# Sahlqvist Correspondence Theory for Second-Order Propositional Modal Logic 

Zhiguang Zhao


#### Abstract

Modal logic with propositional quantifiers (i.e. second-order propositional modal logic (SOPML)) has been considered since the early time of modal logic. Its expressive power and complexity are high, and its van-Benthem-Rosen theorem and Goldblatt-Thomason theorem have been proved by ten Cate (2006). However, the Sahlqvist theory of SOPML has not been considered in the literature. In the present paper, we fill in this gap. We develop the Sahlqvist correspondence theory for SOPML, which covers and properly extends existing Sahlqvist formulas in basic modal logic. We define the class of Sahlqvist formulas for SOMPL step by step in a hierarchical way, each formula of which is shown to have a first-order correspondent over Kripke frames effectively computable by an algorithm ALBA ${ }^{\text {SOMPL }}$. In addition, we show that certain $\Pi_{2}$-rules correspond to $\Pi_{2}$-Sahlqvist formulas in SOMPL, which further correspond to first-order conditions, and that even for very simple SOMPL Sahlqvist formulas, they could already be noncanonical.


Keywords: correspondence theory, second-order propositional modal logic, ALBA algorithm, $\Pi_{2}$-rules, canonicity

## 1 Introduction

Second-Order Propositional Modal Logic (SOMPL). Modal logic with propositional quantifiers has been considered in the literature since Kripke [29], Bull [10], Fine [18, 19], and Kaplan [17]] ${ }^{1}$ This language is of high complexity: its satisfiability problem is not decidable, and indeed not even analytical. In Kaminski and Tiomkin [27], the authors showed that the expressive power for SOMPL whose modalities are S 4.2 or weaker is the same as second-order predicate logic. However, not every second-order formula is equivalent to an SOMPL-formula, since

[^0]SOMPL-formulas are preserved under generated submodels (see van Benthem [37]). In ten Cate [35], the author proved the analogues of the van Benthem-Rosen theorem (on the model level) and Goldblatt-Thomason theorem (on the frame level) for SOMPL. Therefore, a natural question is: on the frame level, can we find a natural fragment of SOPML-formulas such that each formula in this fragment corresponds to a first-order formula, in the sense of Sahlqvist theory (see [33, 37])? This is what we will answer in the paper.

Correspondence Theory. Typically, modal correspondence theory [37] concerns the correspondence of modal formulas and first-order formulas over Kripke frames, via the tools of standard translation. Syntactic classes (e.g. Sahlqvist formulas [33], inductive formulas [24], etc.) of modal formulas are identified to have first-order correspondents and are canonical, i.e. their validity are closed under taking canonical extensions.
In the present paper, we identify the Sahlqvist formulas of SOMPL, which cover and properly extend the Sahlqvist fragment in basic modal logic. We show that each Sahlqvist SOMPL formula corresponds to a first-order formula by an algorithm ALBA ${ }^{\text {SOPML }}$. In particular, we have the following observations: the SOMPL Sahlqvist formula $\forall p(\square p \wedge \forall q(q \rightarrow \diamond \diamond q \vee p) \rightarrow p)$ corresponds to $\forall x \forall y(R x y \wedge$ $R y x \rightarrow R x x$ ), which is not modally definable since this property is not preserved under taking bounded morphic image (see Example 7.3); the SOMPL Sahlqvist formula $\forall q(\forall p(p \rightarrow \diamond p \vee q) \rightarrow q)$ is not canonical (see Example 7.2), which is in contrast to the basic modal logic setting where each Sahlqvist formula is canonical.

Non-standard Rules. Another topic that is related to the present paper is nonstandard rules, starting from Gabbay [21] where a non-standard rule for irreflexivity is introduced. These rules have been used in temporal logic [11, 22], regionbased theories of space [2, 36] and are used to prove completeness results for modal logic systems with non- $\xi$-rules [38]. In particular, the so-called $\Pi_{2}$-rules [7, 8, 34] which generalize both the irreflexivity rule of Gabbay [21] and the non- $\xi$-rules of Venema [38], have their natural $\forall \exists$-counterparts, which are essentially $\forall \exists$-SOMPL formulas, fit naturally into the language of SOMPL. We use the correspondence algorithm to compute the first-order correspondents of a subclass of $\Pi_{2}$-rules whose $\forall \exists$-counterparts are SOMPL $\Pi_{2}$-Sahlqvist formulas.

Our methodology. The present paper use the same methodology as [15, 12]. In the present paper, inspired by the Sahlqvist rules in Santoli [34], we identify the Sahlqvist formulas of SOMPL, which are generalizations of Sahlqvist formulas in basic modal logic and have first-order correspondents. The Sahlqvist fragment of

SOPML is defined in a step-by-step way, and we give an algorithm ALBA ${ }^{\text {SOPML }}$ (Ackermann Lemma Based Algorithm) which can successfully reduce Sahlqvist formulas in SOPML to first-order formulas and is sound with respect to Kripke semantics.

Structure of the paper. The structure of the paper is as follows: Section 2 gives the necessary preliminaries. Section 3 gives the definition of Sahlqvist SOPML formulas step by step. Section 4 defines the algorithm ALBA ${ }^{\text {SOPML }}$. Section 5 shows the soundness of the algorithm with respect to Kripke frames. Section 6 shows that the algorithm succeeds on all Sahlqvist SOPML formulas. Section 7 gives some examples and connect them with non-standard rules, and one example shows that even for very simple Sahlqvist SOPML formulas, they can already be non-canonical. Section 8 gives some final remarks and conclusion.

## 2 Preliminaries

### 2.1 Language and semantics

In the present paper we consider the unimodal language. Given a set Prop of propositional variables, the second-order propositional modal formulas are defined as follows:

$$
\varphi::=p|\perp| \top|\neg \varphi| \varphi \wedge \varphi|\varphi \vee \varphi| \varphi \rightarrow \varphi|\square \varphi| \diamond \varphi|\forall p \varphi| \exists p \varphi
$$

where $p \in$ Prop. We use the notation $\vec{p}$ to denote a set of propositional variables and use $\varphi(\vec{p})$ to indicate that the propositional variables occur in $\varphi$ are all in $\vec{p}$. We say that an occurrence of a propositional variable $p$ in a formula $\varphi$ is positive (resp. negative) if it is in the scope of an even (resp. odd) number of negations (here $\alpha \rightarrow \beta$ is regarded as $\neg \alpha \vee \beta$ ).
The semantics of the second-order propositional modal formulas are defined as follows:

Definition 1. A Kripke frame is a pair $\mathbb{F}=(W, R)$ where $W \neq \varnothing$ is the domain of $\mathbb{F}$, the accessibility relation $R$ is a binary relation on $W$. A Kripke model is a pair $\mathbb{M}=(\mathbb{F}, V)$ where $V: \operatorname{Prop} \rightarrow P(W)$ is a valuation on $\mathbb{F} . V_{X}^{p}$ denote a valuation which is the same as $V$ except that $V_{X}^{p}(p)=X \subseteq W$.
Now the satisfaction relation can be defined as follows: given any Kripke model $\mathbb{M}=(W, R, V)$, any $w \in W$, the basic and Boolean cases are standard, and for modalities and propositional quantifiers,

| $\mathbb{M}, w \Vdash \square \varphi$ | iff | for any $v$ such that $R w v, \mathbb{M}, v \Vdash \varphi ;$ |
| :--- | :--- | :--- |
| $\mathbb{M}, w \Vdash \diamond \diamond$ | iff | there exists $v$ such that $R w v$ and $\mathbb{M}, v \Vdash \varphi ;$ |
| $\mathbb{M}, w \Vdash \forall p \varphi$ | iff | for all $X \subseteq W,\left(W, R, V_{X}^{p}\right), w \Vdash \varphi ;$ |
| $\mathbb{M}, w \Vdash \exists p \varphi$ | iff | there exists $X \subseteq W$ such that $\left(W, R, V_{X}^{p}\right), w \Vdash \varphi$. |

In order to use the algorithm to compute the first-order correspondents of Sahlqvist SOPML formulas, we will need the following expanded modal language which is defined as follow :

$$
\begin{gathered}
\varphi::=p|\mathbf{i}| \perp|\top| \neg \varphi|\varphi \wedge \varphi| \varphi \vee \varphi|\varphi \rightarrow \varphi| \\
\square \varphi|\diamond \varphi| ■ \varphi|\diamond \varphi| \forall p \varphi|\exists p \varphi| \forall \mathbf{i} \varphi|\exists \mathbf{i} \varphi| \mathbf{l}(\varphi, \varphi)
\end{gathered}
$$

where $p \in \operatorname{Prop}, \mathbf{i} \in$ Nom is a nominal, $\llbracket$ and are the backward-looking box and diamond respectively, $\forall \mathbf{i}$ and $\exists \mathbf{i}$ are nominal quantifiers, and $\mathbf{I}$ is a binary modality. We call a formula pure if it does not contain propositional variables or propositional quantifiers (it can contain nominals, nominal quantifiers and the binary modality I).

The interpretation of the expanded modal language is given as follows: For a valuation $V$, it is defined as $V$ : Prop $\cup$ Nom $\rightarrow P(W)$ such that $V(\mathbf{i})$ is a singleton for all $\mathbf{i} \in$ Nom. The additional satisfaction clauses are given as follows (here $V_{v}^{\mathbf{i}}$ denote a valuation which is the same as $V$ except that $\left.V_{v}^{\mathbf{i}}(\mathbf{i})=\{v\} \subseteq W.\right)$ :

$$
\begin{array}{lll}
\mathbb{M}, w \Vdash \mathbf{i} & \text { iff } & V(\mathbf{i})=\{w\} ; \\
\mathbb{M}, w \Vdash \llbracket \varphi & \text { iff } & \text { for any } v \text { such that } R v w, \mathbb{M}, v \Vdash \varphi ; \\
\mathbb{M}, w \Vdash \varphi & \text { iff } & \text { there exists } v \text { such that } R v w \text { and } \mathbb{M}, v \Vdash \varphi ; \\
\mathbb{M}, w \Vdash \forall \mathbf{i} \varphi & \text { iff } & \text { for all } v \in W,\left(W, R, V_{v}^{\mathbf{i}}\right), w \Vdash \varphi ; \\
\mathbb{M}, w \Vdash \exists \mathbf{i} \varphi & \text { iff } & \text { there exists } v \in W \text { such that }\left(W, R, V_{v}^{\mathbf{i}}\right), w \Vdash \varphi ; \\
\mathbb{M}, w \Vdash \mathbf{l}(\varphi, \psi) & \text { iff } & \text { for all } v \in W(\text { if } \mathbb{M}, v \Vdash \varphi, \text { then } \mathbb{M}, v \Vdash \psi) .
\end{array}
$$

We can extend $V$ to a map from the set of formulas to $P(W)$ in the natural way.

### 2.2 Inequalities and complex inequalities

We will find it convenient to use the inequality notation $\varphi \leq \psi$ where $\varphi$ and $\psi$ are formulas. We use Ineq to denote the set of all inequalities in the expanded modal language. We define complex inequalities as follows:

[^1]\[

$$
\begin{gathered}
\text { Comp ::= Ineq | Comp \& Comp | Comp } \Rightarrow \text { Comp | } \\
\forall p \text { Comp | ヨpComp | } \forall \mathbf{i C o m p} \mid \exists i C o m p
\end{gathered}
$$
\]

Here we assume that the quantifiers have a higher precedence than \&, and \& is higher than $\Rightarrow$.
Complex inequalities are interpreted in models $\mathbb{M}=(W, R, V)$ instead of pointed models $(\mathbb{M}, w)$. The semantics of complex inequalities is defined as follows:

- An inequality is interpreted as follows:

$$
(W, R, V) \Vdash \varphi \leq \psi \text { iff }
$$

(for all $w \in W$, if $(W, R, V), w \Vdash \varphi$, then $(W, R, V), w \Vdash \psi)$;

- $(W, R, V) \Vdash \mathrm{Comp}_{1} \& \mathrm{Comp}_{2}$ iff $(W, R, V) \Vdash \mathrm{Comp}_{1}$ and $(W, R, V) \Vdash \mathrm{Comp}_{2}$;
- $(W, R, V) \Vdash$ Comp $_{1} \Rightarrow$ Comp $_{2}$ iff $\left((W, R, V) \Vdash\right.$ Comp $_{1}$ implies $(W, R, V) \Vdash$ $\mathrm{Comp}_{2}$ );
- $(W, R, V) \Vdash \forall p$ Comp iff for all $X \subseteq W,\left(W, R, V_{X}^{p}\right) \Vdash$ Comp;
- $(W, R, V) \Vdash \exists p$ Comp iff there exists an $X \subseteq W$ such that $\left(W, R, V_{X}^{p}\right) \Vdash$ Comp;
- $(W, R, V) \Vdash \forall \mathbf{i C o m p}$ iff for all $v \in W,\left(W, R, V_{v}^{\mathbf{i}}\right) \Vdash$ Comp;
- $(W, R, V) \Vdash \exists i C o m p$ iff there exists an $v \in W$ such that $\left(W, R, V_{v}^{\mathbf{i}}\right) \Vdash$ Comp.


### 2.3 Standard translation

In the correspondence language which is second-order due to the existence of propositional quantifiers in SOPML, we have a binary predicate symbol $R$ corresponding to the binary relation, a set of constant symbols $i$ corresponding to each nominal $\mathbf{i}$, a set of unary predicate symbols $P$ corresponding to each propositional variable $p$.

