Bijective Faithful Translations among Default Logics

Paolo Liberatore*

Abstract

In this article, we study translations between variants of defaults logics such that the extensions of the theories that are the input and the output of the translation are in a bijective correspondence. We assume that a translation can introduce new variables and that the result of translating a theory can either be produced in time polynomial in the size of the theory or its output is of size polynomial in the size of the theory; we restrict to the case in which the original theory has extensions. This study fills a gap between two previous pieces of work, one studying bijective translations among restrictions of default logics, and the other one studying non-bijective translations between default logics variants.

1 Introduction

A translation from one logic to another is *faithful* if it preserves not only the consequences but also the models of the original theory. What in modal logic is a model, in default logic [Rei80, Bes89, Ant99] is an extension; therefore, a faithful translation involving two default logics is a translation preserving the extensions.

The existence and non-existence of faithful translations among various logics are known [Imi87, Kon88, ET93, Got95]. Recently, some effort has been devoted to translations that introduce new variables [Jan98, Jan01, Jan03, DS03, DS05]: these translations generate theories which may contain new variables in addition to the ones of the corresponding original theories. The addition of new variables allows for translations that would otherwise be impossible: for example, no translation that exactly preserves the extensions exists from justified default logic to Reiter default logic; this is because $E_1 \subset E_2$ cannot hold for two Reiter extensions E_1 and E_2 of the same theory, while this situation is instead possible for two justified extensions [Lib05]. Introducing new variables can however circumvent this difficulty, because two justified extensions E_1 and E_2 such that $E_1 \subset E_2$ can be translated into $E_1 \cup E'_1$ and $E_2 \cup E'_2$, respectively, provided that $E_1 \cup E'_1 \not\subset E_2 \cup E'_2$.

The possibility of adding new variables is therefore of interest because it allows for translations that would otherwise be impossible. These translations are not defined in terms of logical equivalence between extensions $E_1 \equiv E_2$, but in terms of var-equivalence $E_1 \equiv_X E_2$, where X is the set of variables of the original theory and $E_1 \equiv_X E_2$ means that E_1 and E_2 have the same consequences when restricted over the alphabet X [LLM03].

^{*}Dipartimento di Informatica e Sistemistica, Università di Roma La Sapienza, Via Ariosto 25, 00185, Rome, Italy. Email: paolo@liberatore.org

Faithful translations can be defined in two ways, which are equivalent when new variables are not allowed. In particular, a translation is faithful if each theory T_1 is translated into a theory T_2 such that either:

- 1. there is a bijection between the extensions of T_1 and the extensions of T_2 such that the associated extensions of T_1 and T_2 are equivalent, or
- 2. for every extension of T_1 there exists an equivalent extension of T_2 and vice versa.

These two definitions can also be given when new variables are allowed, by replacing "equivalence" with "var-equivalence". However, they no longer coincide. Indeed, the second definition allows a single extension of T_1 to be associated to several extensions of T_2 . For example, if T_1 is build on variables $\{x\}$ and T_2 on $\{x,y\}$, the second definition allows the same extension $Cn(\{x\})$ to be associated to the two extensions $Cn(\{x,y\})$ and $Cn(\{x,\neg y\})$. These two extensions are indeed var-equivalent to the original one, but are not classically equivalent to it or to each other. This translation is faithful according to the second definition but not according to the first.

This example shows that that the two considered definitions of faithfulness do not coincide. In this paper, we call translations satisfying the first definition bijective faithful and translations satisfying the second faithful. This choice is motivated by the fact that all translations obeying the first definition also obey the second but not vice versa, that is, a bijection between the extensions is an additional requirement over the translation.

Translations among default theories producing new variables have been studied by Janhunen [Jan98, Jan01, Jan03] and Delgrande and Schaub [DS03, DS05]. All these authors considered faithful translations, but using two differing definitions: the former author studies bijective faithful translations, the latter authors do not require a bijection between extensions.

In particular, Delgrande and Schaub [DS03, DS05] have shown faithful polynomial-time translations from some default logic variants into Reiter default logic. Some of their faithful reductions produce a bijection between the extensions only using a definition of extensions that include the justification of the applied defaults. In particular, if one defines an extension to be the deductive closure of the consequences of the applied defaults, their translation from justified default logic into Reiter default logic is bijective, while their translations from constrained and rational to Reiter default logic are faithful but not bijective. All of their reductions are bijective if one takes an extension to be include also the justification of the applied defaults.

Janhunen [Jan98, Jan01, Jan03] has instead studied bijective faithful translations, but not between default logics variants but between default logics restrictions, and between default logics and other logics. As a result, the study of bijective translations between default logics variants is still largely open, and is the subject of this article.

The results about the existence of polynomial-time and polynomial-size bijective faithful translations are shown in Table 1. The existence of polynomial-time bijective faithful translations from constrained or rational default logic to Reiter or justified default logic would have some consequences on complexity classes, whenever extensions are considered to be the deductive closure of the consequences of the applied default only (i.e., not including the justifications). Ideally, a negative result should be unconditioned (e.g., there is no bijective polynomial time faithful translations from constrained to Reiter default logics) or at

least conditioned to the collapse of the polynomial hierarchy (e.g., if there exists a bijective polynomial-time faithful translations from constrained to Reiter default logics then the polynomial hierarchy collapses); unfortunately, none of these two claims could be proved. We however show some consequences on complexity classes of the existence of a bijective polynomial time faithful translations from constrained to Reiter default logics.

Faithful translations cannot exist from semantics where a theory may have no extension to semantics where this is not possible. On the other hand, it can be shown that in some cases theories having no extensions are the only ones that cannot be translated. For example, Reiter default logic (which allows a theory to have no extensions) cannot be in general translated into normal default logic (in which every theory has at least an extension). However, if one restricts to theories having at least one extension, then Reiter default logic can be faithfully translated into normal default logic [Lib06]. Translations that work in the assumption of existence of extensions are of interest because theories can be modified in a very simple way so that they are added a single known extension. Many problems, such as entailment, number of extensions, etc. can therefore be solved via such translations.

In this paper, we show bijective faithful translations from rational and Reiter to constrained default logic and from Reiter to justified default logic. These translations are polynomial-time but require not only the original theory to have an extension, but also that a formula equivalent to one of the strongest (i.e., minimal w.r.t. set containment) extension is given. Such translations are of interest because an extension, being the deductive closure of a set of consequences of some defaults in the theory, can always be represented by a polynomially sized formula. The result of these translations are therefore of size polynomial in the size of the original theory. In other words, for every theory in the original semantics (provided it has extensions) there exists a theory in the resulting semantics that has size polynomial in that of the original theory. More concisely, what can be expressed in the first semantics can also be expressed in the second one in comparable space. Size-preserving translations of this kind are called polysize because they produce a result that is polynomial in size w.r.t. the size of the input theory. Finally, we show some consequences of the existence of a bijective faithful polysize translation from constrained or rational to Reiter or justified default logic on the counting hierarchy.

2 Definitions

2.1 Default Logics

We use the operational semantics for default logics. Two slightly different, but equivalent, operational semantics for default logics have been given independently by Antoniou and Sperschneider [AS94, Ant99] and by Froidevaux and Mengin [FM92, FM94]. A default is a rule of the form:

$$d = \frac{\alpha : \beta}{\gamma}$$

The formulae α , β , and γ are called the precondition, the justification, and the consequence of d, and are denoted as prec(d), just(d), and cons(d), respectively. This notation is extended to sets and sequences of defaults in the obvious way. A default is applicable if its precondition is true and its justification is consistent; if this is the case, its consequence should be considered true.

↓From To →	Reiter	Justified	Rational	Constrained
Reiter	yes^1	no^2	yes^3	no^2
Justified	yes^3		yes^4	yes
Rational	${\rm open^5}$	no^2		no^2
Constrained	open^5	open^5	yes	

Bijective faithful polytime translations between semantics

↓From To –	Reiter	Justified	Rational	Constrained
Reiter	yes^1	yes	yes^3	yes
	(polytime)	(strongest extension)	(polytime)	(strongest extension)
Justified	yes^3		yes^4	yes
	(polytime)		(polytime)	(polytime)
Rational	open^5	open^5		yes
				(strongest extension)
Constrained	open^5	open^5	polytime	

Bijective faithful polysize translations between semantics.

Table 1: Faithful Translations between Semantics

A default theory is a pair $\langle D, W \rangle$ where D is a set of defaults and W is a consistent theory, called the background theory. The assumption that W is consistent is not standard; however, all known semantics give the same evaluation when the background theory is inconsistent. We also make some other assumptions about the default theory: all formulae are propositional, the alphabet and the set D are finite, and all defaults have a single justification. The latter assumption is irrelevant for some semantics (such as constrained default logic) but not for other ones (such as justified default logic.)

We use semantics of default logics based on sequences of defaults. We typically denote such sequences by Π , Π' , etc. We also denote by $\Pi \cdot \Pi'$ the sequence composed of Π followed by Π' . When Π' is composed of a single default d, we also denote this concatenation by $\Pi \cdot d$. Given a sequence Π and one of its defaults d, we denote by $\Pi[d]$ the sequence of defaults preceeding d in Π . We define a process to be a sequence of defaults that can be applied starting from the background theory.

Definition 1 A process of a default theory $\langle D, W \rangle$ is a sequence of defaults Π such that $W \cup cons(\Pi)$ is consistent and $W \cup cons(\Pi[d]) \models prec(d)$ for every default $d \in \Pi$.

The definition of processes only takes into account the preconditions and the consequences of defaults. This is because the interpretation of the justifications depends on the semantics.

¹ Delgrande and Schaub [DS03, DS05] proved that Reiter default logic can "simulate itself", i.e., there is a non-trivial polynomial time bijective faithful translation from Reiter default logic to itself.

² Unless the polynomial hierarchy collapses.

³ Proved by Delgrande and Schaub [DS03, DS05].

⁴ Trivially entailed by the reductions from justified to Reiter default logic and from Reiter to rational default logic by Delgrande and Schaub [DS03, DS05].

⁵ Effects of the existence of a polysize translation on the counting hierarchy are shown in this article.

All semantics select a set of processes that satisfy two conditions: success and closure. Intuitively, success means that the justifications of the applied defaults are not contradicted; closure means that no other default should be applied.

The particular definitions of success and closure depend on the specific semantics; in turn, closure can be defined in terms of applicability of a default. The following are the definitions used by the variants of default logic considered in this paper.

Success:

Local: for each $d \in \Pi$, the set $W \cup cons(\Pi) \cup \{just(d)\}$ is consistent;

Global: $W \cup cons(\Pi) \cup just(\Pi)$ is consistent.

Closure:

Inapplicability: no default $d \notin \Pi$ is applicable to Π ; applicability of a default d in Π is defined as $W \cup cons(\Pi) \models prec(d)$ and:

Local Applicability: $W \cup cons(\Pi) \cup just(d)$ is consistent;

Global Applicability: $W \cup cons(\Pi) \cup just(\Pi) \cup just(d)$ is consistent.

Maximality: for any $d \notin \Pi$, the sequence $\Pi \cdot [d]$ is not a successful process.

Reiter default logic uses local success and local inapplicability closure; justified default logic uses local success and maximality closure; rational default logic uses global success and global inapplicability closure; constrained default logic uses global success and maximality closure.

The definition of processes by Antoniou and Sperschneider [AS94, Ant99] and that by Froidevaux and Mengin [FM92, FM94] differ mainly in "when justifications are checked". In terms of our definition or processes, Antoniou and Sperschneider do not allow a sequence of defaults to be a process if the justification of a default is not consistent with the background theory and the consequences of the previous defaults. On the contrary, this is allowed by our definition and that by Froidevaux and Mengin. To the aim of automated deduction, the first definition may allow reducing the width of the tree of processes; on the other hand, the second definition is slightly simpler from a formal point of view.

