A General Approach to Aggregation Problems

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1 Introduction

There is a new field emerging around issues concerning the aggregation of a collection of individual "judgments" of a group of agents. An individual "judgment" is represented by a set of sentences in some logical language. One looks for a procedure that has as its output a "social" judgment. A key result in this area is List and Pettit's impossibility result [13]. By generalizing the well-known doctrinal paradox [12], List and Pettit were able to show an "Arrow"-style [1] impossibility theorem: for judgment sets that are subsets of a sufficiently rich collection of sentences there is no "well-behaved" aggregation procedure. There have been a number of refinements and generalizations of this elegant result [3, 7, 8, 16, 17, 18].

As our starting point we take work of Dietrich and List, which builds on List and Pettit's result and clarifies the connection between judgment aggregation impossibility results and Arrow's famous impossibility result. In the setting studied by these authors, an **agenda** is a collection of sentences in some logical language.[1] A judgment set is a subset of the agenda. The impossibility results arise out of assumptions made about 1. the agenda, 2. the possible judgment sets and 3. the aggregation functions. Formal details can be found in [4].

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[1]Whether the underlying logic is propositional, first-order, modal, etc. is not important for this discussion. See [3] for a discussion.

Virtually all proofs of impossibility results follow a similar line of reasoning. The main idea is to show that assumptions about the aggregation procedure and the structure of the agenda force the set of so-called "winning coalitions" to have particular algebraic properties. Intuitively, a *winning coalition* is a set of agents that can *force* the aggregation procedure to select a certain proposition. More formally, let N be the set of agents. A set $M \subseteq N$ is said to be a winning coalition if and only if for all propositions φ from the agenda, and in any situation, the aggregation procedure selects φ whenever exactly the agents in M select φ.

Given an aggregation procedure F, let Ω_F be the set of winning coalitions associated with F. The proof of the main impossibility result in [4] amounts to showing that Ω_F is an *ultrafilter*, that is, a collection closed under intersections and supersets and that satisfies for all $M \subseteq N$, either $M \in \Omega_F$ or $N - M \in \Omega_F$. Any ultrafilter over a finite set must contain a singleton. Under the assumption that the set of agents is finite, the impossibility result immediately follows, i.e., the set of winning coalitions contains a singleton. Thus there is a formal connection between properties of the agenda and aggregation procedure and properties of the set of winning coalitions.

We have two goals in this paper. Our principal goal is to introduce a general framework to systematically investigate the connection between properties of the agenda and aggregation procedure one the one hand, and properties of the winning coalitions on the other. Our framework is abstract and algebraic in nature; and the heart of our paper is formed by three sections in which we investigate how this perspective relates to more familiar concepts in the literature. The next section introduces our basic framework. In section 3, we show how the algebraic structures that we consider arise from more classical perspectives, i.e., preference or judgement aggregation. In section 4, we discuss how some of the traditional axioms of social choice theory can be generalized in our setting.

Our second goal is to prove an Arrow-style impossibility result in our general setting. Section 5 contains such a result. We conclude in Section 6.

2 Our Setting

Algebraic Preliminaries. To introduce our general framework we will need some formal machinery. Much of this terminology is well known and the reader is referred to [2] for an extensive discussion. An **partially ordered set** ("poset") is a pair $(Z, \leq)$ where $\leq$ is a reflexive, transitive and anti-symmetric relation on Z. We write $z < z'$ if $z \leq z'$ and $z \neq z'$. With a slight abuse of notation we will use Z to denote the ordered set $(Z, \leq)$. Given an element $z \in Z$, we write $\downarrow z := \{y \in Z \mid y \leq z\}$.

For $S \subseteq Z$, we write $\bigvee S$ for the least upper bound of S, if it exists; $\bigvee S$ is called the **join** of S. Similarly, we write $\bigwedge S$ for the greatest lower bound of S, which is called the **meet**. If every pair x, y of elements of Z has a join and a meet, then we call Z a **lattice**. The **join** of x and y is denoted $x \vee y$ and the **meet** is denoted $x \wedge y$.

A lattice is said to be **topped** if there is some element, denoted $\mathbf{1}$, such that for all $z \in Z$, $z \leq \mathbf{1}$; equivalently $\mathbf{1} = \bigvee Z$. A lattice has a **zero** if there is an element $\mathbf{0}$ such that for all $z \in Z$, $\mathbf{0} \leq z$, that is, $\mathbf{0} = \bigwedge Z$. A topped lattice with a zero is called **bounded**.

We write $x \prec y$ if $x < y$ and $x \leq z < y$ implies $x = z$. An **atom** of a lattice Z with a zero element $\mathbf{0}$ is any element $a \in Z$ such that $\mathbf{0} \prec a$. Let $\mathcal{A}(Z)$ denote the set of atoms of Z. Given $z \in Z$, let $\mathcal{A}(z)$ denote the set of atoms below z, i.e., $\mathcal{A}(z) := \{a \in \mathcal{A}(Z) \mid a \leq z\}$. Elements $m \neq \mathbf{1}$ such that if $m \prec z$ then $z = \mathbf{1}$ are called **co-atoms**. Let $\mathcal{M}(Z)$ denote the set of all co-atoms in Z. Finally, given $z \in Z$, a **complement** of z is an element $y \in Z$ such that $z \vee y = \mathbf{1}$ and $z \wedge y = \mathbf{0}$. A number of lattices with special properties will be relevant for this paper:

- A lattice is **atomic** if there is some $a \in \mathcal{A}(z)$ for every $z \neq \mathbf{0}$.
- If every element of Z can be written as the join of its atoms, that is $z = \bigvee \mathcal{A}(z)$, then Z is called **atomistic**.[2]
- If for every $z \neq \mathbf{1}$ there exists $m \in \mathcal{M}(Z)$ such that $z \leq m$, then Z is called **co-atomic**.
- A lattice is **complete** if *every* set $S \subseteq Z$ has a join, i.e., $\bigvee S$ exists.
- A lattice is **compact** if for every non-empty set $S \subseteq Z$, if $\bigvee S = z$ and $y \leq z$, then $y \leq \bigvee T$ for some *finite* $T \subseteq S$.[3]
- A **Boolean algebra**, or **Boolean lattice**, is a distributive lattice[4] in which every element has a (necessarily unique[5]) complement.
- We say a lattice is **dichotomic** if it is (i). atomistic (and hence atomic) and (ii). co-atomic and (iii). has the following property: For all $a \in$

[2]Of course, every atomistic lattice is atomic. But the converse is not true — consider the set of natural numbers with its natural ordering, $(\mathbf{N}, \leq)$.

[3]This means in particular that z itself equals $\bigvee T$ for a finite subset $T \subseteq S$.

[4]A lattice is distributive if for all $x, y, z \in Z$, $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$.

[5]In a distributive lattice, if an element has a complement then it is unique.

$\mathcal{A}(Z)$, there is $a^c \in Z$ such that for every $m \in \mathcal{M}(Z)$, $a \vee m = \mathbf{1}$ iff $a^c \leq m$.

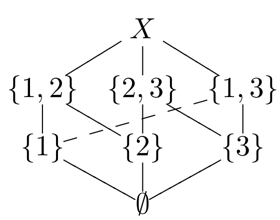
As will become apparent below, the join operator plays a special role in our framework. A crucial notion that appears throughout the text is *semi-order embedabbility*, which makes precise when a lattice Y is "contained in" another lattice Z in a join-preserving way.

Definition 1 ***"semi order-embeddable":*** *Let Y and Z be lattices. Y is semi-order embeddable in Z if there exists a map f such that:*
(a) $f(\mathbf{1}_y) = f(\mathbf{1}_z)$; (b) $f(y_1) \vee f(y_2) = f(y_1 \vee y_2)$. (where $y_1, y_2 \in Y$ are arbitrary elements of Y, and $\mathbf{1}_y = \bigvee Y$ and $\mathbf{1}_z = \bigvee Z$).

Semi order-embeddability is a weaker variant of a more often considered notion of **order-embeddability**. A map f is an order-embedding if it preserves meets and zero in addition to joins and $\mathbf{1}$.

To illustrate the above concepts, consider the following simple example of a lattice.

The powerset of 3 lattice. Suppose that X is a set with three elements, i.e., $X = \{1, 2, 3\}$. The poset $(\wp(X), \subseteq)$ forms a lattice which can be pictured as follows:



We will denote this lattice by $\mathbf{3}$. The join and meet operations are given by $\cup$ and $\cap$ respectively. The lattice is topped by X and the zero is $\emptyset$. In fact, the lattice is a boolean algebra as it is distributive and each element has a complement. For example, the complement of $\{x_1\}$ is $\{x_2, x_3\}$.