Definition 2. The standard translation of the expanded SOPML language is defined as follows:

- $S T_{x}(p):=P x$;
- $S T_{x}(\mathbf{i}):=x=i$;
- $S T_{x}(\perp):=\perp ;$
- $S T_{x}(\mathrm{~T}):=\mathrm{T}$;
- $S T_{x}(\neg \varphi):=\neg S T_{x}(\varphi)$;
- $S T_{x}(\varphi \wedge \psi):=S T_{x}(\varphi) \wedge S T_{x}(\psi)$;
- $S T_{x}(\varphi \vee \psi):=S T_{x}(\varphi) \vee S T_{x}(\psi)$;
- $S T_{x}(\varphi \rightarrow \psi):=S T_{x}(\varphi) \rightarrow S T_{x}(\psi) ;$
- $S T_{x}(\square \varphi):=\forall y\left(R x y \rightarrow S T_{y}(\varphi)\right)$;
- $S T_{x}(\diamond \varphi):=\exists y\left(R x y \wedge S T_{y}(\varphi)\right) ;$
- $S T_{x}(■ \varphi):=\forall y\left(R y x \rightarrow S T_{y}(\varphi)\right)$;
- $S T_{x}(\varphi):=\exists y\left(R y x \wedge S T_{y}(\varphi)\right)$;
- $S T_{x}(\forall p \varphi):=\forall P S T_{x}(\varphi)$;
- $S T_{x}(\exists p \varphi):=\exists P S T_{x}(\varphi)$;
- $S T_{x}(\forall \mathbf{i} \varphi):=\forall i S T_{x}(\varphi) ;$
- $S T_{x}(\exists \mathbf{i} \varphi):=\exists i S T_{x}(\varphi)$;
- $S T_{x}(\mathbf{l}(\varphi, \psi)):=\forall y\left(S T_{y}(\varphi) \rightarrow S T_{y}(\psi)\right)$.

The following proposition states that this translation is correct:
Proposition 3. For any Kripke model $\mathbb{M}$, any $w \in W$ and any expanded $\operatorname{SOPML}$ formula $\varphi$,

$$
\mathbb{M}, w \Vdash \varphi \text { iff } \mathbb{M} \vDash S T_{x}(\varphi)[x:=w] .
$$

For inequalities and complex inequalities, the standard translation is given in a global way:

Definition 4. - $S T(\varphi \leq \psi):=\forall x\left(S T_{x}(\varphi) \rightarrow S T_{x}(\psi)\right)$;

- $S T\left(\mathrm{Comp}_{1} \& \mathrm{Comp}_{2}\right)=S T\left(\mathrm{Comp}_{1}\right) \wedge S T\left(\mathrm{Comp}_{2}\right) ;$
- $S T\left(\mathrm{Comp}_{1} \Rightarrow \mathrm{Comp}_{2}\right)=S T\left(\mathrm{Comp}_{1}\right) \rightarrow S T\left(\mathrm{Comp}_{2}\right)$;
- $S T(\forall p($ Comp $)):=\forall P(S T($ Comp $))$;
- $S T(\exists p(\mathrm{Comp})):=\exists P(S T(\mathrm{Comp})) ;$
- $S T(\forall i(C o m p)):=\forall i(S T(C o m p)) ;$
- $S T(\exists i(C o m p)):=\exists i(S T(C o m p))$.

Proposition 5. For any Kripke model $\mathbb{M}$, any inequality Ineq, any complex inequality Comp,

$$
\begin{aligned}
& \mathbb{M} \Vdash \text { Ineq iff } \mathbb{M} \vDash S T \text { (Ineq); } \\
& \mathbb{M} \Vdash \text { Comp iff } \mathbb{M} \vDash S T \text { (Comp). }
\end{aligned}
$$

## 3 Sahlqvist formulas in second-order propositional modal logic

In this section, we define Sahlqvist formulas of second-order propositional modal logic step by step.
We first define (quantifier-free) positive formulas $\operatorname{POS}(\vec{p})$ whose propositonal variables are among $\vec{p}$ :
$\operatorname{POS}(\vec{p})::=p|\perp| \mathrm{T}|\operatorname{POS}(\vec{p}) \wedge \operatorname{POS}(\vec{p})| \operatorname{POS}(\vec{p}) \vee \operatorname{POS}(\vec{p})|\square \mathrm{POS}(\vec{p})| \diamond \operatorname{POS}(\vec{p})$
where $p$ is in $\vec{p}$. These positive formulas have similar roles to the positive consequent part in Sahlqvist formulas in basic modal logic, which are going to receive minimal valuations. The reason why we do not allow propositional quantifiers in positive formulas is that we want the formula after receiving the minimal valuations to be translated into a first-order formula, while propositional quantifiers will make it second-order.

### 3.1 The $\Pi_{1}$-fragment: Sahlqvist formulas in basic modal logic

We define the $\Pi_{1}$-Sahlqvist antecedent $\operatorname{Sahl}_{1}(\vec{p})$ whose propositonal variables are among $\vec{p}$ :

$$
\operatorname{Sahl}_{1}(\vec{p})::=\square^{n} p|\perp| \mathrm{T}|\neg \operatorname{POS}(\vec{p})| \operatorname{Sahl}_{1}(\vec{p}) \wedge \operatorname{Sahl}_{1}(\vec{p}) \mid \diamond \operatorname{Sahl}_{1}(\vec{p})
$$

where $p$ is in $\vec{p}$.
Then the $\Pi_{1}$-Sahlqvist formulas are defined as $\forall \vec{p}\left(\operatorname{Sahl}_{1}(\vec{p}) \rightarrow \mathrm{POS}(\vec{p})\right)$. Indeed, Sahlqvist formula $\sqrt[3]{ }$ in the basic modal logic setting can be treated as universally quantified by propositional quantifiers which bind all occurrences of propositional variables, so in this sense the $\Pi_{1}$-Sahlqvist formulas can be taken as the Sahlqvist formulas in basic modal logic.

[^2]
### 3.2 The $\Pi_{2}$-fragment

We define the PIA formula $\operatorname{PIA}(\vec{q}, \vec{p})$ as follows:

$$
\operatorname{PIA}(\vec{q}, \vec{p})::=p|\square \operatorname{PIA}(\vec{q}, \vec{p})| \operatorname{PIA}(\vec{q}, \vec{p}) \wedge \operatorname{PIA}(\vec{q}, \vec{p}) \mid \operatorname{POS}(\vec{q}) \vee \operatorname{PIA}(\vec{q}, \vec{p})
$$

where $p$ is in $\vec{p}$. Here the PIA formula has two bunches of propositional variables: $\vec{q}$ is to receive minimal valuations for $\vec{q}$ from somewhere else, and $\vec{p}$ is used to compute minimal valuations for $\vec{p}$. Then it is easy to see that $\operatorname{PIA}(\vec{q}, \vec{p})$ is equivalent to the form $\wedge \square(\operatorname{POS}(\vec{q}) \vee \square(\operatorname{POS}(\vec{q}) \vee \ldots p))$, where $p$ is in $\vec{p}$.
Now we can define $\Pi_{2}$-Sahlqvist antecedents as follows:

$$
\operatorname{Sahl}_{2}(\vec{p})::=\operatorname{Sahl}_{1}(\vec{p})\left|\forall \vec{q}\left(\operatorname{Sahl}_{1}(\vec{q}) \rightarrow \operatorname{PIA}(\vec{q}, \vec{p})\right)\right| \operatorname{Sahl}_{2}(\vec{p}) \wedge \operatorname{Sahl}_{2}(\vec{p}) \mid \diamond \operatorname{Sahl}_{2}(\vec{p})
$$

Then $\Pi_{2}$-Sahlqvist formulas are defined as $\forall \vec{p}\left(\operatorname{Sahl}_{2}(\vec{p}) \rightarrow \operatorname{POS}(\vec{p})\right)$.
It is easy to see that formulas of the form $\forall \vec{p}\left(\operatorname{Sahl}_{1}(\vec{p}) \wedge \forall \vec{q}\left(\operatorname{Sahl}_{1}(\vec{q}) \rightarrow \operatorname{PIA}(\vec{q}, \vec{p})\right) \rightarrow\right.$ $\operatorname{POS}(\vec{p})$ ) are in the $\Pi_{2}$-hierarchy.

### 3.3 The $\Pi_{n}$-fragment

Now for the $\Pi_{n}$-fragment, assume that we have already defined $\Pi_{n-1}$-Sahlqvist antecedents $\operatorname{Sahl}_{n-1}(\vec{p})$ and $\Pi_{n-1}-$ Sahlqvist formulas $\forall \vec{p}\left(\operatorname{Sahl}_{n-1}(\vec{p}) \rightarrow \operatorname{POS}(\vec{p})\right)$, then we can define $\Pi_{n}$-Sahlqvist antecedents as follows:
$\operatorname{Sahl}_{n}(\vec{p})::=\operatorname{Sahl}_{n-1}(\vec{p})\left|\forall \vec{q}\left(\operatorname{Sahl}_{n-1}(\vec{q}) \rightarrow \operatorname{PIA}(\vec{q}, \vec{p})\right)\right| \operatorname{Sahl}_{n}(\vec{p}) \wedge \operatorname{Sahl}_{n}(\vec{p}) \mid \diamond \operatorname{Sahl}_{n}(\vec{p})$
Then $\Pi_{n}$-Sahlqvist formulas are defined as $\forall \vec{p}\left(\operatorname{Sahl}_{n}(\vec{p}) \rightarrow \operatorname{POS}(\vec{p})\right)$.

## 4 The Algorithm ALBA ${ }^{\text {SOMPL }}$

In the present section, we define the correspondence algorithm ALBA ${ }^{\text {SOMPL }}$ for second-order propositional modal logic, in the style of [13, 14]. The algorithm receives a $\Pi_{n}$-Sahlqvist formula $\forall \vec{p}\left(\operatorname{Sahl}_{n}(\vec{p}) \rightarrow \operatorname{POS}(\vec{p})\right)$ as input and goes in three stages.

## 1. Preprocessing and first approximation:

The algorithm receives a $\Pi_{n}$-Sahlqvist formula $\forall \vec{p}\left(\operatorname{Sahl}_{n}(\vec{p}) \rightarrow \mathrm{POS}(\vec{p})\right)$ as input, and then apply the rewriting rule:

$$
\frac{\forall \vec{p}\left(\operatorname{Sahl}_{n}(\vec{p}) \rightarrow \operatorname{POS}(\vec{p})\right)}{\forall \vec{p}\left(\operatorname{Sahl}_{n}(\vec{p}) \leq \operatorname{POS}(\vec{p})\right)}
$$

Then apply the first-approximation rule:

$$
\frac{\forall \vec{p}\left(\operatorname{Sahl}_{n}(\vec{p}) \leq \operatorname{POS}(\vec{p})\right)}{\forall \vec{p} \forall \mathbf{i}_{0}\left(\mathbf{i}_{0} \leq \operatorname{Sahl}_{n}(\vec{p}) \Rightarrow \mathbf{i}_{0} \leq \operatorname{POS}(\vec{p})\right)}
$$

## 2. The reduction stage:

In this stage, we aim at reducing $\mathbf{i} \leq \operatorname{Sahl}_{n}(\vec{p})$ to a complex inequality in which $p$ occurs either in the form $\varphi \leq p$ where $\varphi$ is pure or in the form $\mathbf{j} \leq \neg \operatorname{POS}(\vec{p})$.
(a) The commutativity rule and the associativity rule for $\&$;
(b) The rules for nominals:
i. Splitting rule:

$$
\frac{\mathbf{i} \leq \alpha \wedge \beta}{\mathbf{i} \leq \alpha \& \mathbf{i} \leq \beta}(S p l-N o m)
$$

ii. Separation rule:

$$
\frac{\mathbf{i} \leq \alpha \rightarrow \beta}{\mathbf{i} \leq \alpha \Rightarrow \mathbf{i} \leq \beta}(\text { Sep }- \text { Nom })
$$

iii. Quantifier rule:

$$
\frac{\mathbf{i} \leq \forall q \alpha}{\forall q(\mathbf{i} \leq \alpha)}(\text { Quant }-N o m)
$$

iv. Approximation rule:

$$
\frac{\mathbf{i} \leq \diamond \alpha}{\exists \mathbf{j}(\mathbf{j} \leq \alpha \& \mathbf{i} \leq \diamond \mathbf{j})}(\text { Approx }- \text { Nom })
$$

The nominals introduced by the approximation rule must not occur in the whole complex inequality before applying the rule.
(c) The residuation rules:

$$
\frac{\alpha \leq \square \beta}{\diamond \alpha \leq \beta}(\text { Res }-\square) \quad \frac{\alpha \leq \beta \vee \gamma}{\alpha \wedge \neg \beta \leq \gamma}(\text { Res }-\vee)
$$

(d) The splitting rule:

$$
\frac{\alpha \leq \beta \wedge \gamma}{\alpha \leq \beta \& \alpha \leq \gamma}(S \text { plitting })
$$

(e) The quantifier rules:

$$
\frac{\exists \mathbf{j}\left(\text { Comp }_{1}\right) \& \text { Comp }_{2}}{\exists \mathbf{j}\left(\text { Comp }_{1} \& \text { Comp }_{2}\right)}(\text { Scope }-\&) \quad \frac{\exists \mathbf{j}\left(\text { Comp }_{1}\right) \Rightarrow \text { Comp }_{2}}{\forall \mathbf{j}\left(\text { Comp }_{1} \Rightarrow \text { Comp }_{2}\right)}(\text { Scope }-\Rightarrow)
$$

where Comp 2 does not have free occurrences of $\mathbf{j}$.

$$
\begin{array}{ll}
\frac{\forall q \forall p(\text { Comp })}{\forall p \forall q(\text { Comp })}(E x-p q) & \frac{\forall \mathbf{i} \forall p(\text { Comp })}{\forall p \forall \mathbf{i}(\text { Comp })}(E x-p \mathbf{i}) \\
\frac{\forall p \forall \mathbf{i}(\text { Comp })}{\forall \mathbf{i} \forall p(\text { Comp })}(E x-\mathbf{i} p) & \frac{\forall \mathbf{i} \forall \mathbf{j}(\text { Comp })}{\forall \mathbf{j} \forall \mathbf{i}(\text { Comp })}(E x-\mathbf{j i})
\end{array}
$$

$$
\frac{\forall p\left(\mathrm{Comp}_{1} \Rightarrow\left(\mathrm{Comp}_{2} \& \mathrm{Comp}_{3}\right)\right)}{\forall p\left(\mathrm{Comp}_{1} \Rightarrow \mathrm{Comp}_{2}\right) \& \forall p\left(\mathrm{Comp}_{1} \Rightarrow \mathrm{Comp}_{3}\right)}(S p l-\text { Quant }-p)
$$

$$
\frac{\forall \mathbf{i}\left(\mathrm{Comp}_{1} \Rightarrow\left(\mathrm{Comp}_{2} \& \mathrm{Comp}_{3}\right)\right)}{\forall \mathbf{i}\left(\mathrm{Comp}_{1} \Rightarrow \operatorname{Comp}_{2}\right) \& \forall \mathbf{i}\left(\mathrm{Comp}_{1} \Rightarrow \mathrm{Comp}_{3}\right)}(S p l-\text { Quant }-\mathbf{i})
$$

