# Complexity of the Guarded Two-Variable Fragment with Counting Quantifiers

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#### Abstract

We show that the finite satisfiability problem for the guarded two-variable fragment with counting quantifiers is in EXPTIME. The method employed also yields a simple proof of the result obtained in Kazakov [6], that the satisfiability problem for the guarded two-variable fragment with counting quantifiers is in EXPTIME.

### 1 Introduction

The two-variable fragment with counting quantifiers, here denoted  $C^2$ , is the set of function-free, first-order formulas containing at most two variables, but with the counting quantifiers  $\exists_{\leq C}$ ,  $\exists_{\geq C}$  and  $\exists_{=C}$  (for every C>0) allowed. The quarted two-variable fragment with counting quantifiers, here denoted  $\mathcal{GC}^2$ , is the subset of  $\mathcal{C}^2$  whose formulas contain no individual constants, and where all quantifiers appear only in guarded patterns (explained below). Both  $\mathcal{C}^2$  and  $\mathcal{GC}^2$ are assumed to contain equality. Neither fragment has the finite model property, so that their respective satisfiability problems and finite satisfiability problems do not coincide. It was shown in Pratt-Hartmann [11] that the satisfiability and finite satisfiability problems for  $\mathcal{C}^2$  are both in NEXPTIME. It was shown in Kazakov [6] that the satisfiability problem for  $\mathcal{GC}^2$  is in EXPTIME; however the method employed in that paper yields no information about the finite satis fiability problem for  $\mathcal{GC}^2$ . In this paper, we show that the finite satisfiability problem for  $\mathcal{GC}^2$  is in EXPTIME; furthermore, the method employed here also yields a simple proof of Kazakov's result. All complexity results mentioned in this paper assume succinct (binary) coding of numerical quantifier subscripts.

The fragment  $\mathcal{GC}^2$  is a proper superset of the "description logic"  $\mathcal{ALCQI}$  (Calvanese [2]), which in turn is a proper superset of the description logic  $\mathcal{ALC}$  with general concept inclusion (essentially, multimodal K with universal quantification of formulas). This latter fragment has the finite model property, and its satisfiability problem (= finite satisfiability problem) is EXPTIME-hard. Hence, the complexity bounds reported here for  $\mathcal{GC}^2$  are tight. Furthermore, it was shown in Lutz et al. [8, 9] that the finite satisfiability problem for  $\mathcal{ALCQI}$ 

is in EXPTIME. The present paper thus extends that result to the whole of  $\mathcal{GC}^2$ .

By contrast,  $\mathcal{GC}^2$  is a proper subset of  $\mathcal{C}^2$ , whose satisfiability and finite satisfiability problems are NEXPTIME-hard. In fact, even the 2-variable fragment of first-order logic without counting quantifiers, which has the finite model property, exhibits a NEXPTIME-hard satisfiability problem, as can be shown by using its formulas to encode exponentially large grids (see, e.g. Börger et al. [1], pp. 253 ff.). In such encodings, the quantifiers ( $\forall$  and  $\exists$ ) appear in unguarded patterns. On the other hand, in the presence of counting quantifiers, a similar encoding is possible using only guarded quantification, provided that just one individual constant is admitted to the language. Thus, both the lack of individual constants and the restriction to guarded patterns of quantification are essential to the comparatively low complexity of  $\mathcal{GC}^2$  (assuming, of course, that EXPTIME  $\neq$  NEXPTIME).

It was shown in Pratt-Hartmann [11], Corollary 1 that, if a formula  $\phi$  of  $\mathcal{C}^2$  is finitely satisfiable, then it has a model whose size is bounded by a doubly exponential function of the number of symbols in  $\phi$ . This bound is optimal, even for the fragment  $\mathcal{GC}^2$ , in the sense that there exists a sequence  $\{\phi_i\}$  of finitely satisfiable formulas of  $\mathcal{GC}^2 \subseteq \mathcal{C}^2$  whose size grows as a polynomial function of i, but whose smallest satisfying structures grow as a doubly exponential function of i (Grädel et al. [5], p. 317). In view of this lower bound on the sizes of smallest satisfying structures, the upper complexity bound on the finite satisfiability problem for  $\mathcal{GC}^2$  proved here is noteworthy.

The plan of this paper is as follows. Section 2 defines the fragment  $\mathcal{GC}^2$  and establishes some basic results. Section 3 describes a procedure for transforming any  $\mathcal{GC}^2$ -formula into an (exponentially large) constraint satisfaction problem with variables ranging over  $\mathbb{N}$ . Section 4 uses this transformation to prove that the finite satisfiability problem for  $\mathcal{GC}^2$  is in EXPTIME. Section 5 is dessert.

### 2 Preliminaries

We restrict consideration to first-order languages whose only primitive symbols are the variables x and y, the usual Boolean connectives, the quantifiers  $\forall$ ,  $\exists$ ,  $\exists_{\leq C}$ ,  $\exists_{\geq C}$ ,  $\exists_{=C}$  (for all C > 0), the equality predicate (here written as  $\approx$ ) and a non-logical signature of 0-ary, unary and binary predicates. There are no individual constants or function-symbols in these languages. If p is any binary predicate (including  $\approx$ ), we call an atomic formula having either of the forms p(x,y) or p(y,x) a guard-atom. The two-variable guarded fragment with counting quantifiers,  $\mathcal{GC}^2$ , can then be defined as the smallest set of formulas satisfying the following conditions:

- 1.  $\mathcal{GC}^2$  contains all atomic formulas and is closed under Boolean combinations:
- 2. if  $\phi$  is a formula of  $\mathcal{GC}^2$  with at most one free variable, and u is a variable (i.e. either x or y), then the formulas  $\forall u\phi$  and  $\exists u\phi$  are in  $\mathcal{GC}^2$ ;

3. if  $\phi$  is a formula of  $\mathcal{GC}^2$ ,  $\gamma$  a guard-atom, u a variable, and Q any of the quantifiers  $\exists$ ,  $\exists_{\leq C}$ ,  $\exists_{\geq C}$ ,  $\exists_{=C}$  (for C > 0), then the formulas  $\forall u(\gamma \to \phi)$  and  $Qu(\gamma \land \phi)$  are in  $\mathcal{GC}^2$ .

We take the semantics to be the usual semantics of first-order logic, with counting quantifiers interpreted in the obvious way. If  $\phi$  is any formula of  $\mathcal{GC}^2$ , we denote the number of symbols in  $\phi$  by  $\|\phi\|$ , assuming binary coding of numerical quantifier subscripts.

According to the above syntax, the non-counting quantifiers  $\exists$  and  $\forall$  may apply without restriction to formulas with at most one free variable; however, they may apply to formulas with two free variables only in the presence of a guard-atom. By contrast, the counting quantifiers  $\exists_{\leq C}$ ,  $\exists_{\geq C}$ ,  $\exists_{=C}$  may only every apply in the presence of a guard atom (which by definition has two free variables). Note in particular that the formula  $\exists_{=1} xp(x)$  is not in  $\mathcal{GC}^2$ . In fact, the next lemma shows that no formula of  $\mathcal{GC}^2$  can force a predicate p to be uniquely instantiated in its models.

**Definition 1.** Let  $\mathfrak{A}$ ,  $\mathfrak{B}$  be structures over disjoint domains A, B, respectively, interpreting a common signature  $\Sigma$  with no individual constants. The *union*  $\mathfrak{A} \cup \mathfrak{B}$  of  $\mathfrak{A}$  and  $\mathfrak{B}$  is the structure with domain  $A \cup B$  and interpretations  $\sigma^{\mathfrak{A} \cup \mathfrak{B}} = \sigma^{\mathfrak{A}} \cup \sigma^{\mathfrak{B}}$  for every  $\sigma \in \Sigma$ .

**Lemma 1.** Let  $\phi$  be a formula of  $\mathcal{GC}^2$  and  $\mathfrak A$  a structure over the signature of  $\phi$ . For  $N \geq 1$ , let  $N \cdot \mathfrak A$  denote the union of N disjoint copies of  $\mathfrak A$ . If  $\phi$  is satisfied in  $\mathfrak A$ , then it is satisfied in  $N \cdot \mathfrak A$ .

*Proof.* If  $\theta : \{x, y\} \to A$  is any variable assignment over A and  $1 \le i \le N$ , let  $\theta^i$  be the variable assignment over  $N \cdot \mathfrak{A}$  which maps x and y to the corresponding elements in the ith copy of  $\mathfrak{A}$ . A routine structural induction on  $\phi$  shows that  $\mathfrak{A} \models_{\theta} \phi$  if and only if, for some (= for all) i ( $1 \le i \le N$ ),  $N \cdot \mathfrak{A} \models_{\theta^i} \phi$ .

It follows immediately from Lemma 1 that, if a formula of  $\mathcal{GC}^2$  has a finite model, then it has arbitrarily large finite models, and indeed infinite models.

The following lemma, which is a cosmetic modification of Kazakov [6], Lemma 2, establishes a normal form for  $\mathcal{GC}^2$ -formulas.