The Framework. Let N be a non-empty set of agents. For much of what we say in this text it will not matter whether or not N is finite. However, as usual, we assume there are at least two agents (i.e., $|N| \geq 2$). The key idea is that agents are assumed to select elements of some topped lattice Z. Intuitively, the elements of Z represent judgment sets and if $z' \leq z$

then acceptance of z implies acceptance of z'. A **profile** is any function[6] $\pi : N \to Z$. If $z \in Z$, we write $\pi[A] = z$ if all agents in the set A choose z (i.e., for all $i \in A$, $\pi(i) = z$). If $z \leq \pi(i)$ then we say that agent i **accepts** z. Hence we assume that if $\pi(i) = z$ then agent i accepts all z' such that $z' \leq z$, i.e., i accepts each element in the set $\downarrow\pi(i)$.

Let $Z^N := \{\pi \mid \pi : N \to Z\}$ be the set of all profiles. An **aggregation function** is a map $F : Z^N \to Z$. Given a profile π, $F(\pi)$ is the *socially* accepted element of Z. Note that, in general, F may be a partial function. We write $dom(F)$ for the domain of F.

The next section contains an extended discussion of how to interpret this framework. For now, we give the basic intuitions. The idea is that the elements of the lattice are the possible judgment sets. The ordering can be interpreted as follows: if $z' \leq z$ then z' contains *less information*[7] than z. It is now easy to understand why the crucial operation is *combining*—joining—two or more elements of Z; after all, we are looking for ways to combine the judgment sets of individual agents. A special role is played by $\mathbf{1}$, which intuitively is the set of all propositions (*the* inconsistent set). If $z \vee z' \neq \mathbf{1}$, then z and z' are consistent with each other. This role of $\mathbf{1}$ motivates the following assumption (both on profiles and aggregation functions) that is made throughout the paper.

Consistency: For all $\pi \in Z^N$ and all $i \in N$, $\pi(i) < \mathbf{1}$ and $F(\pi) < \mathbf{1}$. That is, all agents and society are consistent.

Note that this property can be viewed as two properties on F. The first is a *range restriction* ($\forall \pi \in Z^N,\ F(\pi) < \mathbf{1}$). The second is a *domain restriction* on F. That is, we assume that the domain of F is restricted to consistent profiles. Let Π be the set of all consistent profiles, i.e., $\Pi = \{\pi \in Z^N \mid \forall i \in N,\ \pi(i) < \mathbf{1}\}$. A standard assumption in the literature is that with respect to Π, F is a total function.

Definition 2 ***"Universal Domain":*** *An aggregation function* $F : Z^N \to Z$ *satisfies* **universal domain** *(with respect to consistent profiles) if for all* $\pi \in \Pi$*,* $F(\pi)$ *exists, i.e.,* $\Pi \subseteq dom(F)$*.*

[6] When N is finite, a profile is often represented as a element of $Z^n = Z \times Z \times \cdots Z$ (n-fold product) where n is the size of N. We have chosen to use the function notation since, unless explicitly stated, we do not assume that N is finite.

[7] This information-theoretic interpretation of a lattice is very important in theoretical computer science. In this setting, elements of the lattice represent (finite) information about an algorithm. See [2] Chapters 8 and 9 for details.

When the function F satisfies universal domain as defined above, we write $F : \Pi \to Z$.

Decisive sets. Fix a profile $\pi \in \Pi$. Each element $z \in Z$ partitions N into three sets. Informally, these are the sets that any social aggregation function can "take into account" when making the social choice. The first set is the set of agents that accepts z, formally, let $\llbracket z \rrbracket_\pi = \{i \in N \mid z \leq \pi(i)\}$ be this set. Second, the set of agents that would like to "block" z from being socially accepted, because their judgment is incompatible with it. This is the set $\llbracket \not{z} \rrbracket_\pi = \{i \in N \mid z \vee \pi(i) = \mathbf{1}\}$. Finally, there is the set of agents that do not have an opinion about z, i.e., the set $N - (\llbracket z \rrbracket_\pi \cup \llbracket \not{z} \rrbracket_\pi)$.

A key notion for our paper is a **decisive** subset of N.

Definition 3 ***"Decisive subset":*** *Let F be an aggregation function. Suppose $z \in Z$ and let $M \subseteq N$. M is* ***decisive for*** *F* ***with respect to*** *z iff for all $\pi \in \Pi$ the following holds. Whenever $z \leq \pi(i)$ for all $i \in M$ and $z \vee \pi(j) = \mathbf{1}$ for all $j \in N - M$, then $z \leq F(\pi)$. A set $M \subseteq N$ is* ***decisive*** *if for all $z \in Z$, M is decisive for F with respect to z.*

Our discussion above suggests two (dual) notions connected with the definition of decisiveness:

1. A set M forces F to accept z if for all π if $\llbracket z \rrbracket_\pi = M$ then $z \leq F(\pi)$.

2. A set M blocks F from accepting z if for all π if $\llbracket \not{z} \rrbracket_\pi = M$, then $z \not\leq F(\pi)$.

Of course, forcing F to accept z is dual to blocking F from accepting z. The notion of decisiveness we use is weaker—it states that in order for M to force F to accept z, everyone in M must accept z *and* everyone outside of M must block z. That is, in the face of direct opposition, the group M still manages to force F to accept z. Formally, using the above notation, according to the above definition M is decisive for F with respect to z if for all π if $\llbracket z \rrbracket_\pi = M$ and $\llbracket \not{z} \rrbracket_\pi = N - M$ then $z \leq F(\pi)$.

We may slightly weaken the antecedent by saying that M is *weakly decisive* for F with respect to z, if for all π if $\llbracket z \rrbracket_\pi = M$ and $\llbracket \not{z} \rrbracket_\pi \cap N - M \neq \emptyset$ then $z \leq F(\pi)$. All proofs will go through with this weaker notion of decisiveness.

3 Examples

In this section we indicate how the more familiar judgment and preference aggregation settings fit into our algebraic framework. We begin with a general (and well-known) fact. Let W be a non-empty set and $\wp(W)$ the powerset of W. A **closure operator** on W is a function $C : \wp(W) \to \wp(W)$ satisfying the following three conditions:

1. For all $X \subseteq W$, $X \subseteq C(X)$
2. For all $X, Y \subseteq W$, if $X \subseteq Y$ then $C(X) \subseteq C(Y)$
3. For all $X \subseteq W$, $C(C(X)) \subseteq C(X)$

A set $X \subseteq W$ is said to be **closed** if $C(X) = X$. Let $W_C = \{X \mid X \text{ is closed}\}$. We will make use of the following well-known fact (see [2, Proposition 7.2] for details).

Fact 1 *For any set $W \neq \emptyset$ and closure operator $C : \wp(W) \to \wp(W)$, W_C is a complete lattice with*

$$\bigwedge_{i \in I} C(X_i) = \bigcap_{i \in I} X_i$$

and

$$\bigvee_{i \in I} C(X_i) = C(\bigcup_{i \in I} X_i),$$

for any index set I.

We now illustrate how each component of the judgement aggregation framework is represented in our framework.

The Alternatives and Judgement Sets. The basic premise of the judgement aggregation setting is that a group of agents is making collective judgments about *interconnected propositions*. Typically, it is assumed that the propositions are expressions in some formal language and the "interconnection" is derived from a *consequence relation* (cf. for example, the setting in [4]). More formally, let $\mathcal{L}$ be a formal language with a negation symbol ($\neg$). For simplicity, we might work with the language of propositional calculus although this is not crucial (cf. [3]). Abstractly, a **consequence relation** is any relation $\vdash \subseteq \wp(\mathcal{L}) \times \mathcal{L}$ satisfy the following properties:[8]

1. $\{p\} \vdash p$
2. Suppose $A \subseteq B$. Then if $A \vdash p$ then $B \vdash p$.

[8]In fact, the second property follows from the other two. However, we include it to highlight that we are interested in monotonic consequence relations. Consult [11] for references and an overview of this algebraic approach to logic.

3. Suppose $A \vdash p$ and for all $q \in A$, $B \vdash q$. Then $B \vdash p$

Dietrich [3] provides an extensive analysis of impossibility results in this setting.[9] We first need some more terminology:

- A set $X \subseteq \mathcal{L}$ is **$\vdash$-inconsistent** (or more simply **inconsistent** if no confusion about the consequence relation will arise) if there is some $p \in \mathcal{L}$ such that $X \vdash p$ and $X \vdash \neg p$.
- We say $X \subseteq \mathcal{L}$ is **consistent**[10] if X is not inconsistent.
- A set X is **deductively closed** provided X contains all of its consequences, i.e., $X = \{p \mid X \vdash p\}$.
- Finally, a set X is **complete** if for every pair $p, \neg p \in \mathcal{L}$, either $p \in X$ or $\neg p \in X$.

A **judgement set** is any set $X \subseteq \mathcal{L}$ where $p \in X$ is intended to mean "the agent (or group) accepts p." Typically it is assumed that judgement sets are consistent, complete and deductively closed (cf. [13]), but weaker assumptions have also been discussed (cf. [8, 6, 5]).