(f) The Ackermann rule:

In this step, we compute the minimal valuation for propositional variables and use the Ackermann rule to eliminate all the propositional variables.
$\frac{\forall q\left(\alpha_{1} \leq \beta_{1} \& \ldots \& \alpha_{n} \leq \beta_{n} \& \psi_{1} \leq q \& \ldots \& \psi_{m} \leq q \Rightarrow \alpha \leq \beta\right)}{\alpha_{1}[\bigvee \psi / q] \leq \beta_{1}[\bigvee \psi / q] \& \ldots \& \alpha_{n}[\bigvee \psi / q] \leq \beta_{n}[\bigvee \psi / q] \Rightarrow \alpha[\bigvee \psi / q] \leq \beta[\bigvee \psi / q]}$ where:
i. $\varphi[\theta / p]$ means uniformly replace occurrences of $p$ in $\varphi$ by $\theta$;
ii. $\vee \psi=\psi_{1} \vee \ldots \vee \psi_{m}$;
iii. Each $\alpha_{i}$ is positive, and each $\beta_{i}$ negative in $q$, for $1 \leq i \leq n$;
iv. $\alpha$ is negative in $q$ and $\beta$ is positive in $q$;
v. Each $\psi_{i}$ is pure (therefore $q$ does not occur in $\psi_{i}$ ).
(g) The packing rule:

$$
\frac{\forall \mathbf{i}\left(\alpha_{1} \leq \beta_{1} \& \ldots \& \alpha_{n} \leq \beta_{n} \Rightarrow \alpha \leq \beta\right)}{\exists \mathbf{i}\left(\mathbf{l}\left(\alpha_{1}, \beta_{1}\right) \wedge \ldots \wedge \mathbf{l}\left(\alpha_{n}, \beta_{n}\right) \wedge \alpha\right) \leq \beta}
$$

where $\beta$ does not contain occurrences of $\mathbf{i}$.
3. Output: By the execution of the algorithm, we can guarantee (see Theorem 6.1) that given a $\Pi_{n}$-Sahlqvist formula as input, we can rewrite it into a pure complex inequality. Then we use standard translation to translate it into a first-order formula.

From the design of the algorithm, we can see that it is specifically designed for $\Pi_{n^{-}}$ Sahlqvist formulas. Therefore, when we try to extend the $\Pi_{n}$-Sahlqvist fragment, we need to revise the rules accordingly.

## 5 Soundness of ALBA ${ }^{\text {SOPML }}$

In the present section, we will prove the soundness of the algorithm $A L B A^{S O P M L}$. The basic proof structure is similar to [14].

Theorem 5.1 (Soundness). If $\mathrm{ALBA}^{\mathrm{SOPML}}$ runs successfully on an input $\Pi_{n}$-Sahlqvist formula $\forall \vec{p}\left(\operatorname{Sahl}_{n}(\vec{p}) \rightarrow \operatorname{POS}(\vec{p})\right)$ and outputs a first-order formula $\mathrm{FO}\left(\forall \vec{p}\left(\operatorname{Sahl}_{n}(\vec{p}) \rightarrow\right.\right.$ $\operatorname{POS}(\vec{p}))$ ), then for any Kripke frame $\mathbb{F}=(W, R)$,

$$
\mathbb{F} \Vdash \forall \vec{p}\left(\operatorname{Sahl}_{n}(\vec{p}) \rightarrow \operatorname{POS}(\vec{p})\right) \text { iff } \mathbb{F} \vDash \mathrm{FO}\left(\forall \vec{p}\left(\operatorname{Sahl}_{n}(\vec{p}) \rightarrow \operatorname{POS}(\vec{p})\right)\right)
$$

Proof. The proof goes similarly to [14, Theorem 8.1]. Let $\forall \vec{p}\left(\operatorname{Sahl}_{n}(\vec{p}) \leq \operatorname{POS}(\vec{p})\right)$ denote the complex inequality after the first rewrite rule, $\forall \vec{p} \forall \mathbf{i}_{0}\left(\mathbf{i}_{0} \leq \operatorname{Sahl}_{n}(\vec{p}) \Rightarrow\right.$ $\left.\mathbf{i}_{0} \leq \operatorname{POS}(\vec{p})\right)$ denote the complex inequality after the first approximation rule, $\operatorname{Comp}\left(\forall \vec{p} \forall \mathbf{i}_{0}\left(\mathbf{i}_{0} \leq \operatorname{Sahl}_{n}(\vec{p}) \Rightarrow \mathbf{i}_{0} \leq \mathrm{POS}(\vec{p})\right)\right)$ denote the complex inequality after Stage 2, and $\mathrm{FO}\left(\forall \vec{p}\left(\operatorname{Sahl}_{n}(\vec{p}) \rightarrow \mathrm{POS}(\vec{p})\right)\right)$ denote the standard translation of the complex inequality obtained after Stage 2 , then it suffices to show the equivalence from (1) to (5) given below:

$$
\begin{align*}
& \mathbb{F} \Vdash \forall \vec{p}\left(\operatorname{Sahl}_{n}(\vec{p}) \rightarrow \operatorname{POS}(\vec{p})\right)  \tag{1}\\
& \mathbb{F} \Vdash \forall \vec{p}\left(\operatorname{Sahl}_{n}(\vec{p}) \leq \operatorname{POS}(\vec{p})\right)  \tag{2}\\
& \mathbb{F} \Vdash \forall \vec{p} \forall \mathbf{i}_{0}\left(\mathbf{i}_{0} \leq \operatorname{Sahl}_{n}(\vec{p}) \Rightarrow \mathbf{i}_{0} \leq \operatorname{POS}(\vec{p})\right)  \tag{3}\\
& \mathbb{F} \Vdash \operatorname{Comp}\left(\forall \vec{p} \forall \mathbf{i}_{0}\left(\mathbf{i}_{0} \leq \operatorname{Sahl}_{n}(\vec{p}) \Rightarrow \mathbf{i}_{0} \leq \operatorname{POS}(\vec{p})\right)\right)  \tag{4}\\
& \mathbb{F} \vDash \operatorname{FO}\left(\forall \vec{p}\left(\operatorname{Sahl}_{n}(\vec{p}) \rightarrow \operatorname{POS}(\vec{p})\right)\right) \tag{5}
\end{align*}
$$

- the equivalence between (1) and (2) follows from Proposition 6:
- the equivalence between (2) and (3) follows from Proposition 7,
- the equivalence between (3) and (4) follows from Proposition 8 ;
- the equivalence between (4) and (5) follows from Proposition 5 ,

In the remainder of this section, we prove the soundness of the rules in Stage 1 and 2.

Proposition 6 (Soundness of the first rewrite rule in Stage 1). The first rewrite rule is sound in both directions in $\mathbb{F}$, i.e. the formula before the rule is valid in $\mathbb{F}$ iff the complex inequality after the rule is valid in $\mathbb{F}$.

Proof.

$$
\mathbb{F} \Vdash \forall \vec{p}\left(\operatorname{Sahl}_{n}(\vec{p}) \rightarrow \operatorname{POS}(\vec{p})\right)
$$

iff for all $V,(\mathbb{F}, V) \Vdash \forall \vec{p}\left(\operatorname{Sahl}_{n}(\vec{p}) \rightarrow \operatorname{POS}(\vec{p})\right)$
iff for all $V$, for all $\vec{X} \subseteq W,\left(\mathbb{F}, V_{\vec{X}}^{\vec{p}}\right) \Vdash \operatorname{Sahl}_{n}(\vec{p}) \rightarrow \operatorname{POS}(\vec{p})$
iff for all $V$, for all $\vec{X} \subseteq W,\left(\mathbb{F}, V_{\vec{X}}^{\vec{p}}\right) \Vdash \operatorname{Sahl}_{n}(\vec{p}) \leq \operatorname{POS}(\vec{p})$
iff for all $V,(\mathbb{F}, V) \Vdash \forall \vec{p}\left(\operatorname{Sahl}_{n}(\vec{p}) \leq \operatorname{POS}(\vec{p})\right)$
iff $\quad \mathbb{F} \Vdash \forall \vec{p}\left(\operatorname{Sahl}_{n}(\vec{p}) \leq \operatorname{POS}(\vec{p})\right)$.

Proposition 7 (Soundness of the first approximation rule in Stage 1). The first approximation rule is sound in both directions in $\mathbb{F}$, i.e. the complex inequality before the rule is valid in $\mathbb{F}$ iff the complex inequality after the rule is valid in $\mathbb{F}$.

Proof. $\mathbb{F} \Vdash \forall \vec{p}\left(\operatorname{Sahl}_{n}(\vec{p}) \leq \operatorname{POS}(\vec{p})\right)$
iff for all $V,(\mathbb{F}, V) \Vdash \forall \vec{p}\left(\operatorname{Sahl}_{n}(\vec{p}) \leq \operatorname{POS}(\vec{p})\right)$
iff for all $V$, for all $\vec{X} \subseteq W,\left(\mathbb{F}, V_{\vec{X}}^{\vec{p}}\right) \Vdash \operatorname{Sahl}_{n}(\vec{p}) \leq \operatorname{POS}(\vec{p})$
iff for all $V$, all $\vec{X} \subseteq W$, all $w \in W,\left(\mathbb{F}, V_{\vec{X}}^{\vec{p}}\right), w \Vdash \operatorname{Sahl}_{n}(\vec{p})$ implies $\left(\mathbb{F}, V_{\vec{X}}^{\vec{p}}\right), w \Vdash$ POS( $\vec{p}$ )
iff for all $V$, all $\vec{X} \subseteq W$, all $w \in W,\left(\mathbb{F}, V_{\vec{X}, w}^{\vec{p}, \mathbf{i}_{0}}\right) \Vdash \mathbf{i}_{0} \leq \operatorname{Sahl}_{n}(\vec{p})$ implies $\left(\mathbb{F}, V_{\vec{X}, w}^{\vec{p} \mathbf{i}_{0}}\right) \Vdash$ $\mathbf{i}_{0} \leq \operatorname{POS}(\vec{p})$
iff for all $V,(\mathbb{F}, V) \Vdash \forall \vec{p} \forall \mathbf{i}_{0}\left(\mathbf{i}_{0} \leq \operatorname{Sahl}_{n}(\vec{p}) \Rightarrow \mathbf{i}_{0} \leq \operatorname{POS}(\vec{p})\right)$
iff $\mathbb{F} \Vdash \forall \vec{p} \forall \mathbf{i}_{0}\left(\mathbf{i}_{0} \leq \operatorname{Sahl}_{n}(\vec{p}) \Rightarrow \mathbf{i}_{0} \leq \operatorname{POS}(\vec{p})\right)$.
Proposition 8 (Soundness of the rules in Stage 2). The rules in Stage 2 are sound in both directions in $\mathbb{F}$, i.e. the complex inequality before the rule is valid in $\mathbb{F}$ iff the complex inequality after the rule is valid in $\mathbb{F}$.

Proof. It suffices to show that each rule in Stage 2 is sound in both directions in $\mathbb{F}$.

- For the commutativity rule and associativity rule for $\&$, by the validity of $\alpha \wedge \beta \leftrightarrow \beta \wedge \alpha$ and $(\alpha \wedge \beta) \wedge \gamma \leftrightarrow \alpha \wedge(\beta \wedge \gamma)$.
- For the splitting rule for nominals and the splitting rule for arbitrary formulas, it follows from the following equivalence: for all Kripke frame $\mathbb{F}$ and all $V, \mathbb{F}, V \Vdash \alpha \leq \beta \wedge \gamma \operatorname{iff}(\mathbb{F}, V \Vdash \alpha \leq \beta$ and $\mathbb{F}, V \Vdash \alpha \leq \gamma)$.
- For the separation rule for nominals, it follows from the following equivalence: for all $\mathbb{F}=(W, R)$ and all $V$,
$\mathbb{F}, V \Vdash \mathbf{i} \leq \alpha \rightarrow \beta$
iff $\mathbb{F}, V, V(\mathbf{i}) \Vdash \alpha \rightarrow \beta$
iff $\mathbb{F}, V, V(\mathbf{i}) \Vdash \alpha$ implies $\mathbb{F}, V, V(\mathbf{i}) \Vdash \beta$
iff $\mathbb{F}, V \Vdash \mathbf{i} \leq \alpha$ implies $\mathbb{F}, V \Vdash \mathbf{i} \leq \beta$
iff $\mathbb{F}, V \Vdash \mathbf{i} \leq \alpha \Rightarrow \mathbf{i} \leq \beta$.
- For the quantifier rule for nominals, it follows from the following equivalence: for all $\mathbb{F}=(W, R)$ and any $V$,

```
F},V\Vdash\mathbf{i}\leq\forallq
iff }\mathbb{F},V,V(\mathbf{i})\Vdash\forallq
iff for all }X\subseteqW,\mathbb{F},\mp@subsup{V}{X}{q},V(\mathbf{i})\Vdash
iff for all }X\subseteqW,\mathbb{F},\mp@subsup{V}{X}{q},\mp@subsup{V}{X}{q}(\mathbf{i})\Vdash
iff for all }X\subseteqW,\mathbb{F},\mp@subsup{V}{X}{q}\Vdash\mathbf{i}\leq
iff \mathbb{F},V\Vdash\forallq(\mathbf{i}\leq\alpha).
```

- For the approximation rule for nominals, it suffices to show that for any $\mathbb{F}=(W, R)$ and any $V$,

1. if $(\mathbb{F}, V) \Vdash \mathbf{i} \leq \diamond \alpha$, then there is a valuation $V^{\mathbf{j}}$ such that $V^{\mathbf{j}}$ is the same as $V$ except $V^{\mathbf{j}}(\mathbf{j})$, and $\left(\mathbb{F}, V^{\mathbf{j}}\right) \Vdash \mathbf{i} \leq \diamond \mathbf{j}$ and $\left(\mathbb{F}, V^{\mathbf{j}}\right) \Vdash \mathbf{j} \leq \alpha$;
2. if $(\mathbb{F}, V) \Vdash \mathbf{i} \leq \diamond \mathbf{j}$ and $(\mathbb{F}, V) \Vdash \mathbf{j} \leq \alpha$, then $(\mathbb{F}, V) \Vdash \mathbf{i} \leq \diamond \alpha$.

