Preservation by fibring of the finite model property

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Abstract

Capitalising on the graph-theoretic account of fibring proposed in [31], we show that fibring preserves the finite model property under mild conditions. Illustrations are provided for modal, deontic, paraconsistent and linear logics.

Keywords: combined logics, fibring, finite model property, preservation.

1 Introduction

It is well know that in favorable conditions the finite model property (in short fmp) can be used for establishing that a logic is decidable[1]. For this reason, the preservation of fmp by mechanisms for combining logics has been in the foreground of those working on the field. Using the so called surrogates technique [18], fmp was shown to be preserved by fusion of modal logics. Fusion is a special case of fibring [12], but the preservation of fmp by fibring in general was not established.

Herein, also using the surrogates technique, we are able to extend this transference result to the fibring of a much wider class of logics by taking advantage of a novel feature of the graph-theoretic semantics of fibring proposed in [31]: every model of the logic obtained by fibring $\mathcal{L}_1$ and $\mathcal{L}_2$ is roughly the Cartesian product of a model of $\mathcal{L}_1$ and a model of $\mathcal{L}_2$. Previous semantics of fibring [13, 29, 34, 32, 8] did not share this essential ingredient of our proof of the fmp transference. Observe also that the graph-theoretic account of fibring encompasses a much wider universe of logics, including paraconsistency and

[1]For instance, if the logic is finitely axiomatizable, then decidability is an immediate corollary of fmp. Furthermore, if the logic is axiomatizable and its models are defined independently of its axioms, again decidability follows from fmp. However, there are examples of axiomatizable logics with fmp that are not decidable, like the one provided by Urquhart [33]. That logic is not finitely axiomatizable and its models are the normal Kripke models that satisfy the axioms.

substructurality, thanks to using an abstraction map from the semantic universe to the language, instead of the traditional interpretation map from the latter to the former.

The reader is expected to be conversant with the concepts and main results of the graph-theoretic account of fibring given in [30, 31], but for convenience we provide a very brief summary in Section 2, together with a straightforward extension to the case of constrained fibring (allowing the sharing of symbols). Graph-theoretic examples of logics with fmp are given in Section 3. The main result is proved in Section 4 and illustrated in Section 5 for establishing the fmp for logics obtained by fibring the logics with fmp discussed in Section 3.

2 Preliminaries on graph-theoretic fibring

Signatures and interpretation structures of logics are seen as m-graphs in the graph-theoretic account of logics and their fibrings proposed in [30, 31].

An *m-graph*, is a tuple $G = (V, E, \mathsf{src}, \mathsf{trg})$ such that V is a set (of *vertexes* or *nodes*), E is a set (of *m-edges*), $\mathsf{src} : E \to V^+$ and $\mathsf{trg} : E \to V$ are maps (the source and the target, respectively), where, as usual, V^+ denotes the set of all finite non-empty sequences of V. We may write $e : s \to v$ or $e \in G(s, v)$ when $e \in E$, $\mathsf{src}(e) = s$ and $\mathsf{trg}(e) = v$.

2.1 Language

A *signature* is an m-graph $\Sigma = (\{\pi, \diamond\}, E, \mathsf{src}, \mathsf{trg})$ such that no m-edge has $\diamond$ as target. Node π is the *formula sort* and node $\diamond$ is the *concrete sort.* The m-edges play the role of *constructors* for building formulas. For instance, the connective $\neg$ is represented by the edge $\neg : \pi \to \pi$. A propositional symbol q is seen as an edge $q : \diamond \to \pi$.

From an m-graph G we can (canonically) generate a category $\mathcal{C}(G)$ with finite products (for details see [30]). The objects of the category are non-empty sequences of nodes. The morphisms are the paths over the m-graph. We represent the product of nodes s_1, s_2 as $s_1 s_2$. Among the morphisms we have projections of the form $\mathsf{p}_i^{s_1 s_2} : s_1 s_2 \to s_i$ and pairings of the form $\langle h_1, h_2 \rangle : w \to s_1 s_2$ whenever $h_i : w \to s_i$.

The *formulas* over Σ are the morphisms of the category $\mathcal{C}(\Sigma)$ with target π. The underlying path identifies the constructors of the formula. We say that a formula is *concrete* whenever the sort π does not occur in the source, hence the name for node $\diamond$. This notion corresponds to the traditional (set-theoretic) notion of language of formulas over Σ. For instance, the concrete formula $(\neg\, q_1) \supset q_2$, where q_1 and q_2 are propositional symbols, is represented by the morphism:

$$\supset \circ \langle \neg \circ q_1, q_2 \rangle : \diamond \to \pi$$

in $\mathcal{C}(\Sigma)$. We may write $\supset(\neg(q_1), q_2)$ for $\supset \circ \langle \neg \circ q_1, q_2 \rangle$. We say that a formula is *schematic* if only part of their structure is known (or determined). More concretely, by a *schema formula* we mean a morphism in $\mathcal{C}(\Sigma)$ with target π

and with a source where π occurs. For instance, the morphism

$$\supset \circ \langle \supset \circ \langle \mathsf{p}_1^{\pi\pi}, \mathsf{p}_2^{\pi\pi} \rangle, \mathsf{p}_2^{\pi\pi} \rangle$$

is a schema formula with two schema variables, usually written $(\xi_1 \supset \xi_2) \supset \xi_2$. When looking at formulas as morphisms, projections play the role of schema variables. From now on we stick to the categorical notation.

Observe that $\supset \circ \langle \supset \circ \langle q_1, q_2 \rangle, q_2 \rangle$ is an instance of $\supset \circ \langle \supset \circ \langle \mathsf{p}_1^{\pi\pi}, \mathsf{p}_2^{\pi\pi} \rangle, \mathsf{p}_2^{\pi\pi} \rangle$, that is,

$$\supset \circ \langle \supset \circ \langle q_1, q_2 \rangle, q_2 \rangle = \supset \circ \langle \supset \circ \langle \mathsf{p}_1^{\pi\pi}, \mathsf{p}_2^{\pi\pi} \rangle, \mathsf{p}_2^{\pi\pi} \rangle \circ \langle q_1, q_2 \rangle.$$

The set of formulas, i.e. the *language* over Σ, is denoted by $L(\Sigma)$. The *main constructor* of a formula is the last edge of the path. The set $\mathrm{SF}(\varphi)$ of *subformulas* of a formula φ is inductively defined as follows:

- $\mathrm{SF}(\varphi) = \{\varphi\}$ if either $\varphi : \diamond \to \pi$ or φ is $\mathsf{p}_i^{\pi^n}$ for every $n \geq 1$ and $i = 1, \ldots, n$;
- $\mathrm{SF}(\varphi) = \cup_{j=1}^{k} \mathrm{SF}(\varphi_j) \cup \{\varphi\}$ assuming that φ is $c \circ \langle \varphi_1, \ldots, \varphi_k \rangle$ where $c \in E$ is an edge.

Letting φ be the formula $\supset \circ \langle \supset \circ \langle \mathsf{p}_1^{\pi\pi}, \mathsf{p}_2^{\pi\pi} \rangle, \mathsf{p}_2^{\pi\pi} \rangle$ we have:

$$\mathrm{SF}(\varphi) = \{\varphi, \langle \supset \circ \langle \mathsf{p}_1^{\pi\pi}, \mathsf{p}_2^{\pi\pi} \rangle, \mathsf{p}_2^{\pi\pi} \rangle, \supset \circ \langle \mathsf{p}_1^{\pi\pi}, \mathsf{p}_2^{\pi\pi} \rangle, \mathsf{p}_1^{\pi\pi}, \mathsf{p}_2^{\pi\pi}\}.$$

We denote by Σ_n the set of all the n-ary constructors, that is, the m-edges from π^n to π in signature Σ, and we denote by Σ_n^+ the set of all morphisms from π^n to π in category $\mathcal{C}(\Sigma)$. The elements of Σ_n^+ are known as *derived constructors* of arity n, while the elements of Σ_n are called *primitive constructors* of arity n.

2.2 Semantics

For the semantics we need the notion of m-graph morphism. An *m-graph morphism* $h : G_1 \to G_2$ is a pair of maps $h^{\mathsf{v}} : V_1 \to V_2$ and $h^{\mathsf{e}} : E_1 \to E_2$ such that $\mathsf{src}_2 \circ h^{\mathsf{e}} = h^{\mathsf{v}} \circ \mathsf{src}_1$ and $\mathsf{trg}_2 \circ h^{\mathsf{e}} = h^{\mathsf{v}} \circ \mathsf{trg}_1$.

An *interpretation structure* I over a signature Σ is a triple

$$(G', \alpha, D)$$

where G' is an m-graph (the *operations graph*), $\alpha : G' \to G$ is an m-graph morphism (the *abstraction morphism*), $\emptyset \subsetneq D \subsetneq (\alpha^{\mathsf{v}})^{-1}(\pi)$ is a non-empty set and $(\alpha^{\mathsf{v}})^{-1}(\diamond) = \{\blacklozenge\}$. The set V' of nodes of the operations graph is called the *universe* of *values*. Observe that V' is partitioned by α: we denote by V'_v the *domain* $(\alpha^{\mathsf{v}})^{-1}(v)$ of values for each sort v. The elements of V'_π are the *truth values* and the element $\blacklozenge$ of $V'_\diamond$ is the *concrete value*. The elements of the set D are the *distinguished truth values*. We say that an interpretation structure I is *finite* whenever V' is a finite m-graph, that is, when V'_π and E' are both finite. Observe that α can be extended to a functor from $\mathcal{C}(G')$ to $\mathcal{C}(\Sigma)$, mapping projections to projections. We refer to this functor also as α.

In order to prove our main result, we need to introduce some notions related with local satisfaction by a path and global satisfaction by a set of paths.

Let $\varphi : \pi^n{}_\diamond{}^m \to \pi$ where $n > 0$ and $m \geq 0$ be a schema formula and I an interpretation structure both over signature Σ. We say that $w' : u_1 \dots u_n{}_\blacklozenge{}^m \to d$ in $\mathcal{C}(G')$ is a *path for* φ in I if $w' \in \alpha^{-1}(\varphi)$. When w' is a path for φ in I and its target is in D we write

$$I, w' \Vdash \varphi$$

and say that path w' of I *satisfies* formula φ. Given a set W' of paths for φ in I, we say that φ is W'*-path satisfiable* in I, denoted by

$$I, W' \Vdash_\exists \varphi,$$

whenever there exists a path $w' \in W'$ such that $I, w' \Vdash \varphi$. We say that W' of I *satisfies* φ, denoted by $I, W' \Vdash_\forall \varphi$ or, simply, by

$$I, W' \Vdash \varphi,$$

if $I, w' \Vdash \varphi$ for every path $w' \in W'$. Finally, we say that W' of I *falsifies* φ whenever $I, W' \nVdash \varphi$.

We say that φ is *path satisfiable* in I, denoted by

$$I \Vdash_\exists \varphi,$$

whenever $I, \alpha^{-1}(\varphi) \Vdash_\exists \varphi$. We say that I *satisfies* φ, denoted by $I \Vdash_\forall \varphi$ or simply, as usual, by

$$I \Vdash \varphi,$$

if $I, \alpha^{-1}(\varphi) \Vdash \varphi$. Finally, we say that I *falsifies* φ whenever $I \nVdash \varphi$.

The denotation of a formula φ over I is

$$[\![\varphi]\!]^I = \{\mathsf{trg}(w') : \alpha(w') = \varphi\}.$$

Clearly, $[\![\varphi]\!]^I \subseteq D$ iff $I \Vdash \varphi$.

In the sequel we also need the following notions and notations concerning paths with selected sources for a given schema formula. Given a formula φ over Σ and s a sequence of values, we denote by $\mathsf{W}^I_{s\varphi}$ the set of all paths for φ with source s in I. Then s is said to be a socket for φ in I. Moreover, we denote by W^I_φ the set of all paths for φ whenever φ is a formula but not a schema formula. Finally, given a path w' for φ and a subformula δ of φ, we denote by

$$w'_{|\delta}$$

the subpath of w' such that $\alpha(w'_{|\delta}) = \delta$.

In order to deal with logics, we need to work with classes of interpretation structures over the same signature. An *interpretation system* is a pair $(\Sigma, \mathfrak{I})$ where Σ is a signature and $\mathfrak{I}$ is a class of interpretation structures over Σ.

A formula φ is *falsifiable* in an interpretation system $(\Sigma, \mathfrak{I})$ if there is $I \in \mathfrak{I}$ such that I falsifies φ.