What is important for this paper is that every $\vdash$ (satisfying the above three properties) defines a closure operator $C_\vdash : \wp(\mathcal{L}) \to \wp(\mathcal{L})$ as follows: for $X \subseteq \mathcal{L}$, $C_\vdash(X) = \{p \mid X \vdash p\}$. Conversely, every closure operator C can be used to define a consequence relations $\vdash_C$ as follows, for $X \subseteq \mathcal{L}$ and $p \in \mathcal{L}$, $X \vdash_C p$ iff $p \in C(X)$. Thus, given a logical language and a closure operator satisfying properties 1–3, using Fact 1, we can construct a complete lattice $L^{dc}_\vdash$ whose elements are the deductively closed subsets of $\mathcal{L}$. Note that the top of this lattice will be *the*[11] inconsistent set (i.e., the set of all propositions). Notice also that the **co-atoms** of this lattice are the maximally consistent subsets (i.e., complete and consistent subsets). Thus, in our setting, fixing the lattice $L^{dc}_\vdash$ amounts to a *rationality assumption* that the agents only choose deductively closed and consistent judgement

[9] Typically, the judgement aggregation problem is studied in the context of propositional logic. Dietrich [3] provides a general and unified framework to study judgement aggregation problems with a variety of underlying logics. The goal of his paper is to highlight the exact properties of the underlying consequence relation that is used to obtain various impossibility results. In particular, 5 properties of a consequence relation (called L1-L5) are highlighted. Each of his conditions L1-L5 is used in this paper.

[10] Dietrich [3]considers a weaker notion of consistency: a set X is **weakly consistent** if it contains at most one member of each pair $p, \neg p \in \mathcal{L}$. In our setting, these are equivalent.

[11] Assuming $\vdash$ satisfies 1–3, the deductive closure of any inconsistent set will be the set of all formulas.

sets. A stronger rationality assumption is that agents choose *complete* and consistent judgement sets. In settings, where judgement sets are assumed to be complete as well, the agents' choices in the lattice must be restricted to the co-atoms.

Weaker rationality assumptions can also been discussed. In particular, it is not hard to see that we can work with judgement sets satisfying *only* consistency (cf. [5] for a recent impossibility result where judgments sets are only assumed to be consistent). However, this lattice arises as the lattice of closed sets of a different closure operator.

Definition 4 *Let $\mathcal{L}$ be a formal language and $\vdash$ a consequence relation for $\mathcal{L}$. Define $C'_{\vdash} : \wp(\mathcal{L}) \to \wp(\mathcal{L})$ as follows: for $X \subseteq \mathcal{L}$,*

$$C'_{\vdash}(X) = \begin{cases} \mathcal{L} & \textit{if } X \textit{ is } \vdash\textit{-inconsistent} \\ X & \textit{otherwise (i.e., } X \textit{ is } \vdash\textit{-consistent)} \end{cases}$$

We first note that $C'_{\vdash}$ is in fact a closure operator.

Fact 2 *Suppose $\mathcal{L}$ is a formal language and $\vdash$ satisfies properties 1 and 2 above. Then $C'_{\vdash}$ is a closure operator.*

Proof Suppose $X, Y \subseteq \mathcal{L}$. Since $C'(X)$ is either $\mathcal{L}$ or X, trivially $X \subseteq C'(X)$. Suppose $X \subseteq Y$. Either X is $\vdash$-consistent or X is $\vdash$-inconsistent. If X is $\vdash$-inconsistent and $\vdash$ satisfies property 2. above, then Y is also $\vdash$-inconsistent. Hence $C'(X) = \mathcal{L} = C'(Y)$. If X is $\vdash$-consistent, then $C'(X) = X$. Therefore, $X = C'(X) \subseteq C'(Y)$, as $C'(Y)$ is either Y or $\mathcal{L}$ and it is assumed that $X \subseteq Y$. To see $C'C'(X) \subseteq C'(X)$, first note X is either consistent or inconsistent. If X is $\vdash$-inconsistent, then $C'(X) = \mathcal{L}$ and since C' is monotonic $C'(\mathcal{L}) = \mathcal{L}$. Thus $C'(C'(X)) = \mathcal{L} = C'(X)$. Suppose that $C'(X)$ is $\vdash$-consistent. Then, by definition $C'(X) = X$, so $X = C'(X) = C'(C'(X))$. ■

Again we use Fact 1 to construct a lattice $L^c_{\vdash}$ of *consistent* judgement sets, where the top element is the inconsistent set of all formulas and the co-atoms are again the maximally consistent subsets.

An alternative algebraic approach followed by Gärdenfors [8] is to assume that the set of *alternatives* are elements of a boolean algebra. Recall that a boolean algebra is a distribute lattice in which every element has a (unique) complement. Note that elements of Gärdenfors' algebra are intended to represent possible alternatives whereas elements of our lattices are intended to represent *sets* of alternatives. In any boolean algebra $\mathcal{B}$, a consequence relation can be defined as follows: given two elements x, y of the boolean algebra, we say y is a **consequence** of x provided $(-x \vee y) = \mathbf{1}$, where $-x$ is the complement of x (alternatively, if x is less than y in the order). This can be lifted to sets by saying that an element y of $\mathcal{B}$ is a consequence of a set X of elements of $\mathcal{B}$ if[12] $(-\bigvee X \vee y) = \mathbf{1}$. Given a consequence relation, defining the lattices as described above is an easy exercise.

To summarize, given a formal language $\mathcal{L}$ we can construct the powerset lattice $L_{\subseteq} = (\wp(\mathcal{L}), \subseteq)$. This lattice makes no rationality assumptions and does not represent any interconnections between the propositions (elements of $\mathcal{L}$). Fixing a consequence relation $\vdash \subseteq \wp(\mathcal{L}) \times \mathcal{L}$ highlights various sublattices of $L_{\subseteq}$. Precisely which sublattices are of interest depends on the rationality assumptions (i.e., consistency, deductive closure, completeness, etc.). Notice that each of the above lattices ($L_{\subseteq}, L^{dc}_{\vdash}$, and $L^{c}_{\vdash}$) satisfy additional properties. For example, all lattices are atomistic (and hence atomic), co-atomic, and complemented, to name a few. Our main goal in this paper is to investigate how these lattice-theoretic assumptions are used when proving Arrow-style impossibility results.

The Agenda. An **agenda** is a set of propositions under consideration, or "on the table". Formally, it is any subset A of the logical language $\mathcal{L}$ (or sub-boolean algebra in Gärdenfors' setting). Once the agenda is fixed, it is assumed that the agents' judgement sets are subsets of the agenda and the above notions of completeness, deductive closure and consistency are relativized to the agenda.[13]

In our setting, fixing an agenda means that we restrict attention to a *sublattice* of the "full lattice" built from the language $\mathcal{L}$ and consequence relations $\vdash$ as described above (i.e., either $L^{c}_{\vdash}$ or $L^{dc}_{\vdash}$ depending on rationality assumptions). Of course, not *every* sublattice will lead to an impossibility result. Consider, for example, the agenda $\mathsf{A} = \{p, q, p \wedge q\}$ and suppose that an agent chooses the set $X = \{p, p \wedge q\}$. Is this agent inconsistent? Of course, the answer depends on how to interpret what it means that "$q \notin X$". If this is taken to mean that the agent accepts $\neg q$, then the agent is inconsistent.[14] One can also argue that, since the agenda does not contain $\neg q$, the agent was not given the ability to express that fact that he is inconsistent. For reasons

[12] We need to assume $\mathcal{B}$ is complete in case X is infinite.

[13] For example, given an agenda A, a judgement set $X \subseteq \mathsf{A}$ is complete with respect to A if for each $p, \neg p \in \mathsf{A}$ either $p \in X$ or $\neg p \in X$.

[14] In the AI and non-monotonic logic literature, this is called the *closed world assumption*.

such as these, it is often assumed that the agenda is *negation closed*.[15] In our setting, this amounts to the following condition on the agenda: every element has a (not necessarily unique) complement (cf. Section 4.2 for more details).

An agenda A in the usual sense is a set of propositions (typically not the full language $\mathcal{L}$). Given any sublattice L of $L_{\subseteq}$, we write $L(\mathsf{A})$ for the sublattice of L where each element is a subset of A. For example, $L^c_{\vdash}(\mathsf{A})$ is the lattice of all $\vdash$-consistent subsets of A.