For item 1 , if $(\mathbb{F}, V) \Vdash \mathbf{i} \leq \diamond \alpha$, then $(\mathbb{F}, V), V(\mathbf{i}) \Vdash \diamond \alpha$, therefore there exists a $w \in W$ such that $(V(\mathbf{i}), w) \in R$ and $(\mathbb{F}, V), w \Vdash \alpha$. Now take $V^{\mathbf{j}}$ such that $V^{\mathbf{j}}$ is the same as $V$ except that $V^{\mathbf{j}}(\mathbf{j})=\{w\}$, then $\left(V^{\mathbf{j}}(\mathbf{i}), V^{\mathbf{j}}(\mathbf{j})\right) \in R$, so $\left(\mathbb{F}, V^{\mathbf{j}}\right) \Vdash \mathbf{i} \leq \diamond \mathbf{j}$ and $\left(\mathbb{F}, V^{\mathbf{j}}\right) \Vdash \mathbf{j} \leq \alpha$.
For item 2 , suppose $(\mathbb{F}, V) \Vdash \mathbf{i} \leq \diamond \mathbf{j}$ and $(\mathbb{F}, V) \Vdash \mathbf{j} \leq \alpha$. Then $(V(\mathbf{i}), V(\mathbf{j})) \in R$ and $(\mathbb{F}, V), V(\mathbf{j}) \Vdash \alpha$, so $(\mathbb{F}, V), V(\mathbf{i}) \Vdash \diamond \alpha$, therefore $(\mathbb{F}, V) \Vdash \mathbf{i} \leq \diamond \alpha$.

- For the residuation rule for $\square$, it suffices to show that for any $\mathbb{F}=(W, R)$ and any $V,(\mathbb{F}, V) \Vdash \alpha \leq \beta$ iff $(\mathbb{F}, V) \Vdash \alpha \leq \square \beta$.
$\Rightarrow$ : if $(\mathbb{F}, V) \Vdash \alpha \leq \beta$, then for all $w \in W$, if $(\mathbb{F}, V), w \Vdash \alpha$, then $(\mathbb{F}, V), w \Vdash$ $\beta$. Our aim is to show that for all $v \in W$, if $(\mathbb{F}, V), v \Vdash \alpha$, then $(\mathbb{F}, V), v \Vdash \square \beta$.

Consider any $v \in W$ such that $(\mathbb{F}, V), v \Vdash \alpha$. Then for any $u \in W$ such that $(v, u) \in R,(\mathbb{F}, V), u \Vdash \alpha$. Since $(\mathbb{F}, V) \Vdash \alpha \leq \beta$, we have that $(\mathbb{F}, V), u \Vdash \beta$, so for any $u \in W$ such that $(v, u) \in R,(\mathbb{F}, V), u \Vdash \beta$, so $(\mathbb{F}, V), v \Vdash \square \beta$.
$\Leftarrow:$ if $(\mathbb{F}, V) \Vdash \alpha \leq \square \beta$, then for all $w \in W$, if $(\mathbb{F}, V), w \Vdash \alpha$, then $(\mathbb{F}, V), w \Vdash$ $\square \beta$. Our aim is to show that for all $v \in W$, if $(\mathbb{F}, V), v \Vdash \alpha$, then $(\mathbb{F}, V), v \Vdash \beta$.
Now assume that $(\mathbb{F}, V), v \Vdash \alpha$. Then there is a $u \in W$ such that $(u, v) \in$ $R$ and $(\mathbb{F}, V), u \Vdash \alpha$. By $(\mathbb{F}, V) \Vdash \alpha \leq \square \beta$, we have that $(\mathbb{F}, V), u \Vdash \square \beta$. Therefore, for $v \in W$, we have $(u, v) \in R$, thus $(\mathbb{F}, V), v \Vdash \beta$.

- For the residuation rule for $\vee$, it follows from the validity of $(\alpha \rightarrow(\beta \vee \gamma)) \leftrightarrow$ $((\alpha \wedge \neg \beta) \rightarrow \gamma)$.
- For the quantifier scope rules, it follows from the validity of $\exists x \alpha \wedge \beta \leftrightarrow$ $\exists x(\alpha \wedge \beta)$ and $(\exists x \alpha \rightarrow \beta) \leftrightarrow \forall x(\alpha \rightarrow \beta)$ (where $x$ does not occur in $\beta$ ).
- For the quantifier exchange rules, it follows from the validity of $\forall P \forall Q \alpha \leftrightarrow$ $\forall Q \forall P \alpha, \forall P \forall x \alpha \leftrightarrow \forall x \forall P \alpha$ and $\forall x \forall y \alpha \leftrightarrow \forall y \forall x \alpha$.
- For the quantifier splitting rules, it follows from the validity of $\forall P(\alpha \rightarrow$ $\beta \wedge \gamma) \leftrightarrow \forall P(\alpha \rightarrow \beta) \wedge \forall P(\alpha \rightarrow \gamma)$ and $\forall x(\alpha \rightarrow \beta \wedge \gamma) \leftrightarrow \forall x(\alpha \rightarrow$ $\beta) \wedge \forall x(\alpha \rightarrow \gamma)$.
- For the Ackermann rule, it suffices to show that for any $\mathbb{F}=(W, R)$ and any $V$,
$\mathbb{F}, V \Vdash \forall q\left(\alpha_{1} \leq \beta_{1} \& \ldots \& \alpha_{n} \leq \beta_{n} \& \psi_{1} \leq q \& \ldots \& \psi_{m} \leq q \Rightarrow \alpha \leq \beta\right)$
iff $\mathbb{F}, V \Vdash \alpha_{1}[\bigvee \psi / q] \leq \beta_{1}[\bigvee \psi / q] \& \ldots \& \alpha_{n}[\bigvee \psi / q] \leq \beta_{n}[\bigvee \psi / q] \Rightarrow \alpha[\bigvee \psi / q] \leq$ $\beta[\bigvee \psi / q]$.
$\Rightarrow$ : Easy, by instantiation of the propositional quantifier.
$\Leftarrow:$ Assmue $\mathbb{F}, V \Vdash \alpha_{1}[\bigvee \psi / q] \leq \beta_{1}[\bigvee \psi / q] \& \ldots \& \alpha_{n}[\bigvee \psi / q] \leq \beta_{n}[\bigvee \psi / q] \Rightarrow$ $\alpha[\bigvee \psi / q] \leq \beta[\bigvee \psi / q]$. Then for any $X \subseteq W$, it suffices to show that if $\mathbb{F}, V_{X}^{q} \Vdash \alpha_{1} \leq \beta_{1} \& \ldots \& \alpha_{n} \leq \beta_{n} \& \psi_{1} \leq q \& \ldots \& \psi_{m} \leq q$, then $\mathbb{F}, V_{X}^{q} \Vdash \alpha \leq$ $\beta$. Now assume $\mathbb{F}, V_{X}^{q} \Vdash \alpha_{1} \leq \beta_{1} \& \ldots \& \alpha_{n} \leq \beta_{n} \& \psi_{1} \leq q \& \ldots \& \psi_{m} \leq q$, then $V_{X}^{q}\left(\alpha_{i}\right) \subseteq V_{X}^{q}\left(\beta_{i}\right)$ for $1 \leq i \leq n$, and $V_{X}^{q}\left(\psi_{j}\right) \subseteq X$ for $1 \leq j \leq m$, therefore $V(\bigvee \psi)=V_{X}^{q}(\bigvee \psi) \subseteq X$. Since each $\alpha_{i}$ is positive and each $\beta_{i}$ is negative in $q$, we have that $V\left(\alpha_{i}[\bigvee \psi / q]\right) \subseteq V_{X}^{q}\left(\alpha_{i}\right) \subseteq V_{X}^{q}\left(\beta_{i}\right) \subseteq V\left(\beta_{i}[\bigvee \psi / q]\right), 1 \leq i \leq n$, so by $\mathbb{F}, V \Vdash \alpha_{1}[\bigvee \psi / q] \leq \beta_{1}[\bigvee \psi / q] \& \ldots \& \alpha_{n}[\bigvee \psi / q] \leq \beta_{n}[\bigvee \psi / q] \Rightarrow$ $\alpha[\bigvee \psi / q] \leq \beta[\bigvee \psi / q]$ we have $V(\alpha[\bigvee \psi / q]) \subseteq V(\beta[\bigvee \psi / q])$, therefore by $\alpha$ is negative and $\beta$ is positive in $q$, we have $V_{X}^{q}(\alpha) \subseteq V(\alpha[\bigvee \psi / q]) \subseteq$ $V(\beta[\bigvee \psi / q]) \subseteq V_{X}^{q}(\beta)$, so $\mathbb{F}, V_{X}^{q} \Vdash \alpha \leq \beta$, which concludes the proof.
- For the packing rule, it follows from the following equivalence: for any $\mathbb{F}=(W, R)$ and any $V$,
$\mathbb{F}, V \Vdash \forall \mathbf{i}\left(\alpha_{1} \leq \beta_{1} \& \ldots \& \alpha_{n} \leq \beta_{n} \Rightarrow \alpha \leq \beta\right)$
iff for all $w \in W,\left(\mathbb{F}, V_{w}^{\mathbf{i}}\right) \Vdash \alpha_{1} \leq \beta_{1} \& \ldots \& \alpha_{n} \leq \beta_{n} \Rightarrow \alpha \leq \beta$
iff for all $w \in W$, if $\left(\mathbb{F}, V_{w}^{\mathbf{i}}\right) \Vdash \alpha_{i} \leq \beta_{i}$ for $1 \leq i \leq n$, then $\left(\mathbb{F}, V_{w}^{\mathbf{i}}\right) \Vdash \alpha \leq \beta$
iff for all $w \in W$, if $\left(\mathbb{F}, V_{w}^{\mathbf{i}}\right) \Vdash \mathbb{I}\left(\alpha_{i}, \beta_{i}\right)$ for $1 \leq i \leq n$, then $\left(\mathbb{F}, V_{w}^{\mathbf{i}}\right) \Vdash \alpha \leq \beta$
iff for all $w \in W$, if $\left(\mathbb{F}, V_{w}^{\mathbf{i}}\right) \Vdash \mathbb{l}\left(\alpha_{i}, \beta_{i}\right)$ for $1 \leq i \leq n$, then for all $v \in W$, if $\left(\mathbb{F}, V_{w}^{\mathbf{i}}\right), v \Vdash \alpha$ then $\left(\mathbb{F}, V_{w}^{\mathbf{i}}\right), v \Vdash \beta$
iff for all $w, v \in W$, if $\left(\mathbb{F}, V_{w}^{\mathbf{i}}\right) \Vdash \mathbb{l}\left(\alpha_{i}, \beta_{i}\right)$ for $1 \leq i \leq n$, then if $\left(\mathbb{F}, V_{w}^{\mathbf{i}}\right), v \Vdash \alpha$ then $\left(\mathbb{F}, V_{w}^{\mathbf{i}}\right), v \Vdash \beta$
iff for all $w, v \in W$, if $\left(\mathbb{F}, V_{w}^{\mathbf{i}}\right) \Vdash \mathbf{l}\left(\alpha_{i}, \beta_{i}\right)$ for $1 \leq i \leq n$ and $\left(\mathbb{F}, V_{w}^{\mathbf{i}}\right), v \Vdash \alpha$, then $\left(\mathbb{F}, V_{w}^{\mathbf{i}}\right), v \Vdash \beta$
iff for all $w, v \in W$, if $\left(\mathbb{F}, V_{w}^{\mathbf{i}}\right), v \Vdash \mathbb{l}\left(\alpha_{i}, \beta_{i}\right)$ for $1 \leq i \leq n$ and $\left(\mathbb{F}, V_{w}^{\mathbf{i}}\right), v \Vdash \alpha$, then $\left(\mathbb{F}, V_{w}^{\mathbf{i}}\right), v \Vdash \beta$
iff for all $w, v \in W$, if $\left(\mathbb{F}, V_{w}^{\mathbf{i}}\right), v \Vdash \mathbf{l}\left(\alpha_{1}, \beta_{1}\right) \wedge \ldots \wedge \mathbf{l}\left(\alpha_{n}, \beta_{n}\right) \wedge \alpha$, then $\left(\mathbb{F}, V_{w}^{\mathbf{i}}\right), v \Vdash$ $\beta$
iff for all $w, v \in W$, if $\left(\mathbb{F}, V_{w}^{\mathbf{i}}\right), v \Vdash \mathbf{l}\left(\alpha_{1}, \beta_{1}\right) \wedge \ldots \wedge \mathbf{l}\left(\alpha_{n}, \beta_{n}\right) \wedge \alpha$, then $(\mathbb{F}, V), v \Vdash$ $\beta$
iff for all $v \in W$, if there exists a $w \in W$ such that $\left(\mathbb{F}, V_{w}^{\mathbf{i}}\right), v \Vdash \mathbf{l}\left(\alpha_{1}, \beta_{1}\right) \wedge \ldots \wedge$ $\mathbf{l}\left(\alpha_{n}, \beta_{n}\right) \wedge \alpha$, then $(\mathbb{F}, V), v \Vdash \beta$
iff for all $v \in W$, if $(\mathbb{F}, V), v \Vdash \exists \mathbf{i}\left(\mathbf{l}\left(\alpha_{1}, \beta_{1}\right) \wedge \ldots \wedge \mathbf{l}\left(\alpha_{n}, \beta_{n}\right) \wedge \alpha\right)$, then $(\mathbb{F}, V), v \Vdash$ $\beta$
iff $\mathbb{F}, V \Vdash \exists \mathbf{i}\left(\mathbf{l}\left(\alpha_{1}, \beta_{1}\right) \wedge \ldots \wedge \mathbf{l}\left(\alpha_{n}, \beta_{n}\right) \wedge \alpha\right) \leq \beta$.