The extensions of a default theory can be defined in two different ways, both based on the set of selected processes. In this paper, we use the following one: if Π is a successful processes, an extension is $Cn(W \cup cons(\Pi))$. This definition is what is actually necessary for defining query answering: the *skeptical* consequences of a default theory are the formulae that are entailed by all its extensions; the *credulous* consequences are those implied by some of its extensions.

Extensions for rational and constrained default logic have been initially defined in a form that is equivalent to the pair $\langle just(\Pi), Cn(W \cup cons(\Pi)) \rangle$, where Π is a successful process. This second definition includes in the extensions also the justification of the applied defaults. In order to distinguish between extensions according to the first or to the second definition, we use different names.

Extension: $Cn(W \cup cons(\Pi))$, where Π is a selected process;

Double Extension: the pair $\langle just(\Pi), Cn(W \cup cons(\Pi)) \rangle$, where Π is a selected process.

According to the second definition, two processes composed of the same defaults in different order always generate two different extensions. The same is not true for the first definition of extensions for constrained and rational default logic. In other words, two processes composed of different defaults can generate the same extension in these two semantics, if the extension is defined from consequences only.

A semantics for default logic is *fail-safe* if, for every default theory, any successful process is the prefix of a successful and closed process. A successful process of a fail-safe semantics cannot "fail": if we can apply a sequence of defaults, then an extension will be eventually generated, possibly after applying some other defaults. In other words, the situation in which we apply some defaults but then find out that we do not generate an extension never occurs. Fail-safeness is a form of commitment to defaults: if we apply a default, we never end up with contradicting its assumption.

Fail-safeness can also be seen as a form of monotonicity of processes w.r.t. to sets of defaults: if a semantics is fail-safe, then adding some defaults to a theory may only extend the successful and closed process of the theory and create new ones. However, this form of monotonicity is not the same as that typically used in the literature, which is defined in terms of consequences, not processes. Froidevaux and Mengin [FM94, Theorem 29] have proved a result that essentially states that every semantics in which closure is defined as maximal success is fail-safe. As a result, justified and constrained default logics are fail-safe.

We assume that the background theory W is consistent. In this case, if a semantics is fail-safe, then every default theory has a successful and closed process: since the process [] is successful, a process Π that is successful and closed exists. The condition of antimonotonicity provides an algorithm for finding this successful and closed process: if Π is successful and closed, all its initial fragments are successful as well. We can therefore obtain a successful and closed process by iteratively adding to [] a default that leads to a successful process.

2.2 Translations

We assume that translations between default logics can introduce new variables. Technically, this is possible thanks to the concept of var-equivalence [LLM03]. In plain terms, two formulae are var-equivalent if and only if their consequences, when restricted to be formulae on a given alphabet, are the same.

Definition 2 (Var-Equivalence) Two formulae α and β are var-equivalent w.r.t. variables X if and only if $\alpha \models \gamma$ iff $\beta \models \gamma$ for every formula γ that only contains variables in X.

The translations we consider may introduce new variables: a theory $\langle D, W \rangle$ built on variables X is translated into a default theory $\langle D', W' \rangle$ built on variables $X \cup Y$. Faithful translations between default logics based on var-equivalence of extensions have been considered by Delgrande and Schaub [DS03, DS05] and by Janhunen [Jan98, Jan03]. These authors use slightly different definitions of faithful translations. Delgrande and Schaub use the following definition.

Definition 3 (Faithful Translation) A translation that maps each default theory $\langle D, W \rangle$ into a default theory $\langle D', W' \rangle$ is faithful if and only if each extension of $\langle D, W \rangle$ is varequivalent w.r.t. the variables of $\langle D, W \rangle$ to at least one extension of $\langle D', W' \rangle$, and vice versa.

Equivalently, the set of the extensions of $\langle D', W' \rangle$, after forgetting [LLM03] the added variables, is exactly the same as the set of extensions of $\langle D, W \rangle$. According to this translation, a single extension E of $\langle D, W \rangle$ may correspond to several extensions of $\langle D', W' \rangle$, all varequivalent to E w.r.t. the variables of $\langle D, W \rangle$.

The translations used by Janhunen [Jan03] are faithful in this sense, but also require a bijection to exists between the extensions of the two theories.

Definition 4 (Bijective Faithful Translation) A translation that maps each default theory $\langle D, W \rangle$ into a default theory $\langle D', W' \rangle$ is bijective faithful if and only if each extension of $\langle D, W \rangle$ is var-equivalent w.r.t. the variables of $\langle D, W \rangle$ to exactly one extension of $\langle D', W' \rangle$, and vice versa.

This definition is only different from the previous one only because "is var-equivalent [...] to at least one extension" is replaced by "is var-equivalent [...] to exactly one extension". The second definition requires a bijection between the sets of extensions to exist. Every translation that satisfies Definition 4 also satisfies Definition 3, but not vice versa. Note that non-bijective faithful translations implicitly requires every extension of $\langle D', W' \rangle$ to be associated with a single extension of $\langle D, W \rangle$ but the converse does not necessarily hold.

A requirement we impose on the translations is that of being polynomial. There are two possible definitions of polynomiality, depending on what is required to be polynomial: the running time or the size of the produced output. This difference is important, as some translations require exponential time but still output a polynomially large theory. In this paper, we consider three kinds of translations:

polynomial: runs in polynomial time;

strongest extension: runs in polynomial time but require one of the strongest extension of the original theory;

polysize: produces a polynomially large result.

The existence of a polynomial-time translation from one semantics to another means that any theory expressed in the first semantics can be translated in polynomial time into an equivalent theory in the second one. Such translations are usually considered good from a computational point of view because they allow solving problems about the first semantics using procedures developed for the second one.

However, polynomial-time translations does not tell everything about the ability of semantics at representing knowledge. The existence or non-existence of polynomial-time translations do not give an answer to the question "is it true that, for every formula in the first semantics, there exists a formula in the second semantics that is equivalent to it and only polynomially larger than it?" A polysize translation from the first semantics to the second instead provides a positive answer to this question.

Finally, strongest-extension translations are a particular kind of polysize translations. They are considered separately from polysize translations because the time required by the translation, not only the size of its produced result, is still bounded by the complexity of finding one of the strongest extensions of the original theory. In turn, finding such an extension might be easy in particular cases such as, for example, when a single extension is introduced by changing the default theory as explained below.

2.3 Theories Having No Extensions

Semantics for default logics differ as for whether a theory might or might not have no extension. For example, the theory $\langle \frac{:a}{\neg a}, \emptyset \rangle$ has no extension in Reiter and rational default logic. All theories have at least one extension in justified and constrained default logics.

This argument has been used to prove that Reiter's default logics cannot be translated into justified or constrained default logic by Delgrande and Schaub [DS03], and that seminormal default theories cannot always be translated into normal default theories by Janhunen [Jan03].

In this paper, we consider translations that work in the assumption that the theory to be translated has some extension. The existence of extensions might be guaranteed because:

- 1. the theory encodes a domain in which it is known that an extension exists; for example, while using default logic for encoding problem of planning, the particular domain might guarantee the existence of a plan;
- 2. theories can be made having extensions by a simple translations that adds a single known extension to them:

$$\left\langle \left\{ \frac{:a}{a}, \frac{:\neg a}{\neg a} \right\} \cup \left\{ \frac{\neg a \land \alpha : \beta}{\gamma} \middle| \frac{\alpha : \beta}{\gamma} \in D \right\}, W \right\rangle$$

For all considered semantics, this theory has exactly the extensions of $\langle D, W \rangle$ with $\neg a$ added to them plus the single extension Cn(a). This proves that a very simple change can make theories guaranteed to have extensions. The resulting theory only have an easy recognizable added extension. Querying this theory produces results that are either identical to those of the original theory (e.g., entailment) or similar (counting the extensions).

This point is relevant to the present article because we prove that some translations are possible if the default theory to translate is assumed to have extensions. In particular, we show translations from rational to constrained default logic, and from Reiter to constrained and justified default logic; such translations would be impossible if the theory to be translated lacks extensions. We can therefore conclude that the possible lack of extensions is the only reason why such a translation is impossible in general.

Since these translations are polysize, checking the existence of extensions does not introduce an additional cost (in terms of size). The translation works even in the case of no extensions by slightly extending the syntax and semantics so that a default theory is either a pair $\langle D, W \rangle$ or the special symbol \bot , which is not assigned any extensions by the semantics. Thanks to this minimal change, the translation from, for example, Reiter to justified default logic can be extended to theories having no extensions by translating such theories into \bot .

A similar change can be done without changing the syntax or semantics but adding an exception to the definition of faithfulness, so that a theory with no extensions can be translated into a theory with a single inconsistent extension. This way, we can translate theories not having extensions into $\langle \emptyset, \{\bot\} \rangle$. This translation preserve the number of extensions unless the original theory have none, and also preserve the skeptical consequences exactly.

3 Polynomial Time Translations

In this section we show two polynomial-time bijective faithful translations, one from constrained to rational default logics, the other from justified to constrained default logics. We also show that the existence of such translations from either Reiter or rational default logic to either justified or constrained default logic implies that $\Sigma_2^p \subseteq \Pi_2^p$. We also prove that the existence of such translations from either constrained or rational default logic to either Reiter or justified default logic implies that $NP^{NP} = UP^{NP}$.

3.1 From Constrained to Rational

Constrained default logic can be translated into rational default logic by simply making all defaults seminormal. In other words, a theory $\langle D, W \rangle$ is translated into the following one.

$$T_{CR}\left(\frac{\alpha:\beta}{\gamma}\right) = \frac{\alpha:\beta \wedge \gamma}{\gamma}$$

$$T_{CR}(\langle D, W \rangle) = \langle \{T_{CR}(d) \mid d \in D\}, W \rangle$$

We prove that the processes of $\langle D, W \rangle$ are translated into the processes of $T_{CR}(\langle D, W \rangle)$.

Lemma 1 There exists a bijection between the constrained processes of $\langle D, W \rangle$ and the rational processes of $T_{CR}(\langle D, W \rangle)$ such that the extensions generated by two associated processes are equivalent.

Proof. We define the translation of a process $[d_1, \ldots, d_n]$ as $T_{CR}([d_1, \ldots, d_n]) = [T_{CR}(d_1), \ldots, T_{CR}(d_n)]$. Let Π be a constrained process of $\langle D, W \rangle$. We show that $\Pi' = T_{CR}(\Pi)$ is a rational process of $T_{CR}(\langle D, W \rangle)$.

- Π' is a process. The fact that $W \cup cons(\Pi'[d]) \models prec(d')$ for every $d' \in \Pi'$ follows from the fact that the same condition holds for the original process Π , and preconditions and consequences are not changed by the translation.
- Π' is globally successful. This condition holds because $just(\Pi') \cup cons(\Pi') = just(\Pi) \cup cons(\Pi)$, and Π is successful.
- Π' is closed. We have to prove that no default is applicable to Π' . Since Π is maximally globally successful, either $W \cup cons(\Pi) \not\models prec(d)$ or $\Pi \cdot [d]$ is not globally successful. In the first case, $d' = T_{CR}(d)$ is not rationally applicable to Π' . In the second case, we have that $W \cup just(\Pi \cdot [d]) \cup cons(\Pi \cdot [d])$ is not consistent. By definition, $just(\Pi' \cdot [d']) \cup cons(\Pi') = just(\Pi \cdot [d]) \cup cons(\Pi \cdot [d])$ because the translation adds the consequence of each default to its justification. As a result, we have that $W \cup just(\Pi') \cup cons(\Pi') \cup just(d')$ is inconsistent; therefore, d' cannot be applied to Π' .