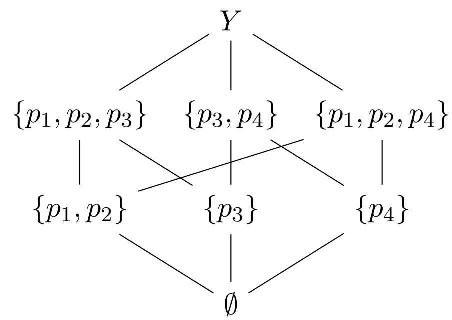
In [13] it was noted that an impossibility result requires certain "richness" conditions on the agenda (in addition to the agenda be negation closed).[16] This has lead to research on *characterization theorems* that identify properties of agendas that *correspond* to impossibility results. It is beyond the scope of this paper to survey all of the properties that have been proposed. We will focus on just one property from [4].

Definition 5 ***"Minimal Connectedness":*** *An agenda* A *is* ***minimally connected*** *if there is an set* $Y \subseteq \mathsf{A}$ *with* $|Y| \geq 3$ *such that (i)* Y *is a* ***minimally inconsistent set*** *(that is,* Y *is inconsistent, but every subset of* Y *is consistent) and (ii) there is an even subset* $Z \subseteq Y$ *such that* $Y - Z \cup \{\neg p \mid p \in Z\}$ *is consistent.*

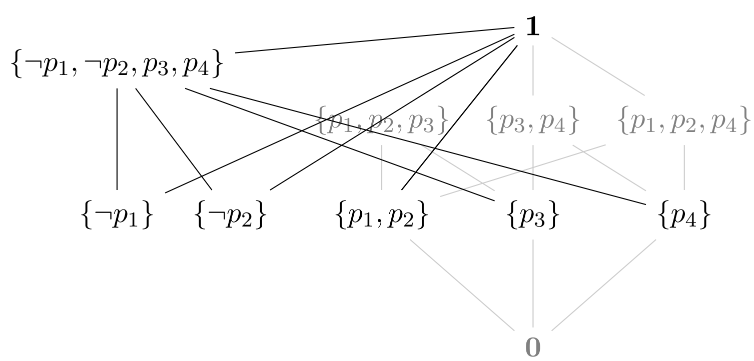
Suppose that the agenda satisfies minimal connectedness and consider $L_{\vdash}(\mathsf{A})$ (the exact rationality assumptions are not important). It is not hard to see that part (i) in the above definition implies that the powerset of 3 lattice is order-embeddable in $L_{\vdash}(\mathsf{A})$. For example, suppose that the set Y has 4 elements (i.e, $Y = \{p_1, p_2, p_3, p_4\}$. Then part (i) says that Y is minimally inconsistent, i.e, every subset of Y is consistent, but Y is inconsistent. Focusing on the elements $\{p_1, p_2\}$, $\{p_3\}$ and $\{p_4\}$ we find the following sublattice in $L_{\vdash}(\mathsf{A})$:

[15] That is, if p is in the agenda then $\sim p$ is in the agenda, where $\sim p$ is $\neg p$ if $p \neq \neg q$, otherwise $\sim p$ is $\neg p$.

[16] In fact, this observation is already present in rudimentary form in Guilbaud [10].



Clearly this is isomorphic to the powerset of 3 lattice described in the previous section. Property (ii) in the above definition implies that additional structure is present in $L_{\vdash}(\mathsf{A})$. Finding a general way of describing this substructure is beyond the scope of this article. Instead we look at an example. Suppose that $Z = \{p_1, p_2\}$ is the witness for property (ii) in the above definition. Then, $\{\neg p_1, \neg p_2, p_3, p_4\}$ is consistent. Hence, we have the following additional structure in the lattic $L_{\vdash}(\mathsf{A})$:



In the above picture, note that $\{\neg p_1\} \vee \{p_1, p_2\} = \mathbf{1}$ and $\{\neg p_2\} \vee \{p_1, p_2\} = \mathbf{1}$. In [4], this property is used to prove that the aggregation function is monotonic. In this paper, Monotonicity of the aggregation function is stated as an explicit axiom (cf. Section 4.3). As such, in the next Sections, we only focus on the first property of the above definition.

Other agenda-richness properites have been proposed in the literature. For example, Gärdenfors [8] assumes a property that implies his agenda is non-atomic (recall that his agenda is a boolean algebra). Recently, there is a growing interest in so-called *characterization theorems* that characterizes the "agenda-richness properties" that are necessary and sufficient for an Arrow-style impossibility result. It is beyond the scope of this article to discuss the details—the interested reader can consult, for example, [7, 17]. In our setting, these agenda-richness assumptions amount to assuming the existence of a sublattice with particular structural properties. In this way, these characterization results can be discussed in our setting; however, this will be left for future work.

Following [4], we can view *preference* aggregation as a special case of judgement aggregation (cf. also [14] for a comparison between judgement and preference aggregation). Suppose C is a finite set of candidates and $\mathcal{P}$ the set of **strict linear orders** over C. In the standard Arrovian setting, agents choose elements of $\mathcal{P}$. In our setting, consider the lattice of subsets of $C \times C$ that are consistent with the order being transitive and irreflexive (the top element is the set $C \times C$). Assuming the agents are selecting connected orders amounts to assuming the agents choose co-atoms of this lattice. Note that this lattice is dichotomic: if (x, y) is inconsistent with a complete strict preference relation, then the relation must contain (y, x), and vice-versa. The lattice **3** also appears here in the form of the **Condorcet triple**:

$$\begin{tikzcd}
 & \mathbf{1} \arrow[dl, dash] \arrow[d, dash] \arrow[dr, dash] & \\
x > y > z \arrow[d, dash] \arrow[dr, dash] & z > x > y \arrow[dl, dash] \arrow[dr, dash] & y > z > x \arrow[dl, dash] \arrow[d, dash] \\
x > y \arrow[dr, dash] & y > z \arrow[d, dash] & z > x \arrow[dl, dash] \\
 & \mathbf{0} &
\end{tikzcd}$$

In the lattice above, for example, $x > y > z$ means $\{(x, y), (x, z), (y, z)\}$ and $x > y$ means $\{(x, y)\}$. In this simplified setting, one immediately sees a conflict between Arrow's Indpendence of Irrelevant Alternatives (IIA) and forcing agents to choose the co-atoms. Arguing very informally, IIA states that the *only* information that the aggregation procedure can use when deciding whether to make $x > y > z$ the social preference are the atoms below $x > y > z$ (in this case, $x > y$ and $y > z$). Now, in the context of preference aggregation, Arrows' IIA property implies *Neutrality*[17]. Informally, *Neutrality* says that the social choice with respect to some basic element of the agenda depends only on the pattern of the choices of the voters with respect to this particular element. But this means that, using neutrality, if

[17]See, for example, [9] for a proof of this fact. In the context of judgement aggregation, it can be shown that IIA implies Neutrality under certain richness assumptions on the agenda, such as path connectedness [7, 17, 4].

$x > y > z$ is socially accepted then a symmetric argument can be used to force the procedure to accept $y > z > x$. Since the join of these elements is $\mathbf{1}$, this gives a contradiction.[18]

In the rest of this paper, we shift the focus from the formal languages and consequence relations to the lattices of possible judgement sets. The lattices we have considered in this Section ($L^c_\vdash(\mathsf{A})$ and $L^{dc}_\vdash(\mathsf{A})$) satisfy various lattice-theoretic proeprties. Our goal in this paper is to relax these assumptions about the lattice as much as possible while still being able to prove an Arrow-style impossibility result. For instance, in this paper we will typically not assume atomicity of the lattice. While in many contexts, assuming atomicity is very natural, in this text we are interested in the minimal amount of structure needed to prove an impossibility result (cf. [3] for results similar in spirit). We hope that this level of generality can lead to new insights about formal relationships between properties of the agenda, rationality assumptions and properties of the aggregation function.

4 Axioms

Now that we have seen how the lattice-theoretic framework fits in with the rest of the social choice literature, we begin by examining a few axioms familiar from this literature to see how they generalize to our new setting.

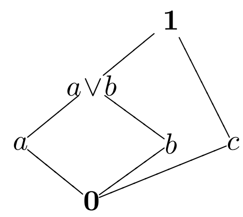
4.1 Neutrality

Neutrality of the aggregation function—in the literature on judgement aggregation this property is also known as systematicity—is instrumental to many Arrow-style results. In atomistic lattices, the corresponding property is stated as follows:

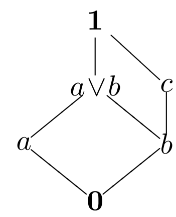
Definition 6 ***"Neutrality$^{\mathbf{A}}$"***: *For all π and π' and $a, b \in \mathcal{A}(Z)$ if $[\![a]\!]_\pi = [\![b]\!]_{\pi'}$ then $a \leq F(\pi)$ iff $b \leq F(\pi')$.*

If Z is an atomistic lattice, each $z \in Z$ can be written as the join of the atoms below it, and hence in such lattices Neutrality$^{\mathbf{A}}$ "grounds" the way the aggregation function behaves on non-atomic ("compound") elements in terms of the atoms below them—a fact which underlies many classical proofs in the literature on social choice. It is interesting to think about possible generalizations of this notion to non-atomistic lattices, where such a reduction

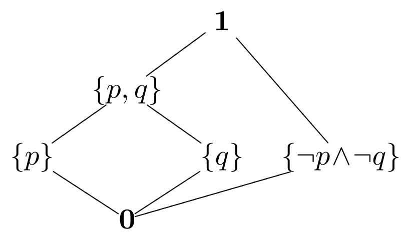
[18] A similar point is made more formally in number of papers by Donald Saari. See, for example, [20].