## 6 Success of ALBA ${ }^{\text {SOPML }}$ on $\Pi_{n}$-Sahlqvist formulas

By success of ALBA ${ }^{\text {SOPML }}$ on $\Pi_{n}$-Sahlqvist formulas we mean that the algorithm ALBA ${ }^{\text {SOPML }}$ can transform any input $\Pi_{n}$-Sahlqvist formula into a pure complex inequality which does not contain any propositional variables or any propositional quantifiers (here we allow nominal quantifiers to occur). We prove this by induction on $n$ that ALBA ${ }^{\text {SOPML }}$ successfully transforms $\mathbf{i} \leq \operatorname{Sahl}_{n}(\vec{p})$ into given shapes.

Proposition 9. In the reduction stage, by running the algorithm ALBA ${ }^{\text {SOPML }}, \mathbf{i} \leq$ $\mathrm{Sahl}_{1}(\vec{p})$ can be reduced to the following complex inequality:

$$
\exists \overrightarrow{\mathbf{j}}(\text { NEG \& NOM \& MinVal) }
$$

where

- $\exists \overrightarrow{\mathbf{j}}$ is a (possibly empty) bunch of nominal quantifiers;
- NEG is a (possibly empty) meta-conjunction of inequalities of the form $\mathbf{j} \leq$ $\neg \operatorname{POS}(\vec{p})$, where $\mathbf{j}$ is either $\mathbf{i}$ or in $\overrightarrow{\mathbf{j}}$,
- NOM is a (possibly empty) meta-conjunction of inequalities of the form $\mathbf{j} \leq$ $\diamond \mathbf{k}$, where $\mathbf{j}, \mathbf{k}$ are either $\mathbf{i}$ or in $\overrightarrow{\mathbf{j}}$,
- MinVal is a (possibly empty) meta-conjunction of inequalities of the form $\psi \leq p$, where $\psi$ is pure and $p$ is in $\vec{p}$.

Proof. We prove by induction on the formula complexity of $\operatorname{Sahl}_{1}(\vec{p})$.

- For the case where $\operatorname{Sahl}_{1}(\vec{p})=\perp, \top$, trivial.
- For the case where $\operatorname{Sahl}_{1}(\vec{p})=\square^{n} p$, by applying the residuation rule for $\square$, we get ${ }^{n} \mathbf{i} \leq p$, which belongs to MinVal.
- For the case where $\operatorname{Sahl}_{1}(\vec{p})=\neg \operatorname{POS}(\vec{p})$, it already belongs to NEG.
- For the case where $\operatorname{Sahl}_{1}(\vec{p})=\operatorname{Sahl}_{1}^{a}(\vec{p}) \wedge \operatorname{Sahl}_{1}^{b}(\vec{p})$, we first apply (SplNom) to $\mathbf{i} \leq \operatorname{Sahl}_{1}(\vec{p})$ and get $\mathbf{i} \leq \operatorname{Sahl}_{1}^{a}(\vec{p})$ and $\mathbf{i} \leq \operatorname{Sahl}_{1}^{b}(\vec{p})$. Then we apply the induction hypothesis and get

$$
\exists \overrightarrow{\mathfrak{j}}^{a}\left(\mathrm{NEG}^{a} \& \mathrm{NOM}^{a} \& \operatorname{MinVal}^{a}\right) \& \exists \overrightarrow{\mathbf{j}^{b}}\left(\mathrm{NEG}^{b} \& \mathrm{NOM}^{b} \& \mathrm{MinVal}^{b}\right)
$$

By applying the (Scope-\&) rule and commutativity and associativity rules for $\&$, we get the desired shape.

- For the case where $\operatorname{Sahl}_{1}(\vec{p})=\diamond \operatorname{Sah}_{1}^{a}(\vec{p})$, we first apply (Approx-Nom) for $\diamond$ and get $\exists \mathbf{k}\left(\mathbf{k} \leq \operatorname{Sahl}_{1}^{a}(\vec{p}) \& \mathbf{i} \leq \diamond \mathbf{k}\right)$. Then we apply the induction hypothesis to $\mathbf{k} \leq \operatorname{Sahl}_{1}{ }^{a}(\vec{p})$ and get


## $\exists \mathbf{k}(\exists \overrightarrow{\mathbf{j}}(\mathrm{NEG} \& N O M \& M i n V a l) \& \mathbf{i} \leq \diamond \mathbf{k})$.

By applying the (Scope-\&) rule and commutativity and associativity rules for $\&$, we get the desired shape ( $\mathbf{i} \leq \diamond \mathbf{k}$ is merged into NOM).

Proposition 10. In the reduction stage, by running the algorithm ALBA ${ }^{\text {SOPML }}$, for any formula $\psi$ such that

- $\psi$ contains no propositional quantifiers;
- $\psi$ contains propositional variables at most from $\vec{q}$;
- all occurrences of $\vec{q}$-variables are negative in $\psi$;
$\psi \leq \mathrm{PIA}(\vec{q}, \vec{p})$ can be reduced to the following complex inequality:

$$
\operatorname{Rel} \operatorname{Min} \operatorname{Val}(\vec{q}, \vec{p}))
$$

where $\operatorname{ReIMin} \operatorname{Val}\left(\vec{q}, \vec{p} \|^{4}\right.$ is a meta-conjunction of inequalities of the form $\varphi \leq p, \varphi$ has the three properties for $\psi$ stated above, and $p$ is in $\vec{p}$.
Especially, this proposition holds for $\psi=\mathbf{i}$.
Proof. We prove by induction on the complexity of $\operatorname{PIA}(\vec{q}, \vec{p})$.

- For the basic case where $\operatorname{PIA}(\vec{q}, \vec{p})=p$, trivial.
- For the case where $\operatorname{PIA}(\vec{q}, \vec{p})=\square \operatorname{PIA}^{a}(\vec{q}, \vec{p})$, we first apply the (Res- - ) rule and get $\psi \leq \operatorname{PIA}^{a}(\vec{q}, \vec{p})$. Then by induction hypothesis, it is transformed into $\operatorname{Rel} \operatorname{Min} \operatorname{Val}(\vec{q}, \vec{p})$ ) of the required shape.
- For the case where $\operatorname{PIA}(\vec{q}, \vec{p})=\operatorname{PIA}^{a}(\vec{q}, \vec{p}) \wedge \operatorname{PIA}^{b}(\vec{q}, \vec{p})$, we first apply (Splitting) and get $\psi \leq \operatorname{PIA}^{a}(\vec{q}, \vec{p})$ and $\psi \leq \operatorname{PIA}^{b}(\vec{q}, \vec{p})$. Then by induction hypothesis, these two inequalities can be transformed into $\operatorname{RelMinVal}^{a}(\vec{q}, \vec{p})$ ) and $\operatorname{ReIMinVal}^{b}(\vec{q}, \vec{p})$ ) of the required shape, which put together is also of the required shape.
- For the case where $\operatorname{PIA}(\vec{q}, \vec{p})=\operatorname{POS}(\vec{q}) \vee \operatorname{PIA}^{a}(\vec{q}, \vec{p})$, by applying (Res-$\left.-\vee\right)$, we get $\psi \wedge \neg \operatorname{POS}(\vec{q}) \leq \operatorname{PIA}^{a}(\vec{q}, \vec{p})$. Then $\psi \wedge \neg \operatorname{POS}(\vec{q})$ satisfies the conditions required in the proposition, so we can apply the induction hypothesis and get the $\operatorname{Rel} \operatorname{Min} \operatorname{Val}(\vec{q}, \vec{p})$ ) of the required shape.

Proposition 11. In the reduction stage, by running the algorithm ALBA ${ }^{\text {SOPML }}$, $\mathbf{i} \leq \forall \vec{q}\left(\operatorname{Sahl}_{1}(\vec{q}) \rightarrow \mathrm{PIA}(\vec{q}, \vec{p})\right)$ can be reduced to the following complex inequality:

$$
\forall \overrightarrow{\mathbf{j}}(\text { PURE } \Rightarrow \operatorname{MinVal}(\vec{p}))
$$

where

- PURE is a meta-conjunction of pure inequalities,
- MinVal( $(\vec{p})$ is a meta-conjunction of inequalities of the form $\psi \leq p$, where $\psi$ is pure and $p$ is in $\vec{p}$.

[^3]Therefore, $\mathbf{i} \leq \forall \vec{q}\left(\operatorname{Sahl}_{1}(\vec{q}) \rightarrow \operatorname{PIA}(\vec{q}, \vec{p})\right)$ can be reduced to the form

$$
\operatorname{Min} \operatorname{Val}(\vec{p}),
$$

where $\operatorname{Min} \operatorname{Val}(\vec{p})$ is a meta-conjunction of inequalities of the form $\psi \leq p$, where $\psi$ is pure and $p$ is in $\vec{p}$.

Proof. We first apply (Quant-Nom) on

$$
\mathbf{i} \leq \forall \vec{q}\left(\operatorname{Sahl}_{1}(\vec{q}) \rightarrow \mathrm{PIA}(\vec{q}, \vec{p})\right),
$$

then apply (Sep-Nom) we get

$$
\forall \vec{q}\left(\mathbf{i} \leq \operatorname{Sahl}_{1}(\vec{q}) \Rightarrow \mathbf{i} \leq \operatorname{PIA}(\vec{q}, \vec{p})\right) .
$$

By Proposition 9 , we have

$$
\forall \vec{q}(\exists \overrightarrow{\mathbf{j}}(\text { NEG \& NOM \& MinVal) } \Rightarrow \mathbf{i} \leq \operatorname{PIA}(\vec{q}, \vec{p})) .
$$

By Proposition 10, we have

$$
\forall \vec{q}(\exists \overrightarrow{\mathbf{j}}(\mathrm{NEG} \& \mathrm{NOM} \& \operatorname{MinVal}) \Rightarrow \operatorname{ReIMin} \operatorname{Val}(\vec{q}, \vec{p})) .
$$

Then by applying (Scope- $\Rightarrow$ ) and repeatedly applying (Ex-ip), we have

$$
\forall \overrightarrow{\mathbf{j}} \forall \vec{q}(\operatorname{NEG} \& \operatorname{NOM} \& \operatorname{MinVal} \Rightarrow \operatorname{Rel} \operatorname{Min} \operatorname{Val}(\vec{q}, \vec{p})) .
$$

Then by applying the Ackermann rule for each propositional variable in $\vec{q}$, NEG receives the minimal valuation from MinVal and become a meta-conjunction of pure inequalities, NOM remains pure, MinVal disappears, and $\operatorname{ReIMinVal}(\vec{q}, \vec{p}))$ becomes a meta-conjunction of inequalities of the form $\psi \leq p$ where $\psi$ is pure and $p$ is in $\vec{p}$. Now what we have is the following shape, as required by the proposition:

$$
\forall \overrightarrow{\mathbf{j}}(\text { PURE } \Rightarrow \operatorname{Min} \operatorname{Val}(\vec{p})) .
$$

Then apply (Spl-Quant-i) and the packing rule, one get a complex inequality $\operatorname{Min} \operatorname{Val}(\vec{p})$ of the required form.

Proposition 12. In the reduction stage, by running the algorithm ALBA ${ }^{\text {SOPML }}$, $\mathbf{i} \leq \operatorname{Sahl}_{2}(\vec{p})$ can be reduced to the following complex inequality:
$\exists \overrightarrow{\mathbf{j}}($ NEG \& NOM \& MinVal)
where $\exists \overrightarrow{\mathbf{j}}$, NEG, NOM, MinVal are described as in Proposition 9

Proof. We prove by induction on the complexity of $\operatorname{Sahl}_{2}(\vec{p})$.

- For the case where $\operatorname{Sahl}_{2}(\vec{p})=\operatorname{Sahl}_{1}(\vec{p})$, see Proposition 9
- For the case where $\operatorname{Sahl}_{2}(\vec{p})=\forall \vec{q}\left(\operatorname{Sahl}_{1}(\vec{q}) \rightarrow \operatorname{PIA}(\vec{q}, \vec{p})\right)$, by Proposition $11 \mathbf{i} \leq \forall \vec{q}\left(\operatorname{Sahl}_{1}(\vec{q}) \rightarrow \operatorname{PIA}(\vec{q}, \vec{p})\right)$ is reduced to $\forall \overrightarrow{\mathbf{j}}($ PURE $\Rightarrow \operatorname{MinVal}(\vec{p}))$. Now apply (Spl-Quant-i) and the packing rule, we have a meta-conjunction of inequalities of the form $\varphi \leq p$ where $\varphi$ is pure and $p$ is in $\vec{p}$, so it belongs to MinVal.
- For the case where $\operatorname{Sahl}_{2}(\vec{p})=\operatorname{Sahl}_{2}^{a}(\vec{p}) \wedge \operatorname{Sahl}_{2}^{b}(\vec{p})$, similar to the $\operatorname{Sahl}_{1}(\vec{p})=$ $\operatorname{Sahl}_{1}^{a}(\vec{p}) \wedge \operatorname{Sahl}_{1}^{b}(\vec{p})$ case in the proof of Proposition 9
- For the case where $\operatorname{Sahl}_{2}(\vec{p})=\diamond \operatorname{Sahl}_{2}^{a}(\vec{p})$, similar to the $\operatorname{Sahl}_{1}(\vec{p})=\diamond \operatorname{Sah}_{1}^{a}(\vec{p})$ case in the proof of Proposition 9

Proposition 13. In the reduction stage, by running the algorithm ALBA ${ }^{\text {SOPML }}$, $\mathbf{i} \leq \operatorname{Sahl}_{n}(\vec{p})$ can be reduced to the following complex inequality:

## ヨ $\overrightarrow{\mathbf{j}}($ NEG \& NOM \& MinVal)

where $\exists \overrightarrow{\mathbf{j}}$, NEG, NOM, MinVal are described as in Proposition 9
Proof. We prove by induction on $n$. For $n=1,2$, they are already proved in Proposition 9 and 12. Now we assume that for $n=k$ the property holds, then by an argument similar to Proposition 11, we have that $\mathbf{i} \leq \forall \vec{q}\left(\operatorname{Sahl}_{k}(\vec{q}) \rightarrow \operatorname{PIA}(\vec{q}, \vec{p})\right)$ can be reduced to the following complex inequality:

$$
\forall \overrightarrow{\mathbf{j}}(\text { PURE } \Rightarrow \operatorname{MinVal}(\vec{p}))
$$

where PURE and $\operatorname{MinVal}(\vec{p})$ are as described in Proposition 11. Then by an argument similar to Proposition [12, $\mathbf{i} \leq \operatorname{Sahl}_{k+1}(\vec{p})$ can be reduced to the following complex inequality:

## $\exists \overrightarrow{\mathbf{j}}($ NEG \& NOM \& MinVal)

where $\exists \overrightarrow{\mathbf{j}}$, NEG, NOM, MinVal are described as in Proposition 9 , hence the property holds for $n=k+1$.