**Lemma 2.** Let  $\psi$  be a formula in  $\mathcal{GC}^2$ . We can construct, in time bounded by a polynomial function of  $\|\psi\|$ , a formula

$$\phi := \forall x \alpha \land \bigwedge_{1 \le h \le l} \forall x \forall y (e_h(x, y) \to (\beta_h \lor x \approx y)) \land$$

$$\bigwedge_{1 \le i \le m} \forall x \exists_{=C_i} y (f_i(x, y) \land x \not\approx y)$$

$$(1)$$

such that: (i)  $\alpha$  is a quantifier-free formula not involving  $\approx$  with x as its only variable; (ii) l and m are positive integers; (iii) for all h  $(1 \le h \le l)$ ,  $e_h$  is a binary predicate other than  $\approx$ , and  $\beta_h$  is a quantifier-free formula not involving  $\approx$  with x and y as its only variables; (iv) for all i  $(1 \le i \le m)$ ,  $C_i$  is a positive

integer, and  $f_i$  is a binary predicate other than  $\approx$ ; (v)  $\phi$  is satisfiable if and only if  $\psi$  is satisfiable; (vi)  $\phi$  is finitely satisfiable if and only if  $\psi$  is finitely satisfiable.

*Proof.* Standard transformation to Scott normal form, using Lemma 1. See Kazakov  $op.\ cit.$  for details.

Hence, to show that the (finite) satisfiability problem for  $\mathcal{GC}^2$  is in EXP-TIME, it suffices to consider only formulas of the form (1). Furthermore, we may assume without loss of generality that no 0-ary predicates (proposition letters) occur in  $\phi$ , since we can consider each of the (at most  $2^{\|\phi\|}$ ) truth-value assignments to the 0-ary predicates of  $\phi$  in turn, replacing each 0-ary predicate with  $\top$  or  $\bot$  according to its truth-value in the considered assignment.

Accordingly, fix  $\phi$  to be some formula of the form (1) over a signature of unary and binary predicates. Set  $C = \max_{1 \leq i \leq m} C_i$ , and let  $\Sigma$  be the signature of  $\phi$  together with  $\log((mC)^2 + 1)$  (rounded up) new unary predicates. Thus,  $|\Sigma|$  is bounded by a polynomial (actually, linear) function of  $||\phi||$ . Since  $\Sigma$  is the only signature we shall be concerned with in the sequel, we generally suppress reference to it. Thus, 'predicate' henceforth means 'predicate in  $\Sigma \cup \{\approx\}$ ', 'structure' henceforth means 'structure interpreting  $\Sigma$ ', and so on. We keep the meanings of the symbols  $\alpha$ , l, m,  $e_h$ ,  $\beta_h$ ,  $(1 \leq h \leq l)$ ,  $C_i$ ,  $f_i$   $(1 \leq i \leq l)$ ,  $\phi$ , C,  $\Sigma$  fixed throughout this paper. The predicates  $f_1, \ldots, f_m$  will play a key role in the ensuing argument; we refer to them as the counting predicates. There is no restriction on these predicates' occurring in other parts of  $\phi$ : in particular, they may feature as guards.

We review some standard concepts. A *literal* is an atomic formula or the negation of an atomic formula. A *1-type* is a maximal consistent set of equality-free literals involving only the variable x. A *2-type* is a maximal consistent set of equality-free literals involving only the variables x and y. If  $\tau$  is a 2-type, then the result of transposing the variables x and y in  $\tau$  will also be a 2-type, denoted  $\tau^{-1}$ . If  $\mathfrak A$  is any structure, and  $a \in A$ , then there exists a unique 1-type  $\pi(x)$  such that  $\mathfrak A \models \pi[a]$ ; we denote  $\pi$  by  $\operatorname{tp}^{\mathfrak A}[a]$ . If, in addition,  $b \in A$  is distinct from a, then there exists a unique 2-type  $\tau(x,y)$  such that  $\mathfrak A \models \tau[a,b]$ ; we denote  $\tau$  by  $\operatorname{tp}^{\mathfrak A}[a,b]$ . We do not define  $\operatorname{tp}^{\mathfrak A}[a,b]$  if a=b.

**Notation 1.** Any 2-type  $\tau$  includes a unique 1-type, denoted  $\operatorname{tp}_1(\tau)$ ; in addition, we write  $\operatorname{tp}_2(\tau)$  for  $\operatorname{tp}_1(\tau^{-1})$ .

**Remark 1.** Let  $\mathfrak{A}$  be a structure, and let a, b be distinct elements of A. If  $\operatorname{tp}^{\mathfrak{A}}[a,b] = \tau$ , then  $\operatorname{tp}^{\mathfrak{A}}[b,a] = \tau^{-1}$ ,  $\operatorname{tp}^{\mathfrak{A}}[a] = \operatorname{tp}_{1}(\tau)$ , and  $\operatorname{tp}^{\mathfrak{A}}[b] = \operatorname{tp}_{2}(\tau)$ .

**Definition 2.** Let  $\tau$  be a 2-type. We say that  $\tau$  is a message-type if, for some counting predicate  $f_i$   $(1 \le i \le m)$ ,  $f_i(x,y) \in \tau$ . If  $\tau$  is a message-type such that  $\tau^{-1}$  is also a message-type, we say that  $\tau$  is invertible; otherwise,  $\tau$  is non-invertible. Finally, if  $\tau$  is a 2-type such that neither  $\tau$  nor  $\tau^{-1}$  is a message-type, we say that  $\tau$  is silent.

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**Remark 2.** Let  $\mathfrak{A}$  be a structure, and let a, b be distinct elements of A. Then  $\operatorname{tp}^{\mathfrak{A}}[a,b]$  is a message-type just in case  $\mathfrak{A} \models f_i[a,b]$  for some i  $(1 \leq i \leq m)$ ; if so, then this message-type is invertible just in case  $\mathfrak{A} \models f_{i'}[b,a]$  for some i'  $(1 \leq i' \leq m)$ .

The terminology is meant to suggest the following imagery. If  $\operatorname{tp}^{\mathfrak{A}}[a,b]$  is a message-type  $\mu$ , then we may imagine that a sends a message (of type  $\mu$ ) to b. If  $\mu$  is invertible, then b replies by sending a message (of type  $\mu^{-1}$ ) back to a. If  $\operatorname{tp}^{\mathfrak{A}}[a,b]$  is silent, then neither element sends a message to the other.

**Definition 3.** Let  $\mathfrak{A}$  be a structure. We say that  $\mathfrak{A}$  is *chromatic* if distinct elements connected by a chain of 1 or 2 invertible message-types have distinct 1-types. That is,  $\mathfrak{A}$  is chromatic just in case, for all  $a, a', a'' \in A$ :

- 1. if  $a \neq a'$  and  $\operatorname{tp}^{\mathfrak{A}}[a, a']$  is an invertible message-type, then  $\operatorname{tp}^{\mathfrak{A}}[a] \neq \operatorname{tp}^{\mathfrak{A}}[a']$ ; and
- 2. if a, a', a'' are pairwise distinct and both  $\operatorname{tp}^{\mathfrak{A}}[a, a']$  and  $\operatorname{tp}^{\mathfrak{A}}[a', a'']$  are invertible message-types, then  $\operatorname{tp}^{\mathfrak{A}}[a] \neq \operatorname{tp}^{\mathfrak{A}}[a'']$ .

**Lemma 3.** If  $\phi$  has a model, then it has a chromatic model over the same domain.

*Proof.* Suppose  $\mathfrak{A} \models \phi$ , and consider the (undirected) graph G on A whose edges are the pairs of distinct elements connected by a chain of 1 or 2 invertible message-types. That is,  $G = (A, E^1 \cup E^2)$ , where

$$E^{1} = \{(a, a') \mid a \neq a' \text{ and tp}^{\mathfrak{A}}[a, a'] \text{ is an invertible message-type} \}$$

$$E^{2} = \{(a, a'') \mid a \neq a'' \text{ and for some } a' \in A, (a, a') \text{ and } (a', a'') \text{ are both in } E^{1} \}.$$

Since  $C = \max_{1 \leq i \leq m} C_i$ , the degree of G (in the normal graph-theoretic sense) is at most  $(mC)^2$ . Now use the standard (greedy) algorithm to colour the nodes of G with  $(mC)^2 + 1$  colours in such a way that no edge joins two nodes of the same colour. By interpreting the  $\log((mC)^2 + 1)$  (rounded up) unary predicates of  $\Sigma$  not occurring in  $\phi$  to encode these colours, we obtain the desired chromatic model.

In the sequel, we shall need to refer to sets of invertible message-types indexed by bit-strings as follows. Let the 1-types be enumerated as

$$\Pi = \pi_0, \dots, \pi_{P-1}.$$

Evidently, P is a power of 2, so  $p = \log P$  is an integer. (Actually,  $p = |\Sigma|$ .) Now let s be any bit string  $(0 \le |s| \le p)$ , and denote the string of length 0 by  $\epsilon$ . We inductively define the sub-sequence  $\Pi_s$  of  $\Pi$  by setting  $\Pi_{\epsilon}$  to be the whole of  $\Pi$ , and setting  $\Pi_{s0}$  and  $\Pi_{s1}$  to be the left and right halves of  $\Pi_s$ , respectively. Formally:

$$\Pi_{\epsilon} = \pi_0, \dots, \pi_{P-1},$$

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and if  $\Pi_s = \pi_j, \dots, \pi_{k-1}$ , with |s| < p,

$$\Pi_{s0} = \pi_j, \dots, \pi_{\frac{k-j}{2}}$$

$$\Pi_{s1} = \pi_{\frac{k-j}{2}+1}, \dots, \pi_{k-1}.$$

Thus, if |s| = p, then  $\Pi_s$  is a one-element sequence  $\pi_j$ , where j is the integer  $(0 \le j < P)$  encoded by the bit-string s in the usual way. To avoid clumsy circumlocutions, we occasionally equivocate between bit-strings and the integers they encode, thus, for example, writing  $\pi_s$  instead of  $\pi_j$  in this case. But we will only ever write  $\pi_s$  if |s| = p.