We now show the converse: if $\Pi' = T_{CR}(\Pi)$ is a rational process of $T_{CR}(\langle D, W \rangle)$, then Π is a constrained process of $\langle D, W \rangle$. As before, we denote by d' the result of translating the single default d.

 Π is a process. As in the previous case, since preconditions and consequences are not changed by the reduction, if Π' is a process so is Π .

 Π is successful. $W \cup cons(\Pi) \cup just(\Pi)$ is consistent because it is equal to $W \cup cons(\Pi') \cup just(\Pi')$, which is consistent because Π' is a globally successful process.

 Π is maximally successful. We have to prove that, if $W \cup cons(\Pi) \models prec(d)$, then $\Pi \cdot [d]$ is not a successful process. Since Π' is closed according to global applicability, if $W \cup cons(\Pi') \models prec(d')$, then $W \cup cons(\Pi') \cup just(\Pi) \cup just(d')$ is inconsistent. Since the latter is equal to $W \cup cons(\Pi') \cup just(\Pi) \cup cons(d') \cup just(d') = W \cup cons(\Pi \cdot [d]) \cup just(\Pi \cdot [d])$, its inconsistency implies that $\Pi \cdot [d]$ is not maximally successful.

The following corollary easily follows.

Corollary 1 The constrained extensions of $\langle D, W \rangle$ and the rational extensions of $T_{CR}(\langle D, W \rangle)$ are the same.

In this very simple case we were able to show a faithful translation that does not introduce new variables, but this is not generally possible. Remarkably, beside the empty set of extensions, constrained and rational default logic are able to express exactly the same sets of extensions [Lib05].

3.2 From Justified to Constrained

The semantics of justified default logic is based on a local consistency check, in which each justification is checked against the combined consequences of all defaults in the process. This kind of consistency check can be simulated in constrained default logic by using a separate alphabet for each justification. Assume that the original theory contains m defaults d_1, \ldots, d_m and its variables are those in a set X. The i-th default is translated as follows.

$$T_{JC}\left(\frac{\alpha:\beta}{\gamma},i\right) = \frac{\alpha:\beta[X/X_i]}{\gamma \wedge \gamma[X/X_1] \wedge \dots \wedge \gamma[X/X_m]}$$

This translation assumes a total ordering over the defaults of the original theory. Processes are translated in the obvious way, while a default theory is translated as follows.

$$T_{JC}(\langle \{d_1,\ldots,d_m\},W\rangle = \langle \{T_{JC}(d_1,1),\ldots,T_{JC}(d_m,m)\},\ W\wedge W[X/X_1]\wedge\cdots\wedge W[X/X_m]\rangle$$

For every default d_i we have an alphabet X_i . The justification of each default d_i is translated into its associated alphabet X_i . Whenever a default is applied its consequence γ is drawn on all alphabets, so that each justification is checked separately.

Lemma 2 There exists a bijection between the justified processes of $\langle D, W \rangle$ and the constrained processes of $T_{JC}(\langle D, W \rangle)$ such that the extensions generated by two associated processes are var-equivalent w.r.t. the variables of $\langle D, W \rangle$.

Proof. We show a correspondence between each justified process Π of $\langle D, W \rangle$ and its corresponding process $\Pi' = T_{JC}(\Pi)$ of $T_{JC}(\langle D, W \rangle)$. By definition, $W \cup cons(\Pi)$ is var-equivalent to $W \wedge W[X/X_1] \wedge \cdots \wedge W[X/X_m] \cup cons(\Pi')$ because the background theory and the consequences of translated defaults contain the corresponding formulae of the original one and other ones that do not affect the value of the variables X.

We prove that Π is a justified process of the original theory if and only if $\Pi' = T_{JC}(\Pi)$ is a constrained process of the translated theory. The following sequence of equations relates Π and Π' :

$$\begin{split} W' \cup cons(\Pi') \cup just(\Pi') &= \\ &= W \cup \bigcup_{i=1,\dots,m} W[X/X_i] \cup \bigcup_{d_i' \in \Pi'} just(d_i') \cup \bigcup_{d_i' \in \Pi'} cons(d_i') \\ &= W \cup \bigcup_{d_i \notin \Pi} W[X/X_i] \cup \bigcup_{d_i \in \Pi} W[X/X_i] \cup \\ &\cup \bigcup_{d_i \in \Pi} just(d_i)[X/X_i] \cup \bigcup_{d_i \in \Pi} cons(d_i) \cup \bigcup_{\substack{d_j \in \Pi \\ d_i \notin \Pi}} cons(d_i) \cup \\ &\cup \bigcup_{d_i \notin \Pi} W[X/X_i] \cup \bigcup_{\substack{d_j \in \Pi \\ d_i \notin \Pi}} cons(d_j)[X/X_i] \\ &\cup \bigcup_{d_i \in \Pi} \left(W[X/X_i] \cup just(d_i)[X/X_i] \cup \bigcup_{d_j \in \Pi} cons(d_j)[X/X_i] \right) \\ &= W \cup \bigcup_{d_i \in \Pi} cons(d_i) \cup \\ &\cup \bigcup_{d_i \notin \Pi} \left(W \cup \bigcup_{d_j \in \Pi} cons(d_j) \right) [X/X_i] \\ &\cup \bigcup_{d_i \notin \Pi} \left(W \cup just(d_i) \cup \bigcup_{d_j \in \Pi} cons(d_j) \right) [X/X_i] \end{split}$$

The consistency of such formula is equivalent to the consistency of all formulae $W \cup just(d_i) \cup \bigcup_{d_i \in \Pi} cons(d_i)$ for every $d_i \in \Pi$, because all these formulae are on different alphabets and their consistency entails the consistency of $W \cup \bigcup_{d_i \in \Pi} cons(d_i)$ and of all its variants on the alphabets X_i for $d_i \notin \Pi$. The proof of the lemma is based on this fact: the global successfulness of Π' is equivalent to the local successfulness of Π .

Let us first assume that Π is a justified process of $\langle D, W \rangle$, and show that Π' is a constrained process of $T_{JC}(\langle D, W \rangle)$.

- Π' is a process. This is because the precondition of the defaults are not changed by the translation and the background theory and the consequence of each translated default d' are var-equivalent to the consequence of the original default d on the variables X.
- Π' is globally successful. This fact holds because the global successfulness of Π' is equivalent to the local successfulness of Π , as shown above.
- Π' is maximally globally successful. We have to prove that, for every default d'_i such that $W' \cup cons(\Pi') \models prec(d'_i)$, the formula $W' \cup cons(\Pi' \cdot [d'_i]) \cup just(\Pi' \cdot [d'_i])$ is inconsistent. Let us therefore assume that $W' \cup cons(\Pi') \models prec(d'_i)$, which implies that $W \cup cons(\Pi) \models prec(d_i)$. Since Π is a maximal locally successful process, we have that $\Pi \cdot [d_i]$ is not locally successful. As a result, $\Pi' \cdot [d'_i]$ is not globally successful.

Let us now assume that Π' is a constrained process of $T_{JC}(\langle D, W \rangle)$, and prove that Π is a justified process of $\langle D, W \rangle$.

- Π is a process. As in the previous case, since Π' is a process, we have that $W' \cup cons(\Pi'[d']) \models prec(d')$. Since prec(d') = prec(d) and $W' \cup cons(\Pi'[d'])$ is var-equivalent to $W \cup cons(\Pi)$ on the variables X, we have that $W \cup cons(\Pi) \models prec(d)$.
- Π is locally successful. This is because Π' is globally successful, and this condition implies the local success of Π' .
- Π is maximally locally successful. We have to show that, if $W \cup cons(\Pi) \models prec(d_i)$, then $\Pi \cdot [d_i]$ is not successful. The condition that $W \cup cons(\Pi) \models prec(d_i)$ implies that $W' \cup cons(\Pi') \models prec(d'_i)$. As a result, $\Pi' \cdot [d'_i]$ is not globally successful. As a result $\Pi \cdot [d_i]$ is not locally successful.

We have therefore proved that the justified processes of the original theory correspond to the constrained processes of the translated theory. The var-equivalence is guaranteed by the fact that the alphabets are disjoint. \Box

This lemma allows proving that the translation from justified to constrained default logic is faithful.

Corollary 2 There exists a bijection from the justified extensions of $\langle D, W \rangle$ to the constrained extensions of $T_{JC}(\langle D, W \rangle)$ such that two associated extensions are var-equivalent w.r.t. the variables of $\langle D, W \rangle$.

3.3 From Reiter or Rational to Justified or Constrained

Regarding translations from Reiter or rational default logic into justified or constrained default logic, of course the translation is in general impossible, as the first two semantics might not have extensions while the latter always has. We consider the specific case in which theories are known to have extensions.

We show that no bijective-faithful polynomial-time translation from rational default logic to any failsafe semantics (such as constrained or justified default logic) exists even if the original theory is known to have extensions, unless the polynomial hierarchy collapses. The same claim has been proved for Reiter semantics in another paper [Lib06].

This claim is proved by showing that the problem of entailment in theories having a single extension is $\Sigma_2^p \cap \Pi_2^p$ -hard for the rational semantics. The same problem is in Δ_2^p for every failsafe semantics because it amounts to generating a single extension, and this generation is in Δ_2^p because it can be done by applying one of the applicable defaults and iterating.

Lemma 3 The problem of deciding the existence of rational extensions of a theory having at most one rational extension and an empty background theory is Σ_2^p -hard.

Proof. Given $\exists X \forall Y.F$, we build the following theory.

$$\left\langle \left\{ \frac{: x_i z_i}{z_i \wedge (a \to x_i)}, \frac{: \neg x_i z_i}{z_i \wedge (a \to \neg x_i)} \right\} \cup \left\{ \frac{z_1 \dots z_n \wedge (a \to F) :}{\neg a}, \frac{z_1 \dots z_n : a}{\mathsf{false}} \right\}, \emptyset \right\rangle$$

The two defaults corresponding to the variable x_i have mutually inconsistent justifications $x_i z_i$ and $\neg x_i z_i$. Once one of them is applied, its justification disallows the application of the other one.

Since no other default can be applied until all z_i 's are derived, a process is not closed until either one of the two defaults associated to x_i is applied. What results from this application is a truth interpretation over the variables X conditioned to the variable a.

If this interpretation entails F regardless of the value of the variables Y, we can apply the default having $\neg a$ as a conclusion. The result of this application is that of making all formulae $a \to x_i$ and $a \to \neg x_i$ derived so far vacuous, and the last default not applicable.

On the other hand, unless a is derived by the application of this default, the last default can be applied generating a failure. This means that the theory has at most one extension, and that happens exactly when $\exists X \forall Y.F.$

This lemma can be used for deriving the complexity of the problem of entailment for theories having a single extension.

Theorem 1 The problem of entailment is $\Sigma_2^p \cap \Pi_2^p$ -hard for rational default logic even if the default theory is guaranteed to have exactly one extension.

