $1 \leq i \leq n$, so by \mathbb{F} , $V \Vdash \alpha_1[\bigvee \psi/q] \leq \beta_1[\bigvee \psi/q] \& \dots \& \alpha_n[\bigvee \psi/q] \leq \beta_n[\bigvee \psi/q] \Rightarrow \alpha[\bigvee \psi/q] \leq \beta[\bigvee \psi/q]$ we have $V(\alpha[\bigvee \psi/q]) \subseteq V(\beta[\bigvee \psi/q])$, therefore by α is negative and β is positive in q, we have $V_X^q(\alpha) \subseteq V(\alpha[\bigvee \psi/q]) \subseteq V(\beta[\bigvee \psi/q])$

• For the packing rule, it follows from the following equivalence: for any $\mathbb{F} = (W, R)$ and any V,

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\mathbb{F}, V \Vdash \forall \mathbf{i}(\alpha_1 \leq \beta_1 \& \dots \& \alpha_n \leq \beta_n \Rightarrow \alpha \leq \beta)
 iff for all w \in W, (\mathbb{F}, V_w^i) \Vdash \alpha_1 \leq \beta_1 \& \dots \& \alpha_n \leq \beta_n \Rightarrow \alpha \leq \beta
iff for all w \in W, if (\mathbb{F}, V_w^{\mathbf{i}}) \Vdash \alpha_i \leq \beta_i for 1 \leq i \leq n, then (\mathbb{F}, V_w^{\mathbf{i}}) \Vdash \alpha \leq \beta
iff for all w \in W, if (\mathbb{F}, V_w^i) \Vdash \mathbf{l}(\alpha_i, \beta_i) for 1 \le i \le n, then (\mathbb{F}, V_w^i) \Vdash \alpha \le \beta
 iff for all w \in W, if (\mathbb{F}, V_w^i) \Vdash \mathbf{l}(\alpha_i, \beta_i) for 1 \le i \le n, then for all v \in W, if
(\mathbb{F}, V_w^{\mathbf{i}}), v \Vdash \alpha \text{ then } (\mathbb{F}, V_w^{\mathbf{i}}), v \Vdash \beta
iff for all w, v \in W, if (\mathbb{F}, V_w^{\mathbf{i}}) \Vdash \mathbf{l}(\alpha_i, \beta_i) for 1 \leq i \leq n, then if (\mathbb{F}, V_w^{\mathbf{i}}), v \Vdash \alpha
 then (\mathbb{F}, V_w^{\mathbf{i}}), v \Vdash \beta
iff for all w, v \in W, if (\mathbb{F}, V_w^{\mathbf{i}}) \Vdash \mathbf{l}(\alpha_i, \beta_i) for 1 \leq i \leq n and (\mathbb{F}, V_w^{\mathbf{i}}), v \Vdash \alpha, then
 (\mathbb{F}, V_w^{\mathbf{i}}), v \Vdash \beta
 iff for all w, v \in W, if (\mathbb{F}, V_w^i), v \Vdash \mathbf{l}(\alpha_i, \beta_i) for 1 \le i \le n and (\mathbb{F}, V_w^i), v \Vdash \alpha,
 then (\mathbb{F}, V_w^{\mathbf{i}}), v \Vdash \beta
iff for all w, v \in W, if (\mathbb{F}, V_w^{\mathbf{i}}), v \Vdash \mathbf{l}(\alpha_1, \beta_1) \land \ldots \land \mathbf{l}(\alpha_n, \beta_n) \land \alpha, then (\mathbb{F}, V_w^{\mathbf{i}}), v \Vdash \mathbf{l}(\alpha_1, \beta_1) \land \ldots \land \mathbf{l}(\alpha_n, \beta_n) \land \alpha
iff for all w, v \in W, if (\mathbb{F}, V_w^i), v \Vdash \mathbf{l}(\alpha_1, \beta_1) \land \ldots \land \mathbf{l}(\alpha_n, \beta_n) \land \alpha, then (\mathbb{F}, V), v \Vdash \mathbf{l}(\alpha_1, \beta_1) \land \ldots \land \mathbf{l}(\alpha_n, \beta_n) \land \alpha
iff for all v \in W, if there exists a w \in W such that (\mathbb{F}, V_{w}^{\mathbf{i}}), v \Vdash \mathbf{l}(\alpha_{1}, \beta_{1}) \land \ldots \land
\mathbf{l}(\alpha_n, \beta_n) \wedge \alpha, then (\mathbb{F}, V), v \Vdash \beta
iff for all v \in W, if (\mathbb{F}, V), v \Vdash \exists \mathbf{i}(\mathbf{l}(\alpha_1, \beta_1) \land ... \land \mathbf{l}(\alpha_n, \beta_n) \land \alpha), then (\mathbb{F}, V), v \Vdash
iff \mathbb{F}, V \Vdash \exists \mathbf{i}(\mathbf{l}(\alpha_1, \beta_1) \land \ldots \land \mathbf{l}(\alpha_n, \beta_n) \land \alpha) \leq \beta.
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6 Success of ALBA SOPML on Π_n -Sahlqvist formulas

By success of ALBA^{SOPML} on Π_n -Sahlqvist formulas we mean that the algorithm ALBA^{SOPML} can transform any input Π_n -Sahlqvist formula into a pure complex inequality which does not contain any propositional variables or any propositional quantifiers (here we allow nominal quantifiers to occur). We prove this by induction on n that ALBA^{SOPML} successfully transforms $\mathbf{i} \leq \text{Sahl}_n(\vec{p})$ into given shapes.

Proposition 9. *In the reduction stage, by running the algorithm* $ALBA^{SOPML}$, $\mathbf{i} \leq Sahl_1(\vec{p})$ *can be reduced to the following complex inequality:*

 $\exists \vec{j}$ (NEG & NOM & MinVal)

where

- $\exists \vec{j}$ is a (possibly empty) bunch of nominal quantifiers;
- NEG is a (possibly empty) meta-conjunction of inequalities of the form $\mathbf{j} \leq \neg \mathsf{POS}(\vec{p})$, where \mathbf{j} is either \mathbf{i} or in $\vec{\mathbf{j}}$,
- NOM is a (possibly empty) meta-conjunction of inequalities of the form $\mathbf{j} \leq \Diamond \mathbf{k}$, where \mathbf{j} , \mathbf{k} are either \mathbf{i} or in \mathbf{j} ,
- MinVal is a (possibly empty) meta-conjunction of inequalities of the form $\psi \leq p$, where ψ is pure and p is in \vec{p} .