We may use the sets  $\Pi_s$  to define sets of invertible message-types indexed by bit-strings as follows. Fix any 1-type  $\pi$ , and denote by  $\Lambda_{\pi}$  the set of invertible message-types  $\lambda$  such that  $\operatorname{tp}_1(\lambda) = \pi$ . If s is any bit-string such that  $|s| \leq p$ , let

$$\Lambda_{\pi,s} = \{ \lambda \in \Lambda_{\pi} \mid \text{tp}_2(\lambda) \in \Pi_s \}.$$

Thus, the  $\Lambda_{\pi,s}$  are sets of invertible message-types identified purely by their terminal 1-types. Except in very special cases, these sets will contain more than one member, even when |s| = p. However, for chromatic models, we have the following important fact.

**Lemma 4.** If  $\mathfrak{A}$  is chromatic,  $\pi = \operatorname{tp}^{\mathfrak{A}}[a]$ , and s is a bit-string with |s| = p, then there can be at most one element  $b \in A \setminus \{a\}$  such that  $\operatorname{tp}^{\mathfrak{A}}[a,b] \in \Lambda_{\pi,s}$ .

*Proof.* Any two such elements would be connected by a chain of two invertible message-types, and would both have the 1-type  $\pi_s$ .

Finally, we use bit strings to index other sets of 2-types as follows. Again, fix any 1-type  $\pi$ , and consider the set of non-invertible message-types  $\mu$  such that  $\operatorname{tp}_1(\mu) = \pi$ . Let these be enumerated in some way as a sequence

$$\mu_{\pi,0},\ldots,\mu_{\pi,R-1}.$$

Furthermore, consider the set of silent 2-types  $\mu$  such that  $\operatorname{tp}_1(\mu) = \pi$ . Let these be enumerated in some way as a sequence

$$\mu_{\pi,R},\ldots,\mu_{\pi,Q-1}.$$

Thus, the sequence

$$M_{\pi} = \mu_{\pi,0}, \dots, \mu_{\pi,Q-1}.$$

is a list of precisely those 2-types  $\tau$  such that  $\operatorname{tp}_1(\tau) = \pi$  and  $\tau^{-1}$  is not a message-type. Evidently, R and Q are independent of the choice of  $\pi$ ; moreover, Q is a power of 2, so  $q = \log Q$  is an integer. (We remark that R is not a power of 2.) Let t be any bit string  $(0 \le |t| \le q)$ . We inductively define the sub-sequence  $M_{\pi,t}$  of  $M_{\pi}$  by setting  $M_{\pi,\epsilon}$  to be the whole of  $M_{\pi}$ , and setting  $M_{\pi,t0}$  and  $M_{\pi,t1}$  to be the left and right halves of  $M_{\pi,t}$ , respectively. Formally:

$$M_{\pi,\epsilon} = \mu_{\pi,0}, \dots, \mu_{\pi,Q-1},$$

and if  $M_{\pi,t} = \mu_{\pi,j}, \dots, \mu_{\pi,k-1}$ , with |t| < q,

$$\begin{array}{rcl} M_{\pi,t0} & = & \mu_{\pi,j}, \dots, \mu_{\pi,\frac{k-j}{2}} \\ M_{\pi,t1} & = & \mu_{\pi,\frac{k-j}{2}+1}, \dots, \mu_{\pi,k-1}. \end{array}$$

Thus, if |t| = q, then  $M_{\pi,t}$  is a one-element sequence  $\mu_{\pi,j}$ , where j is the integer  $(0 \le j < Q)$  encoded by the bit-string t in the usual way. Again we may for convenience write  $\mu_{\pi,t}$  instead of  $\mu_{\pi,j}$  in this case, but here too we only ever write  $\mu_{\pi,t}$  if |t| = q.

# 3 Transformation into an integer constraint problem

Henceforth, *vector* means "m-dimensional vector over  $\mathbb{N}$ ". If  $\mathbf{u}$  and  $\mathbf{v}$  are vectors, we write  $\mathbf{u} \leq \mathbf{v}$  if every component of  $\mathbf{u}$  is less than or equal to the corresponding component of  $\mathbf{v}$ ; we write  $\mathbf{u} < \mathbf{v}$  if  $\mathbf{u} \leq \mathbf{v}$  and  $\mathbf{u} \neq \mathbf{v}$ . Similarly for  $\geq$  and >. The number of vectors  $\mathbf{u}$  such that  $\mathbf{u} \leq \mathbf{C}$  is bounded by  $(C+1)^m$ , and hence by an exponential function of  $\|\phi\|$ .

Referring to the formula (1), denote the vector  $(C_1, \ldots, C_m)$  by  $\mathbf{C}$  and the vector  $(0, \ldots, 0)$  by  $\mathbf{0}$ . Moreover, given any 2-type  $\tau$ , we write  $\mathbf{C}_{\tau}$  for the vector  $(C_{\tau,1}, \ldots, C_{\tau,m})$  where, for all i  $(1 \le i \le m)$ ,

$$C_{\tau,i} = \begin{cases} 1 & \text{if } f_i(x,y) \in \tau, \\ 0 & \text{otherwise.} \end{cases}$$
 (2)

Note that, if  $\tau$  is not a message-type—in particular, if  $\tau$  is a silent 2-type—we have  $\mathbf{C}_{\tau} = \mathbf{0}$ .

Let  $\tau$  be any 2-type. Since  $\tau$  is a finite set of formulas with free variables x and y, we may write  $\bigwedge \tau$  to denote their conjunction. Referring again to the formula (1), we say that  $\tau$  is *forbidden*, if the formula

$$\alpha(x) \wedge \alpha(y) \wedge \bigwedge_{1 \le h \le l} (e_h(x, y) \to \beta_h) \wedge \bigwedge \tau$$
 (3)

is unsatisfiable. Thus, if  $\mathfrak{A} \models \phi$  and a, b are distinct elements of A, then  $\operatorname{tp}^{\mathfrak{A}}[a, b]$  cannot be forbidden. Since (3) is purely Boolean, we can evidently identify the forbidden 2-types in time bounded by an exponential function of  $\|\phi\|$ .

In the sequel, we take  $\pi$  to vary over the set of 1-types,  $\lambda$  to vary over the set of invertible message-types, s to vary over the set of bit-strings of length at most p, t to vary over the set of bit-strings of length at most q, and  $\mathbf{u}$ ,  $\mathbf{v}$  and  $\mathbf{w}$  to vary over the set of vectors  $\leq \mathbf{C}$ . (Similarly for their primed counterparts  $\pi'$ ,  $\lambda'$ , s', t',  $\mathbf{u'}$ ,  $\mathbf{v'}$  and  $\mathbf{w'}$ .) We refer to these sets as the *standard ranges* of the respective letters. Occasionally, additional restrictions on these ranges will be imposed.

Now let V be the set whose elements are the following (distinct) symbols, where the indices  $\lambda$ ,  $\pi$ , s, t,  $\mathbf{u}$ ,  $\mathbf{v}$ ,  $\mathbf{w}$  vary over their standard ranges:

$$x_{\lambda}$$
,  $y_{\pi,s,\mathbf{u}}$ ,  $z_{\pi,t,\mathbf{u}}$ ,  $\hat{y}_{\pi,s,\mathbf{v},\mathbf{w}}$  whenever  $|s| < p$ ,  $\hat{z}_{\pi,t,\mathbf{v},\mathbf{w}}$  whenever  $|t| < q$ .

The symbols  $\hat{y}_{\pi,s,\mathbf{v},\mathbf{w}}$  and  $\hat{z}_{\pi,t,\mathbf{v},\mathbf{w}}$  are not defined when |s|=p and |t|=q. The cardinality of V is evidently bounded by an exponential function of  $\|\phi\|$ . We impose some arbitrary order on V, and refer to its elements as variables. If U is a non-empty set of variables, enumerated, in order, as  $\{u_1,\ldots,u_k\}$ , let  $\sum U$  denote the term  $u_1+\cdots+u_k$ ; if U is the empty set, let  $\sum U$  denote the (constant) term 0. In the sequel, we take a constraint to be an equation or inequality involving arithmetical terms over V, or a conditional statement formed from two such inequalities. A solution of a set of constraints over some numerical domain  $\mathbb D$  is simply a function  $\theta:V\to\mathbb D$  under which all the constraints in question evaluate (in the obvious way) to true.