2.3 Fibring of interpretation systems

For introducing the notion of constrained fibring (sharing some constructors), it is convenient to make use of signature morphisms for specifying which symbols are to be shared. As expected, by a *signature morphism* we understand an m-graph morphism with the identity map on $\{\pi, \diamond\}$ as the vertex component.

But let us start first with the notion of unconstrained fibring (no constructor is shared). At the signature level, unconstrained fibring amounts to the disjoint union of the two given signatures. In the category of signatures introduced in [31], this construction is a co-product. Let Σ_1 and Σ_2 be signatures. Their *unconstrained fibring* or *disjoint fibring*, denoted by

$$\Sigma_1 \uplus \Sigma_2$$

is the signature $(\{\pi, \diamond\}, E, \mathsf{src}, \mathsf{trg})$ endowed with injections i_1 and i_2 such that:

- $E = E_1 \uplus E_2 = \mathsf{i}_1^{\mathsf{e}}(E_1) \cup \mathsf{i}_2^{\mathsf{e}}(E_2)$;
- src and trg are such that $\mathsf{src} \circ \mathsf{i}_j^{\mathsf{e}} = \mathsf{src}_j$ and $\mathsf{trg} \circ \mathsf{i}_j^{\mathsf{e}} = \mathsf{trg}_j$.

At the semantic level things are a bit more involved even in the case of unconstrained fibring. Let (Σ_1, I_1) and (Σ_2, I_2) be interpretation structures. Given $v_1' \in V_{1\pi}'$ and $v_2' \in V_{2\pi}'$, we say that v_1' and v_2' *agree*, written $v_1' \approx v_2'$, if they are both distinguished or both non distinguished. This notion is pointwise extended to sequences of the same length. The *unconstrained fibring* or *disjoint fibring* of interpretation structures (Σ_1, I_1) and (Σ_2, I_2), denoted by

$$(\Sigma_1, I_1) \uplus (\Sigma_2, I_2),$$

is the co-product of (Σ_1, I_1) and (Σ_2, I_2) in the category of interpretation structures introduced in [31]. More concretely, it is the interpretation structure $(\Sigma_1 \uplus \Sigma_2, I_1 \uplus I_2)$ where $I_1 \uplus I_2 = ((V', E', \mathsf{src}', \mathsf{trg}'), \alpha, D)$ with:

- $V' = (D_1 \times D_2) \cup ((V_{1\pi}' \setminus D_1) \times (V_{2\pi}' \setminus D_2)) \cup \{(\blacklozenge_1, \blacklozenge_2)\}$;
- E' is composed of
  - each $\mathsf{i}_1^{\mathsf{e}}(e_1)_{e_1', u', v'}$ with $e_1 \in E_1, e_1' \in E_1', u' \in V_2'^{+}, v' \in V_2'$ such that
  $$\begin{cases} e_1 = \alpha_1^{\mathsf{e}}(e_1') \\ u' \approx \mathsf{src}_1'(e_1') \text{ and } v' \approx \mathsf{trg}_1'(e_1'), \end{cases}$$
  - each $\mathsf{i}_2^{\mathsf{e}}(e_2)_{u', v', e_2'}$ with $e_2 \in E_2, u' \in V_1'^{+}, v' \in V_1', e_2' \in E_2'$ such that
  $$\begin{cases} e_2 = \alpha_2^{\mathsf{e}}(e_2') \\ u' \approx \mathsf{src}_2'(e_2') \text{ and } v' \approx \mathsf{trg}_2'(e_2'); \end{cases}$$
- src' is such that
$$\begin{cases} \mathsf{src}'(\mathsf{i}_1^{\mathsf{e}}(e_1)_{e_1', u', v'}) = (\mathsf{src}_1'(e_1'), u') \\ \mathsf{src}'(\mathsf{i}_2^{\mathsf{e}}(e_2)_{u', v', e_2'}) = (u', \mathsf{src}_2'(e_2')); \end{cases}$$

- trg' is such that

$$\begin{cases}\mathsf{trg}'(\mathsf{i}^{\mathsf{e}}_1(e_1)_{e'_1,u',v'}) = (\mathsf{trg}'_1(e'_1), v') \\ \mathsf{trg}'(\mathsf{i}^{\mathsf{e}}_2(e_2)_{u',v',e'_2}) = (v', \mathsf{trg}'_2(e'_2));\end{cases}$$

- α is such that

$$\begin{cases}\alpha^{\mathsf{v}}((u'_1, u'_2)) = \begin{cases}\diamond & \text{if } (u'_1, u'_2) = (\blacklozenge_1, \blacklozenge_2) \\ \pi & \text{otherwise}\end{cases} \\ \alpha^{\mathsf{e}}(\mathsf{i}^{\mathsf{e}}_1(e_1)_{e'_1,u',v'}) = \mathsf{i}^{\mathsf{e}}_1(e_1) \\ \alpha^{\mathsf{e}}(\mathsf{i}^{\mathsf{e}}_2(e_2)_{u',v',e'_2}) = \mathsf{i}^{\mathsf{e}}_2(e_2);\end{cases}$$

- $D = D_1 \times D_2$.

It is worthwhile to comment upon the nature of the above construction. The unconstrained fibring of two interpretation structures is roughly their Cartesian product, although presented as a co-product in order to follow the combination at the signature level where it is a disjoint union. Clearly, the set V' of the vertexes of G' is roughly the Cartesian product of V'_1 and V'_2 (respecting typing and distinguishedness). Less obviously, the set E' of the edges of G' also has a product nature. In this case the edges of G'_1 are paired with all compatible 'mute' edges in G'_2 and vice versa. By a 'mute' edge we mean a virtual edge that is not mapped to the signature. The relevance of this notion suggests the use of partial abstraction functions instead of abstraction maps, partial on the edge component, but the categorical development of this idea is outside the scope of this paper.

In the case of constrained fibring (when some constructors are shared), we use signature morphisms for specifying which symbols are to be shared. Let $h_1 : \Sigma_0 \to \Sigma_1$ and $h_2 : \Sigma_0 \to \Sigma_2$ be injective signature morphisms. Given such a source diagram, we say that constructors e_1 and e_2 are *shared* if there is a constructor e in Σ_0 such that $e_1 = h^{\mathsf{e}}_1(e)$ and $e_2 = h^{\mathsf{e}}_2(e)$. Furthermore, we say that Σ_1 and Σ_2 *share* Σ_0 *via* h_1 and h_2.

The *constrained fibring* of signatures Σ_1 and Σ_2 *sharing* Σ_0 *via* h_1 and h_2, denoted by

$$\Sigma_1 \uplus^{\Sigma_0}_{h_1 h_2} \Sigma_2$$

is the target of the push-out of h_1 and h_2. Equivalently, it is the target of the co-equalizer

$$q : \Sigma_1 \uplus \Sigma_2 \to \Sigma_1 \uplus^{\Sigma_0}_{h_1 h_2} \Sigma_2$$

of $\mathsf{i}_1 \circ h_1$ and $\mathsf{i}_2 \circ h_2$. More concretely, it is the signature $(\{\pi, \diamond\}, E^*, \mathsf{src}^*, \mathsf{trg}^*)$ with:

- $E^* = \mathsf{i}^{\mathsf{e}}_1(E_1) \cup (\mathsf{i}^{\mathsf{e}}_2(E_2) \setminus \mathsf{i}^{\mathsf{e}}_2(h^{\mathsf{e}}_2(E_0)))$;
- src^* and trg^* are such that $\mathsf{src}^* \circ \mathsf{i}^{\mathsf{e}}_j = \mathsf{src}_j$ and $\mathsf{trg}^* \circ \mathsf{i}^{\mathsf{e}}_j = \mathsf{trg}_j$.

The *constrained fibring* of interpretation structures (Σ_1, I_1) and (Σ_2, I_2) *sharing* Σ_0 *via* h_1 and h_2, denoted by

$$(\Sigma_1, I_1) \uplus^{\Sigma_0}_{h_1 h_2} (\Sigma_2, I_2),$$

is the co-Cartesian lifting by the forgetful functor (from the category of interpretation structures to the category of signatures) of q on the interpretation structure

$$(\Sigma_1, I_1) \uplus (\Sigma_2, I_2) = (\Sigma_1 \uplus \Sigma_2, ((V', E', \mathsf{src}', \mathsf{trg}'), \alpha, D)).$$

More concretely, it is the interpretation structure $(\Sigma_1 \uplus^{\Sigma_0}_{h_1 h_2} \Sigma_2, I_1 \uplus^{\Sigma_0}_{h_1 h_2} I_2)$ where $I_1 \uplus^{\Sigma_0}_{h_1 h_2} I_2 = ((V^{*\prime}, E^{*\prime}, \mathsf{src}^{*\prime}, \mathsf{trg}^{*\prime}), \alpha^*, D^*)$ with:

- $V^{*\prime} = V'$;
- $E^{*\prime}$ is composed of
  - each $\mathsf{i}^{\mathsf{e}}_1(e_1)_{e'_1,u',v'}$ with $e_1 \in E_1, e'_1 \in E'_1, u' \in V'^{+}_2, v' \in V'_2$ such that
$$\begin{cases} e_1 = \alpha^{\mathsf{e}}_1(e'_1) \notin h^{\mathsf{e}}_1(E_0) \\ u' \approx \mathsf{src}'_1(e'_1) \text{ and } v' \approx \mathsf{trg}'_1(e'_1), \end{cases}$$
  - each $\mathsf{i}^{\mathsf{e}}_2(e_2)_{u',v',e'_2}$ with $e_2 \in E_2, u' \in V'^{+}_1, v' \in V'_1, e'_2 \in E'_2$ such that
$$\begin{cases} e_2 = \alpha^{\mathsf{e}}_2(e'_2) \notin h^{\mathsf{e}}_2(E_0) \\ u' \approx \mathsf{src}'_2(e'_2) \text{ and } v' \approx \mathsf{trg}'_2(e'_2), \end{cases}$$
  - each $\mathsf{i}^{\mathsf{e}}_1(h^{\mathsf{e}}_1(e))_{e'_1,e'_2}$ with $e \in E_0, e'_1 \in E'_1, e'_2 \in E'_2$ such that
$$\begin{cases} \alpha^{\mathsf{e}}_1(e'_1) = h^{\mathsf{e}}_1(e) \text{ and } \alpha^{\mathsf{e}}_2(e'_2) = h^{\mathsf{e}}_2(e) \\ \mathsf{src}'_1(e'_1) \approx \mathsf{src}'_2(e'_2) \text{ and } \mathsf{trg}'_1(e'_1) \approx \mathsf{trg}'_2(e'_2); \end{cases}$$
- $\mathsf{src}^{*\prime}$ is such that
$$\begin{cases} \mathsf{src}^{*\prime}(\mathsf{i}^{\mathsf{e}}_1(e_1)_{e'_1,u',v'}) = (\mathsf{src}'_1(e'_1), u') \\ \mathsf{src}^{*\prime}(\mathsf{i}^{\mathsf{e}}_2(e_2)_{u',v',e'_2}) = (u', \mathsf{src}'_2(e'_2)) \\ \mathsf{src}^{*\prime}(\mathsf{i}^{\mathsf{e}}_1(h^{\mathsf{e}}_1(e))_{e'_1,e'_2}) = (\mathsf{src}'_1(e'_1), \mathsf{src}'_2(e'_2)); \end{cases}$$
- $\mathsf{trg}^{*\prime}$ is such that
$$\begin{cases} \mathsf{trg}^{*\prime}(\mathsf{i}^{\mathsf{e}}_1(e_1)_{e'_1,u',v'}) = (\mathsf{trg}'_1(e'_1), v') \\ \mathsf{trg}^{*\prime}(\mathsf{i}^{\mathsf{e}}_2(e_2)_{u',v',e'_2}) = (v', \mathsf{trg}'_2(e'_2)) \\ \mathsf{trg}^{*\prime}(\mathsf{i}^{\mathsf{e}}_1(h^{\mathsf{e}}_1(e))_{e'_1,e'_2}) = (\mathsf{trg}'_1(e'_1), \mathsf{trg}'_2(e'_2)); \end{cases}$$

- α^* is such that

$$\begin{cases} \alpha^{*\mathsf{v}} = \alpha^{\mathsf{v}} \\ \alpha^{*\mathsf{e}}(\mathsf{i}_1^{\mathsf{e}}(e_1)_{e_1',u',v'}) = \mathsf{i}_1^{\mathsf{e}}(e_1) \\ \alpha^{*\mathsf{e}}(\mathsf{i}_2^{\mathsf{e}}(e_2)_{u',v',e_2'}) = \mathsf{i}_2^{\mathsf{e}}(e_2) \\ \alpha^{*\mathsf{e}}(\mathsf{i}_1^{\mathsf{e}}(h_1^{\mathsf{e}}(e))_{e_1',e_2'}) = \mathsf{i}_1^{\mathsf{e}}(h_1^{\mathsf{e}}(e)); \end{cases}$$

- $D^* = D$.