Proof. We prove by induction on the formula complexity of $Sahl_1(\vec{p})$.

- For the case where $Sahl_1(\vec{p}) = \bot, \top$, trivial.
- For the case where $Sahl_1(\vec{p}) = \Box^n p$, by applying the residuation rule for \Box , we get $\blacklozenge^n \mathbf{i} \le p$, which belongs to MinVal.
- For the case where $Sahl_1(\vec{p}) = \neg POS(\vec{p})$, it already belongs to NEG.
- For the case where $Sahl_1(\vec{p}) = Sahl_1^a(\vec{p}) \wedge Sahl_1^b(\vec{p})$, we first apply (SplNom) to $\mathbf{i} \leq Sahl_1(\vec{p})$ and get $\mathbf{i} \leq Sahl_1^a(\vec{p})$ and $\mathbf{i} \leq Sahl_1^b(\vec{p})$. Then we apply the induction hypothesis and get

$$\exists \vec{j}^a (NEG^a \& NOM^a \& MinVal^a) \& \exists \vec{j}^b (NEG^b \& NOM^b \& MinVal^b).$$

By applying the (Scope-&) rule and commutativity and associativity rules for &, we get the desired shape.

• For the case where $Sahl_1(\vec{p}) = \diamondsuit Sahl_1^a(\vec{p})$, we first apply (Approx-Nom) for \diamondsuit and get $\exists \mathbf{k} (\mathbf{k} \leq Sahl_1^a(\vec{p}) \& \mathbf{i} \leq \diamondsuit \mathbf{k})$. Then we apply the induction hypothesis to $\mathbf{k} \leq Sahl_1^a(\vec{p})$ and get

$$\exists \mathbf{k} (\exists \mathbf{j} (\mathsf{NEG} \& \mathsf{NOM} \& \mathsf{MinVal}) \& \mathbf{i} \leq \Diamond \mathbf{k}).$$

By applying the (Scope-&) rule and commutativity and associativity rules for &, we get the desired shape ($\mathbf{i} \leq \Diamond \mathbf{k}$ is merged into NOM).

Proposition 10. *In the reduction stage, by running the algorithm* $ALBA^{SOPML}$, *for any formula* ψ *such that*

• ψ contains no propositional quantifiers;

- ψ contains propositional variables at most from \vec{q} ;
- all occurrences of \vec{q} -variables are negative in ψ ;

 $\psi \leq \mathsf{PIA}(\vec{q}, \vec{p})$ can be reduced to the following complex inequality:

$RelMinVal(\vec{q}, \vec{p}))$

where RelMinVal $(\vec{q}, \vec{p})^4$ is a meta-conjunction of inequalities of the form $\varphi \leq p$, φ has the three properties for ψ stated above, and p is in \vec{p} . Especially, this proposition holds for $\psi = \mathbf{i}$.

Proof. We prove by induction on the complexity of $PIA(\vec{q}, \vec{p})$.

- For the basic case where $PIA(\vec{q}, \vec{p}) = p$, trivial.
- For the case where $\mathsf{PIA}(\vec{q}, \vec{p}) = \Box \mathsf{PIA}^a(\vec{q}, \vec{p})$, we first apply the (Res- \Box) rule and get $\bullet \psi \leq \mathsf{PIA}^a(\vec{q}, \vec{p})$. Then by induction hypothesis, it is transformed into $\mathsf{RelMinVal}(\vec{q}, \vec{p})$) of the required shape.
- For the case where $\mathsf{PIA}(\vec{q}, \vec{p}) = \mathsf{PIA}^a(\vec{q}, \vec{p}) \wedge \mathsf{PIA}^b(\vec{q}, \vec{p})$, we first apply (Splitting) and get $\psi \leq \mathsf{PIA}^a(\vec{q}, \vec{p})$ and $\psi \leq \mathsf{PIA}^b(\vec{q}, \vec{p})$. Then by induction hypothesis, these two inequalities can be transformed into $\mathsf{RelMinVal}^a(\vec{q}, \vec{p})$) and $\mathsf{RelMinVal}^b(\vec{q}, \vec{p})$ of the required shape, which put together is also of the required shape.
- For the case where $\mathsf{PIA}(\vec{q}, \vec{p}) = \mathsf{POS}(\vec{q}) \vee \mathsf{PIA}^a(\vec{q}, \vec{p})$, by applying (Res- \vee), we get $\psi \land \neg \mathsf{POS}(\vec{q}) \leq \mathsf{PIA}^a(\vec{q}, \vec{p})$. Then $\psi \land \neg \mathsf{POS}(\vec{q})$ satisfies the conditions required in the proposition, so we can apply the induction hypothesis and get the $\mathsf{RelMinVal}(\vec{q}, \vec{p})$) of the required shape.

Proposition 11. In the reduction stage, by running the algorithm $ALBA^{SOPML}$, $\mathbf{i} \leq \forall \vec{q}(Sahl_1(\vec{q}) \rightarrow PIA(\vec{q}, \vec{p}))$ can be reduced to the following complex inequality:

$$\forall \vec{j}(PURE \Rightarrow MinVal(\vec{p}))$$

where

- PURE is a meta-conjunction of pure inequalities,
- MinVal (\vec{p}) is a meta-conjunction of inequalities of the form $\psi \leq p$, where ψ is pure and p is in \vec{p} .

⁴Here RelMinVal means relative minimal valuation.

Therefore, $\mathbf{i} \leq \forall \vec{q}(\mathsf{Sahl}_1(\vec{q}) \to \mathsf{PIA}(\vec{q}, \vec{p}))$ can be reduced to the form $\mathsf{MinVal}(\vec{p}),$

where $MinVal(\vec{p})$ is a meta-conjunction of inequalities of the form $\psi \leq p$, where ψ is pure and p is in \vec{p} .