With this notation, let  $\mathcal{E}_1$  be the following set of constraints involving the variables V, where  $\pi$ ,  $\mathbf{u}$ ,  $\mathbf{v}$ ,  $\mathbf{w}$  again vary over their standard ranges, and s, t vary over bit-strings such that |s| < p and |t| < q:

$$z_{\pi,\epsilon,\mathbf{u}} = y_{\pi,\epsilon,\mathbf{C}-\mathbf{u}} \tag{4}$$

$$y_{\pi,s,\mathbf{u}} = \sum \{\hat{y}_{\pi,s,\mathbf{v}',\mathbf{w}'} \mid \mathbf{v}' + \mathbf{w}' = \mathbf{u}\}$$
 (5)

$$z_{\pi,t,\mathbf{u}} = \sum \{\hat{z}_{\pi,t,\mathbf{v}',\mathbf{w}'} \mid \mathbf{v}' + \mathbf{w}' = \mathbf{u}\}$$
 (6)

$$y_{\pi,s0,\mathbf{v}} = \sum \{\hat{y}_{\pi,s,\mathbf{v},\mathbf{w}'} \mid \mathbf{v} + \mathbf{w}' \le \mathbf{C}\}$$
 (7)

$$y_{\pi,s1,\mathbf{w}} = \sum \{\hat{y}_{\pi,s,\mathbf{v}',\mathbf{w}} \mid \mathbf{v}' + \mathbf{w} \leq \mathbf{C}\}$$
 (8)

$$z_{\pi,t0,\mathbf{v}} = \sum \{\hat{z}_{\pi,t,\mathbf{v},\mathbf{w}'} \mid \mathbf{v} + \mathbf{w}' \le \mathbf{C}\}$$
 (9)

$$z_{\pi,t1,\mathbf{w}} = \sum \{\hat{z}_{\pi,t,\mathbf{v}',\mathbf{w}} \mid \mathbf{v}' + \mathbf{w} \le \mathbf{C}\}$$
 (10)

$$1 \leq \sum \{ y_{\pi',\epsilon,\mathbf{u}'} \mid \pi' \text{ a 1-type, } \mathbf{u}' \leq \mathbf{C} \}.$$
 (11)

Let  $\mathcal{E}_2$  consist of the following constraints, where  $\lambda$ ,  $\pi$  vary over their standard ranges, s, t vary over bit-strings such that |s| = p, |t| = q, and  $\mathbf{u}$  varies over vectors such that  $\mathbf{0} < \mathbf{u} \le \mathbf{C}$ :

$$y_{\pi,s,\mathbf{u}} = \sum \{x_{\lambda'} \mid \lambda' \in \Lambda_{\pi,s} \text{ and } \mathbf{C}_{\lambda'} = \mathbf{u}\}$$
 (12)

$$z_{\pi,t,\mathbf{u}} = 0$$
 whenever  $\mathbf{u}$  is not a scalar multiple of  $\mathbf{C}_{\tau}$  for  $\tau = \mu_{\pi,t}$  (13)

$$x_{(\lambda^{-1})} = x_{\lambda} \tag{14}$$

$$x_{\lambda} = 0$$
 whenever  $\operatorname{tp}_{1}(\lambda) = \operatorname{tp}_{2}(\lambda)$  (15)

$$x_{\lambda} = 0$$
 whenever  $\lambda$  is forbidden (16)

$$z_{\pi,t,\mathbf{u}} = 0$$
 whenever  $\mu_{\pi,t}$  is forbidden. (17)

Note that, in (13),  $\tau = \mu_{\pi,t}$  is a 2-type, and the vector  $\mathbf{C}_{\tau}$  is defined according to (2). If the integer encoded by t is less than R,  $\tau = \mu_{\pi,t}$  will be a (non-invertible) message-type, and we will have  $\mathbf{C}_{\tau} > \mathbf{0}$ . If, on the other hand, the integer encoded by t is greater than or equal to R,  $\tau$  will be a silent 2-type, and we will have  $\mathbf{C}_{\tau} = \mathbf{0}$ . In this latter case, no vector  $\mathbf{u}$  such that  $\mathbf{u} > \mathbf{0}$  can be a multiple of  $\mathbf{C}_{\tau}$ , whence,  $\mathcal{E}_{2}$  contains the constraint  $z_{\pi,t,\mathbf{u}} = 0$  for all  $\mathbf{u}$  such that  $\mathbf{0} < \mathbf{u} \leq \mathbf{C}$ .

Let  $\mathcal{E}_3$  consist of the following constraints, where  $\pi$  varies over all 1-types, t varies over bit-strings such that |t| = q, and  $\mathbf{u}$  varies over vectors such that  $\mathbf{0} < \mathbf{u} \le \mathbf{C}$ :

$$z_{\pi,t,\mathbf{u}} > 0 \quad \Rightarrow \quad \sum \{y_{\pi',\epsilon,\mathbf{u}'} \mid \pi' = \operatorname{tp}_2(\mu_{\pi,t}) \text{ and } \mathbf{u}' \le \mathbf{C}\} \ge 3mC. \quad (18)$$

Again, if the integer encoded by t is greater than or equal to R, we have already argued that  $\mathcal{E}_2$  contains the constraint,  $z_{\pi,t,\mathbf{u}} = 0$  for all  $\mathbf{u}$  such that  $\mathbf{0} < \mathbf{u} \leq \mathbf{C}$ , rendering the corresponding instances of (18) trivial.

Finally, let  $\mathcal{E} = \mathcal{E}_1 \cup \mathcal{E}_2 \cup \mathcal{E}_3$ .

### 4 Main result

**Lemma 5.** Let  $\phi$  and  $\mathcal{E}$  be as above. If  $\phi$  is finitely satisfiable, then  $\mathcal{E}$  has a solution over  $\mathbb{N}$ .

*Proof.* Suppose  $\phi$  is finitely satisfiable. By Lemma 3, let  $\mathfrak{A}'$  be a finite, chromatic model of  $\phi$ , and by Lemma 1, let  $\mathfrak{A} = 3mC \cdot \mathfrak{A}'$ . Thus,  $\mathfrak{A}$  is also chromatic. Let  $\mathfrak{A}$  have domain A. If  $\pi$  is a 1-type, let  $A_{\pi} = \{a \in A \mid \operatorname{tp}^{\mathfrak{A}}[a] = \pi\}$ . Now suppose  $a \in A_{\pi}$ . For any bit-string s  $(0 \leq |s| \leq p)$ , define the s-spectrum of a, denoted  $\operatorname{sp}_{s}^{\mathfrak{A}}[a]$ , to be the vector whose ith component  $(1 \leq i \leq m)$  is given by

$$|\{b \in A : \mathfrak{A} \models f_i[a, b], b \neq a \text{ and } \operatorname{tp}^{\mathfrak{A}}[a, b] \in \Lambda_{\pi, s}\}|.$$

For any bit-string t  $(0 \le |t| \le q)$ , define the t-tally of a, denoted  $\operatorname{tl}_t^{\mathfrak{A}}[a]$ , to be the vector whose ith component  $(1 \le i \le m)$  is given by

$$|\{b \in A : \mathfrak{A} \models f_i[a, b], b \neq a \text{ and } \operatorname{tp}^{\mathfrak{A}}[a, b] \in M_{\pi, t}\}|.$$

These quantities are easiest to understand when the strings s and t take the value  $\epsilon$ . For any  $a \in A$ ,  $\operatorname{sp}^{\mathfrak{A}}_{\epsilon}[a]$  is the vector whose ith component records the number of elements b to which a sends a message of invertible type containing the atom  $f_i(x,y)$ . Likewise,  $\operatorname{tl}^{\mathfrak{A}}_{\epsilon}[a]$  is the vector whose ith component records the number of elements b to which a sends a message of non-invertible type containing the atom  $f_i(x,y)$ . If  $\operatorname{tp}^{\mathfrak{A}}[a] = \pi$  and  $0 < |s| \leq p$ , then  $\operatorname{sp}^{\mathfrak{A}}_{s}[a]$  is obtained in the same way as  $\operatorname{sp}^{\mathfrak{A}}_{\epsilon}[a]$ , except that we discount all messages whose type is not a member of  $\Lambda_{\pi,s}$ . Likewise, if  $\operatorname{tp}^{\mathfrak{A}}[a] = \pi$  and  $0 < |t| \leq q$ , then  $\operatorname{tl}^{\mathfrak{A}}_{t}[a]$  is obtained in the same way as  $\operatorname{tl}^{\mathfrak{A}}_{t}[a]$ , except that we discount all messages

whose type is not a member of  $M_{\pi,t}$ . It is then easy to see that, for all  $a \in A$ , all bit-strings s (|s| < p) and all bit-strings t (|t| < q),

$$\operatorname{sp}_{\epsilon}^{\mathfrak{A}}[a] + \operatorname{tl}_{\epsilon}^{\mathfrak{A}}[a] = \mathbf{C} \tag{19}$$

$$\operatorname{sp}_{s0}^{\mathfrak{A}}[a] + \operatorname{sp}_{s1}^{\mathfrak{A}}[a] = \operatorname{sp}_{s}^{\mathfrak{A}}[a] \tag{20}$$

$$\operatorname{tl}_{t0}^{\mathfrak{A}}[a] + \operatorname{tl}_{t1}^{\mathfrak{A}}[a] = \operatorname{tl}_{t}^{\mathfrak{A}}[a]. \tag{21}$$

Let  $\theta: V \to \mathbb{N}$  be defined by:

$$\begin{array}{lll} \theta(x_{\lambda}) &=& |\{a \in A: \text{ there exists } b \in A \setminus \{a\} \text{ such that } \operatorname{tp}^{\mathfrak{A}}[a,b] = \lambda\}| \\ \theta(y_{\pi,s,\mathbf{u}}) &=& |\{a \in A_{\pi}: \operatorname{sp}_{s}^{\mathfrak{A}}[a] = \mathbf{u}\}| \\ \theta(z_{\pi,t,\mathbf{u}}) &=& |\{a \in A_{\pi}: \operatorname{tl}_{t}^{\mathfrak{A}}[a] = \mathbf{u}\}| \\ \theta(\hat{y}_{\pi,s,\mathbf{v},\mathbf{w}}) &=& |\{a \in A_{\pi}: \operatorname{sp}_{s0}^{\mathfrak{A}}[a] = \mathbf{v} \text{ and } \operatorname{sp}_{s1}^{\mathfrak{A}}[a] = \mathbf{w}\}| \quad \text{whenever } |s|$$

where the indices  $\lambda$ ,  $\pi$ , s, t,  $\mathbf{u}$ ,  $\mathbf{v}$  and  $\mathbf{w}$  vary over their standard ranges.