It is worthwhile to notice that operation edges mapped to a shared constructor are paired only with the operation edges of the other graph mapped to that shared constructor. Only the operation edges that are mapped to unshared constructors are paired with the 'mute' edges of the other graph.

Observe also that if (Σ_1, I_1) and (Σ_2, I_2) are both finite then so is their (unconstrained or constrained) fibring.

Contrarily to previous approaches to fibring, the graph-theoretic account provides very simple notions of unconstrained and constrained fibring of interpretation systems, once these notions are established for interpretation structures. The *unconstrained fibring* of interpretation systems $(\Sigma_1, \mathfrak{I}_1)$ and $(\Sigma_2, \mathfrak{I}_2)$, denoted by

$$(\Sigma_1, \mathfrak{I}_1) \uplus (\Sigma_2, \mathfrak{I}_2),$$

is the interpretation system $(\Sigma_1 \uplus \Sigma_2, \mathfrak{I}_1 \uplus \mathfrak{I}_2)$ where:

$$\mathfrak{I}_1 \uplus \mathfrak{I}_2 = \{I_1 \uplus I_2 : I_1 \in \mathfrak{I}_1, I_2 \in \mathfrak{I}_2\}.$$

The *constrained fibring* of $(\Sigma_1, \mathfrak{I}_1)$ and $(\Sigma_2, \mathfrak{I}_2)$ *sharing* Σ_0 *via* h_1 and h_2, denoted by

$$(\Sigma_1, \mathfrak{I}_1) \uplus_{h_1 h_2}^{\Sigma_0} (\Sigma_2, \mathfrak{I}_2),$$

is the interpretation system $(\Sigma_1 \uplus_{h_1 h_2}^{\Sigma_0} \Sigma_2, \mathfrak{I}_1 \uplus_{h_1 h_2}^{\Sigma_0} \mathfrak{I}_2)$ where:

$$\mathfrak{I}_1 \uplus_{h_1 h_2}^{\Sigma_0} \mathfrak{I}_2 = \{I_1 \uplus_{h_1 h_2}^{\Sigma_0} I_2 : I_1 \in \mathfrak{I}_1, I_2 \in \mathfrak{I}_2\}.$$

3 Finite model property

An interpretation system $(\Sigma, \mathfrak{I})$ is said to have the *finite model property*, abbreviated by fmp, if for every formula φ falsifiable in $(\Sigma, \mathfrak{I})$ there is a finite interpretation J in $\mathfrak{I}$ such that J falsifies φ. The next section is dedicated to proving a sufficient condition for preservation of the fmp by fibring. Here, we introduce a couple of useful notions and provide graph-theoretic accounts of logics with fmp that are used in the sequel for illustration purposes.

An interpretation system $(\Sigma, \mathfrak{I})$ is said to *have strong negation* if there is either a primitive or a derived unary constructor $\neg : \pi \to \pi$ such that, for every interpretation structure I in $\mathfrak{I}$:

- $I, \mathsf{W}^I_{s\varphi} \Vdash_\exists \varphi$ iff $I, \mathsf{W}^I_{s\neg\varphi} \nVdash \neg(\varphi)$ whenever φ is a schema formula and s is a socket for it;

- $I, \mathsf{W}^I_\varphi \Vdash_\exists \varphi$ iff $I, \mathsf{W}^I_{\neg\varphi} \not\Vdash \neg(\varphi)$ whenever φ is a concrete formula.

An interpretation system $(\Sigma, \mathfrak{I})$ is said to *have disjunction* if there is either a primitive or a derived binary constructor $\vee : \pi\pi \to \pi$ such that

$$I \not\Vdash \vee(\varphi_1, \varphi_2) \text{ iff } I \not\Vdash \varphi_1 \text{ and } I \not\Vdash \varphi_2$$

for every interpretation structure I in $\mathfrak{I}$. The following result is used later on.

Proposition 3.1 An interpretation system $(\Sigma, \mathfrak{I})$ with disjunction has the fmp iff, for any finite set of formulas $\{\varphi_1, \ldots, \varphi_n\}$, if there is I in $\mathfrak{I}$ such that $I \not\Vdash \varphi_i$ for $i = 1, \ldots, n$ then there is a finite J in $\mathfrak{I}$ such that $J \not\Vdash \varphi_i$ for $i = 1, \ldots, n$.

3.1 Modal logic

The first example concerns normal modal logic (for details see [10] and [6]) endowed with semantics given by modal algebras. Let Π_1 be a set of propositional symbols. Consider the *modal signature* Σ_1 with the following m-edges:

- $p_1 : \diamond \to \pi$ for each p_1 in Π_1;
- $\neg_1 : \pi \to \pi$;
- $\supset_1 : \pi\pi \to \pi$;
- $\Box_1 : \pi \to \pi$.

Let $\mathcal{A} = (A, \sqcap, \sqcup, -, \bot, \top, \Box)$ be a modal algebra (see [20, 21] and also [10] for a more recent presentation and completeness proofs) for a normal modal logic ML, either K or an enrichment of K, with signature Σ_1, and v a valuation over the algebra (that is, a map from Π_1 to A). The (graph-theoretic) interpretation structure $I_1(\mathcal{A}, v) = (G'_1, \alpha_1, D_1)$ over Σ_1 induced by $\mathcal{A}$ and v is as follows:

- G' is such that:
  - $V'_1 = A \cup \{\blacklozenge\}$;
  - E'_1 is composed by the following edges:
    * $p'_1 : \blacklozenge \to v(p_1)$ for $p_1 \in \Pi_1$;
    * $\neg_{1a} : a \to -a$ for each a in A;
    * $\supset_{1a_1a_2} : a_1\, a_2 \to ((-a_1) \sqcup a_2)$ for each a_1 and a_2 in A;
    * $\Box_{1a} : a \to \Box a$ for each a in A.
- $\alpha : G' \to G$ is such that:
  - $\alpha_1^{\mathsf{v}}(a) = \pi$ for each $a \in A$;
  - $\alpha_1^{\mathsf{v}}(\blacklozenge) = \diamond$;
  - $\alpha_1^{\mathsf{e}}(p'_1) = p_1$ for each $p_1 \in \Pi_1$;
  - $\alpha_1^{\mathsf{e}}(\neg_a) = \neg_1$;
  - $\alpha_1^{\mathsf{e}}(\supset_{a_1a_2}) = \supset_1$;

- $\alpha_1^{\mathsf{e}}(\Box_a) = \Box_1$.

- $D_1 = \{\top\}$.

It is not hard to prove the following result, where by $\mathcal{A}, v \Vdash \varphi$ we mean that the modal algebra $\mathcal{A}$ and the valuation v satisfy φ.

Proposition 3.2 $\mathcal{A}, v \Vdash \varphi$ iff $I_1(\mathcal{A}, v) \Vdash \varphi$.

The (graph-theoretic) interpretation system for modal logic ML is the pair composed by the signature Σ_1 and the class $\mathfrak{I}_1$ of the interpretation structures induced by the modal algebras for ML and the valuations over them. It is straightforward to verify that the following results hold.

Proposition 3.3 Let φ be a modal formula. Then, φ is a theorem of ML iff $I \Vdash \varphi$ for every $I \in \mathfrak{I}_1$.

Proof: Since every normal modal logic is complete with respect to the algebraic semantics (see Theorems 7.2 and 7.43 of [10]) we conclude that $I \Vdash \varphi$ for every $I \in \mathfrak{I}_1$. QED

Proposition 3.4 If modal logic ML has the fmp with respect to the algebraic semantics then $(\Sigma_1, \mathfrak{I}_1)$ also has the fmp.

It should be stressed that a normal modal logic has the fmp with respect to the algebraic semantics iff it has the fmp with respect to Kripke frames (the interested reader can see Theorem 17 of [20] where the correspondence between finite Kripke frames and finite modal algebras is established). Thus, the result also holds when we consider modal logic ML endowed with Kripke semantics.

These results are easily extended to multimodal logics. Later on, we shall need to work with bimodal logics for dealing with fusions of (mono)modal logics.

3.2 Paraconsistent logic

Consider now the paraconsistent logic mbC (see [9]). Let Π_2 be a set of propositional symbols. The mbC *signature* Σ_2 has the following m-edges[2]:

- $p_2 : \diamond \to \pi$ for each p_2 in Π_2;

- $\neg_2, \odot_2 : \pi \to \pi$;

- $\supset_2, \wedge_2, \vee_2 : \pi\pi \to \pi$.

Let $\mathcal{M} = (T, D, O)$ be the non deterministic matrix for mbC, where $T = \{\mathbf{t}, \mathbf{I}, \mathbf{f}\}$, $D = \{\mathbf{t}, \mathbf{I}\}$ and O is the set of three-valued truth-tables for the connectives (for details see [4]). We refrain from giving these truth-tables, but the reader will be able to recover them from the graph-theoretic denotation of the connectives provided below. Let $v : \Pi_2 \to T$ be a map. The interpretation structure $I_2(v) = (G_2', \alpha_2, D_2)$ over Σ_2 induced by v is as follows:

[2] In order to avoid confusion with composition, we use $\odot$ for the consistency operator.

- G_2' is such that:
  - $V' = \{\mathbf{t}, \mathbf{I}, \mathbf{f}\} \cup \{\blacklozenge\}$;
  - E' is composed of the following m-edges:
    * $p_2' : \blacklozenge \to v(p_2)$ for $p_2 \in \Pi_2$;
    * $\neg_{u_1'u_2'} : u_1' \to u_2'$ where u_1' is in $\{\mathbf{I}, \mathbf{f}\}$ and u_2' is in D_2;
    * $\neg_{\mathbf{tf}} : \mathbf{t} \to \mathbf{f}$;
    * $\odot_{u_1'u_2'} : u_1' \to u_2'$ where u_1' is in $\{\mathbf{t}, \mathbf{f}\}$ and u_2' is in V_π';
    * $\odot_{\mathbf{If}} : \mathbf{I} \to \mathbf{f}$;
    * $\supset_{u_1'u_2'v'} : u_1'\, u_2' \to v'$ where u_1' is $\mathbf{f}$ or u_2' is in D_2, and v' is in D_2;
    * $\supset_{v'\mathbf{ff}} : v'\, \mathbf{f} \to \mathbf{f}$ for v' is in D_2;
    * $\wedge_{u_1'u_2'v'} : u_1'\, u_2' \to v'$ where u_1', u_2' and v' are in D_2;
    * $\wedge_{u_1'u_2'\mathbf{f}} : u_1'\, u_2' \to \mathbf{f}$ where u_1' is $\mathbf{f}$ or u_2' is $\mathbf{f}$;
    * $\vee_{u_1'u_2'v'} : u_1'\, u_2' \to v'$ where u_1' or u_2' are in D_2, and v' is in D_2;
    * $\vee_{\mathbf{fff}} : \mathbf{f}\,\mathbf{f} \to \mathbf{f}$.
- $\alpha_2 : G' \to G$ is such that:
  - $\alpha_2^{\mathsf{v}}(u') = \pi$ with u' in $\{\mathbf{t}, \mathbf{I}, \mathbf{f}\}$;
  - $\alpha_2^{\mathsf{v}}(\blacklozenge) = \diamond$;
  - $\alpha_2^{\mathsf{e}}(p_2') = p_2$ for each $p_2 \in \Pi_2$;
  - $\alpha_2^{\mathsf{e}}(\neg_{u_1'u_2'}) = \neg_2$ for every $\neg_{u_1'u_2'}$ in E';
  - $\alpha_2^{\mathsf{e}}(\odot_{u_1'u_2'}) = \odot_2$ for every $\odot_{u_1'u_2'}$ in E';
  - $\alpha_2^{\mathsf{e}}(\supset_{u_1'u_2'b}) = \supset_2$ for every $\supset_{u_1'u_2'b}$ in E';
  - $\alpha_2^{\mathsf{e}}(\wedge_{u_1'u_2'b}) = \wedge_2$ for every $\wedge_{u_1'u_2'b}$ in E';
  - $\alpha_2^{\mathsf{e}}(\vee_{u_1'u_2'b}) = \vee_2$ for every $\vee_{u_1'u_2'b}$ in E'.
- $D_2 = \{\mathbf{t}, \mathbf{I}\}$.