Proof. We have shown that every problem in Σ_2^p can be reduced to checking whether a theory having zero or one extension has in fact one extension in rational default logic. Now, consider that for every instance of a problem in $\Sigma_2^p \cap \Pi_2^p$ we can produce two theories, the first having 0/1 extensions, and the second having 1/0 extensions, depending on the instance of the problem. We can then use a single variable separating the two cases: we combine these two theories $\langle D, \emptyset \rangle$ and $\langle D', \emptyset \rangle$ by introducing a new variable b, which is either true or false in each extension of the following default theory:

$$\left\langle \left\{ \frac{:b}{b} , \frac{:\neg b}{\neg b} \right\} \cup \left\{ \frac{b \wedge \alpha : \beta}{\gamma} \middle| \frac{\alpha : \beta}{\gamma} \in D \right\} \cup \left\{ \frac{\neg b \wedge \alpha : \beta}{\gamma} \middle| \frac{\alpha : \beta}{\gamma} \in D' \right\}, \emptyset \right\rangle$$

This theory has all extensions of the two original theories. In this case, it has a single extension, implying something or something else depending on the original instance. \Box

What has been proved so far using complexity classes is that there is no poly-time reduction from rational default logic to any fail-safe semantics.

Corollary 3 If there exists a bijective faithful polynomial-time translation from either Reiter or rational default logic to either justified or constrained default logics then $\Sigma_2^p \subseteq \Pi_2^p$.

3.4 From Constrained and Rational to Reiter and Justified

Delgrande and Schaub [DS03, DS05] showed polynomial-time faithful translations from all four considered semantics to Reiter default logic. The translations from constrained and rational default logic are bijective only when double extensions are used. This is because these two translations copy the justifications of defaults into their consequence. As a result, two constrained or rational processes generating the same extension correspond to two different Reiter extensions. The correspondence is instead a bijection when double extensions are used instead. Consider the following theory in either constrained or rational default logic.

$$\left\langle \left\{ \frac{:a}{b}, \frac{:\neg a}{b} \right\}, \emptyset \right\rangle$$

Constrained default logic selects two processes, each one composed of a single default. These two processes generate the same extension Cn(b), but two different double extensions $\langle a, Cn(b) \rangle$ and $\langle \neg a, Cn(b) \rangle$. This theory is translated into the following theory in Reiter default logic:

$$\left\langle \left\{ \frac{: a' \wedge b'}{b \wedge a' \wedge b'}, \frac{: \neg a' \wedge b'}{b \wedge \neg a' \wedge b'} \right\}, \emptyset \right\rangle$$

The Reiter processes of this theory are still composed of one default each. However, these processes generate not only two different double extensions $\langle a' \wedge b', Cn(b \wedge a' \wedge b') \rangle$ and $\langle \neg a' \wedge b', Cn(b \wedge \neg a' \wedge b') \rangle$, but also two different extensions $Cn(b \wedge a' \wedge b')$ and $Cn(b \wedge \neg a' \wedge b')$. This translation is therefore bijective on double extensions but not on extensions, as a single extension of the original theory corresponds to two extensions of the resulting theory.

An open question is therefore whether bijective faithful polynomial-time reductions from constrained and rational default logic to Reiter exist, when extensions, rather than double extensions, are considered. We show the effects of existence of such translations on complexity classes.

Checking whether a formula is equivalent to an extension of a default theory in the constrained or rational semantics is Σ_2^p complete [Lib05]. The following lemma proves that the problem remains hard even if the theory is known to have either one or two extensions, both known in advance.

Lemma 4 For any formula F over variables $X \cup Y$ one can build in polynomial time a default theory $\langle D, W \rangle$ whose rational and constrained extensions are $Cn(\neg a \land \neg b)$ and, if $\exists X \forall Y.F$ is valid, $Cn(\neg a \land b)$.

Proof. Let $X = \{x_1, \ldots, x_n\}$. The default theory corresponding to $\exists X \forall Y.F$ is $\langle D, \emptyset \rangle$ where:

$$D = \left\{ \frac{: x_i}{a \to x_i}, \ \frac{: \neg x_i}{a \to \neg x_i} \middle| \ 1 \le i \le n \right\} \cup \left\{ \frac{a \to F : \neg ab}{\neg ab}, \ \frac{: \neg a \neg b}{\neg a \neg b} \right\}$$

Applying the last default generates $\neg a \neg b$, which makes the second-last default inapplicable and entails all consequences of all other default. The formula $\neg a \neg b$ is therefore always an extension of this theory.

The application of the defaults in the first set in either the rational or the constrained semantics lead to a partial truth evaluation over the variables x_i conditioned to a. We can then apply the second-last default and generate b if and only if F is valid for this partial truth evaluation. As a result, this theory has always the extension $Cn(\neg a \land \neg b)$, and also has the extension $Cn(\neg a \land b)$ if and only if $\exists X \forall Y.F$ is valid.

The complexity of some problems easily follow from this lemma.

Corollary 4 Checking whether E is a rational or constrained extension of a default theory is Σ_2^p -hard, and this result holds even if the theory has either one or two extensions.

Corollary 5 Checking whether a default theory has at least two constrained or rational extensions is Σ_2^p -hard even if the theory has either one or two extensions.

Corollary 6 Checking whether a default theory skeptically entails a formula in the constrained or rational semantics is Π_2^p -hard even if the theory has either one or two extensions.

Assume that a bijective faithful polynomial time translation from constrained or rational default logic into Reiter or justified default logic exists. By Lemma 4, the validity of $\exists X \forall Y.F$ can be translated in polynomial time into the question of whether $\neg a \land b$ is equivalent to an extension of the theory $\langle D, W \rangle$ of Lemma 4. This question can in turn be translated into the question of whether the translated theory has an extension equivalent to $\neg a \land b \land G$ for some formula G not mentioning the variables in $\{a,b\} \cup X \cup Y$; this is indeed required for this extension to be var-equivalent to $Cn(\neg a \land b)$. On the other hand, the assumption that the translation is bijective ensures that at most one such formula G exists. This means that the problem can be solved in Reiter or justified default logic with an unambiguous Turing machine. Formally, we have the following result.

Theorem 2 If there exists a polynomial time bijective faithful translation from either constrained or rational default logic into Reiter or justified default logic then $NP^{NP} \subseteq UP^{NP}$.

Proof. The problem of whether E is equivalent to a constrained or rational extension of a default theory is Σ_2^p -hard even if the theory is known to have only one other extension which is inconsistent with E.

Assuming that a translation such in the statement of the theorem exists, the problem can be translated into checking the existence of a subset of the consequences of the translated theory E' such that EE' is an extension of the translated theory. On the other hand, if such an E' exists is unique. Therefore, the test can be done by unambiguously guessing a subset of the consequences E' and then checking whether EE' is a Reiter or justified extension of the translated theory; since the latter problem can be solved with a polynomial number of calls to an NP oracle [Ros99, Lib05], the whole problem would be in UP^{NP}.

4 Strongest-Extension Translations

In this section, we show some bijective faithful reductions that require polynomial time only once given one of the strongest extensions E of the original theory is known. Such translations are polynomial-time given a formula that is equivalent to E; since E is deductive closure of the consequences of some defaults in the theory, a formula of polynomial size that is equivalent to E exists. Since these translations produce a polynomially sized result, they are polynomial-size.

4.1 From Rational to Constrained

We show a reduction from rational to constrained default logic. This translation is proved to work by relating each rational process of the original theory with a constrained process of the translated theory. In the sequel, we refer to the rational process of the original theory as the "simulated process" and the constrained process of the translated theory as the "simulating process". We also refer to a formula $a \to \gamma$ as " γ conditioned to a". A simulated process is related to its simulating process as follows:

- 1. the consequences of the simulated process are derived conditioned to a (that is, prepended with $a \rightarrow$) in the simulating process;
- 2. the justifications of the simulated process are derived conditioned to b (that is, prepended with $b \rightarrow$) in the simulating process;
- 3. both consequences and justifications of the simulated process are in the justifications of the simulating process using a different alphabet.

Formally, every process Π is related to its simulating process Π' as follows:

$$cons(\Pi') = \{a \to \gamma \mid \gamma \in cons(\Pi)\} \cup \{b \to \beta \mid \gamma \in just(\Pi)\}$$

$$just(\Pi') = \{\gamma[X/X'] \mid \gamma \in cons(\Pi)\} \cup \{\beta[X/X'] \mid \gamma \in just(\Pi)\}$$

Let $\langle D, W \rangle$ be the original default theory, where $D = \{d_1, \ldots, d_m\}$. The background theory W is translated into $a \to W$. This means that [] and its simulated process satisfy the relationship above. We now define the defaults of the translated theory in such a way the above relationship remains satisfied.

For every default $d_i = \frac{\alpha:\beta}{\gamma}$ of the original theory, we have two defaults which correspond to different conditions of applicability of the original default in the original theory in the rational semantics.

$$\frac{a \to \alpha : \beta[X/X'] \land \gamma[X/X']}{z_i(a \to \gamma)(b \to \beta)}$$

The precondition of this default is $a \to \alpha$ is entailed in the simulating process if and only if the precondition of the original default is entailed by the simulated process.

The justification of this default is consistent if and only if the original default is applicable and does not produce a failure in the simulated default. Indeed, its justification is consistent with $just(\Pi')$ if and only if the set of all consequences and justifications of the simulated process is consistent with $\gamma \wedge \beta$.

Finally, the consequence of this default satisfies the condition that the consequences and justifications of the original default are added to the consequences of the simulating process with the assumptions a and b.

$$\frac{(a \to \alpha)(ab \to \neg \beta):}{z_i}$$

This default is only applicable when the precondition of the original default is entailed but its justification is inconsistent with the current set of justifications and consequences. This is because all justifications and consequences of the simulated process can be obtained by assuming both a and b. Therefore, if assuming both a and b then β is false, then we are in the situation in which the original default cannot be applied because of its justification.

These two defaults can be applied only if the precondition of the original default is entailed. Moreover, the first default can be applied if the simulated default can be applied without generating a failure. The second default can be applied only if the simulated default cannot be applied. Therefore, if $a \to \alpha$ is true, z_i can be produced unless the simulated default can be applied but produces a failure. The idea is that we generate a whenever we are in the condition in which all applicable defaults are applied but a failure is not generated. This is obtained by the following default.

$$\frac{\bigwedge_{d_i \in D} ((a \to prec(d_i)) \to z_i) : a \neg E}{a \neg b z_1 \dots z_m}$$

This default can be applied only if, for each default whose precondition is entailed ($(a \rightarrow prec(d_i))$ is true), either the default cannot be applied or it can be applied without generating a failure (z_i is true). Therefore, this default can be applied only if the simulated process is successful and closed (the justification $\neg E$ is explained below.)

The consequence of this default include a, thus making all consequences that have been derived so far unconditioned. It also includes $\neg bz_1 \dots z_m$; this formula entails all formulae conditioned to b and all formulae z_i derived so far. This way, the resulting extension does not depend on which formulae $b \to \beta$ and z_i have already been generated.

In order for the theory to simulate the original one we have to generate E whenever the simulated process ends up in a failure. To simplify the matter, we allow E to be generated at any time unless the above default has been applied. In the other way around, if we can arrive to a point in which the above default is applied, we do not generate E; in all other cases E can be generated.

$$\frac{:}{\neg a \neg bEz_1 \dots z_m}$$

The consequence of this default include $\neg a$, $\neg b$, and $z_1 \dots z_m$, which entail all consequences of the defaults that have already been applied. This way, the generated extension does not depend on which other defaults have been applied.

The only case in which this default cannot be applied is when a has been already derived. Such a derivation cannot be accomplished by the defaults above because we always check the consistency of a formula γ before generating $a \to \gamma$. Therefore, this default is blocked only if a has not be generated by the previous default.

We can therefore conclude that, if the simulated process if successful and closed we can apply the previous default and generate the corresponding extension. In all other cases, this last default is applicable, and generates the known extension E.

The precondition $\neg E$ of the second-last default is used to avoid E to be generated both by this and the last previous default. In order for this to work, we require E to be an extension such that $E' \models E$ does not hold for any other extension E'. In this case, the inconsistency of E and the extension to be different from E are the same condition. As a result, if generating E a produces an extension that is different from E, then the default cannot be applied because the generated extension would be inconsistent with its justification.