We show that  $\theta$  is a solution of  $\mathcal{E}$  by checking the forms (4)–(18) in turn. The constraints (4)–(6) follow easily from Equations (19)–(21), respectively. The constraints (7)–(10) are immediate. In the (single) constraint (11), the sum on the right-hand side evaluates under  $\theta$  to the cardinality of A, which is positive by definition. To understand the constraints (12), fix any  $\pi$  and s (with |s| = p), and write, for any  $\lambda \in \Lambda_{\pi,s}$ ,

$$A_{\lambda} = \{a \in A \mid \text{ there exists } b \in A \setminus \{a\} \text{ such that } \operatorname{tp}^{\mathfrak{A}}[a,b] = \lambda\}.$$

Thus,  $A_{\lambda} \subseteq A_{\pi}$ , and  $|A_{\lambda}| = \theta(x_{\lambda})$ . By Lemma 4, for any  $a \in A_{\pi}$ , there can be at most one element  $b \in A \setminus \{a\}$  such that  $\operatorname{tp}^{\mathfrak{A}}[a,b] \in \Lambda_{\pi,s}$ . It follows that the sets  $A_{\lambda}$ , for  $\lambda$  varying over  $\Lambda_{\pi,s}$ , are pairwise disjoint, and, moreover, that  $a \in A_{\pi}$  has s-spectrum  $\mathbf{u} > \mathbf{0}$  if any only if it is a member of some (hence, exactly one) of these sets  $A_{\lambda}$ , with  $\mathbf{C}_{\lambda} = \mathbf{u}$ . That is, for all  $\mathbf{u} > \mathbf{0}$ ,

$$\{a \in A_{\pi} \mid \operatorname{sp}_{s}^{\mathfrak{A}}[a] = \mathbf{u}\} = \bigcup \{A_{\lambda'} \mid \lambda' \in \Lambda_{\pi,s} \text{ and } \mathbf{C}_{\lambda'} = \mathbf{u}\},$$

with the sets on the right-hand side pairwise disjoint. The relevant instance of (12) is then immediate from the definition of  $\theta$ . To see why the constraints (13) hold, note that, if |t| = q, then  $M_{\pi,t}$  is a set containing precisely one 2-type  $\tau = \mu_{\pi,t}$ , which is either a non-invertible message-type or a silent 2-type. Either way, for every  $a \in A_{\pi}$ ,  $\operatorname{tl}_t^{\mathfrak{A}}[a]$  must be a scalar multiple (possibly zero) of  $\mathbf{C}_{\tau}$ . In other words, if  $\mathbf{u} > \mathbf{0}$  is not a scalar multiple of  $\mathbf{C}_{\tau}$ , then  $\theta(z_{\pi,t,\mathbf{u}}) = 0$ . To understand the constraints (14), observe that, since  $\mathfrak{A}$  is chromatic,  $\theta(x_{\lambda})$  is actually the total number of messages of (invertible) type  $\lambda$  sent by elements of  $\mathfrak{A}$ , and similarly for  $\theta(x_{(\lambda^{-1})})$ ; and these numbers are obviously equal. The constraints (15) are immediate given that  $\mathfrak{A}$  is chromatic. The constraints (16) and (17) are immediate given that  $\mathfrak{A} \models \phi$ . To understand the constraints (18), fix any  $\pi$ , t and  $\mathbf{u}$  such that |t| = q and  $\mathbf{u} > \mathbf{0}$ . If  $z_{\pi,t,\mathbf{u}} > 0$  holds under  $\theta$ , then  $\mu_{\pi,t}$  is a (non-invertible) message-type; and furthermore, at least one message

of that type must be sent in  $\mathfrak{A}$ , so that  $\mathfrak{A}$  contains at least one element whose 1-type is  $\operatorname{tp}_2(\mu_{\pi,t})$  and hence—since  $\mathfrak{A} = 3mC \cdot \mathfrak{A}'$ —at least 3mC such elements. But the exact number of elements in A whose 1-type is  $\operatorname{tp}_2(\mu_{\pi,t})$  is given by the value, under  $\theta$ , of  $\sum \{y_{\pi',\epsilon,\mathbf{u}'} \mid \pi' = \operatorname{tp}_2(\mu_{\pi,t}) \text{ and } \mathbf{u}' \leq \mathbf{C}\}$ .

In establishing the converse of Lemma 5, the following technical result concerning solutions of  $\mathcal{E}$  will prove useful. To avoid notational clutter, we use the variable names  $x_{\lambda}$ ,  $y_{\pi,s,\mathbf{u}}$ ,  $z_{\pi,t,\mathbf{u}}$ ,  $\hat{y}_{\pi,s,\mathbf{v},\mathbf{w}}$ ,  $\hat{z}_{\pi,t,\mathbf{v},\mathbf{w}}$  to stand for the corresponding natural numbers in some such solution (and similarly for terms involving these variables).

**Lemma 6.** Let  $x_{\lambda}$ ,  $y_{\pi,s,\mathbf{u}}$ ,  $z_{\pi,t,\mathbf{u}}$ ,  $\hat{y}_{\pi,s,\mathbf{v},\mathbf{w}}$ ,  $\hat{z}_{\pi,t,\mathbf{v},\mathbf{w}}$  (with indices having the appropriate ranges) be natural numbers satisfying the constraints  $\mathcal{E}$  given above. Fix any 1-type  $\pi$ , and let  $A_{\pi}$  be a set of cardinality  $\sum \{y_{\pi,\epsilon,\mathbf{u}'} \mid \mathbf{u}' \leq \mathbf{C}\}$ . Then there exists a system of functions on  $A_{\pi}$ 

$$f_{\pi,s}: A_{\pi} \to \{\mathbf{u} \mid \mathbf{u} \leq \mathbf{C}\}$$
  $g_{\pi,t}: A_{\pi} \to \{\mathbf{u} \mid \mathbf{u} \leq \mathbf{C}\},$ 

where the indices s and t vary over their standard ranges, such that, for all vectors  $\mathbf{u} \leq \mathbf{C}$ ,

$$|f_{\pi,s}^{-1}(\mathbf{u})| = y_{\pi,s,\mathbf{u}} \tag{22}$$

$$|g_{\pi,t}^{-1}(\mathbf{u})| = z_{\pi,t,\mathbf{u}}, \tag{23}$$

and such that, for all  $a \in A_{\pi}$ ,

$$\sum \{ f_{\pi,s'}(a) : |s'| = p \} + \sum \{ g_{\pi,t'}(a) : |t'| = q \} = \mathbf{C}.$$
 (24)

*Proof.* Decompose the set  $A_{\pi}$  into pairwise disjoint (possibly empty) sets  $A_{\mathbf{u}}$  such that  $|A_{\mathbf{u}}| = y_{\pi,\epsilon,\mathbf{u}}$ , where the index  $\mathbf{u}$  varies over all vectors  $\leq \mathbf{C}$ . This is possible by the cardinality of  $A_{\pi}$ . For all  $\mathbf{u} \leq \mathbf{C}$ , and all  $a \in A_{\mathbf{u}}$ , set

$$f_{\pi,\epsilon}(a) = \mathbf{u}$$
  $q_{\pi,\epsilon}(a) = \mathbf{C} - \mathbf{u}.$ 

This assignment evidently satisfies (22) for  $s = \epsilon$ ; and by the constraints (4), it also satisfies (23) for  $t = \epsilon$ . We observe in passing that, for all  $a \in A_{\pi}$ ,

$$f_{\pi,\epsilon}(a) + g_{\pi,\epsilon}(a) = \mathbf{C}. \tag{25}$$

We now construct the functions  $f_{\pi,s}$ , where  $0 < |s| \le p$ , by induction on s. Assume that, for some s  $(0 \le |s| < p)$ ,  $f_{\pi,s}$  has been defined and satisfies (22). For every vector  $\mathbf{u} \le \mathbf{C}$ , decompose  $f_{\pi,s}^{-1}(\mathbf{u})$  into pairwise disjoint (possibly empty) sets  $A_{\mathbf{v},\mathbf{w}}$  such that  $|A_{\mathbf{v},\mathbf{w}}| = \hat{y}_{\pi,s,\mathbf{v},\mathbf{w}}$ , where the indices  $\mathbf{v}$ ,  $\mathbf{w}$  vary over all vectors satisfying  $\mathbf{v} + \mathbf{w} = \mathbf{u}$ . This is possible by the constraints (5) together with the assumption that  $f_{\pi,s}$  satisfies (22). Having thus decomposed the sets

 $f_{\pi,s}^{-1}(\mathbf{u})$  (for all  $\mathbf{u} \leq \mathbf{C}$ ), we see that, for any  $a \in A_{\pi}$ , there is precisely one (ordered) pair of vectors  $\mathbf{v}$ ,  $\mathbf{w}$  such that  $a \in A_{\mathbf{v},\mathbf{w}}$ ; hence we may set

$$f_{\pi,s0}(a) = \mathbf{v}$$
  $f_{\pi,s1}(a) = \mathbf{w}$ .