The interpretation system for paraconsistent logic mbC is the pair composed by the signature Σ_2 and the class $\mathfrak{I}_2$ of all interpretation structures $I_2(v)$ for every map $v : \Pi_2 \to T$.

Observe that in this case, it is possible to have in $I_2(v)$ more than one path mapped by α_2 into the same formula. For instance, assuming that $p_2' : \blacklozenge \to \mathbf{I}$, the paths $\neg_{\mathbf{II}}\, p_2'$ and $\neg_{\mathbf{It}}\, p_2'$ are both denotations of $\neg\, p_2$.

In order to prove that mbC is weakly sound and complete for $(\Sigma_2, \mathfrak{I}_2)$, we need to state a previous result. Let $v : L(\Sigma_2) \to T$ be a legal valuation over $\mathcal{M}$ (see [4]). We will denote by $\bar{v} : \Pi_2 \to T$ the restriction of v to the set Π_2. By the very definition of $(\Sigma_2, \mathfrak{I}_2)$ and $\mathcal{M}$, it is not hard to prove the following result:

Proposition 3.5
(i) For every map $v' : \Pi_2 \to T$, every formula $\varphi \in L(\Sigma_2)$ and every $d \in [\![\varphi]\!]^{I_2(v')}$, there exists a legal valuation $v : L(\Sigma_2) \to T$ over $\mathcal{M}$ such that $\bar{v} = v'$ and $v(\varphi) = d$.
(ii) For every legal valuation $v : L(\Sigma_2) \to T$ over $\mathcal{M}$ and every formula $\varphi \in L(\Sigma_2)$, $v(\varphi) \in [\![\varphi]\!]^{I_2(\bar{v})}$.

Using this and the soundness and completeness of mbC with respect to the non deterministic semantics $\mathcal{M}$ we can prove that:

Proposition 3.6 φ is a theorem of mbC iff φ is valid in $\mathfrak{I}_2$.

Proof: Suppose that φ is not valid in $\mathfrak{I}_2$. Then, $I_2(v') \not\Vdash \varphi$ for some map $v' : \Pi_2 \to T$. That is, there exists $d \in [\![\varphi]\!]^{I_2(v')}$ such that $d \notin D_2$. Thus, by Proposition 3.5(i), there exists a legal valuation v over $\mathcal{M}$ such that $v(\varphi) = d$, that is, such that $v(\varphi) \notin D_2$, and so φ is not valid in $\mathcal{M}$. Since mbC is sound with respect to the non deterministic semantics $\mathcal{M}$, it follows that φ is not a theorem of mbC.

Conversely, suppose that φ is not a theorem of mbC. Then, by completeness of mbC with respect to the non deterministic semantics $\mathcal{M}$, there is some legal valuation v such that $v(\varphi) \notin D_2$. Since $v(\varphi) \in [\![\varphi]\!]^{I_2(\bar{v})}$ (by Proposition 3.5(ii)) it follows that $I_2(\bar{v}) \not\Vdash \varphi$ and so φ is not valid in $\mathfrak{I}_2$. QED

Proposition 3.7 The interpretation system $(\Sigma_2, \mathfrak{I}_2)$ has the fmp.

The result above is an immediate consequence of the fact that mbC has the fmp with respect to the three-valued nondeterministic semantics. This fact is an immediate corollary of the lemma of absent propositional symbols that is known to hold in mbC (see [4]).

3.3 Deontic logic

Consider now the deontic logics DPM (see [15]). These are in fact two logics DPM.1 and DPM.2 having the same kind of neighborhood semantics but each with specific properties. From this point on, we use DPM to refer to any of them because they behave in an analogous way for the properties we have in mind.

Let $\mathcal{F} = (W, O)$ be a neighborhood frame for DPM where W is a non-empty set and $O = \{O_w\}_{w \in W}$ is a family where $O_w \subseteq \wp W$. Let v be a valuation in $\mathcal{F}$, that is, a map from Π_3 to $\wp W$. The interpretation structure $I_3(\mathcal{F}, v) = (G'_3, \alpha_3, D_3)$ over Σ_1 induced by v is as follows:

- G'_3 is such that:
  - $V' = \wp W$;
  - E' is composed of the following m-edges:
    - $p'_3 : \blacklozenge \to v(p_3)$ for $p_3 \in \Pi_3$;
    - $\neg_U : U \to W \setminus U$ for $U \in \wp W$;
    - $\supset_{U_1 U_2} : U_1 U_2 \to U$ where $U = (W \setminus U_1) \cup U_2$ for $U_1, U_2 \in \wp W$;
    - $\Box_{U_1} : U_1 \to U_2$ where $U_2 = \{w \in W : U_1 \in O_w\}$ for $U_1 \in \wp W$.
- $\alpha_3 : G' \to G$ is such that:
  - $\alpha_3^{\mathsf{v}}(U) = \pi$ with $U \in \wp W$;
  - $\alpha_3^{\mathsf{v}}(\blacklozenge) = \diamond$;

- $\alpha_3^{\mathsf{e}}(p_3') = p_3$ for each $p_3 \in \Pi_3$;
  - $\alpha_3^{\mathsf{e}}(\neg_U) = \neg_1$ for every $\neg_U$ in E';
  - $\alpha_3^{\mathsf{e}}(\supset_{U_1 U_2}) = \supset_1$ for every $\supset_{U_1 U_2}$ in E';
  - $\alpha_3^{\mathsf{e}}(\Box_U) = \Box_1$ for every $\Box_U$ in E'.

- $D_3 = W$.

It is not hard to prove the following result, where by $\mathcal{F}, v \Vdash \varphi$ we mean that the denotation $[\![\varphi]\!]_{(\mathcal{F},v)}$ of φ is W.

Proposition 3.8 $\mathcal{F}, v \Vdash \varphi$ iff $I_3(\mathcal{F}, v) \Vdash \varphi$.

The (graph-theoretic) interpretation system for deontic logics DPM is the pair composed by the signature Σ_3 and the class $\mathfrak{I}_3$ of the interpretation structures induced by the neighborhood frames for DPM and the valuations over them. Using the completeness of DPM with respect to the neighborhood semantics $\mathcal{F}$, the following result holds:

Proposition 3.9 Formula φ is a theorem of DPM iff $I \Vdash \varphi$ for every $I \in \mathfrak{I}_3$.

The logics DPM have the fmp with respect to the neighborhood semantics (see, for instance, [15]). Then, the following result holds:

Proposition 3.10 The interpretation system $(\Sigma_3, \mathfrak{I}_3)$ also has the fmp.

3.4 Fragment of linear logic

Consider now the HL fragment (see [3]) of linear logic (introduced in [14]), including as primitives only the additive conjunction, the multiplicative negation, the multiplicative implication and the multiplicative disjunction.

Let Π_4 be a set of propositional symbols. The HL *signature* Σ_4 is a m-graph with sorts π and $\diamond$ and the following m-edges:

- $p_4 : \diamond \to \pi$ for each p_4 in Π_4;

- $\neg_4 : \pi \to \pi$;

- $\supset_4, \wedge_4, +_4 : \pi\pi \to \pi$.

Let $\mathcal{A} = (A, \sqcap, \sqcup, -, +, 0)$ be a *-autonomous lattice (see [27, 28]) and let v be a valuation over the algebra (that is, a map from Π_4 to A). That is, $\mathcal{A}$ is such that:

- $(A, +, 0)$ is an Abelian monoid;

- $(A, -, \sqcap, \sqcup)$ is an involutive lattice;

- $-0 \leq -a + b$ iff $a \leq b$;

where $\leq$ is the order induced by the lattice.

The interpretation structure $I_4(\mathcal{A}, v) = (G'_4, \alpha_4, D_4)$ over Σ_4 induced by $\mathcal{A}$ and v is as follows:

- G' is such that:
  - $V'_4 = A \cup \{\blacklozenge\}$;
  - E'_4 is composed by the following edges:
    - $p'_4 : \blacklozenge \to v(p_4)$ for $p_4 \in \Pi_4$;
    - $\neg_a : a \to -a$ for each a in A;
    - $\supset_{a_1 a_2} : a_1\, a_2 \to ((-a_1) + a_2)$ for each a_1 and a_2 in A;
    - $\wedge_{a_1 a_2} : a_1 a_2 \to a_1 \sqcap a_2$ for each a_1, a_2 in A;
    - $+_{a_1 a_2} : a_1 + a_2$ for each a_1, a_2 in A.
- $\alpha_4 : G' \to G$ is such that:
  - $\alpha_4^{\mathsf{v}}(a) = \pi_4$ for each $a \in A$;
  - $\alpha_4^{\mathsf{v}}(\blacklozenge) = \diamond$;
  - $\alpha_4^{\mathsf{e}}(p'_4) = p_1$ for each $p_4 \in \Pi_4$;
  - $\alpha_4^{\mathsf{e}}(\neg_a) = \neg_4$;
  - $\alpha_4^{\mathsf{e}}(\supset_{a_1 a_2}) = \supset_4$;
  - $\alpha_4^{\mathsf{e}}(\wedge_{a_1 a_2}) = \wedge_4$;
  - $\alpha_4^{\mathsf{e}}(+_{a_1 a_2}) = +_4$.
- $D_4 = \{\top\}$.

It is straightforward to verify that the following results hold where by $\mathcal{A}, v \Vdash \varphi$ we mean that the *-autonomous algebra and the valuation satisfy φ.

Proposition 3.11 $\mathcal{A}, v \Vdash \varphi$ iff $I_4(\mathcal{A}, v) \Vdash \varphi$.

The (graph-theoretic) interpretation system for the HL logic is the pair composed by the signature Σ_4 and the class $\mathfrak{I}_4$ of the interpretation structures induced by the *-autonomous algebras for HL and the valuations over them. Hence, using the completeness of HL as proved in [3] with respect to the algebraic semantics it follows:

Proposition 3.12 Formula φ is a theorem of HL iff $I \Vdash \varphi$ for any $I \in \mathfrak{I}_4$.

The logic HL has the fmp with respect to the phase semantics [19]. Since, HL is complete with respect to both phase and $*$-autonomous lattice semantics (see [3]) we conclude that HL has the finite model property with respect to $*$-autonomous lattice semantics.

Proposition 3.13 The interpretation system $(\Sigma_4, \mathfrak{I}_4)$ has the fmp.

4 Preservation of fmp

The problem can be stated as follows. Assume that we have two interpretation systems $(\Sigma_1, \mathfrak{I}_1)$ and $(\Sigma_2, \mathfrak{I}_2)$ that have the fmp. We will establish a sufficient condition for the interpretation system $(\Sigma_1 \uplus, \Sigma_1, I_1 \uplus I_2)$ to have the fmp. We need some preliminary notions and results.

We start by defining an encoding of each formula in $\Sigma_1 \uplus \Sigma_2$ in either a formula in Σ_1 or a formula in Σ_2 using projections. The *encoding map*

$$\tau_1 : L(\Sigma_1 \uplus \Sigma_2) \to L(\Sigma_1)$$

is inductively as follows:

- $\tau_1(\varphi) = \mathsf{id}^\pi : \pi \to \pi$ for either $\varphi = c \in E_2$ or $\varphi = c \circ \langle \varphi_1, \ldots, \varphi_n \rangle$ with $c \in E_2$;
- $\tau_1(q) = q : \diamond \to \pi$ for every $q : \diamond \to \pi \in E_1$;
- $\tau_1(\mathsf{p}_k^u) = \mathsf{p}_k^u : u \to u_k$;
- $\tau_1(\varphi) = c \circ \langle \tau_1(\varphi_1) \circ \mathsf{p}_{u^1}^{u^1 \ldots u^n}, \ldots, \tau_1(\varphi_n) \circ \mathsf{p}_{u^n}^{u^1 \ldots u^n} \rangle : u^1 \ldots u^n \to \pi$ for $\varphi = c \circ \langle \varphi_1, \ldots, \varphi_n \rangle$, assuming that:
  - $c : \pi^n \to \pi \in E_1$;
  - u^j is a sequence of sorts for $j = 1, \ldots, n$;
  - $\tau_1(\varphi_j) : u^j \to \pi$ for $j = 1, \ldots, n$.

The encoding map $\tau_2 : L(\Sigma_1 \uplus \Sigma_2) \to L(\Sigma_2)$ is defined in a similar way.