(a) Lattice $\mathbf{K_1}$



(b) Lattice $\mathbf{K_2}$



(c) Consistent sets of $\{p, q, \neg p \wedge \neg q\}$

Figure 1: Neutrality* versus Neutrality$^{\mathbf{A}}$

is not possible. A *prima facie* natural generalization of Neutrality$^{\mathbf{A}}$ is the following formulation, which is obtained by simply deleting the requirement that a and b be atomic from the definition of Neutrality$^{\mathbf{A}}$.

Definition 7 ***"Neutrality*****"***: *For all* π *and* π' *and* z *and* z', *if* $[\![z]\!]_\pi = [\![z']\!]_{\pi'}$ *then* $z \leq F(\pi)$ *iff* $z' \leq F(\pi')$.

As discussed in section 3, Gärdenfors [8] also works with a non-atomistic lattice, and this author uses ideas similar to the definition above to develop a notion of Independence of Irrelevant Alternatives for non-atomistic lattices.[19] However, as we will argue next the definition above is an unduly strengthening of Neutrality$^{\mathbf{A}}$ and imposing this property upon an aggregation function gives some unintuitive consequences.

Consider the lattice $\mathbf{K_1}$ in figure 1(a)—a concrete example of this lattice is the lattice of consistent sets of the agenda $\{p, q, \neg p \wedge \neg q\}$ as pictured in figure 1(c). Suppose Z is any lattice such that $\mathbf{K_1}$ is semi order-embeddable;

[19]Here, we only wish to emphasize the connection between definition 7 and his definition of IIA for non-atomic lattices. As discussed above, when viewed from our perspective, the framework of Gärdenfors actually yields an atomistic lattice.

and let F be any choice function satisfying Neutrality*. (A) *Neutrality* forces the collection of decisive sets of F to be closed under non-empty intersections.* This condition on the agenda needed to prove closure under intersections, that is, the semi order-embeddability of $\mathbf{K_1}$, is much weaker than those one needs to prove the same thing in the traditional judgment aggregation setting. In particular, the set $\{p, q, \neg p\}$ is not *minimally inconsistent* in the sense of Dietrich and List (as in definition 5).

The lattice $\mathbf{K_2}$ is perhaps even more ubiquitous; it sits, for instance, inside the lattice **3**, but also in the lattice of consistent sets of the agenda $\{p, q, \neg q\}$. Under the condition that $\mathbf{K_2}$ is semi order-embeddable in Z, (B) *Neutrality* forces the collection of decisive sets of F to be closed under non-empty intersections, whenever it is closed under supersets.*

Proof of (A) and (B) Let F be a profile and A and B decisive sets such that $A \cap B \neq \emptyset$. Consider first $\mathbf{K_1}$ and consider the profile π such that $\pi[A] = a$, $\pi[B] = b$, and $\pi[N - A - B] = c$. Since A and B are decisive, $a \vee b \leq F(\pi)$. Now consider $\mathbf{K_2}$ and the profile $\pi[N - (B - A)] = a$, and $\pi[B] = b$. Since B and every superset of A are decisive, $a \vee b \leq F(\pi)$.

Now consider both lattices and any profile π' such that $\pi[A \cap B] = a \vee b$ and $\pi[N - (A \cap B)] = c$. In both lattices

$$[\![a \vee b]\!]_\pi = (A \cap B) = [\![c]\!]_{\pi'},$$

and thus we find $a \vee b \leq F(\pi')$ by Neutrality*. π' was arbitrary and so $A \cap B$ is decisive. ■

Note that we could not have made the same argument based on Neutrality$^{\mathbf{A}}$, since the element $a \vee b$, which figures crucially in it, is not atomic. In the example above, F chooses the element $a \vee b$ because there is sufficient support for choosing a and b individually, but not because there is support for the $a \vee b$. The main issue is that in our more general setting, determining whether a aggregation function accepts an arbitrary element z of the lattice should depend not only on the acceptance set of z but also on the acceptance sets of elements whose acceptance is implied by z (i.e., elements of $\downarrow z$). What we are looking for is some way to make precise what it means for the choices of the agents on $\downarrow z$ to be "the same" as the choices on some other downset $\downarrow z'$. Since the structures $\downarrow z$ and $\downarrow z'$ may be wildly different, it is not *a priori* clear to go about.

The idea of a Neutral Simulation (which is inspired by a similar construction known from modal logic) forms the heart of the notion of Neutrality that we will employ in the balance of this text. The following definitions make this notion precise.

Definition 8 ***"Neutral Simulation"***: *Let $S \subseteq Z \times Z$ be a symmetric binary relation on the lattice Z, and let π and π' be profiles. The relation S is called a* Neutral Simulation *on Z between π and π' if and only if:*
(a) whenever zSz', $[\![z]\!]_\pi = [\![z']\!]_{\pi'}$;
(b) for all $y \leq z$, there exists $y' \leq z'$ such that ySy';
(c) for all $y' \leq z'$, there exists $y \leq z$ such that ySy'.

Definition 9 *"$\overset{\downarrow}{\equiv}$": We write $[\![z]\!]_\pi \overset{\downarrow}{\equiv} [\![z']\!]_{\pi'}$ if there exists a Neutral Simulation on Z between π and π' such that zSz'.*

Definition 10 ***"Neutrality"***: *F satisfies Neutrality (or "is Neutral") if, and only if, for all $z, z' \in Z$ and for all $\pi, \pi' \in dom(F)$, if $[\![z]\!]_\pi \overset{\downarrow}{\equiv} [\![z']\!]_{\pi'}$ then $z \leq F(\pi)$ iff $z' \leq F(\pi')$.*

It may be verified that in the examples based on $\mathbf{K_1}$ and $\mathbf{K_2}$, it is impossible to find a Neutral Simulation between π and π' with respect to the element $a \vee b$ that allows us to make the same argument as we made based on Neutrality$^{\mathbf{A}}$. In section 5 we will show that under this notion of Neutrality, impossibility results emerge only if a richer substructure than $\mathbf{K_1}$ or $\mathbf{K_2}$, *viz.* the lattice $\mathbf{3}$, is semi order-embeddable in Z—a structure that is closely related to an agenda condition that figures prominently in related literature. For now we conclude our discussion of various forms of the Neutrality axiom with a proof that Neutrality$^{\mathbf{A}}$ coincides with our preferred notion of Neutrality in the traditional, atomistic, context, and so is a true generalization.

Lemma 11 *Suppose Z is an atomistic lattice. Then F satisfies Neutrality$^{\mathbf{A}}$ if and only if F satisfies Neutrality.*

Proof. ($\Rightarrow$). Suppose $[\![x]\!]_\pi \overset{\downarrow}{\equiv} [\![y]\!]_{\pi'}$, let S be a witnessing Neutral Simulation between π and π' such that xSy and let F satisfy Neutrality$^{\mathbf{A}}$. Pick any atom $a \leq x$. By the Neutral Simulation, there exists $y' \leq y$ such that aSy' and thus $[\![a]\!]_\pi = [\![y']\!]_{\pi'}$.

Moreover, for all $y'' \leq y'$ there exists $x' \leq a$ such that $[\![x']\!]_\pi = [\![y'']\!]_{\pi'}$. In particular this is true for the atoms that make up y'.
Case 1. For all atoms b below y', $[\![b]\!]_{\pi'} = N$. Then $[\![y']\!] = N$, and hence $[\![a]\!]_\pi = N$, since aSy'. Pick b arbitrarily from the atoms below y'.
Case 2. There exists $b \leq y'$ such that $[\![b]\!]_{\pi'} \neq N$. Now, by Neutral Simulation, there exists $x' \leq a$ such that $[\![x']\!]_\pi = [\![b]\!]_{\pi'}$. Since $x' < a$ implies $x' = \mathbf{0}$, and $[\![\mathbf{0}]\!]_\pi = N$, it must be that $x' = a$.

(a) z and z' are complements

(b) A complement need not be unique

Figure 2: Complements

In both cases $[\![a]\!]_\pi = [\![b]\!]_{\pi'}$. Now suppose $y \leq F(\pi')$. Then $y' \leq F(\pi')$ and hence $b \leq F(\pi')$. By Neutrality$^{\mathbf{A}}$, $a \leq F(\pi)$. Since a was arbitrary and x is the join of its atoms, $x \leq F(\pi)$, and hence we have proved that $y \leq F(\pi') \implies x \leq F(\pi)$.