This defines the functions  $f_{\pi,s0}$  and  $f_{\pi,s1}$ . We observe in passing that, for all  $a \in A_{\pi}$ ,

$$f_{\pi,s0}(a) + f_{\pi,s1}(a) = f_{\pi,s}(a).$$
 (26)

To see that  $f_{\pi,s0}$  and  $f_{\pi,s1}$  both satisfy Equation (22), note that  $f_{\pi,s0}(a) = \mathbf{v}$  if and only if, for some vector  $\mathbf{w}'$  such that  $\mathbf{v} + \mathbf{w}' \leq \mathbf{C}$ ,  $a \in A_{\mathbf{v},\mathbf{w}'}$ . Similarly,  $f_{\pi,s1}(a) = \mathbf{w}$  if and only if, for some vector  $\mathbf{v}'$  such that  $\mathbf{v}' + \mathbf{w} \leq \mathbf{C}$ ,  $a \in A_{\mathbf{v}',\mathbf{w}}$ . That is,

$$f_{\pi,s0}^{-1}(\mathbf{v}) = \bigcup \{A_{\mathbf{v},\mathbf{w}'} \mid \mathbf{v} + \mathbf{w}' \le \mathbf{C}\}$$
  
$$f_{\pi,s1}^{-1}(\mathbf{w}) = \bigcup \{A_{\mathbf{v}',\mathbf{w}} \mid \mathbf{v}' + \mathbf{w} \le \mathbf{C}\},$$

with the collections of sets on the respective right-hand sides being pairwise disjoint. By the constraints (7)–(8), together with the fact that  $|A_{\mathbf{v},\mathbf{w}}| = \hat{y}_{\pi,s,\mathbf{v},\mathbf{w}}$  for all  $\mathbf{v},\mathbf{w}$ , we have:

$$|f_{\pi,s0}^{-1}(\mathbf{v})| = y_{\pi,s0,\mathbf{v}}$$
  
 $|f_{\pi,s1}^{-1}(\mathbf{w})| = y_{\pi,s1,\mathbf{w}},$ 

which establishes (22) for the functions  $f_{\pi,s0}$  and  $f_{\pi,s1}$ . This completes the induction. The construction of the functions  $g_{\pi,t}$  proceeds completely analogously, using Constraints (6), (9) and (10). In carrying out this latter construction, we obtain, in a parallel way to (26),

$$g_{\pi,t0}(a) + g_{\pi,t1}(a) = g_{\pi,t}(a),$$
 (27)

for all bit-strings t (|t| < q) and all  $a \in A_{\pi}$ .

It remains to establish (24). We prove the stronger result that, for all  $a \in A_{\pi}$ ,  $j \ (0 \le j \le p)$  and  $k \ (0 \le k \le q)$ ,

$$\sum \{f_{\pi,s'}(a) : |s'| = j\} + \sum \{g_{\pi,t'}(a) : |t'| = k\} = \mathbf{C}, \tag{28}$$

using a double induction on j and k. If j = k = 0, then the left-hand side of (28) is simply  $f_{\pi,\epsilon}(a) + g_{\pi,\epsilon}(a)$ , which is equal to  $\mathbf{C}$  by (25). Suppose now that the result holds for the pair j, k, with j < p. Then

$$\sum \{f_{\pi,s'}(a) : |s'| = (j+1)\} + \sum \{g_{\pi,t'}(a) : |t'| = k\}$$

$$= \sum \{f_{\pi,s'0}(a) + f_{\pi,s'1}(a) : |s'| = j\} + \sum \{g_{\pi,t'}(a) : |t'| = k\}$$

$$= \sum \{f_{\pi,s}(a) : |s'| = j\} + \sum \{g_{\pi,t'}(a) : |t'| = k\} \quad \text{by (26)}$$

$$= \mathbf{C} \quad \text{by inductive hypothesis.}$$

This establishes the result for the pair j+1, k. An analogous argument using (27) applies when k < m, completing the induction.

Before we come to the promised converse of Lemma 5, we remark on the (exponentially many) choices made during the construction of the various functions  $f_{\pi,s}$  and  $g_{\pi,t}$  in the proof of Lemma 6—specifically, in the decomposition of certain sets into collections of subsets. It is because of this large number of independent choices that solutions of  $\mathcal{E}$  typically encode not one, but many, models of  $\phi$ .

**Lemma 7.** Let  $\phi$  and  $\mathcal{E}$  be as above. If  $\mathcal{E}$  has a solution over  $\mathbb{N}$ , then  $\phi$  is finitely satisfiable.

*Proof.* Suppose  $\mathcal{E}$  has a solution over  $\mathbb{N}$ . Again, we use the variable names  $x_{\lambda}$ ,  $y_{\pi,s,\mathbf{u}}$ ,  $z_{\pi,t,\mathbf{u}}$ ,  $\hat{y}_{\pi,s,\mathbf{v},\mathbf{w}}$ ,  $\hat{z}_{\pi,t,\mathbf{v},\mathbf{w}}$  to stand for the corresponding values in some such solution. Our task is to construct a model  $\mathfrak{A}$  of  $\phi$ .

For each 1-type  $\pi$ , let  $A_{\pi}$  be a set of cardinality  $\sum \{y_{\pi,\epsilon,\mathbf{u}} \mid \mathbf{u} \leq \mathbf{C}\}$ , with the  $A_{\pi}$  pairwise disjoint; and let  $A = \bigcup \{A_{\pi} \mid \pi \text{ a 1-type}\}$ . Think of  $A_{\pi}$  as the set of elements of A which 'want' to have 1-type  $\pi$ . By the constraint (11),  $A \neq \emptyset$ . For every 1-type  $\pi$ , let the functions  $f_{\pi,s}$  and  $g_{\pi,t}$  be constructed as in Lemma 6; we are interested only in those  $f_{\pi,s}$  and  $g_{\pi,t}$  where |s| = p, and |t| = q. For all such  $\pi$ , s, t, and all  $a \in A_{\pi}$ , think of  $f_{\pi,s}(a)$  as the s-spectrum which a 'wants' to have, and think of  $g_{\pi,t}(a)$  as the t-tally which a 'wants' to have. Finally, consider any set  $f_{\pi,s}^{-1}(\mathbf{u})$ , where  $\mathbf{0} < \mathbf{u} \leq \mathbf{C}$  and |s| = p. Using the constraints (12) and Equation (22), we can decompose  $f_{\pi,s}^{-1}(\mathbf{u})$  into pairwise disjoint (possibly empty) sets  $A_{\lambda}$  with  $|A_{\lambda}| = x_{\lambda}$ , where  $\lambda$  varies over the set of invertible message-types such that  $\lambda \in \Lambda_{\pi,s}$  and  $\mathbf{C}_{\lambda} = \mathbf{u}$ . It follows that, if  $a \in A_{\lambda}$ , with  $\lambda \in \Lambda_{\pi,s}$ , then  $\mathbf{C}_{\lambda} = f_{\pi,s}(a)$ . Think of  $A_{\lambda}$  as the set of elements of  $A_{\pi}$  which 'want' to send a single message of (invertible) type  $\lambda$ .

Before proceeding, we pause to consider the construction just described in respect of any of the sets  $A_{\pi}$ . Fixing, for the moment, some bit-string s with |s| = p, we see that  $A_{\pi}$  is decomposed into the pairwise disjoint sets  $f_{\pi,s}^{-1}(\mathbf{u})$  (as  $\mathbf{u}$  varies over vectors such that  $\mathbf{u} \leq \mathbf{C}$ ), and that each of the sets  $f_{\pi,s}^{-1}(\mathbf{u})$ , where  $\mathbf{0} < \mathbf{u} \leq \mathbf{C}$ , is further decomposed into the pairwise disjoint subsets  $A_{\lambda}$  (as  $\lambda$  varies over the elements of  $\Lambda_{\pi,s}$  such that  $\mathbf{C}_{\lambda} = \mathbf{u}$ ). Note that the set  $f_{\pi,s}^{-1}(\mathbf{0})$  is not subject to this further stage of decomposition. This process is performed for every bit string s with |s| = p, so that different values of s lead to independent—and possibly overlapping—decompositions, as illustrated in Fig. 1. Likewise, for every bit-string t with |t| = q,  $A_{\pi}$  is decomposed into the pairwise disjoint sets  $g_{\pi,t}^{-1}(\mathbf{u})$  (as  $\mathbf{u}$  varies over vectors such that  $\mathbf{u} \leq \mathbf{C}$ ). Again, decompositions corresponding to different values of t should be thought of as independent of each other.

We now proceed to construct, for every  $a \in A$ , a data-structure recording a collection of messages sent by a, each of which is labelled with some (invertible or non-invertible) message-type. (These data-structures will later be combined to form a structure  $\mathfrak{A}$  on A.) Recall that, if  $\pi$  is any 1-type, then  $\mu_{\pi,0}, \ldots, \mu_{\pi,R-1}$  is an enumeration of the non-invertible message-types  $\mu$  such that  $\operatorname{tp}_1(\mu) = \pi$ . Fix  $a \in A$ , and let  $\pi$  be the unique 1-type such that  $a \in A_{\pi}$ . The messages sent by a shall be as follows. (i) For every bit-string a such that  $a \in A_{\pi}$  is  $a \in A_{\pi,s}$  be the invertible message-type  $a \in A_{\pi,s}$  such that  $a \in A_{\pi,s}$ 

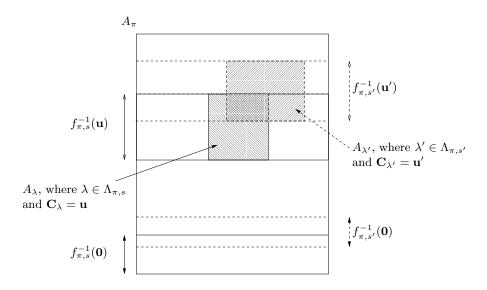


Figure 1: The decompositions of  $A_{\pi}$  for the strings s and s'.