Example 4.1 Let γ be the formula

$$\neg_1 \circ \vee_2 \circ \langle q_1, \neg_2 \circ q_2 \rangle : \diamond^2 \to \pi$$

in $\Sigma_1 \uplus \Sigma_2$. Then

- $\tau_1(\gamma) = \neg_1 \circ \mathsf{id}_\pi : \pi \to \pi$;
- $\tau_2(\vee_2(q_1, \neg_2(q_2))) = \vee_2 \circ \langle \mathsf{p}_1^{\pi\diamond}, \neg_2 \circ q_2 \circ \mathsf{p}_2^{\pi\diamond} \rangle : \diamond\pi \to \pi$. ▷

Let

$$\mathrm{SF}_i(\varphi)$$

be the set of all subformulas of φ whose main constructor is in Σ_i. Then $\mathrm{SF}(\varphi) = \mathrm{SF}_1(\varphi) \cup \mathrm{SF}_2(\varphi)$. For each $\psi \in \mathrm{SF}(\varphi)$ and $i = 1, 2$, we denote by ψ^i the formula $\tau_i(\psi)$.

The next result says that from a path for a formula in $\Sigma_1 \uplus \Sigma_2$ it is possible to extract a path for the encoding in Σ_i of each of its subformulas preserving the satisfaction of the subformula, for $i = 1, 2$.

Lemma 4.2 Let $\varphi \in L(\Sigma_1 \uplus \Sigma_2)$, $I_1 \in \mathfrak{I}_1$, $I_2 \in \mathfrak{I}_2$ and w a path in $I_1 \uplus I_2$ such that $\alpha(w) = \varphi$. Then for each $\psi \in \mathrm{SF}_i(\varphi)$ and $i = 1, 2$ there a path w_ψ^i in I_i such that:

- $\alpha_i(w^i_\psi) = \psi^i$;
- $\mathsf{trg}(w^1_\psi) \in D_1$ iff $\mathsf{trg}(w^2_\psi) \in D_2$;
- $I_1 \uplus I_2, w_{|_\psi} \Vdash \psi \quad \text{iff} \quad I_1, w^1_\psi \Vdash \psi^1,\ I_2, w^2_\psi \Vdash \psi^2$;
- If $\mathsf{trg}(w_{|_\psi}) = (a_1, a_2)$ then $\mathsf{trg}_1(w^1_\psi) = a_1$ and $\mathsf{trg}_2(w^2_\psi) = a_2$.

Proof:
Let ψ be a subformula of φ. We prove the result by induction on the structure of ψ.
Base case: ψ is $e : \diamond \to \pi \in E_1$.
Then ψ^1 is e and ψ^2 is id_π. Assume that $w_{|_e} : \blacklozenge \to (a_1, a_2)$. Take $w^1_\psi : \blacklozenge \to a_1$ and $w^2_\psi = \mathsf{id}_{a_2}$. Then

$$a_1 \in D_1 \quad \text{iff} \quad a_2 \in D_2$$

and

$$\begin{array}{rll} I_1 \uplus I_2, w_{|_e} \Vdash e & \text{iff} & (a_1, a_2) \in D \\ & \text{iff} & a_1 \in D_1 \quad \text{and} \quad a_2 \in D_2 \\ & \text{iff} & I_1, w^1 \Vdash e \quad \text{and} \quad I_2, w^2 \Vdash \mathsf{p}^\pi_1. \end{array}$$

Induction step: ψ is $e \circ \langle \psi_1, \ldots, \psi_k \rangle$.
Then, by the induction hypothesis, there are paths $w^1_{\psi_j}$ and $w^2_{\psi_j}$ such that, for every $j = 1, \ldots, k$:

- $\alpha_1(w^1_{\psi_j}) = \psi^1_j$ and $\alpha_2(w^2_{\psi_j}) = \psi^2_j$;
- $I_1 \uplus I_2, w_{|_{\psi_j}} \Vdash \psi_j \quad \text{iff} \quad I_1, w^1_{\psi_j} \Vdash \psi^1_j,\ I_2, w^2_{\psi_j} \Vdash \psi^2_j$;
- if $\mathsf{trg}(\psi_j) = (a_{1j}, a_{2j})$ then $\mathsf{trg}_1(w^1_{\psi_j}) = a_{1j}$ and $\mathsf{trg}_2(w^2_{\psi_j}) = a_{2j}$.

Hence,

- $I_1 \uplus I_2, w_{|_{\psi_j}} \nVdash \psi_j \quad \text{iff} \quad I_1, w^1_{\psi_j} \nVdash \psi^1_j,\ I_2, w^2_{\psi_j} \nVdash \psi^2_j$.

Assume that $w_{|_\psi} : \blacklozenge \to (a_1, a_2)$. Then edge e' such that $\alpha(e') = e$ is $e' : (a_{11}, a_{21}) \ldots (a_{1n}, a_{2n}) \to (a_1, a_2)$. Assuming that $e \in E_1$ then there is $e'_1 : a_{11} \ldots a_{1n} \to a_1$ in I_1 such that $\alpha_1(e'_1) = e$. Take

$$w^1_\psi = e'_1 \circ \langle w^1_{\psi_1} \circ \mathsf{p}^{b^1 \ldots b^k}_{b_1}, \ldots, w^1_{\psi_k} \circ \mathsf{p}^{b_1 \ldots b_k}_{b_k} \rangle : b^1 \ldots b^k \to a_1$$

assuming that $\mathsf{src}(w^1_{\psi_j}) = b^j$ for $j = 1, \ldots, k$. On the other hand, $w^2_\psi = \mathsf{id}_{a_2}$. Then

$$a_1 \in D_1 \quad \text{iff} \quad a_2 \in D_2$$

and

$$\begin{array}{rll} I_1 \uplus I_2, w_{|\psi} \Vdash \psi & \text{iff} & (a_1, a_2) \in D \\ & \text{iff} & a_1 \in D_1 \quad \text{and} \quad a_2 \in D_2 \\ & \text{iff} & I_1, w^1_\psi \Vdash \psi^1 \quad \text{and} \quad I_2, w^2_\psi \Vdash \psi^2. \end{array}$$

Moreover,

$$\begin{aligned}
\alpha_1(w^1) &= e \circ \langle \alpha_1(w^1_{\psi_1} \circ \mathsf{p}_{b_1}^{b^1 \ldots b^k}), \ldots, \alpha_1(w^1_{\psi_k} \circ \mathsf{p}_{b_k}^{b_1 \ldots b_k})\rangle \\
&= e \circ \langle \alpha_1(w^1_{\psi_1}) \circ \alpha_1(\mathsf{p}_{b_1}^{b^1 \ldots b^k}), \ldots, \alpha_1(w^1_{\psi_k}) \circ \alpha_1(\mathsf{p}_{b_k}^{b_1 \ldots b_k})\rangle \\
&= e \circ \langle \psi^1_1 \circ \mathsf{p}_{u_1}^{u^1 \ldots u^k}, \ldots, \psi^1_k \circ \mathsf{p}_{u_k}^{u_1 \ldots u_k}\rangle \\
&= \tau_1(\psi) = \psi^1.
\end{aligned}$$

The proof is similar when $e \in E_2$. QED

Given a formula $\varphi \in L(\Sigma_1 \uplus \Sigma_2)$ with main constructor in E_1, the sequence

$$\{\delta_k\}_{k=1,\ldots,m}$$

of its *decodings* is inductively defined as follows:

- $\delta_0 = \varphi^1$;
- $\delta_{k+1} = \begin{cases} \delta_k \circ \langle \mathsf{id}_{u_1}, \ldots, \mathsf{id}_{u_{\ell_1 - 1}}, \psi^2_{\ell_1}, \ldots, \psi^2_{\ell_j}, \mathsf{id}_{u_{\ell_j + 1}}, \ldots, \mathsf{id}_{u_{|u|}}\rangle & \text{if } k \text{ even} \\ \delta_k \circ \langle \mathsf{id}_{u_1}, \ldots, \mathsf{id}_{u_{\ell_1 - 1}}, \psi^1_{\ell_1}, \ldots, \psi^1_{\ell_j}, \mathsf{id}_{u_{\ell_j + 1}}, \ldots, \mathsf{id}_{u_{|u|}}\rangle & \text{otherwise} \end{cases}$

  assuming that

  - $\delta_k : u \to \pi$;
  - $\mathsf{p}^u_{\ell_i}$ for $i = 1, \ldots, j$ are the projections in δ_k with target π;
  - there exists a subformula γ_i of φ, for $i = 1, \ldots, j$, such that:
    * $\tau_1(\gamma_i) = \mathsf{p}^u_{\ell_i}$ and $\tau_2(\gamma_i) = \psi^2_{\ell_i}$, for k even;
    * $\tau_2(\gamma_i) = \mathsf{p}^u_{\ell_i}$ and $\tau_2(\gamma_i) = \psi^1_{\ell_i}$, for k odd.

We say that δ_{k+1} is the *direct decoding* of δ_k via $\psi^1_{\ell_1}, \ldots, \psi^1_{\ell_j}$. Observe that each decoding is a formula in $\Sigma_1 \uplus \Sigma_2$ and, moreover,

$$\delta_{k+1} : [u]^{u_{\ell_1} \ldots u_{\ell_j}}_{\mathsf{src}(\psi^m_{\ell_1}) \ldots \mathsf{src}(\psi^m_{\ell_j})} \to \pi.$$

Example 4.3 Consider formula $\neg_1 \circ \vee_2 \circ \langle q_1, \neg_1 \circ q_2\rangle : \diamond^2 \to \pi$ in $\Sigma_1 \uplus \Sigma_2$. Then

- δ_0 is $\neg_1 : \pi \to \pi$;
- δ_1 is $\neg_1 \circ \vee_2 \circ \langle \mathsf{p}_1^{\pi\pi}, \mathsf{p}_2^{\pi\pi}\rangle : \pi\pi \to \pi$;
- δ_2 is $\neg_1 \circ \vee_2 \circ \langle q_1 \circ \mathsf{p}_1^{\diamond\pi}, \neg_1 \circ \mathsf{p}_2^{\diamond\pi}\rangle : \diamond\pi \to \pi$;
- δ_3 is $\neg_1 \circ \vee_2 \circ \langle q_1, \neg_1 \circ q_2\rangle : \diamond\diamond \to \pi$. ▷

We need some notation. Given a path w^1 in I_1 we denote by

$$w^1?$$

the quasi-path in $I_1 \uplus I_2$ obtained from w^1 as follows:

- for each edge $e : u_1 \ldots u_k \to u$ in the operation graph of I_1 we add an edge $e : (u_1, ?_1) \ldots (u_k, ?_k) \to (u, ?)$ in $w^1?$ such that

  each $?_i$ has to be a distinguished truth value whenever u_i is distinguished and not distinguished otherwise,

  ? has to be a distinguished truth value whenever u is distinguished and not distinguished otherwise,

  and each $?_i$ has to be the concrete value whenever u_i is the concrete value;

- for each projection $\mathsf{p}_j^{u_1 \ldots u_n} : u_1 \ldots u_n \to u_j$ in I_1 we add the morphism $\mathsf{p}_j^{u_1 \ldots u_n} : (u_1, ?_1) \ldots (u_n, ?_n) \to (u_j, ?_j)$.

It is a quasi-path since the sources and the targets of the morphisms are not fully defined. Analogously, the quasi-path $?w^2$ in $I_1 \uplus I_2$ is defined from a path w^2 in I_2.

The following result is somehow the reverse of Lemma 4.2. It states how to get a path for a formula out of the paths of the encodings of its subformulas.

Lemma 4.4 Let $\varphi \in L(\Sigma_1 \uplus \Sigma_2)$, $I_1 \in \mathfrak{I}_1$, $I_2 \in \mathfrak{I}_2$. Assume that for each subformula ψ of φ in $\mathrm{SF}_i(\varphi)$:

- there is a path w'_{ψ^i} in I_i for $i = 1, 2$;
- $\mathsf{trg}_1(w'_{\psi^1}) \in D_1$ iff $\mathsf{trg}_2(w'_{\psi^2}) \in D_2$.

Then there is a path w'_φ in $I_1 \uplus I_2$ such that $\alpha(w'_\varphi) = \varphi$ and

$$I_1 \uplus I_2, w'_\varphi \Vdash \varphi \quad \text{iff} \quad I_i, w'_{\varphi^i} \Vdash \varphi^i$$

where $i = 1$ if the main constructor of φ is in E_1 and $i = 2$ otherwise.