The reverse direction of Neutrality is proved similarly.

($\Leftarrow$). Suppose a, a' are atoms and $[\![a]\!]_\pi = [\![a']\!]_{\pi'}$. Then $S = \{(a, a'), (\mathbf{0}, \mathbf{0})\}$ clearly forms a Neutral Simulation. Hence by Neutrality, $a \leq F(\pi) \iff a' \leq F(\pi')$, as required in order to show Neutrality$^{\mathbf{A}}$. ■

4.2 Behavior on Complements

Suppose that Z is a bounded lattice, which may or may not be atomic. We say that elements z_1, z_2 are **complements** if $z_1 \wedge z_2 = \mathbf{0}$ and $z_1 \vee z_2 = \mathbf{1}$; see figure 2(a). Elements $z_1, z_2 \in Z - \{\mathbf{0}, \mathbf{1}\}$ are called **quasicomplements** if of these two equations only $z_1 \vee z_2 = \mathbf{1}$ holds—that is z_1 "blocks" z_2 and *vice versa*. We say that Z is **a lattice with complements** if every element $z \in Z$ has at least one complement (in general a complement need not be unique; see figure 2(b)).

In a "classical" context where F always chooses a co-atom, the following fact can easily seen to be true. If A, $N - A$ is some partition of the agents, and all agents in the set A support some element z_1 (say, a proposition p), and all the others support any given complement z_2 of z_1 (perhaps $\neg p$), then F chooses either z_1 or z_2. If an aggregation function F has this property, we will call it Decisive.

Definition 12 ***"Decisive"***: *F is Decisive if, whenever z_1 and z_2 are complements, and the profile π is such that $z_1 \leq \pi(i)$ for all $i \in A$, and $z_2 \leq \pi(i)$ for all $i \in N - A$, then either $z_1 \leq F(\pi)$ or $z_2 \leq F(\pi)$.*

As explained above, our framework allows for "incomplete" selections. We would like to demand of the aggregation function F to use the information provided to it in terms of profiles in an efficient fashion. Decisiveness captures one such form of efficiency. Decisiveness is a weakening of a condition on F called *completeness* in the literature—this is the condition that F always chooses p or $\neg p$.[20] To bring out the conditions imposed by this axiom a little bit more clearly, consider the following example:

The consensus aggregation function:[21] For all π, $F^{\mathbf{c}}(\pi) := \bigwedge_{i \in N} \pi(i)$.

Clearly $F^{\mathbf{c}}$ always selects a consistent element. However, speaking informally, consensus is not a very efficient aggregation procedure, and in our framework this fact formally reflects in that $F^{\mathbf{c}}$ is not Decisive.

4.3 Monotonicity

Another well known axiom from the literature on social choice theory is Monotonicity, which states—roughly—that when the support for some element z increases, z cannot disappear from the social choice. In atomistic lattices, the way Monotonicity affects how the aggregation function behaves on non-atomic ("compound") elements can again be grounded in terms of the atoms below them:

Definition 13 ***"Monotonicity$^{\mathbf{A}}$"***: *For all $a \in \mathcal{A}Z$ and for all π, π', if $[\![a]\!]_\pi \subseteq [\![a]\!]_{\pi'}$, then $a \leq F(\pi)$ implies $a \leq F(\pi')$.*

Generalizing this idea leads to the following axiom, which will be our preferred notion.

Definition 14 ***"Monotonicity"***: *For all $x \in Z$ and for all π, π', if for all $x' \leq x$ $[\![x']\!]_\pi \subseteq [\![x']\!]_{\pi'}$, then $x \leq F(\pi)$ implies $x \leq F(\pi')$.*

Lemma 15 *Let Z be atomistic. F satisfies Monotonicity$^{\mathbf{A}}$ if and only if F satisfies Monotonicity.*

Proof The right to left direction is immediate. Conversely, suppose $x \leq F(\pi)$. Furthermore suppose for all $x' \leq x$, $[\![x']\!]_\pi \subseteq [\![x']\!]_{\pi'}$. Then for all atoms a below x, we have $[\![a]\!]_\pi \subseteq [\![a]\!]_{\pi'}$, and moreover, since $a \leq x \leq F(\pi)$, by Monotonicity$^{\mathbf{A}}$, $a \leq F(\pi')$. Since x is the join of its atoms, $x \leq F(\pi')$. ■

[20] And of course, in our framework, completeness would correspond to the idea that F always picks a co-atom.

[21] This example assumes Z is a complete lattice when N is infinite.

Again in classical contexts, where agents choose elements from $\mathcal{M}(Z)$, Monotonicity has some additional consequences, which we might adopt as separate axioms in our framework of "incomplete" choices. Suppose π and π' are two distinct profiles, where π can be obtained from π' by modifying some agent's choice away from an element that blocks x, towards an element that is consistent with x. Then, intuitively, this change should not make x disappear from the social choice.

Definition 16 ***"Monotonicity$^\blacktriangledown$":*** *If π and π' are profiles such that (i) $[\![\bcancel{x}]\!]_{\pi'} \subseteq [\![\bcancel{x}]\!]_{\pi}$, and (ii) $\pi(i) = \pi(j)$ for all $i \notin [\![\bcancel{x}]\!]_{\pi} - [\![\bcancel{x}]\!]_{\pi'}$, then $x \leq F(\pi)$ implies $x \leq F(\pi')$.*

The following idea is an obvious variant.

Definition 17 ***"Monotonicity$^\blacktriangle$":*** *If π and π' are profiles such that (i) $[\![\bcancel{x}]\!]_{\pi'} \supseteq [\![\bcancel{x}]\!]_{\pi}$, and (ii) $\pi(i) = \pi(j)$ for all $i \notin [\![\bcancel{x}]\!]_{\pi'} - [\![\bcancel{x}]\!]_{\pi}$, then $x \not\leq F(\pi)$ implies $x \not\leq F(\pi')$.*

The following lemma relates these two notions to the Monotonicity axiom.

Lemma 18 *Suppose all agents choose elements in $\mathcal{M}(Z)$. Then Monotonicity implies both Monotonicity$^\blacktriangledown$ and Monotonicity$^\blacktriangle$.*

Proof Suppose $x \leq F(\pi)$. Let $y \leq x$. Let π' be a profile satisfying the antecedent of Monotonicity$^\blacktriangledown$. In this case, for each $i \in N$, $\pi(i) = \pi'(i)$ except when $i \in [\![\bcancel{x}]\!]_{\pi}$ but $i \notin [\![\bcancel{x}]\!]_{\pi'}$. Let $y \leq x$. If $\pi(i), \pi'(i) \in \mathcal{M}(Z)$, either $y \leq \pi(i)$ or $y \vee \pi(i) = \mathbf{1}$, and $y \leq \pi'(i)$. Hence for all $y \leq x$, $[\![y]\!]_{\pi} \subseteq [\![y]\!]_{\pi'}$ and by Monotonicity, $x \leq F(\pi')$.

Suppose $x \not\leq F(\pi)$. Let $y \leq x$. Let π' be a profile satisfying the antecedent of Monotonicity$^\blacktriangle$. In this case, for each $i \in N$, $\pi(i) = \pi'(i)$ except when $i \notin [\![\bcancel{x}]\!]_{\pi}$ but $i \in [\![\bcancel{x}]\!]_{\pi'}$. Let $y \leq x$. If $\pi(i), \pi'(i) \in \mathcal{M}(Z)$, $y \leq \pi(i)$ and either $y \leq \pi'(i)$ or $y \vee \pi'(i) = \mathbf{1}$. Hence for all $y \leq x$, $[\![y]\!]_{\pi'} \subseteq [\![y]\!]_{\pi}$. Suppose $x \leq F(\pi')$. By Monotonicity, $x \leq F(\pi)$, contradicting our assumption on $F(\pi)$. $\blacksquare$

5 An Impossibility Result

In the previous section, we have discussed generalizations of Neutrality and Monotonicity to our lattice-theoretic setting. In this section we investigate what axioms allow us to prove an Arrow-style impossibility theorem. Throughout the section, we consider a lattice Z with complements (and hence Z is topped and has a zero), though we do not assume Z is atomistic or even atomic.

We start by considering the behaviour of F with respect to some arbitrary element $x \in Z$ and one of x's complements. We use the notation $\overline{x}$ to denote some fixed, but arbitrary, element x' such that $x \leq x'$.

Definition 19 *"$\omega(x_1, x_2)$": Let x_1 and x_2 be complements and define $\omega(x_1, x_2) \subseteq \wp(N)$ as follows: for all $A \subseteq N$, $A \in \omega(x_1, x_2)$ iff whenever $\forall i \in A, \pi(i) = \overline{x}_1$ and $\forall i \in N - A, \pi(i) = \overline{x}_2$, then $x_1 \leq F(\pi)$.*

Thus a set of agents A is an element of $\omega(x_1, x_2)$, if A can force x_1 when its opposition $N - A$, accepts the complement x_2.