(hence  $\mathbf{C}_{\lambda} = f_{\pi,s}(a)$ ), and let a send a single message labelled  $\lambda_{a,s}$ . Note that, if  $f_{\pi,s}(a) > \mathbf{0}$ , then  $\lambda_{a,s}$  exists and is unique by the construction of the sets  $A_{\lambda}$ . (ii) For every bit string t such that |t| = q, if t encodes an integer less than R (so that  $\mu = \mu_{\pi,t}$  is a non-invertible message-type and  $\mathbf{C}_{\mu} > \mathbf{0}$ ), let  $n_{a,t}$  be the unique natural number n such that  $g_{\pi,t}(a) = n\mathbf{C}_{\mu}$ , and let a send  $n_{a,t}$  distinct messages labelled  $\mu_{\pi,t}$ . Note that, if  $g_{\pi,t}(a) = \mathbf{0}$ , then  $n_{a,t} = 0$ ; on the other hand, if  $g_{\pi,t}(a) > \mathbf{0}$ , then  $n_{a,t}$  exists by the constraints (13) and Equation (23). The resulting data-structure is depicted in Fig. 2, where, for readability, we have replaced any bit-strings by the integers they conventionally denote. For all  $a \in A$  and all i ( $1 \le i \le m$ ), let  $C_{a,i}$  ( $1 \le i \le m$ ) be the number of messages sent by a having any label  $\nu$  for which  $f_i(x,y) \in \nu$ , and furthermore let  $\mathbf{C}_a$  be the vector  $(C_{a,1}, \ldots, C_{a,m})$ . By inspection of Fig. 2,

$$\mathbf{C}_{a} = \sum \{f_{\pi,s'}(a) : |s'| = p\} + \sum \{g_{\pi,t'}(a) : |t'| = q\},$$
 and so, by Equation (24), 
$$\mathbf{C}_{a} = \mathbf{C}.$$
 (29)

We now build  $\mathfrak A$  in four steps as follows.

**Step 1** (Fixing the 1-types): For all 1-types  $\pi$  and all  $a \in A_{\pi}$ , set  $\operatorname{tp}^{\mathfrak{A}}[a] = \pi$ . Since the  $A_{\pi}$  are pairwise disjoint, no clashes arise.

Step 2 (Fixing the invertible message-types): Let  $\lambda$  be any invertible message-type. By construction, exactly  $|A_{\lambda}| = x_{\lambda}$  elements of A send some message labelled with  $\lambda$ , and each of those elements sends exactly one such message. Hence, the number of messages labelled with  $\lambda$  (over all  $a \in A$ ) is  $x_{\lambda}$ ; likewise,

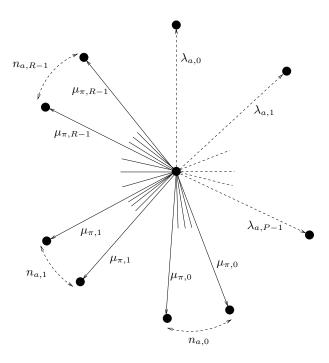


Figure 2: The messages sent by  $a \in A_{\pi}$ . For each j  $(0 \le j < P)$ , a may or may not send a message labelled  $\lambda_{a,j}$  (hence the dotted lines); if it does, then  $\lambda_{a,j} \in \Lambda_{\pi,j}$ . For each k  $(0 \le k < R)$ , a sends  $n_{a,k}$  messages labelled  $\mu_{\pi,k}$ ; but the numbers  $n_{a,k}$  can be zero.

the number of messages labelled with  $\lambda^{-1}$  is  $x_{\lambda^{-1}}$ . By the constraints (14), we may put the  $\lambda$ -labelled messages and the  $\lambda^{-1}$ -labelled messages in 1–1 correspondence. If  $a \in A$  sends a  $\lambda$ -labelled message, let  $b \in A$  send the corresponding  $\lambda^{-1}$ -labelled message, and set  $\operatorname{tp}^{\mathfrak{A}}[a,b] = \lambda$ . For this assignment to make sense, we need to check that a and b are distinct. But, by construction, we must have  $x_{\lambda} > 0$ , whence, by the constraints (15),  $\operatorname{tp}_{1}(\lambda) \neq \operatorname{tp}_{2}(\lambda)$ , so that  $A_{\operatorname{tp}_{1}(\lambda)}$  and  $A_{\operatorname{tp}_{2}(\lambda)}$  are disjoint sets containing a and b, respectively. Thus, the assignment  $\operatorname{tp}^{\mathfrak{A}}[a,b] = \lambda$  makes sense, and does not clash the with 1-type assignments in Step 1. We can think of the element b as 'receiving' the message sent by a (and vice versa). Moreover, by construction, for every 1-type  $\pi'$ , a sends at most one message labelled with an invertible message-type  $\lambda'$  such that  $\operatorname{tp}_{2}(\lambda') = \pi'$ . Therefore, there is no chance that these assignments clash with each other.

Step 3 (Fixing the non-invertible message-types): As a preliminary, for every 1-type  $\pi$ , we decompose  $A_{\pi}$  into three pairwise disjoint (possibly empty) sets  $A_{\pi,0}$ ,  $A_{\pi,1}$  and  $A_{\pi,2}$  satisfying the condition that, if  $|A_{\pi}| \geq 3mC$ , then  $|A_{\pi,j}| \geq mC$  for all j ( $0 \leq j \leq 2$ ). Now let  $\mu$  be any non-invertible message-type, let  $\pi = \text{tp}_1(\mu)$ , and let  $\rho = \text{tp}_2(\mu)$ . (Note that  $\pi$  and  $\rho$  may be identical.) Let t be the bit-string of length q such that  $\mu = \mu_{\pi,t}$ , and suppose some element a sends

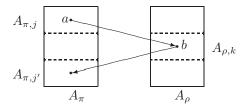


Figure 3: Fixing the non-invertible message-types.

 $n_{a,t} > 0$  messages labelled  $\mu$ . It follows that  $a \in A_{\pi}$ , and also that there is a vector  $\mathbf{u} > \mathbf{0}$  such that  $g_{\pi,t}(a) = \mathbf{u}$ , and hence such that  $g_{\pi,t}^{-1}(\mathbf{u})$  is non-empty. By Equation (23),  $z_{\pi,t,\mathbf{u}} > 0$ , whence, by the constraints (18),  $\sum \{y_{\rho,\epsilon,\mathbf{u}'} \mid \mathbf{u}' \leq$  $\mathbb{C}$ }  $\geq 3mC$ . But recall that, since  $\rho$  is a 1-type,  $|A_{\rho}| = \sum \{y_{\rho,\epsilon,\mathbf{u}'} \mid \mathbf{u}' \leq \mathbf{C}\}$ , so that each of the sets  $A_{\rho,0},\,A_{\rho,1}$  and  $A_{\rho,2}$  contains at least mC elements. Since  $a \in A_{\pi}$ , let  $j \ (0 \le j \le 2)$  be such that  $a \in A_{\pi,j}$ , let  $k = j + 1 \ (\text{mod } 3)$ , and select  $n_{a,t}$  elements b from  $A_{\rho,k}$  which have not yet been chosen to receive any other messages (invertible or non-invertible) sent by a. Since the total number of messages sent by a is certainly at most mC, we never run out of choices. For each of these elements b, set  $\operatorname{tp}^{\mathfrak{A}}[a,b] = \mu$ . Since  $\pi = \operatorname{tp}_1(\mu)$  and  $\rho = \operatorname{tp}_2(\mu)$ , these assignments cannot clash with those made in Step 1, and by construction, they cannot clash with assignments corresponding to other messages sent by a. We need only check that they cannot clash with assignments corresponding to messages sent by b. Specifically, we must ensure that, if  $tp^{\mathfrak{A}}[a,b] = \mu$  is assigned as just described, it is not possible for a to be chosen to receive a  $\mu'$ -labelled message sent by b, where  $\mu'$  is some non-invertible message-type. But any  $\mu'$ labelled message sent by  $b \in A_{\rho,k}$ , with  $\operatorname{tp}_2(\mu') = \pi$ , could only be sent to an element in  $A_{\pi,j'}$ , where  $j' = k+1 \pmod{3}$ ; and by assumption,  $A_{\pi,j}$  and  $A_{\pi,j'}$ are disjoint, (Fig. 3). Observe that this conclusion follows even if  $\pi = \rho$ .

**Step 4** (Fixing the remaining 2-types): Recall that a guard-atom is any atom p(x, y) or p(y, x), where p is a binary predicate. If  $\operatorname{tp}^{\mathfrak{A}}[a, b]$  has not been defined, set it to be the 2-type

$$\pi \cup \rho[y/x] \cup \{\neg \gamma \mid \gamma \text{ is a guard-atom not involving } \approx \},$$

where  $\pi = \operatorname{tp}^{\mathfrak{A}}[a]$ ,  $\rho = \operatorname{tp}^{\mathfrak{A}}[b]$ , and  $\rho[y/x]$  is the result of replacing x by y in  $\rho$ . Note that neither this 2-type nor its inverse is a message-type. Note also that, since a and b certainly send some messages, the constraints (16) and (17) ensure that both  $\alpha \wedge \bigwedge \pi$  and  $\alpha \wedge \bigwedge \rho$  are satisfiable.