Proof:
Assume without loss of generality that the main constructor of φ is in E_1. Let $\delta_0, \ldots, \delta_m$ be the sequence of decodings of φ. Consider the sequence of quasi-paths

$$\{qw'_{\delta_k}\}_{k=1,\ldots,m}$$

in $I_1 \uplus I_2$ inductively defined as follows:

- $qw'_{\delta_0} = w'_{\varphi^1}?$;
- $qw'_{\delta_{k+1}} = qw'_{\delta_k} \circ$
$$\begin{cases} \langle \mathsf{id}_{u'_1}, \ldots, \mathsf{id}_{u'_{\ell_1 - 1}}, (?qw'_{\psi^2_{\ell_1}})^+, \ldots, (?qw'_{\psi^2_{\ell_j}})^+, \mathsf{id}_{u_{\ell_j+1}}, \ldots, \mathsf{id}_{u_{|u|}} \rangle & k \text{ even} \\ \langle \mathsf{id}_{u_1}, \ldots, \mathsf{id}_{u_{\ell_1 - 1}}, (qw'_{\psi^1_{\ell_1}}?)^+, \ldots, (qw'_{\psi^1_{\ell_j}}?)^+, \mathsf{id}_{u_{\ell_j+1}}, \ldots \mathsf{id}_{u_{|u|}} \rangle & k \text{ odd} \end{cases}$$

  assuming that

  - $qw'_{\delta_k} : u' \to v'$;

– $\mathsf{p}^{u'}_{\ell_i}$ for $i = 1, \dots, j$ are the projections in qw'_{δ_k};
– there exists a subformula ψ_i of φ, for $i = 1, \dots, j$, such that:
  * $\alpha_1(\mathsf{p}^{u'}_{\ell_i}) = \psi^1_i$, for k even;
  * $\alpha_2(\mathsf{p}^{u'}_{\ell_i}) = \psi^2_i$, for k odd;
– $(?qw'_{\psi^2_{\ell_i}})^+$ is the quasi-path

$$\left[?qw'_{\psi^2_{\ell_j}}\right]^{\mathsf{trg}_2(?qw'_{\psi^2_{\ell_j}})\,\mathsf{src}_1(\mathsf{p}^{u'}_{\ell_i})}_{(\mathsf{trg}_1(\mathsf{p}^{u'}_{\ell_i}),\mathsf{trg}_2(qw'_{\psi^2_{\ell_i}}))\,(\mathsf{trg}_1(\mathsf{p}^{u'}_{\ell_i}),\mathsf{trg}_2(qw'_{\psi^2_{\ell_i}}))}$$

and similarly for $(qw'_{\psi^1_{\ell_1}}?)^+$.

Note that qw'_{δ_k} is a quasi-path since the definition of the vertexes is yet to be completed. Moreover, $(?qw'_{\psi^2_{\ell_i}})^+$ is well defined since, by hypothesis, $\mathsf{trg}_1(\mathsf{p}^{u'}_{\ell_i}) \in D_1$ iff $\mathsf{trg}_2(w'_{\psi^2_{\ell_i}}) \in D_2$. Moreover

$$\mathsf{trg}(qw'_{\delta_{k+1}}) = \mathsf{trg}(qw'_{\delta_k}).$$

Define the sequence of paths

$$\{w'_{\delta_k}\}_{k=1,\dots,m}$$

in $I_1 \uplus I_2$ by completing each quasi-path qw'_{δ_k} as follows: replace each node $(v'_1, ?)$ by (v'_1, v'_2) where v'_2 is any vertex in I_2 such that $\alpha_2(v'_2) = \pi$ and such that $v'_2 \in D_2$ whenever $v'_1 \in D_1$ and $v'_2 \notin D_2$ whenever $v'_1 \notin D_1$ and replace each node $(?, v'_2)$ by (v'_1, v'_2) where v'_1 is any vertex in I_1 such that $\alpha_1(v'_1) = \pi$ and such that $v'_1 \in D_1$ whenever $v'_2 \in D_2$ and $v'_1 \notin D_1$ whenever $v'_2 \notin D_2$.
Observe that w'_{δ_k} is a path for δ_k and, moreover, w'_{δ_m} is a path for φ.
We now show that

$$I_1 \uplus I_2, w'_{\delta_k} \Vdash \delta_k \quad \text{iff} \quad I_i, w'_{\varphi^i} \Vdash \varphi^i$$

and that

$$\alpha(w'_{\delta_k}) = \delta_k$$

by induction on k.
Base case: The thesis follows directly from the choice of w'_{δ_0}.
Induction step: Assume that $I_1 \uplus I_2, w'_{\delta_k} \Vdash \delta_k$ iff $I_i, w'_{\varphi^i} \Vdash \varphi^i$ and that $\alpha(w'_{\delta_k}) = \delta_k$. Then

$$\begin{aligned}
\alpha(w'_{\delta_{k+1}}) &= \alpha(w'_{\delta_k}) \circ \langle \mathsf{id}_{\mathsf{src}(\alpha(w'_{\delta_k}))_{|1}}, \dots, \mathsf{id}_{\mathsf{src}(\alpha(w'_{\delta_k}))_{|\ell_1 - 1}}, \psi^2_{\ell_1}, \dots, \psi^2_{\ell_j}, \dots \rangle \\
&= \delta_k \circ \langle \mathsf{id}_{\mathsf{src}(\alpha(w'_{\delta_k}))_{|1}}, \dots, \mathsf{id}_{\mathsf{src}(\alpha(w'_{\delta_k}))_{|\ell_1 - 1}}, \psi^2_{\ell_1}, \dots, \psi^2_{\ell_j}, \dots \rangle \\
&= \delta_{k+1}.
\end{aligned}$$

Moreover,

$$\begin{array}{lll} I_1 \uplus I_2, w'_{\delta_{k+1}} \Vdash \delta_{k+1} & \text{iff} & \alpha(w'_{\delta_{k+1}}) = \delta_{k+1} \text{ and } \mathsf{trg}(w'_{\delta_{k+1}}) \in D_1 \times D_2 \\ & \text{iff} & \ldots \text{ and } \mathsf{trg}(w'_{\delta_k}) \in D_1 \times D_2 \\ & \text{iff} & I_1, w'_{\varphi^i} \Vdash \varphi^1. \end{array}$$

Thus, the result holds for φ since φ^1 is δ_m. QED

The idea behind the preservation of the fmp in the present context is as follows. Suppose that $I \not\Vdash \varphi$ where I is in $\mathfrak{I}_1 \uplus \mathfrak{I}_2$. By definition of fibring, $I = I_1 \uplus I_2$ for some $I_1 \in \mathfrak{I}_1$ and $I_2 \in \mathfrak{I}_2$. Then we are able to find formulas $\varphi^1 \in \Sigma_1$ and $\varphi^2 \in \Sigma_2$ such that $I_1 \not\Vdash \varphi^1$ and $I_2 \not\Vdash \varphi^2$. The fmp property for $\mathfrak{I}_1$ and $\mathfrak{I}_2$ introduces finite interpretation structures $J_1 \in \mathfrak{I}_1$ and $J_2 \in \mathfrak{I}_2$ such that $J_1 \not\Vdash \varphi^1$ and $J_2 \not\Vdash \varphi^2$ and so $J_1 \uplus J_2 \not\Vdash \varphi$.

Theorem 4.5 Let $(\Sigma_i, \mathfrak{I}_i)$ be an interpretation system with disjunction $\vee_i$ and strong negation $\neg_i$ for $i = 1, 2$. Then $(\Sigma_1, \mathfrak{I}_1) \uplus (\Sigma_2, \mathfrak{I}_2)$ has the fmp provided that both $(\Sigma_1, \mathfrak{I}_1)$ and $(\Sigma_2, \mathfrak{I}_2)$ have the fmp.

Proof:
Let φ be a formula in $L(\Sigma_1 \uplus \Sigma_2)$ and $I \in \mathfrak{I}_1 \uplus \mathfrak{I}_2$ an interpretation structure over $\Sigma_1 \uplus \Sigma_2$ such that $I \not\Vdash \varphi$. Assume, without loss of generality, that the main constructor in φ belongs to Σ_1. Let $I_1 \in \mathfrak{I}_1$ and $I_2 \in \mathfrak{I}_2$ be such that $I = I_1 \uplus I_2$. Then there is a path w in $I_1 \uplus I_2$ such that:

- $\alpha(w) = \varphi$;
- $\mathsf{trg}(w) \notin D_1 \times D_2$.

That is, $I, w \not\Vdash \varphi$. Hence, by Lemma 4.2, for each $\psi \in \mathrm{SF}_i(\varphi)$, $i = 1, 2$, of φ there a path w^i_ψ in I_i such that:

- $\alpha_i(w_{\psi^i}) = \psi^i$;
- $I_1 \uplus I_2, w_{|_\psi} \Vdash \psi$ iff $I_1, w^1_\psi \Vdash \psi^1$, $I_2, w^2_\psi \Vdash \psi^2$;
- $I_1 \uplus I_2, w_{|_\psi} \not\Vdash \psi$ iff $I_1, w^1_\psi \not\Vdash \psi^1$, $I_2, w^2_\psi \not\Vdash \psi^2$;
- If $\mathsf{trg}(w_{|_\psi}) = (a_1, a_2)$ then $\mathsf{trg}_1(w^1_\psi) = a_1$ and $\mathsf{trg}_2(w^2_\psi) = a_2$.

Let $\hat{\psi}^i$ be defined as follows:

$$\hat{\psi}^i = \begin{cases} \psi^i & \text{if } I_i, w^i_\psi \not\Vdash \psi^i \\ \neg_i \psi^i & \text{if } I_i, w^i_\psi \Vdash \psi^i \end{cases}$$

where $\neg_i$ is the strong negation of $(\Sigma_i, \mathfrak{I}_i)$ for $i = 1, 2$. Then

$$I_i, \hat{w}^i_\psi \not\Vdash \hat{\psi}^i, \text{ for every } \psi \in \mathrm{SF}(\varphi) \text{ and } i = 1, 2$$

where

$$\hat{w}^i_\psi = \begin{cases} w^i_\psi & \text{when } \hat{\psi}^i = \psi^i \\ {w'}^i_\psi & \text{otherwise} \end{cases}$$

where w'^{i}_{ψ} is a path for $\neg_i \psi^i$ with the same source as w^i_{ψ}. So

$$I_i \not\Vdash \hat{\psi}^i, \text{ for every } \psi \in \mathrm{SF}(\varphi) \text{ and } i = 1, 2$$

Hence, by Proposition 3.1 and the fact that $(\Sigma_i, \mathfrak{I}_i)$ has the fmp, there is a finite interpretation structure $J = J_1 \uplus J_2 \in \mathfrak{I}_1 \uplus \mathfrak{I}_2$ such that

$$J_i \not\Vdash \hat{\psi}^i, \text{ for every } \psi \in \mathrm{SF}(\varphi) \text{ and } i = 1, 2.$$

That is, there is a path $w'_{\hat{\psi}^i}$ for each $\psi \in \mathrm{SF}(\varphi)$, such that

$$J_i, w'_{\hat{\psi}^i} \not\Vdash \hat{\psi}^i$$

with $i = 1, 2$. Since $(\Sigma_i, \mathfrak{I}_i)$ has strong negation, there is a path w''_{ψ^i} with the same socket for each $\psi \in \mathrm{SF}(\varphi)$ with $\hat{\psi}^i = \neg_i \psi^i$ such that

$$J_i, w''_{\psi^i} \Vdash \hat{\psi}^i.$$

Therefore,

$$\begin{cases} J_i, w'_{\psi^i} \not\Vdash \hat{\psi}^i & \text{whenever } I_i, w_{\psi_i} \not\Vdash \psi^i \\ J_i, w''_{\psi^i} \Vdash \hat{\psi}^i & \text{whenever } I_i, w_{\psi_i} \Vdash \psi^i \end{cases}$$

for every $\psi \in \mathrm{SF}(\varphi)$ and $i = 1, 2$. In particular

$$J_i, w'_{\psi^i} \not\Vdash \hat{\varphi}^i.$$

Thus, by Lemma 4.4, we conclude that there is a path w'_{φ} such that

$$J_1 \uplus J_2, w'_{\varphi} \Vdash \varphi \quad \text{iff} \quad J_i, w'_{\varphi^i} \Vdash \varphi^i$$

where $i = 1$ if the main constructor of φ is in E_1 and $i = 2$ otherwise. Since $J_i, w'_{\varphi^i} \not\Vdash \varphi^i$ then

$$J_1 \uplus J_2, w'_{\varphi} \not\Vdash \varphi$$

and so $J_1 \uplus J_2, \not\Vdash \varphi$. QED

We now give a detailed example of the construction in the proof of Theorem 4.5.

Example 4.6 Let φ be the mixed formula

$$\wedge_1(\vee_1(\neg_2(p_2), p_1), \supset_2(\neg_1(\vee_2(p_1, \neg_2(p_2))), p_2))$$

where the subscript i refers to symbols from the signature Σ_i for $i = 1, 2$. Suppose that $I = I_1 \uplus I_2$ is such that $I \not\Vdash \varphi$.