Proof. We first apply (Quant-Nom) on

$$\mathbf{i} \leq \forall \vec{q}(\mathsf{Sahl}_1(\vec{q}) \to \mathsf{PIA}(\vec{q}, \vec{p})),$$

then apply (Sep-Nom) we get

$$\forall \vec{q} (\mathbf{i} \leq \mathsf{Sahl}_1(\vec{q}) \Rightarrow \mathbf{i} \leq \mathsf{PIA}(\vec{q}, \vec{p})).$$

By Proposition 9, we have

$$\forall \vec{q}(\exists \vec{j}(NEG \& NOM \& MinVal) \Rightarrow i \leq PIA(\vec{q}, \vec{p})).$$

By Proposition 10, we have

$$\forall \vec{q}(\exists \vec{j}(NEG \& NOM \& MinVal) \Rightarrow RelMinVal(\vec{q}, \vec{p})).$$

Then by applying (Scope- \Rightarrow) and repeatedly applying (Ex-ip), we have

$$\forall \vec{j} \forall \vec{q} (NEG \& NOM \& MinVal \Rightarrow RelMinVal(\vec{q}, \vec{p})).$$

Then by applying the Ackermann rule for each propositional variable in \vec{q} , NEG receives the minimal valuation from MinVal and become a meta-conjunction of pure inequalities, NOM remains pure, MinVal disappears, and RelMinVal(\vec{q} , \vec{p})) becomes a meta-conjunction of inequalities of the form $\psi \le p$ where ψ is pure and p is in \vec{p} . Now what we have is the following shape, as required by the proposition:

$$\forall \vec{\mathbf{j}}(\mathsf{PURE} \Rightarrow \mathsf{MinVal}(\vec{p})).$$

Then apply (Spl-Quant-i) and the packing rule, one get a complex inequality $MinVal(\vec{p})$ of the required form.

Proposition 12. In the reduction stage, by running the algorithm $ALBA^{SOPML}$, $i \leq Sahl_2(\vec{p})$ can be reduced to the following complex inequality:

where $\exists \vec{j}$, NEG, NOM, MinVal are described as in Proposition 9.

Proof. We prove by induction on the complexity of Sahl₂(\vec{p}).

- For the case where $Sahl_2(\vec{p}) = Sahl_1(\vec{p})$, see Proposition 9.
- For the case where $Sahl_2(\vec{p}) = \forall \vec{q}(Sahl_1(\vec{q}) \rightarrow PIA(\vec{q}, \vec{p}))$, by Proposition 11, $\mathbf{i} \leq \forall \vec{q}(Sahl_1(\vec{q}) \rightarrow PIA(\vec{q}, \vec{p}))$ is reduced to $\forall \vec{\mathbf{j}}(PURE \Rightarrow MinVal(\vec{p}))$. Now apply (Spl-Quant- \mathbf{i}) and the packing rule, we have a meta-conjunction of inequalities of the form $\varphi \leq p$ where φ is pure and p is in \vec{p} , so it belongs to MinVal.
- For the case where $Sahl_2(\vec{p}) = Sahl_2^a(\vec{p}) \wedge Sahl_2^b(\vec{p})$, similar to the $Sahl_1(\vec{p}) = Sahl_1^a(\vec{p}) \wedge Sahl_1^b(\vec{p})$ case in the proof of Proposition 9.
- For the case where $Sahl_2(\vec{p}) = \diamondsuit Sahl_2^a(\vec{p})$, similar to the $Sahl_1(\vec{p}) = \diamondsuit Sahl_1^a(\vec{p})$ case in the proof of Proposition 9.

Proposition 13. In the reduction stage, by running the algorithm $ALBA^{SOPML}$, $\mathbf{i} \leq Sahl_n(\vec{p})$ can be reduced to the following complex inequality:

∃i(NEG & NOM & MinVal)

where $\exists \vec{j}$, NEG, NOM, MinVal are described as in Proposition 9.

Proof. We prove by induction on n. For n=1,2, they are already proved in Proposition 9 and 12. Now we assume that for n=k the property holds, then by an argument similar to Proposition 11, we have that $\mathbf{i} \leq \forall \vec{q}(\mathsf{Sahl}_k(\vec{q}) \to \mathsf{PIA}(\vec{q}, \vec{p}))$ can be reduced to the following complex inequality:

$$\forall \vec{j}(PURE \Rightarrow MinVal(\vec{p}))$$

where PURE and MinVal(\vec{p}) are as described in Proposition 11. Then by an argument similar to Proposition 12, $\mathbf{i} \leq \operatorname{Sahl}_{k+1}(\vec{p})$ can be reduced to the following complex inequality:

where $\exists \vec{j}$, NEG, NOM, MinVal are described as in Proposition 9, hence the property holds for n = k + 1.

Theorem 6.1. For any Π_n -Sahlqvist formula, the algorithm $\mathsf{ALBA}^{\mathsf{SOPML}}$ transforms it into a complex inequality which does not contain any occurrences of propositional variables or propositional quantifiers.

Proof. Given a Π_n -Sahlqvist formula $\forall \vec{p}(Sahl_n(\vec{p}) \to POS(\vec{p}))$, we first apply the rules in Stage 1 and get

$$\forall \vec{p} \forall \mathbf{i}_0 (\mathbf{i}_0 \leq \mathsf{Sahl}_n(\vec{p}) \Rightarrow \mathbf{i}_0 \leq \mathsf{POS}(\vec{p})).$$

By Proposition 13, we have

$$\forall \vec{p} \forall \mathbf{i}_0 (\exists \vec{\mathbf{j}} (NEG \& NOM \& MinVal) \Rightarrow \mathbf{i}_0 \leq POS(\vec{p})).$$

Then by applying (Scope- \Rightarrow) and repeatedly applying (Ex-ip), we have

$$\forall \mathbf{i}_0 \forall \mathbf{j} \forall \vec{p} (NEG \& NOM \& MinVal \Rightarrow \mathbf{i}_0 \leq POS(\vec{p})).$$

Now we can apply the Ackermann rule repeatedly for each propositional variable p in \vec{p} , then NEG receives the minimal valuation from MinVal and become a metaconjunction of pure inequalities, NOM remains pure, MinVal disappears, and $\mathbf{i}_0 \leq \mathsf{POS}(\vec{p})$ receives the minimal valuation and becomes pure. Now what we have is the following shape:

$$\forall \mathbf{i}_0 \forall \mathbf{j}' (PURE \Rightarrow PURE'),$$

where PURE is a meta-conjunction of pure inequalities, and PURE' is a pure inequality.