This completes the definition of  $\mathfrak{A}$ ; it remains to show that  $\mathfrak{A} \models \phi$ . Referring to (1), we consider first the conjuncts:

$$\forall x \alpha \wedge \bigwedge_{1 \leq h \leq l} \forall x \forall y (e_h(x, y) \to (\beta_h \vee x \approx y)).$$

We see from the constraints (16) and (17) that no 2-type assignment in Steps 2

and 3 violates these conjuncts. And it is obvious that no assignment in Step 4 does so. (This is where we use the guardedness of  $\phi$ , of course.) Finally, we consider the conjuncts

$$\bigwedge_{1 \le i \le m} \forall x \exists_{=C_i} y (f_i(x, y) \land x \not\approx y).$$

To see that these conjuncts are all satisfied, it suffices to note Equation (29) and the fact that none of the 2-types assigned in Step 4 is a message-type. We remark in passing that  $\mathfrak A$  is chromatic.

The constraints  $\mathcal E$  all have the forms

$$\begin{array}{rcl}
x_1 + \dots + x_n & = & x \\
x_1 + \dots + x_n & \geq & 1 \\
x & = & 0 \\
x > 0 \Rightarrow x_1 + \dots + x_n & \geq & D,
\end{array} \tag{30}$$

where  $n > 0, x, x_1, \ldots, x_n$  are variables, and D is a positive integer. We measure the size  $\|\mathcal{E}\|$  of  $\mathcal{E}$  in the usual way, with binary encoding of the constants D.

The following lemma essentially repeats Lutz, Sattler and Tendera [9], Proposition 11. (Those authors in turn credit Calvanese [3].) We repeat the proof for convenience.

**Lemma 8.** Let  $\phi$  and  $\mathcal{E}$  be as above. An algorithm exists to determine, in time bounded by an exponential function of  $\|\phi\|$ , whether  $\mathcal{E}$  has a solution over  $\mathbb{N}$ .

*Proof.* Suppose  $\mathcal{E}$  has a solution  $\theta: V \to \mathbb{N}$ . We define the integer programming problem  $\mathcal{E}_{\theta}$  by replacing every constraint in  $\mathcal{E}$  having the form  $x > 0 \Rightarrow x_1 + \cdots + x_n \geq D$  with one of two corresponding inequalities as follows:

$$\begin{cases} x = 0 & \text{if } \theta(x) = 0 \\ x_1 + \dots + x_n \ge D & \text{otherwise.} \end{cases}$$

It is easy to check that  $\theta$  is a solution of  $\mathcal{E}_{\theta}$ , and, moreover, that every solution of  $\mathcal{E}_{\theta}$  is a solution of  $\mathcal{E}$ . Since  $\mathcal{E}_{\theta}$  is an integer programming problem and has a solution over  $\mathbb{N}$ , by a well-known theorem (Papadimitriou [10]), it has a solution  $\theta'$  over  $\mathbb{N}$  in which every value is bounded by a (positive) integer H, where H can be computed (as a binary string) in time bounded by a polynomial function of  $\|\mathcal{E}_{\theta}\|$  and hence in time bounded by an exponential function of  $\|\phi\|$ . (Of course, the *integer* H is bounded only by an *doubly* exponential function of  $\|\phi\|$ .) Moreover,  $\theta'$  must also be a solution of  $\mathcal{E}$ . Therefore  $\mathcal{E}$  too has a solution over  $\mathbb{N}$  if and only if it has a solution over  $\mathbb{N}$  in which every value is bounded by H.

Now consider the integer programming problem  $\mathcal{E}_H$  defined by replacing every constraint of the form  $x > 0 \Rightarrow x_1 + \cdots + x_n \geq D$  in  $\mathcal{E}$  by the corresponding inequalities

$$Hy \geq x$$

$$x_1 + \dots + x_n \geq Dy,$$

where y is a new variable. Every solution of  $\mathcal{E}_H$  over  $\mathbb{N}$  is a solution of  $\mathcal{E}$ . Moreover, suppose  $\theta'$  is any solution of  $\mathcal{E}$  over  $\mathbb{N}$  in which all values are bounded by H; and let y be one of the new variables of  $\mathcal{E}_H$ , introduced to eliminate the constraint  $x > 0 \Rightarrow x_1 + \cdots + x_n \geq D$ . Let us extend  $\theta'$  to give a value to y as follows:

$$\theta'(y) = \begin{cases} 0 & \text{if } \theta'(x) = 0\\ 1 & \text{otherwise.} \end{cases}$$

It is routine to check that extending  $\theta'$  in this way for all the new variables y in  $\mathcal{E}_H$  yields a solution of  $\mathcal{E}_H$ . Hence  $\mathcal{E}$  can be transformed, in time bounded by an exponential function of  $\|\phi\|$ , into the equisatisfiable (over  $\mathbb{N}$ ) constraint set  $\mathcal{E}_H$ , in which all constraints are of the forms

$$\begin{array}{rcl} x_1 + \dots + x_n & = & x & x & = & 0 \\ x_1 + \dots + x_n & \geq & 1 & Dx_1 & \geq & x_2 \\ x_1 + \dots + x_n & \geq & Dx, & \end{array}$$

where, again, the D are positive integers. It is obvious that, if  $\mathcal{E}_H$  has a solution over the non-negative rationals, then it has a solution over  $\mathbb{N}$  as well. (Simply multiply by the product of all the denominators.) Hence, we can equivalently regard  $\mathcal{E}_H$  as a linear programming problem. But linear programming is in PTIME, by Khachiyan's theorem [7].

**Theorem 1.** The finite satisfiability problem for  $\mathcal{GC}^2$  is in EXPTIME.

*Proof.* Lemmas 2, 5, 7 and 8.  $\Box$ 

## 5 The Satisfiability Problem

The above technique also provides a simple proof of a result derived in Kazakov [6], namely, that the satisfiability problem for  $\mathcal{GC}^2$  is in EXPTIME.

Notation 2. Let  $\mathbb{N}^*$  denote the set  $\mathbb{N} \cup \{\aleph_0\}$ . We extend the ordering > and the arithmetic operations + and  $\cdot$  from  $\mathbb{N}$  to  $\mathbb{N}^*$  in the obvious way. Specifically, we define  $\aleph_0 > n$  for all  $n \in \mathbb{N}$ ; we define  $\aleph_0 + \aleph_0 = \aleph_0 \cdot \aleph_0 = \aleph_0$  and  $0 \cdot \aleph_0 = \aleph_0 \cdot 0 = 0$ ; we define  $n + \aleph_0 = \aleph_0 + n = \aleph_0$  for all  $n \in \mathbb{N}$ ; and we define  $n \cdot \aleph_0 = \aleph_0 \cdot n = \aleph_0$  for all  $n \in \mathbb{N}$  such that n > 0. Under this extension, > remains a total order, and +,  $\cdot$  remain associative and commutative.

Consider again the constraints  $\mathcal{E}$  given in (4)–(18), but now with the variables ranging over the whole of  $\mathbb{N}^*$ . Using the arithmetic in Notation 2, the reasoning of Lemmas 5–7 works unproblematically even when countably infinite sets are allowed. Thus, we have:

**Lemma 9.** Let  $\phi$  and  $\mathcal{E}$  be as above. Then  $\phi$  is satisfiable if and only if  $\mathcal{E}$  has a solution over  $\mathbb{N}^*$ .

*Proof.* If  $\phi$  is satisfiable, then it has a model which is finite or countably infinite. Now proceed as for Lemma 5. For the converse, proceed as for Lemma 7.

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**Lemma 10.** The set of constraints  $\mathcal{E}$  has a solution over  $\mathbb{N}^*$  if and only if it has a solution over  $\{0,\aleph_0\}$ .

*Proof.* Suppose  $\mathcal{E}$  has a solution  $\theta: V \to \mathbb{N}^*$ . By considering the forms in  $\mathcal{E}$ , we see that  $\theta': V \to \{0, \aleph_0\}$  defined by  $\theta'(v) = \aleph_0 \theta'(v)$  is also a solution. The other direction is trivial.

Since the domain  $\{0,\aleph_0\}$  has only 2-elements, variables interpreted over it are essentially Boolean. If  $x \in V$ , let us write X for the corresponding statement x = 0, so that the constraints  $\mathcal{E}$  are viewed as formulas of propositional logic. For example, a constraint of the form

$$x_1 + \dots + x_n = x$$

becomes the set of Boolean formulas

$$\{X_1 \wedge \cdots \wedge X_n \rightarrow X\} \cup \{X \rightarrow X_i \mid 1 \le i \le n\};$$

a constraint of the form

$$x_1 + \dots + x_n \ge 1$$

becomes the Boolean formula

$$X_1 \wedge \cdots \wedge X_n \to \bot;$$

and a constraint of the form

$$x > 0 \implies x_1 + \dots + x_n \ge D$$

becomes the Boolean formula

$$X_1 \wedge \cdots \wedge X_n \to X$$
.

A quick check reveals that all of the resulting formulas are Horn-clauses. This immediately yields:

**Theorem 2 (Kazakov).** The satisfiability problem for  $\mathcal{GC}^2$  is in EXPTIME.

The proof in Kazakov [6] proceeds by showing that satisfiability in  $\mathcal{GC}^2$  can be reduced in polynomial time to satisfiability in the 3-variable guarded fragment; Theorem 2 then follows by the complexity bound for the latter established by Grädel [4]. The approach taken here is thus somewhat more direct. Moreover, Kazakov's reduction is not conservative, and, as mentioned, yields no complexity bound for the corresponding finite satisfiability problem.

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