Theorem 6.1. For any $\Pi_{n}$-Sahlqvist formula, the algorithm ALBA ${ }^{\text {SOPML }}$ transforms it into a complex inequality which does not contain any occurrences of propositional variables or propositional quantifiers.

Proof. Given a $\Pi_{n}$-Sahlqvist formula $\forall \vec{p}\left(\operatorname{Sahl}_{n}(\vec{p}) \rightarrow \operatorname{POS}(\vec{p})\right)$, we first apply the rules in Stage 1 and get

$$
\forall \vec{p} \forall \mathbf{i}_{0}\left(\mathbf{i}_{0} \leq \operatorname{Sahl}_{n}(\vec{p}) \Rightarrow \mathbf{i}_{0} \leq \operatorname{POS}(\vec{p})\right) .
$$

By Proposition 13, we have

$$
\forall \vec{p} \backslash \mathbf{i}_{0}\left(\exists \overrightarrow{\mathbf{j}}(\mathrm{NEG} \& \mathrm{NOM} \& \mathrm{MinVal}) \Rightarrow \mathbf{i}_{0} \leq \operatorname{POS}(\vec{p})\right) .
$$

Then by applying (Scope- $\Rightarrow$ ) and repeatedly applying (Ex-ip), we have

$$
\forall \mathbf{i}_{0} \forall \overrightarrow{\mathbf{j}} \forall \vec{p}\left(\text { NEG \& NOM \& MinVal } \Rightarrow \mathbf{i}_{0} \leq \operatorname{POS}(\vec{p})\right) .
$$

Now we can apply the Ackermann rule repeatedly for each propositional variable $p$ in $\vec{p}$, then NEG receives the minimal valuation from MinVal and become a metaconjunction of pure inequalities, NOM remains pure, MinVal disappears, and $\mathbf{i}_{0} \leq$ $\operatorname{POS}(\vec{p})$ receives the minimal valuation and becomes pure. Now what we have is the following shape:

$$
\forall \mathbf{i}_{0} \forall \overrightarrow{\mathbf{j}}\left(\text { PURE } \Rightarrow \text { PURE }^{\prime}\right),
$$

where PURE is a meta-conjunction of pure inequalities, and PURE' is a pure inequality.

Corollary 6.2. There is an algorithm such that for any $\Pi_{n}$-Sahlqvist formula $\varphi$, it can be transformed into an equivalent first-order formula.

## 7 Examples, non-standard rules and canonicity

### 7.1 Examples

We give three examples of $\Pi_{2}$-Sahlqvist formulas to show how the ALBA ${ }^{\text {SOPML }}$ algorithm works:

```
Example 7.1. \(\forall p(\diamond \square p \wedge \forall q(\diamond \square q \rightarrow \square(\square q \vee \square p)) \rightarrow \square \diamond \square p)\)
\(\forall p \forall \mathbf{i}(\mathbf{i} \leq \diamond \square p \wedge \forall q(\diamond \square q \rightarrow \square(\square q \vee \square p)) \Rightarrow \mathbf{i} \leq \square \diamond \square p)\)
\(\forall p \forall \mathbf{i}(\mathbf{i} \leq \diamond \square p \& \mathbf{i} \leq \forall q(\diamond \square q \rightarrow \square(\square q \vee \square p)) \Rightarrow \mathbf{i} \leq \square \diamond \square p)\)
\(\forall p \forall \mathbf{i}(\mathbf{i} \leq \diamond \square p \& \forall q(\mathbf{i} \leq \diamond \square q \rightarrow \square(\square q \vee \square p)) \Rightarrow \mathbf{i} \leq \square \diamond \square p)\)
\(\forall p \forall \mathbf{i}(\mathbf{i} \leq \diamond \square p \& \forall q(\mathbf{i} \leq \diamond \square q \Rightarrow \mathbf{i} \leq \square(\square q \vee \square p)) \Rightarrow \mathbf{i} \leq \square \diamond \square p)\)
\(\forall p \forall \mathbf{i}(\mathbf{i} \leq \diamond \square p \& \forall q \forall \mathbf{j}(\mathbf{i} \leq \diamond \mathbf{j} \& \mathbf{j} \leq \square q \Rightarrow \mathbf{i} \leq \square(\square q \vee \square p)) \Rightarrow \mathbf{i} \leq \square \diamond \square p)\)
\(\forall p \forall \mathbf{i}(\mathbf{i} \leq \diamond \square p \& \forall q \forall \mathbf{j}(\mathbf{i} \leq \diamond \mathbf{j} \& \forall \mathbf{j} \leq q \Rightarrow \mathbf{i} \leq \square(\square q \vee \square p)) \Rightarrow \mathbf{i} \leq \square \diamond \square p)\)
\(\forall p \forall \mathbf{i}(\mathbf{i} \leq \diamond \square p \& \forall \mathbf{j}(\mathbf{i} \leq \diamond \mathbf{j} \Rightarrow \mathbf{i} \leq \square(\square \mathbf{j} \vee \square p)) \Rightarrow \mathbf{i} \leq \square \diamond \square p)\)
\(\forall p \forall \mathbf{i}(\mathbf{i} \leq \diamond \square p \& \forall \mathbf{j}(\mathbf{i} \leq \diamond \mathbf{j} \Rightarrow \diamond \mathbf{i} \leq \square \mathbf{j} \vee \square p) \Rightarrow \mathbf{i} \leq \square \diamond \square p)\)
```

```
\forall\forall\mathbf{i}(\mathbf{i}\leq\diamond\squarep& \forall\mathbf{j}(\mathbf{i}\leq\diamond\mathbf{j}=>>\mathbf{i}\wedge\neg\square}\mathbf{\}\mathbf{j}\leq\squarep)=>\mathbf{i}\leq\square\diamond\squarep
\forall p \forall \mathbf { i } ( \mathbf { i } \leq \diamond \square p \& \forall \mathbf { j } ( \mathbf { i } \leq \diamond \mathbf { j } \Rightarrow \diamond ( \diamond \mathbf { i } \wedge \neg \square \diamond \mathbf { j } ) \leq p ) \Rightarrow \mathbf { i } \leq \square \diamond \square p )
\forall p \forall \mathbf { i } ( \mathbf { i } \leq \diamond \square p \& \forall \mathbf { j } ( \mathbf { l } ( \mathbf { i } , \diamond \mathbf { j } ) \wedge * ( \diamond \mathbf { i } \wedge \neg \square \leqslant \mathbf { j } ) \leq p ) \Rightarrow \mathbf { i } \leq \square \diamond \square p )
\forall}\forall\mathbf{i}(\mathbf{i}\leq\diamond\squarep&\exists\mathbf{j}(\mathbf{l}(\mathbf{i},\diamond\mathbf{j})\wedge>(\diamond\mathbf{i}\wedge\neg\square\\mathbf{j}))\leqp=>\mathbf{i}\leq\square\diamond\squarep
```

now denote $\exists \mathbf{j}(\mathbf{l}(\mathbf{i}, \diamond \mathbf{j}) \wedge \diamond(\stackrel{\mathbf{i}}{ } \wedge \neg \square \mathbf{j}))$ as $\varphi$, then
$\forall p \forall \mathbf{i}(\mathbf{i} \leq \diamond \square p \& \varphi \leq p \Rightarrow \mathbf{i} \leq \square \diamond \square p)$
$\forall p \forall \mathbf{i} \forall \mathbf{k}(\mathbf{i} \leq \diamond \mathbf{k} \& \mathbf{k} \leq \square p \& \varphi \leq p \Rightarrow \mathbf{i} \leq \square \diamond \square p)$
$\forall p \forall \mathbf{i} \forall \mathbf{k}(\mathbf{i} \leq \diamond \mathbf{k} \& \forall \mathbf{k} \leq p \& \varphi \leq p \Rightarrow \mathbf{i} \leq \square \diamond \square p)$
$\forall \mathbf{i} \forall \mathbf{k}(\mathbf{i} \leq \diamond \mathbf{k} \Rightarrow \mathbf{i} \leq \square \diamond \square(\diamond \mathbf{k} \vee \varphi))$

Then we can use standard translation to get its first-order correspondence.
Example 7.2. $\forall q(\forall p(p \rightarrow \diamond p \vee q) \rightarrow q)$
$\forall q \forall \mathbf{i}(\mathbf{i} \leq \forall p(p \rightarrow \diamond p \vee q) \Rightarrow \mathbf{i} \leq q)$
$\forall q \forall \mathbf{i}(\forall p(\mathbf{i} \leq p \rightarrow \diamond p \vee q) \Rightarrow \mathbf{i} \leq q)$
$\forall q \forall \mathbf{i}(\forall p(\mathbf{i} \leq p \Rightarrow \mathbf{i} \leq \diamond p \vee q) \Rightarrow \mathbf{i} \leq q)$
$\forall q \forall \mathbf{i}(\mathbf{i} \leq \diamond \mathbf{i} \vee q \Rightarrow \mathbf{i} \leq q)$
$\forall q \forall \mathbf{i}(\mathbf{i} \wedge \neg \diamond \mathbf{i} \leq q \Rightarrow \mathbf{i} \leq q)$
$\forall \mathbf{i}(\mathbf{i} \leq \mathbf{i} \wedge \neg \diamond \mathbf{i})$
$\forall \mathbf{i}(\mathbf{i} \leq \neg \diamond \mathbf{i})$
$\forall x \neg R x x$.
By [9] Example 2.58], the irreflexive property is not preserved under taking ultrafilter extensions, which means that the validity of $\forall q(\forall p(p \rightarrow \diamond p \vee q) \rightarrow q)$ is not preserved under taking canonical extensions, which means that $\forall q(\forall p(p \rightarrow$ $\diamond p \vee q) \rightarrow q$ ) is not canonical.

Example 7.3. The following example is not equivalent to any Sahlqvist formula in the basic modal language:

```
\(\forall p(\square p \wedge \forall q(q \rightarrow \diamond \diamond q \vee p) \rightarrow p)\)
\(\forall p \forall \mathbf{i}(\mathbf{i} \leq \square p \wedge \forall q(q \rightarrow \diamond \diamond q \vee p) \Rightarrow \mathbf{i} \leq p)\)
\(\forall p \forall \mathbf{i}(\mathbf{i} \leq \square p \& \mathbf{i} \leq \forall q(q \rightarrow \diamond \diamond q \vee p) \Rightarrow \mathbf{i} \leq p)\)
\(\forall p \forall \mathbf{i}(\stackrel{i}{\mathbf{i}} \leq p \& \mathbf{i} \leq \forall q(q \rightarrow \diamond \diamond q \vee p) \Rightarrow \mathbf{i} \leq p)\)
\(\forall p \forall \mathbf{i}(\diamond \mathbf{i} \leq p \& \forall q(\mathbf{i} \leq q \rightarrow \diamond \diamond q \vee p) \Rightarrow \mathbf{i} \leq p)\)
\(\forall p \forall \mathbf{i}(\diamond \mathbf{i} \leq p \& \forall q(\mathbf{i} \leq q \Rightarrow \mathbf{i} \leq \diamond \diamond q \vee p) \Rightarrow \mathbf{i} \leq p)\)
\(\forall p \forall \mathbf{i}(\mathbf{i} \leq p \& \mathbf{i} \leq \diamond \diamond \mathbf{i} \vee p \Rightarrow \mathbf{i} \leq p)\)
\(\forall p \forall \mathbf{i}(\stackrel{i}{ } \leq p \& \mathbf{i} \wedge \neg \diamond \diamond \mathbf{i} \leq p \Rightarrow \mathbf{i} \leq p)\)
\(\forall p \forall \mathbf{i}(\checkmark \mathbf{i} \vee(\mathbf{i} \wedge \neg \diamond \diamond \mathbf{i}) \leq p \Rightarrow \mathbf{i} \leq p)\)
\(\forall \mathbf{i}(\mathbf{i} \leq \boldsymbol{i} \vee(\mathbf{i} \wedge \neg \diamond \diamond \mathbf{i}))\)
\(\forall \mathbf{i}(\mathbf{i} \leq \mathbf{i}\) or \(\mathbf{i} \leq \mathbf{i} \wedge \neg \diamond \diamond \mathbf{i})\)
```

$$
\begin{aligned}
& \forall \mathbf{i}(\mathbf{i} \leq \diamond \mathbf{i} \text { or } \mathbf{i} \leq \neg \diamond \diamond \mathbf{i}) \\
& \forall \mathbf{i}(\mathbf{i} \leq \diamond \diamond \mathbf{i} \rightarrow \diamond \mathbf{i}) \\
& \forall x \forall y(R x y \wedge R y x \rightarrow R x x)
\end{aligned}
$$

One can show that this property is not modally definable:
Consider $\mathbb{F}_{1}=\left(W_{1}, R_{1}\right)$ where $W_{1}$ is the set of all integers, $R_{1}=\{(x, x+1) \mid$ $\left.x \in W_{1}\right\}, \mathbb{F}_{2}=\left(W_{2}, R_{2}\right)$ where $W_{2}=\left\{w_{0}, w_{1}\right\}, R_{2}=\left\{\left(w_{0}, w_{1}\right),\left(w_{1}, w_{0}\right)\right\}$, then $\mathbb{F}_{2}$ is a bounded morphic image of $\mathbb{F}_{1}, \mathbb{F}_{1} \vDash \forall x \forall y(R x y \wedge R y x \rightarrow R x x)$, while $\mathbb{F}_{2} \notin$ $\forall x \forall y(R x y \wedge R y x \rightarrow R x x)$.

## $7.2 \Pi_{2}$-formulas and rules

In this section we consider the following kinds of rules, each of which is the generalization of the former one:

- Gabbay's irreflexivity rule [21]:

$$
\vdash \neg(p \rightarrow \diamond p) \rightarrow \varphi \Rightarrow \vdash \varphi
$$

where $p$ does not occur in $\varphi$.