Consider a path w in I such that $I, w \not\Vdash \varphi$. Then w has the form described in Figure 1, where we use for the denotation of a constructor the same name as for the constructor (we use graphical notation for paths in order to make easier the building process):

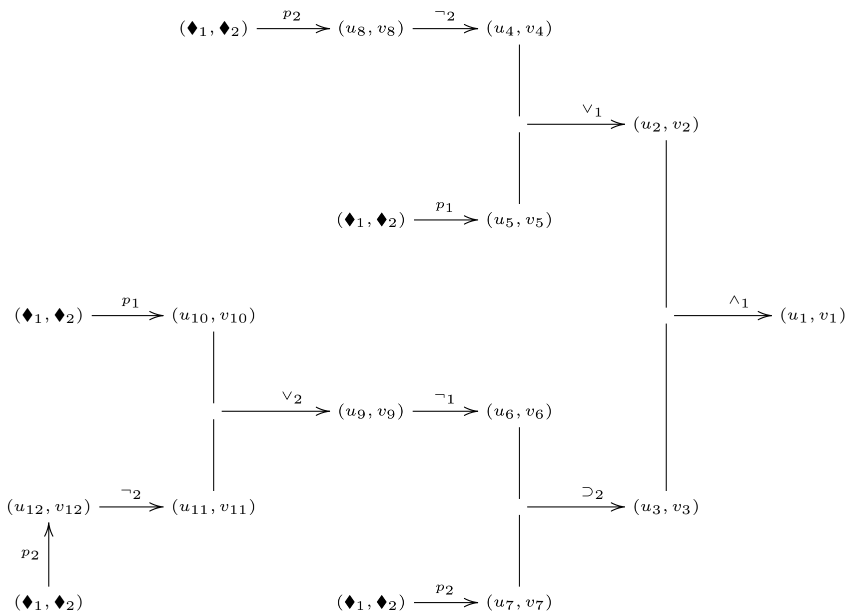

Figure 1: Path w for φ in $I_1 \uplus I_2$.

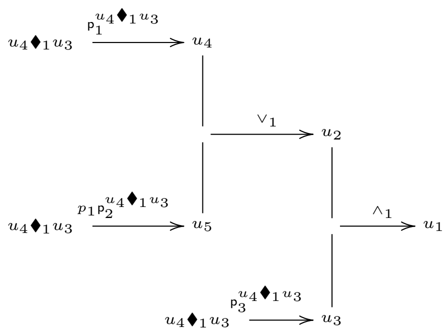

Figure 2: Path $w^1_{\psi_1}$ for ψ^1_1 in I_1.

Since $I, w \not\Vdash \varphi$ then (u_1, v_1) is not distinguished and so $u_1 \notin D_1$ and $v_1 \notin D_2$. Observe that $\mathrm{SF}_1(\varphi) = \{\psi_1, \psi_2, \psi_3, \psi_4\}$ and $\mathrm{SF}_2(\varphi) = \{\delta_1, \delta_2, \delta_3, \delta_4\}$ where

$$\begin{array}{ll}
\psi_1 = \varphi & \delta_1 = \supset_2(\neg_1(\vee_2(p_1, \neg_2(p_2))), p_2) \\
\psi_2 = \vee_1(\neg_2(p_2), p_1) & \delta_2 = \vee_2(p_1, \neg_2(p_2)) \\
\psi_3 = \neg_1(\vee_2(p_1, \neg_2(p_2))) & \delta_3 = \neg_2(p_2) \\
\psi_4 = p_1 & \delta_4 = p_2.
\end{array}$$

(a) Encoding of all formulas in $\mathrm{SF}_1(\varphi)$ and $\mathrm{SF}_2(\varphi)$.

$$
\begin{array}{ll}
\psi_1^1 = \wedge_1(\vee_1(\mathsf{p}_1^{\pi\diamond\pi}, p_1\mathsf{p}_2^{\pi\diamond\pi}), \mathsf{p}_3^{\pi\diamond\pi}) & \delta_1^2 = \supset_2(\mathsf{p}_1^{\pi\diamond}, p_2\mathsf{p}_2^{\pi\diamond}) \\
\psi_2^1 = \vee_1(\mathsf{p}_1^{\pi\diamond}, p_1\mathsf{p}_2^{\pi\diamond}) & \delta_2^2 = \vee_2(\mathsf{p}_1^{\pi\diamond}, \neg_2(p_2)\mathsf{p}_2^{\pi\diamond}) \\
\psi_3^1 = \neg_1 & \delta_3^2 = \neg_2(p_2) \\
\psi_4^1 = p_1 & \delta_4^2 = p_2.
\end{array}
$$

On the other hand, $\psi_i^2 = \mathsf{id}_\pi$ and $\delta_i^1 = \mathsf{id}_\pi$ for $i = 1, \ldots, 4$.

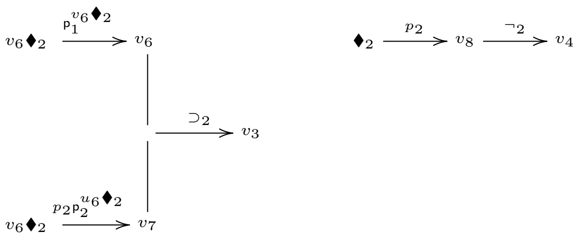

Figure 3: Path $w_{\delta_1}^2$ for δ_1^2 and path $w_{\delta_3}^2$ for δ_3^2 in I_2.

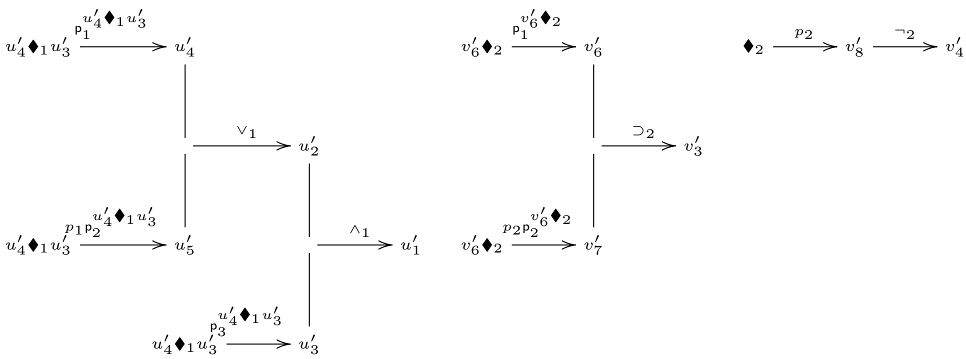

Figure 4: Paths $w'^1_{\psi_1}$ for ψ_1^1 in J_1, $w'^2_{\delta_1}$ for δ_1^2 and $w'^2_{\delta_3}$ for δ_2^3 in J_2.

(b) Path for ψ_i^1 in I_1 and path for δ_i^2 in I_2 for $i = 1, \ldots, 4$ using Lemma 4.2. Path $w_{\psi_1^1}^1$ for $\psi_1^1 : \pi\diamond\pi \to \pi$ is described in Figure 2 using the socket $u_4\blacklozenge u_3$. Observe that:

$$I_1, w_{\psi_1}^1 \not\Vdash \psi_1^1 \quad \text{and} \quad I_2, w_{\psi_1}^2 \not\Vdash \mathsf{id}_\pi$$

where $w_{\psi_1}^2 : v_1 \to v_1$. Path $w_{\delta_1^2}^2$ for $\delta_1^2 : \pi\diamond \to \pi$ and path $w_{\delta_3^2}^2$ for $\delta_3^2 : \diamond \to \pi$ are depicted in Figure 3. Then

$$I_2, w_{\delta_1}^2 \not\Vdash \delta_1^2 \quad \text{and} \quad I_1, w_{\delta_1}^1 \not\Vdash \mathsf{id}_\pi$$

if $(u_3, v_3) \notin D_1 \times D_2$ and $w^1_{\delta_1} : u_3 \to u_3$. Moreover,

$$I_2, w^2_{\delta_3} \not\Vdash \delta^2_3 \quad \text{and} \quad I_1, w^1_{\delta_3} \not\Vdash \mathsf{id}_\pi$$

if $(u_4, v_4) \notin D_1 \times D_2$ and $w^1_{\delta_3} : u_4 \to u_4$.

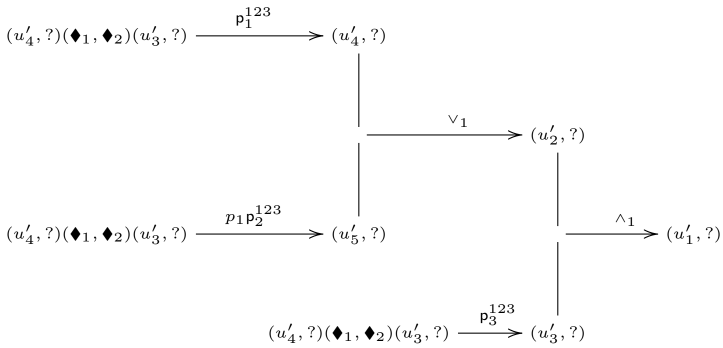

Figure 5: Quasi-path $qw^1_{\psi_1}?$ for ψ^1_1 in $J_1 \uplus J_2$.

Then there are finite interpretation structures J_1 and J_2 such that

- $J_1, w'^1_{\psi_1} \not\Vdash \psi^1_1$ and $J_2, w'^2_{\psi_1} \not\Vdash \mathsf{id}_\pi$;
- $J_2, w'^2_{\delta_1} \not\Vdash \delta^2_1$ and $J_1, w'^1_{\delta_1} \not\Vdash \mathsf{id}_\pi$;
- $J_2, w'^2_{\delta_3} \not\Vdash \delta^2_3$ and $J_1, w'^1_{\delta_3} \not\Vdash \mathsf{id}_\pi$.

(c) Path for φ in $J_1 \uplus J_2$ using Lemma 4.4.
Assume that the path in J_1 for ψ^1_1 and the paths in J_2 for δ^2_1 and δ^2_3 are as depicted in Figure 4. The next step is to transform the paths above in quasi-paths in $J_1 \uplus J_2$. In Figure 5 we provide the quasi-path for ψ^1_1 where projection $\mathsf{p}_i^{(u'_4,?)(\blacklozenge_1,\blacklozenge_2)(u'_3,?)}$ is abbreviated as p^{123}_i.

In Figure 6 we introduce quasi-paths for δ^2_1 and δ^2_3 in $J_1 \uplus J_2$ corresponding to the paths for the same formulas in J_2 introduced in Figure 4.

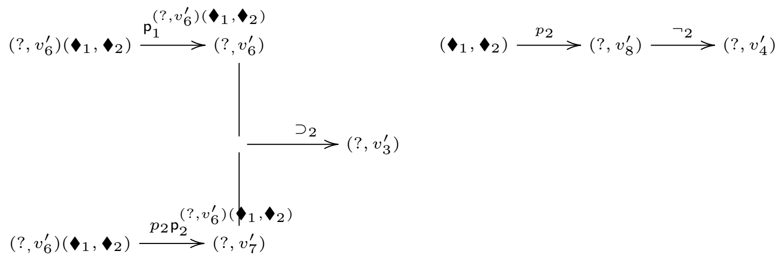

Figure 6: Quasi-path $?qw^2_{\delta_1}$ for δ^2_1 and path $?qw^2_{\delta_3}$ for δ^2_3 in $J_1 \uplus J_2$.

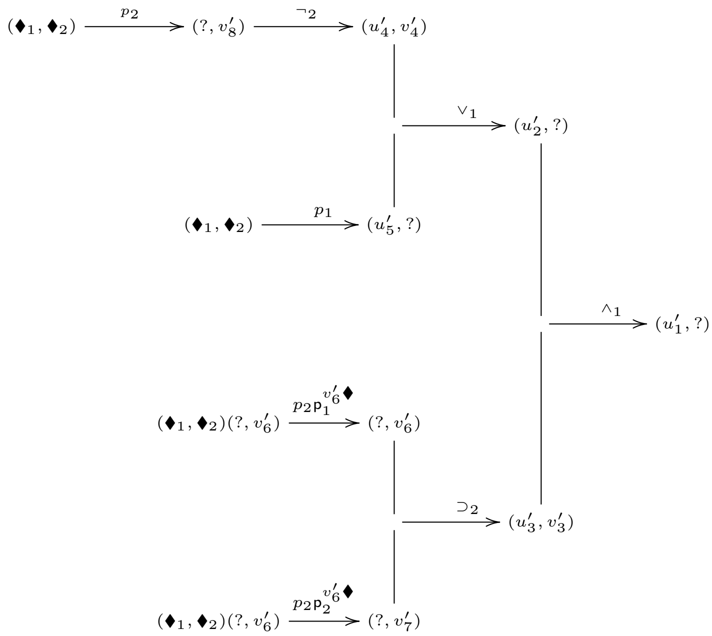

Figure 7: Quasi-path qw'_{δ_1} for δ_1 in $J_1 \uplus J_2$.