The rationale of the proof can be therefore summarized as follows:

1. the consequences of the simulated theory are drawn conditioned to a, and preconditions are checked conditioned to a;

- 2. the justifications of the simulated theory are drawn conditioned to b;
- 3. both consequences and justifications of the original theory are in the justifications of this theory but rewritten in another alphabet;
- 4. we can always generate $\neg a \neg bE$;
- 5. whenever all defaults that can be applied are applied, and that does not result in a failure, we generate $a\neg b$, making all consequences unconditioned and all justifications void:
- 6. we generate $a\neg b$ only when the produced extension is different from E.

4.1.1 Formal Definition of the Translation

Let us now formally prove the correspondence between each theory and its translation. We assume that $\langle D, W \rangle$, where $D = \{d_1, \ldots, d_m\}$, is a theory that has extensions and that E is one of its strongest extensions, that is, for no other extension E' it holds $E' \models E$. We prove some claims relating the processes of the original and translated theory. First, we define the following two functions.

$$T_{RC}^{e}\left(\frac{\alpha:\beta}{\gamma}, i\right) = \frac{a \to \alpha: \beta[X/X'] \land \gamma[X/X']}{z_{i}(a \to \gamma)(b \to \beta)}$$
$$T_{RC}^{n}\left(\frac{\alpha:\beta}{\gamma}, i\right) = \frac{(a \to \alpha)(ab \to \neg \beta):}{z_{i}}$$

The translated default theory is obtained by translating each default separately to two ones and then adding the following two further defaults to it.

$$T_{RC}^{g}(\{d_{1},\ldots,d_{m}\}) = \frac{\bigwedge_{d_{i}\in D}((a\rightarrow prec(d_{i}))\rightarrow z_{i}): a\neg E}{a\neg bz_{1}\ldots z_{m}}$$

$$T_{RC}^{s}(\{d_{1},\ldots,d_{m}\}) = \frac{\vdots}{\neg a\neg bEz_{1}\ldots z_{m}}$$

The translation is defined as follows.

$$T_{RC}(\langle \{d_1, \dots, d_m\}, W \rangle) = \langle T_{RC}(D), T_{RC}(W) \rangle$$
where
$$T_{RC}(D) = \{ T_{RC}^e(d_i, i), T_{RC}^n(d_i, i) \mid 1 \le i \le m \} \cup \{ T_{RC}^g(\{d_1, \dots, d_m\}), T_{RC}^s(\{d_1, \dots, d_m\}) \}$$

$$T_{RC}(W) = (a \to W) \wedge W[X/X']$$

4.1.2 Preliminary Results

In this section we show some general properties of propositional entailment. In particular, we consider formulae in the form $a \to A$, that is, formulae that are "conditioned" to a given variable a. The translation uses formulae conditioned to variables in the background theory and in the preconditions and consequences of some defaults.

Lemma 5 For any triple of formulae A, B, and C not containing the variables a and b, it holds $A \models C$ if and only if $(a \rightarrow A) \land (b \rightarrow B) \models a \rightarrow C$.

Proof. We first assume to the contrary, that $A \models C$ but $(a \to A) \land (b \to B) \not\models (a \to C)$. The latter implies that $(a \to A) \land (b \to B) \land \neg (a \to C)$ is satisfiable. This formula is equivalent to the following other ones:

$$(a \to A) \land (b \to B) \land \neg (a \to C) \equiv (a \to A) \land (b \to B) \land \neg (\neg a \lor C)$$
$$\equiv (a \to A) \land (b \to B) \land a \land \neg C$$
$$\equiv A \land (b \to B) \land a \land \neg C$$

If this formula is consistent, there exists a model that satisfies both A and $\neg C$, thus violating the assumption that $A \models C$.

Let us now prove the converse. We assume that $(a \to A) \land (b \to B) \models (a \to C)$ but $A \not\models C$. The latter condition implies that there exists a model M that satisfies both A and $\neg C$. Let us consider the model M' that extends M by the assignment $a = \mathsf{true}$ and $b = \mathsf{false}$. This model satisfies $a, A, \neg C, \text{ and } b \to B$. As a result, it satisfies $A \land (b \to B) \land a \land \neg C,$ which we proved to be equivalent to $(a \to A) \land (b \to B) \land \neg (a \to C)$. As a result, $(a \to A) \land (b \to B) \not\models a \to C$, contrary to the assumption.

The second lemma is about conditioning with two variables.

Lemma 6 For any triple of formulae A, B, and C not containing the variables a and b, it holds $AB \models C$ if and only if $(a \rightarrow A) \land (b \rightarrow B) \models (ab \rightarrow C)$.

Proof. Assume that $AB \models C$. We have that abAB is equivalent to $ab(a \to A)(b \to B)$. Since AB entails C, we have that $ab(a \to A)(b \to B) \models C$. This condition can be rewritten as $(a \to A)(b \to B) \models (ab \to C)$.

Let us now prove the converse. Assume that $(a \to A) \land (b \to B) \models (ab \to C)$ but $AB \not\models C$. Then, we have a model that satisfies AB and $\neg C$ at the same time. By adding the assignment of $a = \mathsf{true}$ and $b = \mathsf{true}$ we obtain a model that satisfies a, b, A, B, and $\neg C$. This formula therefore satisfies $(a \to A) \land (b \to B)$ but does not satisfy $ab \to C$, contrary to the assumption.

In the following, we use the above lemmas together with the following property.

Property 1 If K is satisfiable and does not share any variables with A and C, then $A \models C$ if and only if $K \land A \models C$.

4.1.3 e-Sequences

We show a correspondence between each sequence of defaults Π of the original theory and the following sequence of defaults of the translated theory:

$$T_{RC}^e([d_{i_1},\ldots,d_{i_k}]) = [T_{RC}^e(d_{i_1},i_1),\ldots,T_{RC}^e(d_{i_k},i_k)]$$

The consequences of Π and $T_{RC}^e(\Pi)$ are related as follows:

$$T_{RC}(W) \wedge cons(T_{RC}^{e}(\Pi)) \equiv$$

$$\equiv (a \to W) \wedge W[X/X'] \wedge cons(T_{RC}^{e}(\Pi))$$

$$\equiv (a \to W) \wedge W[X/X'] \wedge \bigcup_{d_{j} \in \Pi} \left(z_{j} \wedge (a \to cons(d_{j})) \wedge (b \to just(d_{j})) \right)$$

$$\equiv W[X/X'] \wedge \left(\bigcup_{d_{j} \in \Pi} z_{j} \right) \wedge (a \to (W \wedge cons(\Pi))) \wedge (b \to just(\Pi))$$
(1)

Entailment of the precondition of a default from a sequence Π corresponds to the same condition on the translated theory and sequence, as the following lemma shows.

Lemma 7 It holds $W \cup cons(\Pi) \models prec(d_i)$ if and only if $T_{RC}(W) \wedge cons(T_{RC}^e(\Pi)) \models prec(T_{RC}^e(d_i, i))$.

Proof. By the above correspondence between the conclusions of Π and of $T_{RC}^e(\Pi)$, the condition $T_{RC}(W) \wedge cons(T_{RC}^e(\Pi)) \models prec(T_{RC}^e(d_i, i))$ can be rewritten as follows:

$$W[X/X'] \land \left(\bigcup_{d_i \in \Pi} z_j\right) \land (a \to (W \land cons(\Pi))) \land (b \to just(\Pi)) \models a \to prec(d_i)$$

Since W[X/X'] and $\cup z_i$ do not share variables with the other formulae, by Property 1 this condition is equivalent to:

$$(a \to (W \land cons(\Pi))) \land (b \to just(\Pi)) \models a \to prec(d_i)$$

By Lemma 5, this condition is equivalent to
$$W \cup cons(\Pi) \models prec(d_i)$$
.

This lemma proves that the precondition of a default d_i is entailed after the application of Π in the original theory if and only if the same condition holds for the translated theory and defaults. The following result is an immediate consequence of this lemma.

Corollary 7 A sequence of defaults Π is a process of $\langle D, W \rangle$ if and only if $T_{RC}^e(\Pi)$ is a process of $T_{RC}(\langle D, W \rangle)$.

The justifications of Π and of $T_{RC}^e(\Pi)$ are related in a similar way. In particular, we can show the following equivalence about global consistency.

$$T_{RC}(W) \cup just(T_{RC}^{e}(\Pi)) \cup cons(T_{RC}^{e}(\Pi))$$

$$\equiv (a \to W) \wedge W[X/X'] \cup just(T_{RC}^{e}(\Pi)) \cup cons(T_{RC}^{e}(\Pi))$$

$$\equiv (a \to W) \wedge W[X/X'] \cup$$

$$\bigcup_{d_{i} \in D} just(d)[X/X'] \wedge cons(d)[X/X'] \cup$$

$$\bigcup_{d_{i} \in \Pi} z_{i}(b \to just(d))(a \to cons(d))$$

$$\equiv \bigcup_{d_{i} \in \Pi} z_{i} \cup$$

$$W[X/X'] \cup just(\Pi)[X/X'] \cup cons(\Pi)[X/X'] \cup$$

$$(a \to W) \cup (a \to cons(\Pi)) \cup (b \to just(\Pi))$$

$$\equiv \bigcup_{d_i \in \Pi} z_i \cup (W \cup just(\Pi) \cup cons(\Pi))[X/X'] \cup (a \to (W \cup cons(\Pi)) \cup (b \to just(\Pi)))$$
(2)

The consistency of this formula is easy to relate to the corresponding formula of Π .

Lemma 8 A sequence of defaults Π is a globally successful process of $\langle D, W \rangle$ if and only if $T_{RC}^e(\Pi)$ is a globally successful process of $T_{RC}(\langle D, W \rangle)$.

Proof. Formula 2 is consistent if and only if $W \cup just(\Pi) \cup cons(\Pi)$ is consistent. Indeed, Formula 2 contains $W \cup just(\Pi) \cup cons(\Pi)$ rewritten on a new alphabet plus other formulae that are always satisfiable by setting both a and b to false.

This property can be pushed a little further by showing that if $T_{RC}^e(\Pi)$ is a globally successful process, then its justifications and consequences are not only consistent with the background theory but also with any other set of variables from $Z \cup \{a\}$.

Lemma 9 The process $T_{RC}^e(\Pi)$ is globally successful if and only if $T_{RC}(W) \cup cons(T_{RC}^e(\Pi)) \cup just(T_{RC}^e(\Pi)) \cup Z \cup \{a, \neg b\}$ is consistent.

Proof. If $T_{RC}^e(\Pi)$ is globally successful, by Equation (2) $(W \cup just(\Pi) \cup cons(\Pi))[X/X']$ is consistent, which means that $W \cup just(\Pi) \cup cons(\Pi)$ is also consistent, that is, it has a model. By adding the assignment $a = \mathsf{true}, \ b = \mathsf{false}, \ \mathsf{and} \ z_i = \mathsf{true}, \ \mathsf{we} \ \mathsf{obtain} \ \mathsf{a} \ \mathsf{model} \ \mathsf{of} \ T_{RC}(W) \cup cons(T_{RC}^e(\Pi) \cup just(T_{RC}^e(\Pi)) \cup Z \cup \{a, \neg b\}.$

This lemma adds something to the previous one: not only Π is globally successful if and only if $T_{RC}^e(\Pi)$ is globally successful, but the addition of other defaults having a subset of $Z \cup \{a, \neg b\}$ as consequences is irrelevant to the successfulness of $T_{RC}^e(\Pi)$.