Corollary 6.2. There is an algorithm such that for any Π_n -Sahlqvist formula φ , it can be transformed into an equivalent first-order formula.

7 Examples, non-standard rules and canonicity

7.1 Examples

We give three examples of Π_2 -Sahlqvist formulas to show how the ALBA^{SOPML} algorithm works:

```
Example 7.1. \forall p(\Diamond \Box p \land \forall q(\Diamond \Box q \rightarrow \Box (\Box q \lor \Box p)) \rightarrow \Box \Diamond \Box p)
\forall p \forall \mathbf{i} (\mathbf{i} \leq \Diamond \Box p \land \forall q(\Diamond \Box q \rightarrow \Box (\Box q \lor \Box p)) \Rightarrow \mathbf{i} \leq \Box \Diamond \Box p)
\forall p \forall \mathbf{i} (\mathbf{i} \leq \Diamond \Box p \& \mathbf{i} \leq \forall q(\Diamond \Box q \rightarrow \Box (\Box q \lor \Box p)) \Rightarrow \mathbf{i} \leq \Box \Diamond \Box p)
\forall p \forall \mathbf{i} (\mathbf{i} \leq \Diamond \Box p \& \forall q(\mathbf{i} \leq \Diamond \Box q \rightarrow \Box (\Box q \lor \Box p)) \Rightarrow \mathbf{i} \leq \Box \Diamond \Box p)
\forall p \forall \mathbf{i} (\mathbf{i} \leq \Diamond \Box p \& \forall q(\mathbf{i} \leq \Diamond \Box q \Rightarrow \mathbf{i} \leq \Box (\Box q \lor \Box p)) \Rightarrow \mathbf{i} \leq \Box \Diamond \Box p)
\forall p \forall \mathbf{i} (\mathbf{i} \leq \Diamond \Box p \& \forall q \forall \mathbf{j} (\mathbf{i} \leq \Diamond \mathbf{j} \& \mathbf{j} \leq \Box q \Rightarrow \mathbf{i} \leq \Box (\Box q \lor \Box p)) \Rightarrow \mathbf{i} \leq \Box \Diamond \Box p)
\forall p \forall \mathbf{i} (\mathbf{i} \leq \Diamond \Box p \& \forall q \forall \mathbf{j} (\mathbf{i} \leq \Diamond \mathbf{j} \& \mathbf{j} \leq q \Rightarrow \mathbf{i} \leq \Box (\Box q \lor \Box p)) \Rightarrow \mathbf{i} \leq \Box \Diamond \Box p)
\forall p \forall \mathbf{i} (\mathbf{i} \leq \Diamond \Box p \& \forall \mathbf{j} (\mathbf{i} \leq \Diamond \mathbf{j} \Rightarrow \mathbf{i} \leq \Box (\Box \mathbf{q} \lor \Box p)) \Rightarrow \mathbf{i} \leq \Box \Diamond \Box p)
\forall p \forall \mathbf{i} (\mathbf{i} \leq \Diamond \Box p \& \forall \mathbf{j} (\mathbf{i} \leq \Diamond \mathbf{j} \Rightarrow \mathbf{i} \leq \Box (\Box \mathbf{q} \lor \Box p)) \Rightarrow \mathbf{i} \leq \Box \Diamond \Box p)
```

```
\forall p \forall \mathbf{i} (\mathbf{i} \leq \Diamond \Box p \& \forall \mathbf{j} (\mathbf{i} \leq \Diamond \mathbf{j} \Rightarrow \blacklozenge \mathbf{i} \land \neg \Box \blacklozenge \mathbf{j} \leq \Box p) \Rightarrow \mathbf{i} \leq \Box \Diamond \Box p)
\forall p \forall \mathbf{i} (\mathbf{i} \leq \Diamond \Box p \& \forall \mathbf{j} (\mathbf{i} \leq \Diamond \mathbf{j} \Rightarrow \blacklozenge (\spadesuit \mathbf{i} \land \neg \Box \blacklozenge \mathbf{j}) \leq p) \Rightarrow \mathbf{i} \leq \Box \Diamond \Box p)
\forall p \forall \mathbf{i} (\mathbf{i} \leq \Diamond \Box p \& \forall \mathbf{j} (\mathbf{l} (\mathbf{i}, \Diamond \mathbf{j}) \land \blacklozenge (\spadesuit \mathbf{i} \land \neg \Box \spadesuit \mathbf{j}) \leq p) \Rightarrow \mathbf{i} \leq \Box \Diamond \Box p)
\forall p \forall \mathbf{i} (\mathbf{i} \leq \Diamond \Box p \& \exists \mathbf{j} (\mathbf{l} (\mathbf{i}, \Diamond \mathbf{j}) \land \spadesuit (\spadesuit \mathbf{i} \land \neg \Box \spadesuit \mathbf{j})) \leq p \Rightarrow \mathbf{i} \leq \Box \Diamond \Box p)
now \ denote \ \exists \mathbf{j} (\mathbf{l} (\mathbf{i}, \Diamond \mathbf{j}) \land \spadesuit (\spadesuit \mathbf{i} \land \neg \Box \spadesuit \mathbf{j})) \ as \ \varphi, \ then
\forall p \forall \mathbf{i} (\mathbf{i} \leq \Diamond \Box p \& \varphi \leq p \Rightarrow \mathbf{i} \leq \Box \Diamond \Box p)
\forall p \forall \mathbf{i} \forall \mathbf{k} (\mathbf{i} \leq \Diamond \mathbf{k} \& \mathbf{k} \leq \Box p \& \varphi \leq p \Rightarrow \mathbf{i} \leq \Box \Diamond \Box p)
\forall p \forall \mathbf{i} \forall \mathbf{k} (\mathbf{i} \leq \Diamond \mathbf{k} \& \spadesuit \mathbf{k} \leq p \& \varphi \leq p \Rightarrow \mathbf{i} \leq \Box \Diamond \Box p)
\forall \mathbf{i} \forall \mathbf{k} (\mathbf{i} \leq \Diamond \mathbf{k} \Rightarrow \mathbf{i} \leq \Box \Diamond \Box (\spadesuit \mathbf{k} \lor \varphi))
```