In Figure 7, we depict the quasi-path qw'_{δ_1} for

$$\delta_1 = \wedge_1(\vee_1(\neg_2(p_2), p_1)), \supset_2(\mathsf{p}_1^{\pi\Diamond}, p_2\mathsf{p}_2^{\pi\Diamond}).$$

Let w'_{δ_1} be a path obtained from qw'_{δ_1} by filling in ? with arbitrary truth values in such a way that in each pair both are distinguished or both are not distinguished. Then

$$J_1 \uplus J_2, w'_{\delta_1} \Vdash \wedge_1(\vee_1(\neg_2(p_2), p_1)), \supset_2(\mathsf{p}_1^{\pi\Diamond}, p_2\mathsf{p}_2^{\pi\Diamond}).$$

Continuing with this process we obtain a path for φ in $J_1 \uplus J_2$ whose target is a non-distinguished truth value, as required.

Theorem 4.5 can be extended to constrained fibring (with some sharing of constructors). Thus, we have the following result.

Theorem 4.7 Let $(\Sigma_i, \mathfrak{I}_i)$ be an interpretation system with disjunction $\vee_i$ and strong negation $\neg_i$ for $i = 1, 2$ and let $h_i : \Sigma_0 \to \Sigma_i$ be a signature morphisms for $i = 1, 2$. Then $(\Sigma_1, \mathfrak{I}_1) \uplus^{\Sigma_0}_{h_1 h_2} (\Sigma_2, \mathfrak{I}_2)$ has the fmp provided that both $(\Sigma_1, \mathfrak{I}_1)$ and $(\Sigma_2, \mathfrak{I}_2)$ have the fmp.

5 Examples

In this section, we discuss preservation of the fmp by several combinations of the logics described in Section 3. Theorem 4.5 can be applied in all these cases because the logics at hand have both disjunction and strong negation.

Modal paraconsistent logic

Herein, we combine a normal modal logic ML, as described in Subsection 3.1, with the paraconsistent logic mbC, as introduced in Subsection 3.2 (the importance of combining modal logics with paraconsistent logics is discussed in [24]).

Proposition 5.1 A modal paraconsistent logic $(\Sigma_1, \mathfrak{I}_1) \uplus (\Sigma_2, \mathfrak{I}_2)$ has the finite model property, providing that modal logic ML has the fmp.

Proof: The interpretation system $(\Sigma_1, \mathfrak{I}_1)$ has strong negation and disjunction. Additionally, it is possible to define a strong negation as a derived constructor in mbC as follows:

$$\sim \equiv_{\text{abv}} \supset_2 \langle \mathsf{p}_1^\pi, \wedge_2 \langle \odot_2, \wedge_2 \langle \mathsf{p}_1^\pi, \neg_2 \mathsf{p}_1^\pi \rangle\rangle\rangle.$$

On the other hand, $\vee_2$ is a disjunction in our sense. Hence $(\Sigma_2, \mathfrak{I}_2)$ also has strong negation and disjunction. Therefore, by Theorem 4.5, $(\Sigma_1, \mathfrak{I}_1) \uplus (\Sigma_2, \mathfrak{I}_2)$ has the finite model property. QED

Hence ML can be instantiated with any normal modal logic with the fmp (see [7, 16] for several examples). But, it is worthwhile to point out that, as shown in [23, 11], not all normal logics have the fmp. A Hilbert calculus can be given for modal paraconsistent logic by taking the axioms and the rules of both ML and mbC. Then, the following results hold.

Corollary 5.2 The modal paraconsistent logic $(\Sigma_1, \mathfrak{I}_1) \uplus (\Sigma_2, \mathfrak{I}_2)$ is decidable, providing that the modal logic $(\Sigma_1, \mathfrak{I}_1)$ has the fmp.

Proposition 5.3 The modal paraconsistent logic $(\Sigma_1, \mathfrak{I}_1) \uplus (\Sigma_2, \mathfrak{I}_2)$ is weakly complete, assuming that ML is weakly complete.

Proof: By Theorem 7.7 of [31] we conclude the weak completeness by observing that the canonical model of ML is in $\mathfrak{I}_1$ (the Lindenbaum-Tarski algebra — see Section 5.2 of [6]) and the canonical model of mbC is in $\mathfrak{I}_2$ (the latter is an easy consequence of the canonical model in the proof in [4] and Proposition 3.5). QED

Deontic paraconsistent logic

Consider now the combination of the paraconsistent logic mbC, as introduced in Subsection 3.2, with the deontic logic DPM, as described in Subsection 3.3.

Proposition 5.4 The deontic paraconsistent logic $(\Sigma_2, \mathfrak{I}_2) \uplus (\Sigma_3, \mathfrak{I}_3)$ has the finite model property.

Proof: It is clear from the definition that $(\Sigma_3, \mathfrak{I}_3)$ has strong negation $\neg_1$ and disjunction $\vee_1$. On the other hand, $(\Sigma_2, \mathfrak{I}_2)$ also has strong negation and disjunction as shown in the proof of Proposition 5.1. Hence, by Theorem 4.5, $(\Sigma_2, \mathfrak{I}_2) \uplus (\Sigma_3, \mathfrak{I}_3)$ has the finite model property. QED

A Hilbert calculus can be given for deontic paraconsistent logic just by using the axioms and the inference rules of DPM and mbC. Then the following results hold:

Corollary 5.5 The deontic paraconsistent logic $(\Sigma_2, \mathfrak{I}_2) \uplus (\Sigma_3, \mathfrak{I}_3)$ is decidable.

Proposition 5.6 The deontic paraconsistent logic $(\Sigma_2, \mathfrak{I}_2) \uplus (\Sigma_3, \mathfrak{I}_3)$ is weakly complete.

Proof: By Theorem 7.7 of [31] we conclude the weak completeness by observing that the canonical model of DPM is in $\mathfrak{I}_3$ (using the proof in [15]) and the canonical model of mbC is in $\mathfrak{I}_2$. QED

Modal linear logic

Herein, we focus on the combination of a modal normal logic ML, as described in Subsection 3.1, with the fragment HL of linear logic, as presented in Subsection 3.4.

Proposition 5.7 A modal linear logic $(\Sigma_1, \mathfrak{I}_1) \uplus (\Sigma_4, \mathfrak{I}_4)$ has the finite model property, providing that ML has the fmp.

Proof: From [27] we conclude that $(\Sigma_4, \mathfrak{I}_4)$ has strong negation $\neg_4$ and disjunction $\vee_4$. Moreover, in [19] it was proven that the HL logic has the fmp. Hence, by Theorem 4.5, $(\Sigma_1, \mathfrak{I}_1) \uplus (\Sigma_4, \mathfrak{I}_4)$ has the finite model property. QED

Decidability of HL was firstly established in [22] using other techniques. In [3] a Hilbert calculus is presented for HL. Hence, a Hilbert calculus can be given for modal linear logic just by using the axioms and the inference rules of ML and HL. Hence, the following results hold:

Corollary 5.8 A modal linear logic $(\Sigma_1, \mathfrak{I}_1) \uplus (\Sigma_4, \mathfrak{I}_4)$ is decidable, providing that ML has the fmp.

Proposition 5.9 A modal linear logic $(\Sigma_1, \mathfrak{I}_1) \uplus (\Sigma_4, \mathfrak{I}_4)$ is weakly complete, providing that ML is weakly complete.

Proof: By Theorem 7.7 of [31] we conclude the weak completeness by observing that the canonical model of HL is in $\mathfrak{I}_4$ (using the proof in [3]) and the canonical model of ML is in $\mathfrak{I}_1$. QED

Deontic modal logics

Consider the combination of a modal normal logic ML, as described in Subsection 3.1, with the deontic logic DPM, as discussed in Subsection 3.3 (several combinations of deontic, modal and tense operators are of utmost interest in normative contexts, see, for instance, [2]).

Proposition 5.10 A deontic modal logic $(\Sigma_1, \mathfrak{I}_1) \uplus (\Sigma_3, \mathfrak{I}_3)$ has the finite model property, providing that ML has the fmp.

Corollary 5.11 A deontic modal logic $(\Sigma_1, \mathfrak{I}_1) \uplus (\Sigma_3, \mathfrak{I}_3)$ is decidable, providing that ML has the fmp.

Fusion of normal modal logics

Let ML^a and ML^b be two normal modal logics endowed with algebraic semantics. Let $(\Sigma_1, \mathfrak{I}_1^a)$ and $(\Sigma_1, \mathfrak{I}_1^b)$ be interpretation systems, as described in Subsection 3.1, for ML^a and ML^b, respectively. Furthermore, let $\mathrm{ML}^a \oplus \mathrm{ML}^b$ be the fusion of ML^a and ML^b, a bimodal logic.

Let $(\Sigma, \mathfrak{I}) = (\Sigma_1, \mathfrak{I}_1^a) \oplus (\Sigma_1, \mathfrak{I}_1^b)$ be the interpretation system induced by $\mathrm{ML}^a \oplus \mathrm{ML}^b$ as hinted in Subsection 3.1. Satisfaction in $(\Sigma, \mathfrak{I})$ coincides with satisfaction in $\mathrm{ML}^a \oplus \mathrm{ML}^b$ (the multimodal variant of Proposition 3.2).

Thanks to [18] we know that if ML^a and ML^b have the fmp then so has their fusion. We recover now this result as a special case of the transference of fmp by fibring.

To this end, observe first that the signature Σ of the fusion coincides with $\Sigma_1 \uplus^{\Sigma_0}_{h^a,h^b} \Sigma_1$ with $h^a : \Sigma_0 \to \Sigma_1$ and $h^b : \Sigma_0 \to \Sigma_1$ where Σ_0 is the signature with edges $p : \diamond \to \pi$, $\neg : \pi \to \pi$ and $\supset : \pi\pi \to \pi$. Clearly, the propositional symbols and connectives are shared while we keep the two boxes apart, as intended in fusion.

Moreover, it is straightforward to verify that satisfaction in $(\Sigma, \mathfrak{I})$ coincides with satisfaction in $(\Sigma_1, \mathfrak{I}_1^a) \uplus^{\Sigma_0}_{h^a,h^b} (\Sigma_1, \mathfrak{I}_1^b)$. In consequence, we have:

Proposition 5.12 The fusion $\mathrm{ML}^a \oplus \mathrm{ML}^b$ has the finite model property, provided that both ML^a and ML^b have the fmp.

Proof: Again we apply Theorem 4.5 since modal logics have both disjunction and strong negation and fusion coincides satisfaction-wise with the fibring described above. QED

Hence, as expected, the transference of fmp by fibring subsumes the preservation of fmp by fusion of modal logics.

6 Outlook

In this paper, we established sufficient conditions for the preservation by fibring of the finite model property for a large class of logics. We took advantage of a novel feature of the graph-theoretic semantics of fibring proposed in [31]: every model of the logic obtained by fibring $(\Sigma_1, \mathfrak{I}_1)$ and $(\Sigma_2, \mathfrak{I}_2)$ is, roughly speaking, the Cartesian product of a model of $\mathfrak{I}_1$ and a model of $\mathfrak{I}_2$. Observe also that the graph-theoretic account of fibring encompasses a much wider universe of logics, including paraconsistency and substructurality, thanks to using an abstraction map from the model to the language instead of the traditional interpretation map from the language to the model.

Our preservation result of the finite model property covers a wide class of logics: we require only the additional assumption of availability of disjunction and strong negation (either primitive or as abbreviations). It should be noted that intuitionistic logic does not satisfy these requirements. It is our intention to investigate preservation of the fmp when one of the logics is intuitionistic logic by exploring the encoding of intuitionistic logic in the S4 modal logic.

We would like to extend our results to fibring of logics involving fragments of first order logic that have the finite model property such as the $\forall^*\exists$ (see [17]), GF (see [1]) and FO^2 (see [25], among many others). Indeed, the graph-theoretic account of logics is rich enough to deal with quantification logics.

Another topic of interest would be to explore the relationship between fibring and the algebraic Gentzen systems when used to prove the finite model property (see [5], as well as the work in [19, 26]).

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