Having proved that processes correspond to processes and successful processes correspond to successful processes, what remains to be proved is that closed processes correspond to closed processes. This is however not true in the presented reduction, which is based on the idea of treating specially those processes for which a default is globally applicable but would lead to global inconsistency.

4.1.4 n-Sequences

For any sequence of defaults Π of $\langle D, W \rangle$, we consider the following sequence of defaults of $T_{RC}(\langle D, W \rangle)$. In this formula, Π denotes a sequence of elements. The following is not necessarily a process, nor Π has been assume to be a process.

$$T^n_{RC}(\Pi) = T^e_{RC}(\Pi) \cdot \prod_{\substack{W \cup cons(\Pi) \models prec(d_i) \\ W \cup just(\Pi) \cup cons(\Pi) \cup just(d_i) \models \bot}} T^n_{RC}(d_i, i)$$

The following lemma is about the preconditions of the defaults $T_{RC}^n(d_i, i)$ that are in a sequence $T_{RC}^n(\Pi)$.

Lemma 10 If $T_{RC}^n(d_i, i)$ is a default in $T_{RC}^n(\Pi)$, then $T_{RC}(W) \cup cons(T_{RC}^e(\Pi)) \models prec(T_{RC}^n(d_i, i))$ if and only if d_i is not globally applicable to Π .

Proof. The assumption that $T_{RC}^n(d_i, i) \in T_{RC}^n(\Pi)$ implies that $W \cup cons(\Pi) \models prec(d_i)$. Therefore, d_i is globally applicable to Π if and only if $W \cup cons(\Pi) \cup just(\Pi) \cup just(d_i)$ is consistent.

The precondition of $T^n_{RC}(d_i, i)$ is a conjunction of the precondition of $T^e_{RC}(d_i, i)$ and $ab \to \neg just(d_i)$. By Lemma 7, the precondition of $T^e_{RC}(\Pi)$ is entailed by $T^e_{RC}(\Pi)$ if and only if $W \cup cons(\Pi) \cup prec(d_i)$, which is true by assumption.

Regarding the second condition, the entailment of $ab \to \neg just(d_i)$ from $T_{RC}^e(\Pi)$ in formulae is:

$$W[X/X'] \land \Big(\bigcup_{d_j \in \Pi} z_j\Big) \land (a \to (W \land cons(\Pi))) \land (b \to just(\Pi)) \models ab \to \neg just(d_i)$$

The formulae W[X/X'] and $\cup z_i$ can be neglected by Property 1 because they do not share variables with the other formulae. By Lemma 6, the resulting condition $(a \to (W \land cons(\Pi))) \land (b \to just(\Pi)) \models ab \to \neg just(d_i)$ is equivalent to $W \cup cons(\Pi) \cup just(\Pi) \models \neg just(d_i)$, which is the opposite of the global applicability of d_i to Π because $W \cup cons(\Pi) \models prec(d_i)$ holds by assumption.

The fact that the defaults $T_{RC}^n(d_i, i)$ have no justifications and a very simple consequence has the effect that their order in $T_{RC}^n(\Pi)$ does not matter.

Lemma 11 For any sequence Π , the sequence $T_{RC}^n(\Pi)$ is a globally successful process if and only if $T_{RC}^e(\Pi)$ is a globally successful process and $T_{RC}(W) \cup cons(T_{RC}^e(\Pi)) \models prec(T_{RC}^n(d_i, i))$ for every $T_{RC}^n(d_i, i) \in T_{RC}^n(\Pi)$.

Proof. The defaults $T_{RC}^n(d_i, i)$ do not have justifications, and their consequences are contained in Z. As a result, the set of justifications and consequences of $T_{RC}^n(\Pi)$ is exactly the same as that of its first part $T_{RC}^e(\Pi)$ with a subset of Z added to it. By Lemma 9, this set is consistent if and only if $T_{RC}^e(\Pi)$ is globally successful.

Regarding these sequences being processes or not, the consequence of a defaults $T_{RC}^n(d_i, i)$ does not affect the precondition of another default of the same kind $T_{RC}^n(d_j, j)$. Therefore, two defaults of this kind can always be swapped. As a result, if $T_{RC}^n(\Pi)$ is a process then any of its defaults $T_{RC}^n(d_i, i)$ can be moved to be immediately after $T_{RC}^e(\Pi)$. This proves that $T_{RC}^e(\Pi) \cdot [T_{RC}^n(d_i, i)]$ must be a process, which is the same as $T_{RC}(W) \cup cons(T_{RC}^e(\Pi)) \models prec(T_{RC}^n(d_i, i))$ because the default $T_{RC}^n(d_i, i)$ has no justification. The converse is true by the monotonicity of the underlying logic.

These two lemmas can be condensed as follows.

Corollary 8 For any sequence Π , the sequence $T_{RC}^n(\Pi)$ is a globally successful process if and only if $T_{RC}^e(\Pi)$ is a globally successful process and, for any d_i such that $T_{RC}^n(d_i, i) \in T_{RC}^n(\Pi)$, it holds that d_i is not globally applicable to Π .

The condition of $T^n_{RC}(\Pi)$ being a globally successful process can be linked to Π being a rational process.

Lemma 12 The sequence Π is a rational process of $\langle D, W \rangle$ if and only if $T_{RC}^n(\Pi)$ is a globally successful process of $T_{RC}(\langle D, W \rangle)$.

Proof. The sequence Π is a rational process if and only if Π is globally successful and every default not in Π is not globally applicable to Π .

The global success of Π is equivalent to the global success of $T_{RC}^e(\Pi)$ by Lemma 8. We therefore only have to prove that every $d_i \notin \Pi$ is not globally applicable to Π if and only if, for every $T_{RC}^n(d_i,i) \in T_{RC}^n(\Pi)$, it holds that $T_{RC}^e(\Pi) \cdot [T_{RC}^n(d_i,i)]$ is a process. On the other hand, the above corollary proves exactly this claim.

4.1.5 Permutation of Defaults

The correspondence between the processes of the original and the translated theory is not bijective. Indeed, many processes of the translated theory generate the extension E, while the same extension can be generated by one or few processes in the original theory. One reason is that more than one constrained process might generate an extension that is varequivalent to E. On the other hand, we can prove that all such processes generate the same extension.

Lemma 13 All constrained processes of $T_{RC}(\langle D, W \rangle)$ containing $T_{RC}^s(D)$ generate the extension $W[X/X'] \neg a \neg b E z_1 \dots z_m$.

Proof. The formula $W[X/X'] \neg a \neg bEz_1 \dots z_m$ is the conjunction of the background theory and the conclusion of $T_{RC}^s(D)$. If a process contains this default, its generated extension contains this formula. We therefore only have to prove that the generated extension does not include other formulae that are not entailed by this one.

Let Π be a rational process of $T_{RC}(\langle D, W \rangle)$ that contains $T_{RC}^s(D)$. Since this process is successful and this default has $\neg a$ as a conclusion, the process does not contain $T_{RC}^e(D)$, which contains a as a precondition. All other defaults in $T_{RC}(\langle D, W \rangle)$ have consequences that are entailed by $\neg a \neg bEz_1 \dots z_m$; therefore, their presence in the process does not affect the generated extension.

This lemma shows that all processes containing $T_{RC}^s(D)$ generate the same extension, which is var-equivalent to E. Therefore, we can exclude these processes and the extension E from consideration. In other words, we have to prove a bijection between the extensions of the two theories besides the extension E and $W[X/X'] \neg a \neg b E z_1 \dots z_m$. What we actually prove is that there is a bijection between processes modulo permutation on the order of the defaults.

Lemma 14 A constrained processes of $T_{RC}(\langle D, W \rangle)$ not containing $T_{RC}^s(D)$ contains $T_{RC}^g(D)$, and therefore does not generate an extension that is var-equivalent to E.

Proof. The default $T_{RC}^s(D)$ has no precondition and no justification. It is therefore applicable to every process, provided that its consequence is not inconsistent with the conclusions and justifications of the other defaults in the process. On the other hand, the consequence of $T_{RC}^s(D)$ is consistent with all justifications and conclusions of all defaults but $T_{RC}^g(D)$. Therefore, if $T_{RC}^s(D)$ is not in a process, this process must include $T_{RC}^g(D)$. Since $\neg E$ is a justification of this default, the generated extension cannot be var-equivalent to E.

We have therefore divided the constrained processes of $T_{RC}(\langle D, W \rangle)$ into two groups: those containing $T_{RC}^s(D)$ and generating the extension $W[X/X'] \neg a \neg bEz_1 \dots z_m$ and those

including $T_{RC}^g(D)$ and generating an extension that is not var-equivalent to E. The consequences of the translated defaults are all in the form $a \to \gamma$ and $b \to \beta$.

We now prove that processes can be put in a normal form in which defaults $T_{RC}^e(d,i)$ occur first. We first prove that these defaults can always be put before defaults $T_{RC}^n(d,i)$.

Lemma 15 If $T_{RC}(\langle D, W \rangle)$ has a globally successful process in which a default $T_{RC}^e(d, i)$ follows a default $T_{RC}^n(d, i)$, this default theory also has a globally successful process in which the two defaults are swapped.

Proof. The only condition that makes swapping two consecutive defaults in a process impossible is when the precondition of the second default is not entailed without the consequence of the first. It is easy to show that this is not the case in the assumption of the lemma.

Indeed, the consequence of $T_{RC}^n(d_j, j)$ is z_j . The background theory does not contain z_j , while the conclusions of all other defaults either do not contain z_j or are in the form $z_j \wedge A$, for some formula A. Since the precondition of $T_{RC}^e(d_i, i)$ does not contain z_j , Property 1 proves that this precondition is entailed from the previous defaults if and only if it is entailed by the previous defaults minus $T_{RC}^n(d_j, j)$.

We now prove that a default $T_{RC}^e(d,i)$ cannot follow the default $T_{RC}^g(D)$.

Lemma 16 No constrained process of $T_{RC}(\langle D, W \rangle)$ contains $T_{RC}^g(D)$ followed by $T_{RC}^e(d,i)$.