- Venema's non- $\xi$ rules [38]:

$$
\vdash \neg \xi\left(p_{0}, \ldots, p_{n}\right) \rightarrow \varphi \Rightarrow \vdash \varphi
$$

where $p_{0}, \ldots, p_{n}$ does not occur in $\varphi$.

- $\Pi_{2}$ rules [8]:

$$
\vdash F(\vec{\varphi} / \vec{x}, \vec{p}) \rightarrow \chi \Rightarrow \vdash G(\vec{\varphi} / \vec{x}) \rightarrow \chi
$$

where $F, G$ are formulas, $\vec{\varphi}$ is a tuple of formulas, $\chi$ is a formula, and $\vec{p}$ is a tuple of propositional variables which do not occur in $\vec{\varphi}$ and $\chi$.

Gabbay's irreflexivity rule. Now consider Gabbay's irreflexivity rule, its corresponding $\forall \exists$-statement is the following:

$$
\forall q(\forall p(\neg(p \rightarrow \diamond p) \leq q) \Rightarrow \mathrm{\top} \leq q)
$$

therefore, its equivalent SOPML $\forall \exists$-formula is

$$
\forall q(\forall p \mathbf{l}(\neg(p \rightarrow \diamond p), q) \rightarrow \mathbf{l}(\top, q))
$$

now its $A L B A^{\text {SOPML }}$-reduction is as follows $\cdot 5^{5}$

$$
\begin{aligned}
& \forall \mathbf{i}(\mathbf{i} \leq \forall q(\forall p \mathbf{l}(\neg(p \rightarrow \diamond p), q) \rightarrow \mathbf{l}(\top, q))) \\
& \forall \mathbf{i} \forall q((\forall p(\mathbf{i} \leq \mathbf{l}(\neg(p \rightarrow \diamond p), q)) \Rightarrow \mathbf{i} \leq \mathbf{l}(\mathrm{T}, q))) \\
& \forall q(\forall p(\neg(p \rightarrow \diamond p) \leq q) \Rightarrow \mathrm{T} \leq q) \\
& \forall q(\forall p \forall \mathbf{j}(\mathbf{j} \leq \neg(p \rightarrow \diamond p) \Rightarrow \mathbf{j} \leq q) \Rightarrow \mathrm{T} \leq q) \\
& \forall q(\forall p \forall \mathbf{j}(\mathbf{j} \leq p \& \mathbf{j} \leq \neg \diamond p \Rightarrow \mathbf{j} \leq q) \Rightarrow \mathrm{T} \leq q) \\
& \forall q(\forall \mathbf{j}(\mathbf{j} \leq \neg \diamond \mathbf{j} \Rightarrow \mathbf{j} \leq q) \Rightarrow \top \leq q) \\
& \forall q(\forall \mathbf{j}(\mathbf{l} \mathbf{j}, \neg \diamond \mathbf{j}) \wedge \mathbf{j} \leq q) \Rightarrow \mathrm{T} \leq q) \\
& \forall q(\exists \mathbf{j}(\mathbf{l}(\mathbf{j}, \neg \diamond \mathbf{j}) \wedge \mathbf{j}) \leq q \Rightarrow \mathrm{~T} \leq q) \\
& T \leq \exists \mathbf{j}(\mathbf{l} \mathbf{j}, \neg \diamond \mathbf{j}) \wedge \mathbf{j}) \\
& \forall \mathbf{i}(\mathbf{i} \leq \exists \mathbf{j}(\mathbf{l}(\mathbf{j}, \neg \diamond \mathbf{j}) \wedge \mathbf{j})) \\
& \left.\forall x S T_{x}(\exists \mathbf{j}(\mathbf{l} \mathbf{j}, \neg \diamond \mathbf{j}) \wedge \mathbf{j})\right) \\
& \left.\left.\forall x \exists j S T_{x}(\mathbf{(} \mathbf{(} \mathbf{j}, \neg \diamond \mathbf{j}) \wedge \mathbf{j}\right)\right) \\
& \left.\forall x \exists j\left(S T_{x}(\mathbf{l} \mathbf{j}, \neg \diamond \mathbf{j})\right) \wedge S T_{x}(\mathbf{j})\right) \\
& \forall x \exists j(\neg R j j \wedge x=j) \\
& \forall x \neg R x x \text {. }
\end{aligned}
$$

Venema's non- $\xi$ rules. Now consider Venema's non- $\xi$ rules, their corresponding $\forall \exists$-statement is the following:

$$
\forall q(\forall \vec{p}(\neg \xi(\vec{p}) \leq q) \Rightarrow \mathrm{T} \leq q)
$$

When $\xi$ is a Sahlqvist formula $\operatorname{Sahl}_{1}(\vec{p}) \rightarrow \operatorname{POS}(\vec{p})$ in the basic modal language, Venema's rules can be equivalently written in the following SOPML $\forall \exists$-formula:

$$
\forall q\left(\forall \vec{p}\left(\mathbf{l}\left(\neg\left(\operatorname{Sahl}_{1}(\vec{p}) \rightarrow \operatorname{POS}(\vec{p})\right), q\right)\right) \rightarrow \mathbf{l}(\mathrm{T}, q)\right) .
$$

Assume that $\mathbf{i} \leq \operatorname{Sahl}_{1}(\vec{p}) \rightarrow \operatorname{POS}(\vec{p})$ can be reduced to $\mathbf{i} \leq$ Local where Local is pure (which is the modal counterpart of the local frame correspondent of $\operatorname{Sahl}_{1}(\vec{p}) \rightarrow$ $\operatorname{POS}(\vec{p})$ ), then the ALBA ${ }^{\text {SOPML }}$-reduction is as follows:

$$
\begin{aligned}
& \forall q\left(\forall \vec{p}\left(\mathbf{l}\left(\neg\left(\operatorname{Sahl}_{1}(\vec{p}) \rightarrow \operatorname{POS}(\vec{p})\right), q\right)\right) \rightarrow \mathbf{l}(\mathrm{T}, q)\right) \\
& \forall \mathbf{i}\left(\mathbf{i} \leq \forall q\left(\forall \vec{p}\left(\mathbf{l}\left(\neg\left(\operatorname{Sahl}_{1}(\vec{p}) \rightarrow \operatorname{POS}(\vec{p})\right), q\right)\right) \rightarrow \mathbf{l}(\mathrm{T}, q)\right)\right) \\
& \left.\forall \mathbf{i}\left(\mathbf{i} \leq \forall q\left(\forall \vec{p} \mathbf{l}\left(\neg\left(\operatorname{Sahl}_{1}(\vec{p}) \rightarrow \operatorname{POS}(\vec{p})\right), q\right)\right) \rightarrow \mathbf{l}(\mathrm{T}, q)\right)\right) \\
& \forall \mathbf{i} \forall q\left(\mathbf{i} \leq \forall \vec{p}\left(\mathbf{l}\left(\neg\left(\operatorname{Sahl}_{1}(\vec{p}) \rightarrow \operatorname{POS}(\vec{p})\right), q\right)\right) \Rightarrow \mathbf{i} \leq \mathbf{l}(\mathrm{T}, q)\right) \\
& \forall \mathbf{i} \forall q\left(\forall \vec{p}\left(\mathbf{i} \leq \mathbf{l}\left(\neg\left(\operatorname{Sahl}_{1}(\vec{p}) \rightarrow \operatorname{POS}(\vec{p})\right), q\right)\right) \Rightarrow \mathbf{i} \leq \mathbf{l}(\mathrm{T}, q)\right)
\end{aligned}
$$

[^4]```
\(\forall \mathbf{i} \forall q\left(\forall \vec{p}\left(\neg\left(\operatorname{Sahl}_{1}(\vec{p}) \rightarrow \mathrm{POS}(\vec{p})\right) \leq q\right) \Rightarrow \mathbf{i} \leq \mathbf{l}(\mathrm{T}, q)\right)\)
\(\forall q\left(\forall \vec{p}\left(\neg\left(\operatorname{Sahl}_{1}(\vec{p}) \rightarrow \operatorname{POS}(\vec{p})\right) \leq q\right) \Rightarrow \mathrm{T} \leq q\right)\)
\(\forall q\left(\forall \vec{p} \forall \mathbf{j}\left(\mathbf{j} \leq \neg\left(\operatorname{Sahl}_{1}(\vec{p}) \rightarrow \operatorname{POS}(\vec{p})\right) \Rightarrow \mathbf{j} \leq q\right) \Rightarrow \mathrm{T} \leq q\right)\)
\(\left.\forall q\left(\forall \vec{p} \forall \mathbf{j} \mathbf{j} \not \neq\left(\operatorname{Sahl}_{1}(\vec{p}) \rightarrow \operatorname{POS}(\vec{p})\right) \Rightarrow \mathbf{j} \leq q\right) \Rightarrow \mathrm{T} \leq q\right)\)
\(\forall q(\forall \mathbf{j}(\mathbf{j} \not \leq\) Local \(\Rightarrow \mathbf{j} \leq q) \Rightarrow \mathrm{T} \leq q)\)
\(\forall q(\forall \mathbf{j}(\neg \mathbf{l}(\mathbf{j}\), Local \() \wedge \mathbf{j} \leq q) \Rightarrow \mathrm{T} \leq q)\)
\(\forall q(\exists \mathbf{j}(\neg \mathbf{l}(\mathbf{j}\), Local \() \wedge \mathbf{j}) \leq q \Rightarrow \mathrm{~T} \leq q)\)
\(\mathrm{T} \leq \exists \mathbf{j}(\neg \mathbf{l}(\mathbf{j}\), Local \() \wedge \mathbf{j})\)
\(\forall \mathbf{i}(\mathbf{i} \leq \exists \mathbf{j}(\neg \mathbf{l}(\mathbf{j}\), Local \() \wedge \mathbf{j}))\)
\(\forall x S T_{x}(\boldsymbol{\exists} \mathbf{j}(\neg \mathbf{l}(\mathbf{j}\), Local \() \wedge \mathbf{j}))\)
\(\forall x \exists j S T_{x}(\neg(\mathbf{j}\), Local \() \wedge \mathbf{j})\)
\(\forall x \exists j\left(S T_{x}(\neg \mathbf{l}(\mathbf{j}\right.\), Local \(\left.)) \wedge S T_{x}(\mathbf{j})\right)\)
\(\forall x \exists j\left(S T_{x}(\neg \mathbf{l}(\mathbf{j}\right.\), Local \(\left.)) \wedge x=j\right)\)
\(\forall x \exists j\left(\neg S T_{j}\right.\) (Local) \(\left.\wedge x=j\right)\)
\(\forall x \neg S T_{x}\) (Local).
```

$\Pi_{2}$-rules. We first consider the corresponding SOPML-formulas of $\Pi_{2}$-rules. For $\vdash F(\vec{\varphi} / \vec{x}, \vec{p}) \rightarrow \chi \Rightarrow \vdash G(\vec{\varphi} / \vec{x}) \rightarrow \chi$, its corresponding $\forall \exists$-statement is the following:

$$
\forall \vec{x} \forall z(G(\vec{x}) \nsucceq z \Rightarrow \exists \vec{y}(F(\vec{x}, \vec{y}) \not \leq z)),
$$

which is equivalent to

$$
\forall \vec{x} \forall z(\forall \vec{y}(F(\vec{x}, \vec{y}) \leq z) \Rightarrow G(\vec{x}) \leq z),
$$

which is essentially the following SOPML $\forall \exists$-formula:

$$
\forall \vec{p} \forall q(\forall \vec{r}(\mathbf{l}(F(\vec{p}, \vec{r}), q)) \rightarrow \mathbf{l}(G(\vec{p}), q))
$$

When $F(\vec{p}, \vec{r})$ is of the form $\operatorname{Sahl}_{1}(\vec{p}, \vec{r}), G(\vec{p})$ is of the form $\operatorname{POS}(\vec{p})$, the $\mathrm{ALBA}^{\text {SOPML }}$ reduction is as follows:

```
\(\left.\forall \vec{p} \forall q\left(\forall \vec{r} \mathbf{l}\left(\operatorname{Sahl}_{1}(\vec{p}, \vec{r}), q\right)\right) \rightarrow \mathbf{l}(\operatorname{POS}(\vec{p}), q)\right)\)
\(\forall \mathbf{i}\left(\mathbf{i} \leq \forall \vec{p} \forall q\left(\forall \vec{r}\left(\mathbf{l}\left(\operatorname{Sahl}_{1}(\vec{p}, \vec{r}), q\right)\right) \rightarrow \mathbf{l}(\operatorname{POS}(\vec{p}), q)\right)\right)\)
\(\left.\forall \mathbf{i} \forall \vec{p} \forall q\left(\mathbf{i} \leq \forall \vec{r}\left(\mathbf{l} \operatorname{Sahl}_{1}(\vec{p}, \vec{r}), q\right)\right) \Rightarrow \mathbf{i} \leq \mathbf{l}(\operatorname{POS}(\vec{p}), q)\right)\)
\(\forall \mathbf{i} \forall \vec{p} \forall q\left(\forall \vec{r}\left(\mathbf{i} \leq \mathbf{l}\left(\operatorname{Sahl}_{1}(\vec{p}, \vec{r}), q\right)\right) \Rightarrow \mathbf{i} \leq \mathbf{l}(\operatorname{POS}(\vec{p}), q)\right)\)
\(\forall \vec{p} \forall q\left(\forall \vec{r}\left(\operatorname{Sahl}_{1}(\vec{p}, \vec{r}) \leq q\right) \Rightarrow \operatorname{POS}(\vec{p}) \leq q\right)\)
\(\forall \vec{p} \forall q\left(\forall \vec{r} \forall \mathbf{i}\left(\mathbf{i} \leq \operatorname{Sahl}_{1}(\vec{p}, \vec{r}) \Rightarrow \mathbf{i} \leq q\right) \Rightarrow \operatorname{POS}(\vec{p}) \leq q\right)\)
\(\forall \vec{p} \forall q(\forall \vec{r} \forall \mathbf{i}(\exists \overrightarrow{\mathbf{j}}(\operatorname{NEG}(\vec{p}, \vec{r}) \& \operatorname{NOM} \& \operatorname{MinVal}(\vec{p}, \vec{r})) \Rightarrow \mathbf{i} \leq q) \Rightarrow \operatorname{POS}(\vec{p}) \leq q)\)
```

(Here $\operatorname{NEG}(\vec{p}, \vec{r}) \& \operatorname{NOM} \& \operatorname{Min} \operatorname{Val}(\vec{p}, \vec{r})$ are as described in Proposition (9)

$$
\begin{aligned}
& \forall \vec{p} \forall q(\forall \vec{r} \forall \mathbf{i} \forall \overrightarrow{\mathbf{j}}(\mathrm{NEG}(\vec{p}, \vec{r}) \& \operatorname{NOM} \& \operatorname{MinVal}(\vec{p}, \vec{r}) \Rightarrow \mathbf{i} \leq q) \Rightarrow \operatorname{POS}(\vec{p}) \leq q) \\
& \forall \vec{p} \forall q(\forall \vec{r} \forall \mathbf{i} \forall \mathbf{j} \mathbf{~} \mathrm{NEG}(\vec{p}, \vec{r}) \& \operatorname{NOM} \& \operatorname{MinVal}(\vec{p}, \vec{r}) \Rightarrow \mathbf{i} \leq q) \Rightarrow \forall \mathbf{k}(\mathbf{k} \leq \operatorname{POS}(\vec{p}) \Rightarrow \\
& \mathbf{k} \leq q)) \\
& \forall \vec{p} \forall q \forall \mathbf{k}(\forall \vec{r} \forall \mathbf{i} \forall \overrightarrow{\mathbf{j}} \mathbf{\mathrm { j }} \mathrm{NEG}(\vec{p}, \vec{r}) \& \operatorname{NOM} \& \operatorname{Min} \operatorname{Val}(\vec{p}, \vec{r}) \Rightarrow \mathbf{i} \leq q) \& \mathbf{k} \leq \operatorname{POS}(\vec{p}) \Rightarrow \\
& \mathbf{k} \leq q) \\
& \forall \vec{p} \forall q \forall \mathbf{k}(\forall \vec{r} \forall \mathbf{i} \forall \overrightarrow{\mathbf{j}} \mathbf{k} \leq \operatorname{POS}(\vec{p}) \&(\operatorname{NEG}(\vec{p}, \vec{r}) \& \operatorname{NOM} \& \operatorname{MinVal}(\vec{p}, \vec{r}) \Rightarrow \mathbf{i} \leq \\
& q)) \Rightarrow \mathbf{k} \leq q)
\end{aligned}
$$

Then we can apply the Ackermann rule and substitute the minimal valuation of $\vec{p}, \vec{r}$ into $\operatorname{POS}(\vec{p})$ and $\operatorname{NEG}(\vec{p}, \vec{r})$ and make the latter two pure, therefore the complex inequality is equivalent to
$\forall q \forall \mathbf{k}\left(\forall \mathbf{i} \forall \overrightarrow{\mathbf{j}}\left(\right.\right.$ PURE \& $\left(\right.$ PURE $\left.\left.\left.^{\prime} \Rightarrow \mathbf{i} \leq q\right)\right) \Rightarrow \mathbf{k} \leq q\right)$
By the packing rule, PURE $^{\prime} \Rightarrow \mathbf{i} \leq q$ is packed into an inequality $\psi \leq q$ where $\psi$ is pure:

$$
\begin{aligned}
& \forall q \forall \mathbf{k}(\forall \mathbf{i} \forall \overrightarrow{\mathbf{j}} \text { (PURE \& } \psi \leq q) \Rightarrow \mathbf{k} \leq q) \\
& \forall q \forall \mathbf{k}(\forall \mathbf{i} \forall \mathbf{j} \text { (PURE) \& } \forall \mathbf{i} \forall \mathbf{j}(\psi \leq q) \Rightarrow \mathbf{k} \leq q) \\
& \forall q \forall \mathbf{k}(\forall \mathbf{i} \forall \overrightarrow{\mathbf{j}} \text { (PURE) \& } \exists \mathbf{i} \exists \overrightarrow{\mathbf{j}} \psi \leq q \Rightarrow \mathbf{k} \leq q) \\
& \forall \mathbf{k}(\forall \mathbf{i} \forall \overrightarrow{\mathbf{j}} \mathbf{( P U R E}) \Rightarrow \mathbf{k} \leq \exists \mathbf{i} \exists \overrightarrow{\mathbf{j}} \psi) .
\end{aligned}
$$

Then we can perform the standard translation to obtain its corresponding first-order correspondent.

## 8 Conclusion

In this paper, we develop the Sahlqvist correspondence theory for SOPML. We define the class of Sahlqvist formulas for SOMPL, each formula of which is shown to have a first-order correspondent by an algorithm ALBA ${ }^{\text {SOMPL }}$. In addition, we show that certain $\Pi_{2}$-rules correspond to $\Pi_{2}$-Sahlqvist formulas in SOMPL, which further correspond to first-order conditions.
Here we give some final remarks:

- Since the Sahlqvist correspondence theorem talks about frame definability, any propositional variables in the basic modal formulas are already implicitly treated as universally quantified, so what we will do in this paper for SOPML
formulas is to find a Sahlqvist fragment which allows also for existentially quantified proposition variables, not only universally quantified variables. Indeed, this can be seen in the definition of $\Pi_{2}$-Sahlqvist formulas, where universal quantifiers are allowed in the antecedent part.
- This paper can also be seen as looking for a modal counterpart of secondorder quantifier elimination for monadic second-order logic (MSO), as SOPML with global modality is expressively equivalent to MSO (see [31]). Here what we are aiming at is to find a natural fragment in a modal-type language which can be reduced to first-order formulas.

Acknowledgement The research of the author is supported by Taishan University Starting Grant "Studies on Algebraic Sahlqvist Theory" and the Taishan Young Scholars Program of the Government of Shandong Province, China (No.tsqn201909151). The author would like to thank Nick Bezhanishvili for his suggestions and comments on this project, and Balder ten Cate for the detailed comments and remarks which help in improving the paper.

## References

[1] G. A. Antonelli and R. H. Thomason. Representability in second-order propositional poly-modal logic. Journal of Symbolic Logic, 67(3):1039 - 1054, 2002.
[2] P. Balbiani, T. Tinchev, and D. Vakarelov. Modal logics for region-based theories of space. Fundam. Inf., 81(1-3):29-82, Jan. 2007.
[3] F. Belardinelli and W. van der Hoek. Epistemic quantified boolean logic: Expressiveness and completeness results. In IJCAI, pages 2748-2754, 2015.
[4] F. Belardinelli and W. van der Hoek. A semantical analysis of second-order propositional modal logic. In Thirtieth AAAI Conference on Artificial Intelligence, 2016.
[5] F. Belardinelli, H. van Ditmarsch, and W. van der Hoek. Second-order propositional announcement logic. In Proceedings of the 2016 International Conference on Autonomous Agents and Multiagent Systems (AAMAS), International Foundation for Autonomous Agents and Multiagent Systems, pages 635-643, 2016.
[6] P. Besnard, J.-M. Guinnebault, and E. Mayer. Propositional quantification for conditional logic. In Proceedings of the First International Joint Conference on Qualitative and Quantitative Practical Reasoning, ECSQARU/FAPR '97, page 183-197, Berlin, Heidelberg, 1997. Springer-Verlag.
[7] G. Bezhanishvili, N. Bezhanishvili, T. Santoli, and Y. Venema. A strict implication calculus for compact hausdorff spaces. Annals of Pure and Applied Logic, 170(11):102714, 2019.
[8] N. Bezhanishvili, S. Ghilardi, and L. Landi. Model completeness and $\Pi_{2}{ }^{-}$ rules: The case of contact algebras. In N. Olivetti, R. Verbrugge, S. Negri, and G. Sandu, editors, 13th Conference on Advances in Modal Logic, AiML 2020, Helsinki, Finland, August 24-28, 2020, pages 115-132. College Publications, 2020.
[9] P. Blackburn, M. de Rijke, and Y. Venema. Modal logic, volume 53 of Cambridge Tracts in Theoretical Computer Science. Cambridge University Press, 2001.
[10] R. A. Bull. On modal logic with propositional quantifiers. The Journal of Symbolic Logic, 34(2):257-263, 1969.
[11] J. P. Burgess. Decidability for branching time. Studia Logica, 39(2-3):203218, 1980.
[12] W. Conradie, S. Ghilardi, and A. Palmigiano. Unified correspondence. In A. Baltag and S. Smets, editors, Johan van Benthem on Logic and Information Dynamics, volume 5 of Outstanding Contributions to Logic, pages 933-975. Springer International Publishing, 2014.
[13] W. Conradie, V. Goranko, and D. Vakarelov. Algorithmic correspondence and completeness in modal logic. I. The core algorithm SQEMA. Logical Methods in Computer Science, 2:1-26, 2006.
[14] W. Conradie and A. Palmigiano. Algorithmic correspondence and canonicity for distributive modal logic. Annals of Pure and Applied Logic, 163(3):338 376, 2012.
[15] W. Conradie, A. Palmigiano, and S. Sourabh. Algebraic modal correspondence: Sahlqvist and beyond. Journal of Logical and Algebraic Methods in Programming, 91:60-84, 2017.
[16] Y. Ding. On the logics with propositional quantifiers extending s5П. In G. Bezhanishvili, G. D'Agostino, G. Metcalfe, and T. Studer, editors, Advances in Modal Logic 12, proceedings of the 12th conference on "Advances in Modal Logic," held in Bern, Switzerland, August 27-31, 2018, pages 219235. 2018.
[17] D.Kaplan. S5 with quantifiable propositional variables. Journal of Symbolic Logic, 35:355, 1970.
[18] K. Fine. For some proposition and so many possible worlds. PhD thesis, University of Warwick, 1969.
[19] K. Fine. Propositional quantifiers in modal logic. Theoria, 36(3):336-346, 1970.
[20] P. Fritz. Propositional Quantification in Bimodal S5. Erkenntnis, 85(2):455465, 2020.
[21] D. M. Gabbay. An irreflexivity lemma with applications to axiomatizations of conditions on tense frames. In Aspects of philosophical logic, pages 67-89. Springer, 1981.
[22] D. M. Gabbay and I. M. Hodkinson. An Axiomatization of the Temporal Logic with Until and Since over the Real Numbers. Journal of Logic and Computation, 1(2):229-259, 121990.
[23] S. Ghilardi and M. W. Zawadowski. Undefinability of propositional quantifiers in the modal system s4. Studia Logica, 55:259-271, 1995.
[24] V. Goranko and D. Vakarelov. Elementary canonical formulae: Extending Sahlqvist's theorem. Annals of Pure and Applied Logic, 141(1-2):180-217, 2006.
[25] W. H. Holliday. A Note on Algebraic Semantics for S5 with Propositional Quantifiers. Notre Dame Journal of Formal Logic, 60(2):311-332, 2019.
[26] W. H. Holliday and T. Litak. One modal logic to rule them all? In G. Bezhanishvili, G. D'Agostino, G. Metcalfe, and T. Studer, editors, Advances in Modal Logic, volume 12, pages 367-386, London, 2018. College Publications.
[27] M. Kaminski and M. Tiomkin. The expressive power of second-order propositional modal logic. Notre Dame Journal of Formal Logic, 37(1):35-43, 1996.
[28] P. Kremer. Propositional Quantification in the Topological Semantics for S4. Notre Dame Journal of Formal Logic, 38(2):295-313, 1997.
[29] S. A. Kripke. A completeness theorem in modal logic. Journal of Symbolic Logic, 24(1):1-14, 1959.
[30] S. Kuhn. A Simple Embedding of T into Double S5. Notre Dame Journal of Formal Logic, 45(1):13-18, 2004.
[31] A. Kuusisto. A modal perspective on monadic second-order alternation hierarchies. In M. Kracht, M. de Rijke, H. Wansing, and M. Zakharyaschev, editors, Advances in Modal Logic, pages 231-247. CSLI Publications, 2008.
[32] A. Kuusisto. Second-order propositional modal logic and monadic alternation hierarchies. Annals of Pure and Applied Logic, 166(1):1-28, 2015.
[33] H. Sahlqvist. Completeness and correspondence in the first and second order semantics for modal logic. In Studies in Logic and the Foundations of Mathematics, volume 82, pages 110-143. 1975.
[34] T. Santoli. Logics for compact hausdorff spaces via de vries duality. Master's thesis, Universiteit van Amsterdam, 2016.
[35] B. ten Cate. Expressivity of second order propositional modal logic. Journal of Philosophical Logic, 35(2):209-223, 2006.
[36] D. Vakarelov. Region-Based Theory of Space: Algebras of Regions, Representation Theory, and Logics, pages 267-348. Springer New York, New York, NY, 2007.
[37] J. van Benthem. Modal logic and classical logic. Bibliopolis, 1983.
[38] Y. Venema. Derivation rules as anti-axioms in modal logic. Journal of Symbolic Logic, 58(3):1003-1034, 1993.


[^0]:    ${ }^{1}$ For more literature, see [1] 3, 4, 5, , 6, 16, 20, 23, 25, 26, 28, 30, 31, 32].

[^1]:    ${ }^{2}$ Notice that by adding the universal modality A into the language, all of the additional connectives in the expanded modal language can be defined in the language with A . For example, $\mathrm{I}(\varphi, \psi)$ can be rewritten as $\mathrm{A}(\varphi \rightarrow \psi)$, and the backward-looking modality $\bullet$ can be defined by $\varphi \leftrightarrow \exists p(p \wedge$ $\forall q(q \rightarrow \mathrm{~A}(p \rightarrow q))) \wedge \mathrm{E}(\varphi \wedge \diamond p)$ where E is $\neg \mathrm{A} \neg$. The expanded modal language is introduced for the convenience of the algorithm, as what is typically done in algorithmic correspondence theory.

[^2]:    ${ }^{3}$ In [9] Chapter 3], what we call Sahlqvist formulas are called Sahlqvist implications.

[^3]:    ${ }^{4}$ Here RelMinVal means relative minimal valuation.

[^4]:    ${ }^{5}$ Notice that the algorithm here is slightly different from the one defined in the previous sections, due to the introduction of the I connective in the basic language. Similar for the non- $\xi$ rules and the $\Pi_{2}$ rules.