Then we can use standard translation to get its first-order correspondence.

```
Example 7.2. \forall q(\forall p(p \rightarrow \Diamond p \lor q) \rightarrow q)

\forall q \forall \mathbf{i} (\mathbf{i} \leq \forall p(p \rightarrow \Diamond p \lor q) \Rightarrow \mathbf{i} \leq q)

\forall q \forall \mathbf{i} (\forall p(\mathbf{i} \leq p \rightarrow \Diamond p \lor q) \Rightarrow \mathbf{i} \leq q)

\forall q \forall \mathbf{i} (\forall p(\mathbf{i} \leq p \Rightarrow \mathbf{i} \leq \Diamond p \lor q) \Rightarrow \mathbf{i} \leq q)

\forall q \forall \mathbf{i} (\mathbf{i} \leq \Diamond \mathbf{i} \lor q \Rightarrow \mathbf{i} \leq q)

\forall q \forall \mathbf{i} (\mathbf{i} \wedge \neg \Diamond \mathbf{i} \leq q \Rightarrow \mathbf{i} \leq q)

\forall \mathbf{i} (\mathbf{i} \leq \mathbf{i} \wedge \neg \Diamond \mathbf{i})

\forall \mathbf{i} (\mathbf{i} \leq \neg \Diamond \mathbf{i})

\forall x \neg Rxx.
```

By [9, Example 2.58], the irreflexive property is not preserved under taking ultrafilter extensions, which means that the validity of $\forall q(\forall p(p \rightarrow \Diamond p \lor q) \rightarrow q)$ is not preserved under taking canonical extensions, which means that $\forall q(\forall p(p \rightarrow \Diamond p \lor q) \rightarrow q)$ is not canonical.

Example 7.3. The following example is not equivalent to any Sahlqvist formula in the basic modal language:

```
\begin{split} \forall p(\Box p \land \forall q(q \to \Diamond \Diamond q \lor p) \to p) \\ \forall p \forall \mathbf{i}(\mathbf{i} \leq \Box p \land \forall q(q \to \Diamond \Diamond q \lor p) \Rightarrow \mathbf{i} \leq p) \\ \forall p \forall \mathbf{i}(\mathbf{i} \leq \Box p \& \mathbf{i} \leq \forall q(q \to \Diamond \Diamond q \lor p) \Rightarrow \mathbf{i} \leq p) \\ \forall p \forall \mathbf{i}(\blacklozenge \mathbf{i} \leq p \& \mathbf{i} \leq \forall q(q \to \Diamond \Diamond q \lor p) \Rightarrow \mathbf{i} \leq p) \\ \forall p \forall \mathbf{i}(\blacklozenge \mathbf{i} \leq p \& \forall q(\mathbf{i} \leq q \to \Diamond \Diamond q \lor p) \Rightarrow \mathbf{i} \leq p) \\ \forall p \forall \mathbf{i}(\blacklozenge \mathbf{i} \leq p \& \forall q(\mathbf{i} \leq q \Rightarrow \mathbf{i} \leq \Diamond \Diamond q \lor p) \Rightarrow \mathbf{i} \leq p) \\ \forall p \forall \mathbf{i}(\blacklozenge \mathbf{i} \leq p \& \mathbf{i} \leq \Diamond \Diamond \mathbf{i} \lor p \Rightarrow \mathbf{i} \leq p) \\ \forall p \forall \mathbf{i}(\blacklozenge \mathbf{i} \leq p \& \mathbf{i} \land \neg \Diamond \Diamond \mathbf{i} \leq p \Rightarrow \mathbf{i} \leq p) \\ \forall p \forall \mathbf{i}(\blacklozenge \mathbf{i} \lor (\mathbf{i} \land \neg \Diamond \Diamond \mathbf{i}) \leq p \Rightarrow \mathbf{i} \leq p) \\ \forall \mathbf{i}(\mathbf{i} \leq \blacklozenge \mathbf{i} \lor (\mathbf{i} \land \neg \Diamond \Diamond \mathbf{i})) \\ \forall \mathbf{i}(\mathbf{i} \leq \blacklozenge \mathbf{i} or \mathbf{i} \leq \mathbf{i} \land \neg \Diamond \Diamond \mathbf{i}) \end{split}
```

$$\forall \mathbf{i}(\mathbf{i} \leq \mathbf{\Phi} \mathbf{i} \text{ or } \mathbf{i} \leq \neg \diamondsuit \diamond \mathbf{i})$$

$$\forall \mathbf{i}(\mathbf{i} \leq \diamondsuit \diamond \mathbf{i} \rightarrow \mathbf{\Phi} \mathbf{i})$$

$$\forall x \forall y (Rxy \land Ryx \rightarrow Rxx)$$

One can show that this property is not modally definable:

Consider $\mathbb{F}_1 = (W_1, R_1)$ where W_1 is the set of all integers, $R_1 = \{(x, x + 1) \mid x \in W_1\}$, $\mathbb{F}_2 = (W_2, R_2)$ where $W_2 = \{w_0, w_1\}$, $R_2 = \{(w_0, w_1), (w_1, w_0)\}$, then \mathbb{F}_2 is a bounded morphic image of \mathbb{F}_1 , $\mathbb{F}_1 \models \forall x \forall y (Rxy \land Ryx \rightarrow Rxx)$, while $\mathbb{F}_2 \not\models \forall x \forall y (Rxy \land Ryx \rightarrow Rxx)$.