Proof. Consider the first default $T_{RC}^e(d,i)$ that follows $T_{RC}^g(D)$. All defaults between these two are in the form $T_{RC}^n(d,i)$ because this process does not contain $T_{RC}^s(D)$ and $T_{RC}^e(d,i)$ is the first one after $T_{RC}^g(D)$. By Lemma 15, the default $T_{RC}^e(d,i)$ can be moved immediately after the default $T_{RC}^g(D)$. In other words, if there exists a globally successful process in which $T_{RC}^e(d,i)$ follows $T_{RC}^g(D)$, then the following is also a globally successful process:

$$\Pi = \Pi_1 \cdot [T_{RC}^g(D), T_{RC}^e(d_i, i)] \cdot \Pi_2$$

This is a process. Therefore, the precondition of $T_{RC}^e(d_i, i)$ is entailed by $\Pi_1 \cdot [T_{RC}^g(D)]$, which can be rewritten as:

$$\begin{split} T_{RC}(W) \cup cons(\Pi_1) \cup cons(T_{RC}^g(D)) &\models prec(T_{RC}^e(d_i,i)) \\ &\text{iff} \quad (a \to W) \land W[X/X'] \cup \bigcup_{\substack{T_{RC}^e(d_j,j) \in \Pi_1 \\ T_{RC}^n(d_j,j) \in \Pi_1}} cons(T_{RC}^n(d_j,j)) \cup \{a, \neg b, z_1, \dots, z_m\} \models a \to prec(d_i) \\ &\text{iff} \quad (a \to W) \land W[X/X'] \cup \bigcup_{\substack{T_{RC}^e(d_j,j) \in \Pi_1 \\ T_{RC}^n(d_j,j) \in \Pi_1}} z_j \big(\cup \{a, \neg b, z_1, \dots, z_m\} \models a \to prec(d_i) \\ &\text{since a and b are true and false, respectively} \\ &\text{iff} \quad W \land W[X/X'] \cup \bigcup_{\substack{T_{RC}^e(d_j,j) \in \Pi_1 \\ T_{RC}^e(d_j,j) \in \Pi_1}} z_j cons(d_j) \cup \\ &\bigcup_{\substack{T_{RC}^e(d_j,j) \in \Pi_1 \\ T_{RC}^e(d_j,j) \in \Pi_1}} z_j \cup \{a, \neg b, z_1, \dots, z_m\} \models prec(d_i) \\ &\bigcup_{\substack{T_{RC}^e(d_j,j) \in \Pi_1 \\ T_{RC}^e(d_j,j) \in \Pi_1}}} z_j \cup \{a, \neg b, z_1, \dots, z_m\} \models prec(d_i) \end{split}$$

removing subformulae according to Property 1 iff
$$W \cup \bigcup_{T_{RC}^e(d_j,j) \in \Pi_1} cons(d_j) \cup \{a\} \models a \rightarrow prec(d_i)$$
 iff $W \cup \bigcup_{T_{RC}^e(d_j,j) \in \Pi_1} cons(d_j) \models prec(d_i)$ by Lemma 5 iff $(a \rightarrow W) \cup \bigcup_{T_{RC}^e(d_j,j) \in \Pi_1} (a \rightarrow cons(d_j)) \models a \rightarrow prec(d_i)$

The formula preceding \models is a subformula of $T_{RC}(W) \cup cons(\Pi_1)$. As a result, $T_{RC}(W) \cup cons(\Pi_1) \models a \rightarrow prec(d_i)$.

By assumption, $T_{RC}^g(D)$ is applied after Π_1 . Therefore, all its preconditions must be entailed at this point. In particular, $(a \to prec(d_i)) \to z_i$ must be entailed. Since $a \to prec(d_i)$ is true after Π_1 , then z_i must be true as well. Since $T_{RC}^e(d_i, i)$ is not in Π_1 by assumption, the only remaining default having d_i as a consequence is $T_{RC}^n(d_i, i)$. Therefore, we have that Π_1 contains $T_{RC}^n(d_i, i)$.

A consequence of this fact is that the precondition of $T_{RC}^n(d_i, i)$ is entailed from Π_1 . Let us focus on the second part of the precondition:

$$\begin{array}{ll} T_{RC}(W) \cup cons(\Pi_1) \models ab \rightarrow \neg just(d_i) \\ & \text{iff} \quad (a \rightarrow W) \wedge W[X/X'] \cup \\ & \bigcup_{T_{RC}^e(d_j,j) \in \Pi_1} cons(T_{RC}^e(d_j,j)) \cup \bigcup_{T_{RC}^n(d_j,j) \in \Pi_1} cons(T_{RC}^n(d_j,j)) \\ & \models ab \rightarrow \neg just(d_i) \\ & \text{iff} \quad (a \rightarrow W) \wedge W[X/X'] \cup \\ & \bigcup_{T_{RC}^e(d_j,j) \in \Pi_1} z_j \\ & \cup \bigcup_{T_{RC}^e(d_j,j) \in \Pi_1} z_j \\ & \vdash ab \rightarrow \neg just(d_i) \\ & \text{removing the irrelevant parts by Property 1} \\ & \text{iff} \quad (a \rightarrow W) \cup \bigcup_{T_{RC}^e(d_j,j) \in \Pi_1} (a \rightarrow cons(d_j))(b \rightarrow just(d_j)) \models ab \rightarrow \neg just(d_i) \\ & \text{by Lemma 6} \\ & \text{iff} \quad W \cup \bigcup_{T_{RC}^e(d_j,j) \in \Pi_1} cons(d_j) \wedge just(d_j) \models \neg just(d_i) \\ & \text{replacing X with X' everywhere} \\ & \text{iff} \quad W[X/X'] \cup just(\Pi_1) \cup cons(\Pi_1) \models \neg just(d_i)[X/X'] \end{array}$$

The latter formula implies that $W[X/X'] \cup just(\Pi_1) \cup cons(\Pi_1) \cup just(T_{RC}^e(d_i, i))$ is inconsistent. As a result, the process Π is not globally successful, contradicting the assumption. \square

The latter two lemmas, together, implies that every constrained process of $T_{RC}(\langle D, W \rangle)$ can be put in a sort of "normal form".

Corollary 9 For each globally successful process of $T_{RC}(\langle D, W \rangle)$ containing $T_{RC}^g(D)$, there exists another successful process that is composed of the same defaults, but all defaults $T_{RC}^e(d_i, i)$ came first, followed by some defaults $T_{RC}^n(d_i, i)$, followed by $T_{RC}^g(D)$ followed by some other defaults $T_{RC}^n(d_i, i)$.

4.1.6 Correspondence of Extensions

The correspondence between the rational processes of the original theory and the constrained processes of the translated theory is obtained as follows. For each sequence of defaults Π of the original theory, we consider the following sequence of the translated theory.

$$T_{RC}(\Pi) = T_{RC}^n(\Pi) \cdot [T_{RC}^g(D)] \tag{3}$$

We establish the following correspondence: each rational process Π of $\langle D, W \rangle$ not having E as an extension corresponds to the constrained process $T_{RC}(\Pi)$ of $T_{RC}(\langle D, W \rangle)$, and vice versa. The converse is true in the sense that for every constrained process of $T_{RC}(\langle D, W \rangle)$ there is an equivalent constrained process in which all defaults are in the form of $T_{RC}(\Pi)$.

Lemma 17 If $T_{RC}(\Pi)$ is a globally successful process, then there exists a sequence Π' such that $T_{RC}(\Pi) \cdot \Pi'$ is a constrained process of $T_{RC}(\langle D, W \rangle)$ and for all such Π' it holds $cons(T_{RC}(\Pi)) \models cons(\Pi')$.

Proof. If $T_{RC}(\Pi)$ is a globally successful process, it can be a non-constrained process only because it is not maximal. On the other hand, the only applicable defaults are in the form $T_{RC}^n(d_i,i)$ because of Lemma 14 and Lemma 16. The consequences of these defaults are entailed by that of $T_{RC}^g(D)$.

Lemma 18 A formula E' that is not var-equivalent to E is an extension of $T_{RC}(\langle D, W \rangle)$ if and only if $E' = T_{RC}(W) \cup cons(T_{RC}(\Pi))$ and $T_{RC}(\Pi)$ is a globally successful process.

Proof. If $T_{RC}(\Pi)$ is a globally successful process, then it can be completed to form a constrained process by adding to it some defaults whose consequences are already entailed by $T_{RC}(\Pi)$. Since $T_{RC}(\Pi)$ contains the default $T_{RC}^g(D)$, which has $\neg E$ as a justification, the generated extension is not E.

Let us assume that E is a constrained extension of $T_{RC}(\langle D, W \rangle)$ that is not equivalent to E'. By Lemma 13 and Lemma 14, its generating process contains $T_{RC}^g(D)$. By Corollary 9, the default theory contains a process with the same defaults in which the defaults $T_{RC}^e(d_i, i)$ preced all other ones, followed by some defaults $T_{RC}^n(d_i, i)$ followed by $T_{RC}^g(D)$ followed by some other defaults. Denoting by Π the set of defaults d_i such that either $T_{RC}^e(d_i, i)$ or $T_{RC}^n(d_i, i)$ is before $T_{RC}^g(D)$ in this process, we have that this process can be rewritten as $T_{RC}(\Pi) \cdot \Pi'$. Since this is a constrained process, $T_{RC}(\Pi)$ is globally successful.

The following lemma relates the rational process of the original theory with the processes obtained by the function T_{RC} .

Lemma 19 Π is a rational process of $\langle D, W \rangle$ not generating E as an extension if and only if $T_{RC}(\Pi)$ is a globally successful process of $T_{RC}(\langle D, W \rangle)$.

Proof. By Lemma 12, Π is a rational process if and only if $T_{RC}^n(\Pi)$ is globally successful. Since $T_{RC}(\Pi) = T_{RC}^n(\Pi) \cdot [T_{RC}^g(D)]$, if this process if globally successful then $T_{RC}^n(\Pi)$ is globally successful as well. Therefore, we only have to prove that, if Π is a rational process, then $T_{RC}^n(\Pi) \cdot [T_{RC}^g(D)]$ is globally successful. In particular, since $T_{RC}^n(\Pi)$ is globally successful and remains so even if their consequences are added $\{a, \neg b\} \cup Z$ by Lemma 9, what remains to be proved is only that the precondition of $T_{RC}^g(D)$ is entailed from the process $T_{RC}^n(\Pi)$.

Since Π is a rational process, for any default d_i such that $W \cup cons(\Pi) \models prec(d_i)$ it holds that either $d_i \in \Pi$ or that $W \cup cons(\Pi) \cup just(\Pi) \models \neg just(d_i)$. These three conditions can be rephrased in the translated theory as follows.

- $W \cup cons(\Pi) \models prec(d_i)$. This is equivalent to $T_{RC}(W) \cup cons(T_{RC}^n(\Pi)) \models a \rightarrow prec(d_i)$ by Lemma 7;
- $d_i \in \Pi$. This means that $T_{RC}^e(d_i, i) \in T_{RC}^n(\Pi)$, and therefore that $T_{RC}(W) \cup cons(T_{RC}^n(\Pi)) \models z_i$;
- $W \cup cons(\Pi) \cup just(\Pi) \models \neg just(d_i)$. This means that $T_{RC}^n(d_i, i) \in T_{RC}^n(\Pi)$, and therefore that $T_{RC}(W) \cup cons(T_{RC}^n(\Pi)) \models z_i$.

As a result, since Π is a rational process then, for every index i such that $T_{RC}(W) \cup cons(T_{RC}^n(\Pi)) \models a \rightarrow prec(d_i)$ it also holds that $T_{RC}(W) \cup cons(T_{RC}^n(\Pi)) \models z_i$. As a result, the precondition of $T_{RC}^g(D)$ is entailed.

This lemma, together with Lemma 13 and Lemma 14 allows proving the correctness of the translation.

Corollary 10 For every rational process Π of $\langle D, W \rangle$ there are a number of constrained processes of $T_{RC}(\langle D, W \rangle)$ all generating the same extension, which is var-equivalent to the extension generated by Π , and vice versa.

4.2 From Reiter to Justified and Constrained

In order to translate theories from Reiter to justified default logic, we adopt a strategy slightly different from the one used in the previous translation. Namely, we allow the application of a default even if its justification is violated; however, we do not then generate the extension in this case (we generate the known extension instead). We still replace W with $a \to W$. Each default $d_i = \frac{\alpha:\beta}{\gamma}$ is simulated by the two following defaults:

$$\frac{a \to \alpha : \beta[X/X']}{z_i(a \to \gamma)}$$

This default is always applicable whenever the precondition of the simulated default is entailed. In other words, the justification of this default is always consistent at this stage.

$$\frac{(a \to \alpha)(a \to \neg \beta):}{z_i}$$

This default can only be applied if the precondition of the original default is entailed but its justification is inconsistent with the current set of consequences.

These defaults can only be applied if the precondition of the original default is entailed. In particular, if the justification of the original default is contradicted, we have a choice of applying the first or the second default. If the original default is instead applicable, we are forced applying the first default. The fact that the first default can be applied even if the original default cannot will not be a problem, as these processes will be at a later time forced to generate the known extension E.