Lemma 20 *Let F satisfy Universal Domain and Neutrality. Suppose x_1 and x_2 are complements. Then*
(a) for all complements y_1 and y_2, $\omega(x_1, x_2) = \omega(y_1, y_2)$.
(b) $\omega(x_1, x_2) = \omega(x_2, x_1)$.
(c) if $A \in \omega(x_1, x_2)$, then $N - A \notin \omega(x_1, x_2)$.
(d) if, additionally, F is Decisive, then $\omega(x_1, x_2)$ satisfies the "ultraproperty": $A \in \omega(x_1, x_2)$ if and only if $N - A \notin \omega(x_1, x_2)$.

Proof (a). We show $\omega(x_1, x_2) \subseteq \omega(y_1, y_2)$. Let $A \in \omega(x_1, x_2)$ and let π be a profile where $\forall i \in A$, $\pi(i) = \overline{x}_1$ and $\forall i \in N - A$, $\pi(i) = \overline{x}_2$. As F satisfies Universal Domain and $A \in \omega(x_1, x_2)$, we have $x_1 \leq F(\pi)$.

Now consider any profile function π' where $\forall i \in A, \pi'(i) = \overline{y}_1$ and $\forall i \in N - A, \pi'(i) = \overline{y}_2$. For all $x' \leq x_1$, either $x' \not\leq x_2$, and so $[\![x']\!]_\pi = A$ or $x \leq x_2$, and so $[\![x']\!] = N$. By similar reasoning, for all $y' \leq y_1$, $[\![y']\!]_{\pi'} = [\![y_1]\!]_{\pi'} \in \{A, N\}$. Hence, as $[\![\cdot]\!]_.$ is antitonic w.r.t. $\leq$, we see $[\![x_1]\!]_\pi \overset{\downarrow}{\equiv} [\![y_1]\!]_{\pi'}$. By Neutrality $x_1 \leq F(\pi)$ iff $y_1 \leq F(\pi')$. In particular, since $x_1 \leq F(\pi)$ we have $y_1 \leq F(\pi')$. Hence, $A \in \omega(y_1, y_2)$. Symmetric reasoning shows $\omega(y_1, y_2) \subseteq \omega(x_1, x_2)$.

(b). This follows immediately from (a) by taking $(y_1, y_2) = (x_2, x_1)$.

(c). Finally we show that $A \in \omega(x_1, x_2)$ only if $N - A \notin \omega(x_1, x_2)$. Suppose $A \in \omega(x_1, x_2)$. Suppose further $N - A \in \omega(x_1, x_2)$. Then, since by (b) $\omega(x_1, x_2) = \omega(x_2, x_1)$, $N - A \in \omega(x_2, x_1)$. Now let π be any profile such that $\forall i \in A$, $\pi(i) = \overline{x}_1$, and $\forall i \in N - A$, $\pi(i) = \overline{x}_2$. Since $A \in \omega(x_1, x_2)$ it follows that $x_1 \leq F(\pi)$. Since $N - A \in \omega(x_2, x_1)$, it follows $x_2 \leq F(\pi)$. Hence $\mathbf{1} = x_1 \vee x_2 \leq F(\pi)$, a contradiction, since F never selects $\mathbf{1}$. Conclude

$N - A \notin \omega(x_1, x_2)$.

(d). Immediate. ■

We now generalize from a pair of complements to the behaviour of F with respect to some arbitrary element x when this element is blocked by a group of agents.

Definition 21 *: "$\Omega(x)$": Define $\Omega(x) \subseteq \wp(N)$ as follows: for all $A \subseteq N$, $A \in \omega(x)$ iff whenever $[\![x]\!]_\pi = A$ and $[\![x\!\!\backslash]\!]_\pi = N - A$, then $x \leq F(\pi)$.*

In our setting, Ω is the family of decisive sets for the element x. We have the following result.

Lemma 22 *Let F satisfy Monotonicity, Universal Domain, and Neutrality, and let F be Decisive. Then for all $x \in Z$, $\Omega(x) = \Omega$, where $\Omega \subseteq \wp(N)$ satisfying: (i) $A \in \Omega$, $B \supseteq A$ implies $B \in \Omega$;*
(ii) $A \in \Omega$ if and only if $N - A \notin \Omega$.

Proof Let x_1 and x_2 be complements. First we show that $\omega(x_1, x_2)$ is closed upwards. Let $A \subseteq B$. Now let π be such that $\forall i \in A \;\; \pi(i) = \overline{x}_1$ and $\forall i \in N - A \;\; \pi(i) = \overline{x}_2$. Let π' be such that $\forall i \in B \;\; \pi(i) = x_1$ and $\forall i \in N - B \;\; \pi(i) = x_2$. If $A \in \omega(x_1, x_2)$, we have $x_1 \leq F(\pi)$. By Monotonicity $x_2 \leq F(\pi')$ and so by Neutrality, $B \in \omega(x_1, x_2)$.

We show $\Omega(x_1) \supseteq \omega(x_1, x_2)$. Let $A \in \omega(x_1, x_2)$. Let π be any profile such $[\![x_1]\!]_\pi = A$ and $[\![x\!\backslash]\!]_\pi = N - A$. Furthermore, let π' be a profile such that $\forall i \in A \; \pi(i) = x_1$ and $\forall i \in N - A \; \pi(i) = x_2$. Then for all $x' \leq x_1$, $[\![x']\!]_\pi \supseteq [\![x_1]\!]_\pi = [\![x_1]\!]_{\pi'} = [\![x']\!]_{\pi'}$. Since $A \in \omega(x_1, x_2)$, and F satisfies Universal Domain, $x_1 \leq F(\pi')$. By Monotonicity, $x_1 \leq F(\pi)$. Since π was arbitrary, $A \in \Omega(x_1)$.

Next we show $\omega(x_1, x_2) \supseteq \Omega(x_1)$. Suppose $A \in \Omega(x_1)$. Let π be such that $\forall i \in A, \pi(i) = \overline{x}_1$ and $\forall i \in N - A, \pi(i) = \overline{x}_2$, then $x_1 \leq F(\pi)$. Clearly $[\![x_1]\!]_\pi = A$ and $[\![x\!\backslash]\!]_\pi = N - A$. Hence $x_1 \leq F(\pi)$. So $A \in \omega(x_1, x_2)$.

Using the above results and lemma 20(a), we find for any x_1 and y_1 that have complements x_2 and y_2, respectively:

$$\Omega(x_1) \subseteq \omega(x_1, x_2) \subseteq \omega(y_1, y_2) \subseteq \Omega(y_1) \subseteq (y_1, y_2) \subseteq \omega(x_1, x_2) \subseteq \Omega(x_1),$$

proving our claim. ■

The above result shows that $\Omega(x)$, that is, the family of decisive sets, is invariant under the choice of the element x. A family of sets $\Omega \supseteq \wp(N)$ that has property (i) and (ii), and additionally satisfies:

(iii) If $A \in \Omega$ and $B \in \Omega$, then $A \cap B \in \Omega$

is called an **ultrafilter**. We show next that Ω is an ultrafilter if an additional property holds of the lattice Z.

Lemma 23 *Suppose F satisfies the conditions stated in lemma 22. Furthermore suppose $\mathbf{3} = \wp(\{1,2,3\})$ is semi order-embeddable in Z. Then Ω is an ultrafilter.*

Strictly speaking, the condition that $\mathbf{3}$ is semi order-embeddable in Z is weaker than that of minimal inconsistency (cf. definition 5), since the latter amounts to $\mathbf{3}$ being *order*-embeddable. This observation once again illustrates an important feature of our framework, *viz.* that the join-operation is the crucial operation.

Proof We have already shown Ω is closed under supersets and satisfies the ultraproperty (this follows from Lemma 20 and Lemma 22). Thus, we need only show Ω is closed under taking intersections. Let $A, B \in \Omega$. We know $A \cap B \neq \emptyset$. We wish to show that $A \cap B \in \Omega$. Let $C = A \cap B$, and let $A' = A \cup \{i \mid i \notin A, B, C\}$. Note that $A' \supseteq A$ and hence $A' \in \Omega$. Take π such that:

$$
\begin{aligned}
[\![f(\{1\})]\!]_\pi &= A' \\
[\![f(\{3\})]\!]_\pi &= B \\
[\![f(\{2\})]\!]_\pi &= N - C
\end{aligned}
$$

Let f be a semi order-embedding of $\mathbf{3}$, $f(\{1\})$ and $f(\{2,3\})$ are quasi-complements. $[\![f(\{1\})]\!]_\pi = A$ and $[\![f(\{2,3\})]\!] = B \cap (N - C) = N - A$. Since $A' \in \Omega$, $f(\{1\}) \leq F(\pi)$. $f(\{3\})$ and $f(\{1,2\})$ are quasi-complements, hence by analogous reasoning $f(\{3\}) \leq F(\pi)$. Finally, $f(\{1,3\})$ and $f(\{2\})$ are quasi-complements. $[\![f(\{1,3\})]\!]_\pi = C$, and $[\![f(\{2\})]\!]_\pi = N - C$. Since Ω has the ultraproperty either $C \in \Omega$ or $N - C \in \Omega$. Suppose the latter, then $f(\{2\}) \leq F(\pi)$ hence $\bigvee\{f(\{1\}), f(\{2\}), f(\{3\})\} = \mathbf{1} \leq F(\pi)$, a contradiction. So $C \in \Omega$. $\blacksquare$

An agent $i^* \in N$ is called a **dictator** if $x \leq F(\pi)$ whenever $x \leq \pi(i^*)$.