7.2 Π_2 -formulas and rules

In this section we consider the following kinds of rules, each of which is the generalization of the former one:

• Gabbay's irreflexivity rule [21]:

$$\vdash \neg (p \to \Diamond p) \to \varphi \implies \vdash \varphi$$

where p does not occur in φ .

• Venema's non- ξ rules [38]:

$$\vdash \neg \xi(p_0, \dots, p_n) \to \varphi \implies \vdash \varphi$$

where p_0, \ldots, p_n does not occur in φ .

• Π_2 rules [8]:

$$\vdash F(\vec{\varphi}/\vec{x}, \vec{p}) \rightarrow \chi \implies \vdash G(\vec{\varphi}/\vec{x}) \rightarrow \chi$$

where F, G are formulas, $\vec{\varphi}$ is a tuple of formulas, χ is a formula, and \vec{p} is a tuple of propositional variables which do not occur in $\vec{\varphi}$ and χ .

Gabbay's irreflexivity rule. Now consider Gabbay's irreflexivity rule, its corresponding ∀∃-statement is the following:

$$\forall q (\forall p (\neg (p \to \Diamond p) \leq q) \Rightarrow \top \leq q)$$

therefore, its equivalent SOPML \(\formula\) is

$$\forall q (\forall p \ \mathbf{l}(\neg(p \to \Diamond p), q) \to \mathbf{l}(\top, q))$$

now its ALBASOPML-reduction is as follows:5

```
\begin{aligned} &\forall \mathbf{i} (\mathbf{i} \leq \forall q (\forall p \ \mathbf{l} (\neg (p \rightarrow \Diamond p), q) \rightarrow \mathbf{l} (\top, q)))) \\ &\forall \mathbf{i} \forall q ((\forall p (\mathbf{i} \leq \mathbf{l} (\neg (p \rightarrow \Diamond p), q)) \Rightarrow \mathbf{i} \leq \mathbf{l} (\top, q))) \\ &\forall q (\forall p (\neg (p \rightarrow \Diamond p) \leq q) \Rightarrow \top \leq q) \\ &\forall q (\forall p \forall \mathbf{j} (\mathbf{j} \leq \neg (p \rightarrow \Diamond p) \Rightarrow \mathbf{j} \leq q) \Rightarrow \top \leq q) \\ &\forall q (\forall p \forall \mathbf{j} (\mathbf{j} \leq p \ \& \ \mathbf{j} \leq \neg \Diamond p \Rightarrow \ \mathbf{j} \leq q) \Rightarrow \top \leq q) \\ &\forall q (\forall \mathbf{j} (\mathbf{j} (\mathbf{j}, \neg \Diamond \mathbf{j}) \wedge \mathbf{j} \leq q) \Rightarrow \top \leq q) \\ &\forall q (\forall \mathbf{j} (\mathbf{l} (\mathbf{j}, \neg \Diamond \mathbf{j}) \wedge \mathbf{j}) \leq q \Rightarrow \top \leq q) \\ &\forall q (\exists \mathbf{j} (\mathbf{l} (\mathbf{j}, \neg \Diamond \mathbf{j}) \wedge \mathbf{j}) \leq q \Rightarrow \top \leq q) \\ &\forall \tau \leq \exists \mathbf{j} (\mathbf{l} (\mathbf{j}, \neg \Diamond \mathbf{j}) \wedge \mathbf{j}) \\ &\forall \mathbf{i} (\mathbf{i} \leq \exists \mathbf{j} (\mathbf{l} (\mathbf{j}, \neg \Diamond \mathbf{j}) \wedge \mathbf{j})) \\ &\forall x S \ T_x (\exists \mathbf{j} (\mathbf{l} (\mathbf{j}, \neg \Diamond \mathbf{j}) \wedge \mathbf{j})) \\ &\forall x \exists j (S \ T_x (\mathbf{l} (\mathbf{j}, \neg \Diamond \mathbf{j}) \wedge S \ T_x (\mathbf{j})) \\ &\forall x \exists j (\neg R \ j j \wedge x = j) \\ &\forall x \neg R x x. \end{aligned}
```

Venema's non-\xi rules. Now consider Venema's non- ξ rules, their corresponding $\forall \exists$ -statement is the following:

$$\forall q (\forall \vec{p} (\neg \xi(\vec{p}) \le q) \Rightarrow \top \le q)$$

When ξ is a Sahlqvist formula $Sahl_1(\vec{p}) \to POS(\vec{p})$ in the basic modal language, Venema's rules can be equivalently written in the following SOPML $\forall \exists$ -formula:

$$\forall q (\forall \vec{p}(\mathbf{l}(\neg(\mathsf{Sahl}_1(\vec{p}) \to \mathsf{POS}(\vec{p})), q)) \to \mathbf{l}(\top, q)).$$

Assume that $\mathbf{i} \leq \mathsf{Sahl}_1(\vec{p}) \to \mathsf{POS}(\vec{p})$ can be reduced to $\mathbf{i} \leq \mathsf{Local}$ where Local is pure (which is the modal counterpart of the local frame correspondent of $\mathsf{Sahl}_1(\vec{p}) \to \mathsf{POS}(\vec{p})$), then the $\mathsf{ALBA}^{\mathsf{SOPML}}$ -reduction is as follows:

```
\begin{split} &\forall q(\forall \vec{p}(\mathbf{l}(\neg(\mathsf{Sahl}_1(\vec{p}) \to \mathsf{POS}(\vec{p})),q)) \to \mathbf{l}(\top,q)) \\ &\forall \mathbf{i}(\mathbf{i} \leq \forall q(\forall \vec{p}(\mathbf{l}(\neg(\mathsf{Sahl}_1(\vec{p}) \to \mathsf{POS}(\vec{p})),q)) \to \mathbf{l}(\top,q))) \\ &\forall \mathbf{i}(\mathbf{i} \leq \forall q(\forall \vec{p}(\mathbf{l}(\neg(\mathsf{Sahl}_1(\vec{p}) \to \mathsf{POS}(\vec{p})),q)) \to \mathbf{l}(\top,q))) \\ &\forall \mathbf{i} \forall q(\mathbf{i} \leq \forall \vec{p}(\mathbf{l}(\neg(\mathsf{Sahl}_1(\vec{p}) \to \mathsf{POS}(\vec{p})),q)) \to \mathbf{i} \leq \mathbf{l}(\top,q)) \\ &\forall \mathbf{i} \forall q(\forall \vec{p}(\mathbf{i} \leq \mathbf{l}(\neg(\mathsf{Sahl}_1(\vec{p}) \to \mathsf{POS}(\vec{p})),q)) \to \mathbf{i} \leq \mathbf{l}(\top,q)) \end{split}
```

⁵Notice that the algorithm here is slightly different from the one defined in the previous sections, due to the introduction of the I connective in the basic language. Similar for the non- ξ rules and the Π_2 rules.

```
\begin{split} &\forall \mathbf{i} \forall q (\forall \vec{p} (\neg (\mathsf{Sahl}_1(\vec{p}) \to \mathsf{POS}(\vec{p})) \leq q) \implies \mathbf{i} \leq \mathbf{l}(\top,q)) \\ &\forall q (\forall \vec{p} (\neg (\mathsf{Sahl}_1(\vec{p}) \to \mathsf{POS}(\vec{p})) \leq q) \implies \top \leq q) \\ &\forall q (\forall \vec{p} \forall \mathbf{j} (\mathbf{j} \leq \neg (\mathsf{Sahl}_1(\vec{p}) \to \mathsf{POS}(\vec{p})) \implies \mathbf{j} \leq q) \implies \top \leq q) \\ &\forall q (\forall \vec{p} \forall \mathbf{j} (\mathbf{j} \nleq (\mathsf{Sahl}_1(\vec{p}) \to \mathsf{POS}(\vec{p})) \implies \mathbf{j} \leq q) \implies \top \leq q) \\ &\forall q (\forall \mathbf{j} (\mathbf{j} \nleq \mathsf{Local} \implies \mathbf{j} \leq q) \implies \top \leq q) \\ &\forall q (\forall \mathbf{j} (\neg \mathbf{l} (\mathbf{j}, \mathsf{Local}) \land \mathbf{j} \leq q) \implies \top \leq q) \\ &\forall q (\exists \mathbf{j} (\neg \mathbf{l} (\mathbf{j}, \mathsf{Local}) \land \mathbf{j}) \leq q \implies \top \leq q) \\ &\top \leq \exists \mathbf{j} (\neg \mathbf{l} (\mathbf{j}, \mathsf{Local}) \land \mathbf{j})) \\ &\forall x S T_x (\exists \mathbf{j} (\neg \mathbf{l} (\mathbf{j}, \mathsf{Local}) \land \mathbf{j})) \\ &\forall x \exists j S T_x (\neg \mathbf{l} (\mathbf{j}, \mathsf{Local}) \land \mathbf{j})) \\ &\forall x \exists j (S T_x (\neg \mathbf{l} (\mathbf{j}, \mathsf{Local})) \land S T_x (\mathbf{j})) \\ &\forall x \exists j (S T_x (\neg \mathbf{l} (\mathbf{j}, \mathsf{Local})) \land x = j) \\ &\forall x \exists j (\neg S T_j (\mathsf{Local}) \land x = j) \\ &\forall x \neg S T_x (\mathsf{Local}). \end{split}
```

 Π_2 -rules. We first consider the corresponding SOPML-formulas of Π_2 -rules. For $\vdash F(\vec{\varphi}/\vec{x}, \vec{p}) \to \chi \implies \vdash G(\vec{\varphi}/\vec{x}) \to \chi$, its corresponding $\forall \exists$ -statement is the following:

$$\forall \vec{x} \forall z (G(\vec{x}) \nleq z \Rightarrow \exists \vec{y} (F(\vec{x}, \vec{y}) \nleq z)),$$

which is equivalent to

$$\forall \vec{x} \forall z (\forall \vec{y} (F(\vec{x}, \vec{y}) \le z) \implies G(\vec{x}) \le z),$$

which is essentially the following SOPML \(\formula\):

$$\forall \vec{p} \forall q (\forall \vec{r} (\mathbf{l}(F(\vec{p}, \vec{r}), q)) \rightarrow \mathbf{l}(G(\vec{p}), q))$$

When $F(\vec{p}, \vec{r})$ is of the form $Sahl_1(\vec{p}, \vec{r})$, $G(\vec{p})$ is of the form $POS(\vec{p})$, the $ALBA^{SOPML}$ -reduction is as follows:

```
\begin{split} \forall \vec{p} \forall q (\forall \vec{r} (\mathbf{l} (\mathsf{Sahl}_1(\vec{p}, \vec{r}), q)) &\rightarrow \mathbf{l} (\mathsf{POS}(\vec{p}), q)) \\ \forall \mathbf{i} (\mathbf{i} \leq \forall \vec{p} \forall q (\forall \vec{r} (\mathbf{l} (\mathsf{Sahl}_1(\vec{p}, \vec{r}), q)) \rightarrow \mathbf{l} (\mathsf{POS}(\vec{p}), q))) \\ \forall \mathbf{i} \forall \vec{p} \forall q (\mathbf{i} \leq \forall \vec{r} (\mathbf{l} (\mathsf{Sahl}_1(\vec{p}, \vec{r}), q)) &\Rightarrow \mathbf{i} \leq \mathbf{l} (\mathsf{POS}(\vec{p}), q)) \\ \forall \mathbf{i} \forall \vec{p} \forall q (\forall \vec{r} (\mathbf{i} \leq \mathbf{l} (\mathsf{Sahl}_1(\vec{p}, \vec{r}), q)) &\Rightarrow \mathbf{i} \leq \mathbf{l} (\mathsf{POS}(\vec{p}), q)) \\ \forall \vec{p} \forall q (\forall \vec{r} (\mathsf{Sahl}_1(\vec{p}, \vec{r}) \leq q) &\Rightarrow \mathsf{POS}(\vec{p}) \leq q) \\ \forall \vec{p} \forall q (\forall \vec{r} \forall \mathbf{i} (\mathbf{i} \leq \mathsf{Sahl}_1(\vec{p}, \vec{r}) \Rightarrow \mathbf{i} \leq q) &\Rightarrow \mathsf{POS}(\vec{p}) \leq q) \\ \forall \vec{p} \forall q (\forall \vec{r} \forall \mathbf{i} (\exists \mathbf{j} (\mathsf{NEG}(\vec{p}, \vec{r}) \& \mathsf{NOM} \& \mathsf{MinVal}(\vec{p}, \vec{r})) \Rightarrow \mathbf{i} \leq q) &\Rightarrow \mathsf{POS}(\vec{p}) \leq q) \end{split}
```