Theorem 24 *Let F satisfy Universal Domain, Neutrality, Monotonicity, Monotonicity$^{\blacktriangledown}$, and Decisiveness and suppose N is finite; Furthermore suppose* **3** *is semi order-embeddable; then there exists a dictator.*

Proof Since N is finite Ω is a principal ultrafilter with a minimal element $\{i^*\}$. Suppose $x \leq \pi(i)$ and suppose furthermore that $x \not\leq F(\pi)$.

Let X be the set $[\![x]\!]_\pi$, and B be the set $[\![\backslash\!\!\!x]\!]_\pi$. Consider the profile π' where $\pi'(j) = x'$ for all $j \notin X \cup B$. Since at least $i \in X$, we know $N - B \in \Omega$ and so $x \leq F(\pi')$. Comparing π with π', we see that some agents block x under π' but not under π, whereas the opinions of all other agents remain unchanged. By Monotonicity$^{\blacktriangledown}$, $x \leq F(\pi)$. ■

According to Theorem 24 the social choice will be an element z such that $\pi(i^*) \leq z$. In words, everything accepted by the dictator i^* will end up in the social choice. The dictator is thus a bit weaker than a classical Arrovian dictator, who would be able to force the equation $\pi(i^*) = z$ to hold with equality. However, the weaker form of dictatorship arises quite naturally in our framework, because of the possible incompleteness of agents' choices: i^* need not have an opinion on all elements of the lattice Z—it might well happen that $x \not\leq \pi(i^*)$ while at the same time $x \vee \pi(i^*) < \mathbf{1}$. However, if i^* would like to preclude some such $x \in Z$ from the social choice, she has the option of choosing to include a quasicomplement of x below her choice $\pi(i)$, i.e., force an element that blocks x.

The lattice **3** plays a crucial role in the argument, because it forces the collection of decisive sets to be closed under intersections. In fact, a converse of Theorem 24 also holds: if **3** is not embeddable, we can state a possibility result, at least for complete, compact lattices. Let i^* be some designated element of N. The **family of majorities with chair** i^* is the family of sets $\mathcal{F}$ such that for $A \subseteq N$, $A \in \mathcal{F}$ if and only if either $|A| > \frac{1}{2}N$, or $|A| = \frac{1}{2}N$ and $i^* \in A$. The corresponding **majority rule with chair** i^* is the aggregation function:[22]

$$F^{\mathbf{m}}_{i^*}(\pi) := \bigwedge \{z \in Z \mid [\![z]\!]_\pi \in \mathcal{F}\}$$

Theorem 25 *Let N be finite. Let Z be a compact and complete lattice with complements and let F satisfy Universal Domain, Neutrality, Monotonicity, Monotonicity$^{\blacktriangledown}$, and Decisiveness. Then F is necessarily dictatorial if and only if* **3** *is semi order-embeddable into Z.*

[22] Our example of the majority rule with chair i^* solely serves for concreteness. The proof will go through if $\mathcal{F}$ is any strong and proper simple game. For simple game theory see e.g. Taylor and Zwicker [21].

Proof The right to left part of the theorem follows from the previous one. As for the other direction, it is an easy exercise to show $F^{\mathbf{m}}_{i^*}$ satisfies the axioms stated in the theorem. We will prove, by contraposition, that $F^{\mathbf{m}}_{i^*}(\pi) < 1$ for all π (hence is indeed an aggregation function) whenever the lattice **3** is not semi order-embeddable.

Suppose $F^{\mathbf{m}}_{i^*}(\pi) = \mathbf{1}$. Since Z is compact, there exists a finite subset $T \subseteq \{z \in Z \mid [\![z]\!]_\pi \in \mathcal{F}\}$ such that $\bigwedge T = \mathbf{1}$.

Clearly T cannot be empty or contain only a singleton. We will prove T is also "pairwise consistent": let $z_1, z_2 \in T$. Straightforwardly from the properties of the family of majorities with chair i^*, we find $[\![z_1]\!]_\pi \cap [\![z_2]\!]_\pi \neq \emptyset$. Hence there exists $j \in N$, such that $z_1 \vee z_2 \leq \pi(j)$. Now $\pi(j) < \mathbf{1}$, and so by transitivity it follows $z_1 \vee z_2 < \mathbf{1}$.

Now, there must be a set $S \subset T$ such that $\bigvee S < \mathbf{1}$, but $\bigvee S \vee z_1 \vee z_2 = \mathbf{1}$ for some distinct $z_1, z_2 \in T - S$. In particular there is such a set S of smallest cardinality, and moreover we know $|S| \geq 1$, by our pairwise consistency argument. It remains to prove that $\bigvee S \vee z_1 < \mathbf{1}$ and $\bigvee S \vee z_2 < \mathbf{1}$. Suppose one of these inequations fails, say $\bigvee S \vee z_1 = \mathbf{1}$. There is a largest subset $S' \subset S$, such that $S' \vee z_1 < \mathbf{1}$. That is, for every element $y \in S - S'$, $\bigvee S' \vee z_1 \vee y = 1$. But then $S' \subset S$, $y \notin S$, $z_1 \notin S$ contradicts that S is the set of smallest cardinality satisfying the stated assumption.

We have shown that $\bigvee S, z_1, z_2$ are pairwise consistent, yet $\bigvee S \cup \{z_1, z_2\} = \mathbf{1}$, *viz.* **3** is semi order-embeddable. This concludes the proof. ■

6 Conclusion

In this text we have introduced a general framework for studying aggregation problems. Our framework is abstract and lattice-theoretic in nature, the crucial operation being the joining of two elements of the lattice. Our work can be categorized broadly as studying the formal connections between properties of the agenda (or lattice, in our case) and the induced algebraic structure of the set of winning coalitions. There are a number of recent papers closely related to our work, notably Dokow and Holzman [7] and Nehring and Puppe [17]. These authors also work in a very general setting and provide characterization results.

One fruitful aspect of our framework is its unifying ability: in Section 3 we have argued how lattices arise frequently from more traditional approaches to social choice theory (cf. Monjardet [15] for a broader perspective on the use of lattice theory in the social sciences). Additionally, the lattice-theoretic perspective allows us to push the level of abstraction a bit further. We have shown how certain well known axioms might generalize to our setting, and discussed some insights and limitations that arise from this perspective.

Many lattices arising from traditional approaches to social choice theory satisfy very strong conditions. In the penultimate section we have presented an impossibility theorem, while relaxing a number of such strong conditions. Moreover, we have stated a lattice-theoretic property that is necessary to obtain an impossibility result. This property, *viz.* semi order-embeddability of the powerset of 3 lattice, is closely related to the notion of a minimal inconsistent set that appears in the judgment aggregation literature.

Finally, recently there is interest in aggregation of *in*complete orders and *in*complete (but consistent) subsets of an agenda. In the preference aggregation setting see Pini *et al.* [19] and for the judgment aggregation setting see Gärdernfors [8] and Dietrich & List [6]. As we are not assuming that the agents select maximal elements of the lattice, our result contributes to this literature.

In this text we have placed emphasis on exposition of the framework, rather than on extending it in as many directions as possible. From where we stand now, opportunities for further research present themselves in (at least) three directions. First, one might study how other *lattice-theoretic* properties might lead to impossibility results. For instance, one could move from the complemented lattices that we study in section 5 to lattices with pseudo-complements, which arise frequently in logic. Second, one might want to study how other *agenda conditions* would show up in our setting. To give an example, an interesting condition that has appeared in the literature is "path-connectedness". Dietrich and List have shown in the context of judgment aggregation that if the agenda satisfies this condition, then Neutrality of the aggregation function is equivalent to imposing a generalised version of Arrow's Independence of Irrelevant Alternatives axiom [4]. Finally, one might study the consequences of *different axioms*, or perhaps different generalizations of axioms, than those considered in section 4 and 5 inside our setting. We hope to pursue some of these interesting directions in future work.

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