(Here NEG(\vec{p}, \vec{r}) & NOM & MinVal(\vec{p}, \vec{r}) are as described in Proposition 9.)

```
\begin{split} \forall \vec{p} \forall q (\forall \vec{r} \forall \mathbf{i} \forall \mathbf{j} (\mathsf{NEG}(\vec{p}, \vec{r}) \ \& \ \mathsf{NOM} \ \& \ \mathsf{MinVal}(\vec{p}, \vec{r}) \ \Rightarrow \ \mathbf{i} \leq q) \ \Rightarrow \ \mathsf{POS}(\vec{p}) \leq q) \\ \forall \vec{p} \forall q (\forall \vec{r} \forall \mathbf{i} \forall \mathbf{j} (\mathsf{NEG}(\vec{p}, \vec{r}) \ \& \ \mathsf{NOM} \ \& \ \mathsf{MinVal}(\vec{p}, \vec{r}) \ \Rightarrow \ \mathbf{i} \leq q) \ \Rightarrow \ \forall \mathbf{k} (\mathbf{k} \leq \mathsf{POS}(\vec{p}) \ \Rightarrow \\ \mathbf{k} \leq q)) \\ \forall \vec{p} \forall q \forall \mathbf{k} (\forall \vec{r} \forall \mathbf{i} \forall \mathbf{j} (\mathsf{NEG}(\vec{p}, \vec{r}) \ \& \ \mathsf{NOM} \ \& \ \mathsf{MinVal}(\vec{p}, \vec{r}) \ \Rightarrow \ \mathbf{i} \leq q) \ \& \ \mathbf{k} \leq \mathsf{POS}(\vec{p}) \ \Rightarrow \\ \mathbf{k} \leq q) \\ \forall \vec{p} \forall q \forall \mathbf{k} (\forall \vec{r} \forall \mathbf{i} \forall \mathbf{j} (\mathbf{k} \leq \mathsf{POS}(\vec{p}) \ \& \ (\mathsf{NEG}(\vec{p}, \vec{r}) \ \& \ \mathsf{NOM} \ \& \ \mathsf{MinVal}(\vec{p}, \vec{r}) \ \Rightarrow \ \mathbf{i} \leq q)) \\ \Rightarrow \ \mathbf{k} \leq q) \end{aligned}
```

Then we can apply the Ackermann rule and substitute the minimal valuation of \vec{p} , \vec{r} into POS(\vec{p}) and NEG(\vec{p} , \vec{r}) and make the latter two pure, therefore the complex inequality is equivalent to

```
\forall q \forall \mathbf{k} (\forall \mathbf{i} \forall \mathbf{j} (\mathsf{PURE} \& (\mathsf{PURE}' \Rightarrow \mathbf{i} \leq q)) \Rightarrow \mathbf{k} \leq q)
```

By the packing rule, $PURE' \Rightarrow \mathbf{i} \leq q$ is packed into an inequality $\psi \leq q$ where ψ is pure:

```
\begin{split} \forall q \forall \mathbf{k} (\forall \mathbf{i} \forall \mathbf{j} (\mathsf{PURE} \ \& \ \psi \leq q) &\Rightarrow \mathbf{k} \leq q) \\ \forall q \forall \mathbf{k} (\forall \mathbf{i} \forall \mathbf{j} (\mathsf{PURE}) \ \& \ \forall \mathbf{i} \forall \mathbf{j} (\psi \leq q) &\Rightarrow \mathbf{k} \leq q) \\ \forall q \forall \mathbf{k} (\forall \mathbf{i} \forall \mathbf{j} (\mathsf{PURE}) \ \& \ \exists \mathbf{i} \exists \mathbf{j} \psi \leq q \ \Rightarrow \ \mathbf{k} \leq q) \\ \forall \mathbf{k} (\forall \mathbf{i} \forall \mathbf{j} (\mathsf{PURE}) \ \Rightarrow \ \mathbf{k} \leq \exists \mathbf{i} \exists \mathbf{j} \psi). \end{split}
```

Then we can perform the standard translation to obtain its corresponding first-order correspondent.

8 Conclusion

In this paper, we develop the Sahlqvist correspondence theory for SOPML. We define the class of Sahlqvist formulas for SOMPL, each formula of which is shown to have a first-order correspondent by an algorithm ALBA SOMPL . In addition, we show that certain Π_2 -rules correspond to Π_2 -Sahlqvist formulas in SOMPL, which further correspond to first-order conditions.

Here we give some final remarks:

 Since the Sahlqvist correspondence theorem talks about frame definability, any propositional variables in the basic modal formulas are already implicitly treated as universally quantified, so what we will do in this paper for SOPML formulas is to find a Sahlqvist fragment which allows also for existentially quantified proposition variables, not only universally quantified variables. Indeed, this can be seen in the definition of Π_2 -Sahlqvist formulas, where universal quantifiers are allowed in the antecedent part.

This paper can also be seen as looking for a modal counterpart of second-order quantifier elimination for monadic second-order logic (MSO), as SOPML with global modality is expressively equivalent to MSO (see [31]). Here what we are aiming at is to find a natural fragment in a modal-type language which can be reduced to first-order formulas.

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