As above, we have the default that generates the known extension, and which can always be applied:

$$\frac{\vdots}{\neg aEz_1 \dots z_n}$$

This default can be applied provided that a has not been generated. On the converse, if the defaults that have been applied correspond to a successful process, we can instead generate a and produce the extension, by applying the following default:

$$\frac{\bigwedge_{d_i \in D} ((a \to prec(d_i)) \to z_i) : \neg E}{a(X \equiv X') z_1 \dots z_m}$$

Generating a makes all consequences of the defaults that have been applied unconditioned. On the other hand, $X \equiv X'$ makes each formula $\beta[X/X']$ equivalent to β . This is done to check the justifications of all applied defaults of the first kind. If such a default has been applied while its original default could not been applied because of its justification, the addition of $X \equiv X'$ would create a failure. This means that the last default is not applied, as justified default logic does not allow generating a failure. As a result, this last default can only be applied if the simulated process is successful. Otherwise, the only applicable default is the one producing the known extension E.

The justification $\neg E$ forbids E to be generated in two different ways, and work only if E is one of the logically strongest extensions of the original theory.

Since justified default logic can be translated in polynomial time into constrained default logic, it follows that Reiter default logics can be translated into constrained default logic given one of the strongest extensions.

5 Polysize Translations

In this section, we show the effects of the existence of some polynomial-size translations between variants of default logic. Existing translations have been shown in the previous sections: polynomial-time translations and translations that work given a strongest extension are also polysize translations. The following result shows the ability of rational and constrained default logic to express the consistency of a formula with a partial interpretation.

Lemma 20 For any formula F over variables $X \cup Y \cup Z$ it is possible to build in polynomial time a default theory $\langle D, W \rangle$ such that the following hold, where $F|_{\omega_Z}$ is the formula obtained from F by replacing each variable in Z with its truth value in ω_Z :

1. for every truth assignment ω_Z on the variables Z, the formula $\omega_Z \neg a \neg b$ is a rational and constrained extension of $\langle D, W \rangle$;

- 2. for every truth assignment ω_Z on the variables Z, the formula $\omega_Z \neg ab$ is a rational and constrained extension of $\langle D, W \rangle$ if and only if $\exists X \forall Y.F|_{\omega_Z}$ is valid;
- 3. $\langle D, W \rangle$ has no other rational or constrained extension.

Proof. Let $X = \{x_1, \ldots, x_n\}$ and $Z = \{z_1, \ldots, z_m\}$. The default theory that corresponds to F is $\langle D, \emptyset \rangle$, where D is defined as follows; a, b, and $\{k_1, \ldots, k_m\}$ are new variables.

$$D = \left\{ \frac{: z_i k_i}{z_i k_i}, \frac{: \neg z_i k_i}{\neg z_i k_i} \middle| 1 \le i \le m \right\} \cup$$

$$\left\{ \frac{K : x_i}{a \to x_i}, \frac{K : \neg x_i}{a \to \neg x_i} \middle| 1 \le i \le n \right\} \cup$$

$$\left\{ \frac{K(a \to F) : \neg ab}{\neg ab}, \frac{K : \neg a \neg b}{\neg a \neg b} \right\}$$
where $K = k_1 \land \dots \land k_m$

This set of defaults require a choice on all variables z_i to be taken before applying any other default. As a result, every extension of this theory contains a complete truth assignment over the variables Z.

Once such a truth assignment has been obtained, we can apply the default $\frac{K:\neg a \neg b}{\neg a \neg b}$, thus obtaining the extension of the point 1. of the statement.

The only way of blocking this default it to apply the second last default. In turn, this default can be applied only if some of the defaults of the second subset can be applied in such a way the resulting conclusions $a \to \omega_X$ entail $a \to F|_{\omega_Z}$ regardless of the value of Y. Therefore, $\omega_Z \neg ab$ is an extension if and only if $\exists X \forall Y.F|_{\omega_Z}$ is valid. \square

The default theory of the proof does not produce the same Reiter and justified extensions. This is because the defaults $\frac{K:x_i}{a \to x_i}$ and $\frac{K:\neg x_i}{a \to \neg x_i}$ can coexist in the same Reiter or justified process without making it unsuccessful. To make these defaults to contradict the justification of each other one would need to change their justifications to $a \land x_i$ and $a \land \neg x_i$, respectively; this however would make the generation of $\neg a$ by the last two defaults impossible.

This lemma is based on the ability of constrained and rational default logics to collect the justifications of the applied defaults without making them appear in the conclusions. This is the reason why extension-checking is harder in these two semantics than in Reiter and justified default logics.

This idea constitutes the base of a possible proof of non-existence of a bijective translation from rational or constrained default logics to Reiter or justified default logics. Namely, if such a translation existed, then one would be able to solve the set of QBF problems $\exists X \forall Y.F|_{\omega_Z}$ for every ω_Z by first producing a rational or constrained default theory, translating it to Reiter or justified default logic, and then checking for the existence of an extension containing $\omega_Z \neg ab$. For a fixed interpretation ω_Z , such a translation would be certainly feasible in a polynomial amount of space. The point is that a bijective faithful translation would need to produce a theory that has one or two extensions for every interpretation ω_Z over the variables Z.

The problem with this line of proof is that, in the theory that results from the translation, we cannot simply check whether $\omega_Z \neg ab$ is an extension. Indeed, since new variables are allowed, an extension $\omega_Z \neg ab$ is in general translated into an extension $\omega_Z \neg abG$, where G is a formula built over the new variables introduced by the translation.

For this reason, we consider the problem of checking whether a formula is equivalent to part of an extension. This way, we could check $\exists X \forall Y.F|_{\omega_Z}$ by checking whether $\omega_Z \neg ab$ can be extended to form an extension of the theory that results from the translation. We can restrict to the case in which the theory is known to have at most one extension extending the given formula. Indeed, in the lemma above, only a single extension containing $\omega_Z \neg ab$ may possibly exist; the result of a bijective reduction is a single extension, if any, containing $\omega_Z \neg ab$.

A majority Turing machine is a nondeterministic Turing machine that output "yes" if and only if at least half of the computation paths lead to acceptance. The class PP is the class of problems solved by a majority Turing machine that works in polynomial time. Similarly, PP^A is the class of problems solved by a majority Turing machine working in polynomial time and equipped with an oracle that solves the problem A in constant time. The class PP^C , where C is a class of problems, is defined as the union the classes of PP^A for every $A \in C$.

A slightly different characterization of classes defined in terms of oracles and nondeterministic Turing machines is in terms of the counting quantifier C. This quantifier extends both \exists and \forall by allowing the minimal number of assignments making a formula valid to be specified arbitrarily. Wagner [Wag86] and Toran [Tor91] have shown that $PP^K = CK$ for every class K that is defined in terms of quantifiers C, \exists , and \forall . Besides the superficial difference on the bound, this result proves that an oracle majority Turing machine can be restricted to make exactly one call to the oracle in each path of computation without a power loss.

Theorem 3 Deciding whether $|ext(T)| \ge k$ is $C \forall \exists P$ complete for constrained and rational default logic.

Proof. The problem can be solved by counting the number of processes that generate an extension. Since two processes can generate the same extension, we define an ordering over processes and only count the minimal one for each extension. Given a default theory T, we define proc(T) to be its selected processes and minproc(T) its minimal selected processes.

Given a default theory $\langle D, W \rangle$, we add an arbitrary linear ordering < on the set of the defaults. A linear ordering can be then defined over the processes: $\Pi < \Pi'$ if and only if either Π is shorter than Π' , or $\Pi(i) < \Pi'(i)$ where i is the first index for which $\Pi(i) \neq \Pi'(i)$. Counting the extensions can be done by counting the minimal processes:

$$\Pi \in minproc(T) \quad \text{iff} \quad \Pi \in proc(T) \\ \forall \Pi' : \Pi' \not\in proc(T) \ \lor \ (cons(\Pi') \not\equiv cons(\Pi)) \ \lor \ \Pi < \Pi'$$

A process Π is in this set if and only if it is the minimal process generating the extension $W \cup cons(\Pi)$. As a result, we can count the number of extension of T by counting the number of processes in minproc(T). Since deciding whether a process is in minproc(T) is in $\forall \exists P$, deciding whether their number is greater than a number k is in $C \forall \exists P$.

We prove the hardness of the problem by showing a reduction from the problem of establishing whether the number of truth assignments ω_Z over variables Z such that a formula $\exists X \forall Y.F|_{\omega_Z}$ is valid is greater than or equal to a given bound. This problem is $\mathsf{C}\exists \forall \mathsf{P}$ complete [Wag86].

By Lemma 20, the formula $\exists X \forall Y.F|_{\omega_Z}$ is valid for $Z = \omega_Z$ exactly when $\omega_Z \neg ab$ is an extension of the theory $\langle D, W \rangle$ of the lemma. Besides these extensions, the theory $\langle D, W \rangle$ has also exactly $2^{|Z|}$ extensions. Therefore, checking whether the number of truth assignments ω_Z satisfying the condition above is greater than or equal to k is equivalent to checking whether $\langle D, W \rangle$ has at least $|ext(T)| \geq 2^{|Z|} + k$ extensions.

The same problem for Reiter and justified default logic is slightly simpler.

Theorem 4 Checking whether $|ext(T)| \ge k$ is in C \exists for Reiter and justified default logic.

Proof. We show that the problem is in PP^{NP}, which is equal to C \exists . Checking whether a subset of cons(D) is an extension of a theory for the considered two semantics is in Δ_2^p : checking whether $E \in ext(T)$ can be solved by a polynomial number of calls to an NP oracle (actually, a logarithmic number suffices). Counting the number of extensions of T can be solved by counting the number of nondeterministic paths of a Turing machine that has one such path for every $D' \subseteq D$ and calls the oracle for checking whether E = cons(D') is an extension. Some nondeterministic paths have to be added to make the bound k to correspond exactly to one half of the nondeterministic paths.

The complexity of the problem is therefore lower for Reiter and justified default logics than for constrained and rational default logics. Of course, this result is not useful by itself, as the non-existence of a polynomial-time translation is already established.

We denote by $T \rightsquigarrow T'$ the condition of existence of a bijective faithful translation from T to T'. This condition can be formalized as follows, where \equiv_T indicates var-equivalence over the variables of T.

$$T \rightsquigarrow T'$$
 iff $\forall E : E \in ext(T) \leftrightarrow \exists E' : EE' \in ext(T')$
 $\forall E'E'' : E' \in ext(T') \ E'' \in ext(T') \ (E' \equiv_T E'') \rightarrow E' = E''$

Checking the first line of the right-hand side of this equation is in Π_3^p because $EE' \in ext(T')$ is Δ_2^p and therefore in Σ_2^p : as a result, $\exists E'$. $EE' \in ext(T')$ is in Σ_2^p as well. In the second check, the dominating operation is to check the opposite of $E' \equiv_T E''$, and checking var-equivalence is in Π_2^p .

The idea is as follows: assume that, for every T, there exists a theory T' of polynomial size such that $T \rightsquigarrow T'$. If this is the case, we can check the number of extensions of T by first guessing a theory T' of polynomial size, and then checking whether $T \rightsquigarrow T'$ and doing the check on the number of extensions on T'. Formally:

$$|ext(T)| \ge k \iff \exists T' . (T \leadsto T') \land |ext(T')| \ge k$$

 $\Leftrightarrow \forall T' . (T \leadsto T') \rightarrow (|ext(T')| \ge k)$

The first line reformulates the problem with an existential quantifier \exists and a formula that is in $\forall \exists \forall P$ and one that is in $C\exists P$. The second line gives a similar result; note that $T \leadsto T'$ is this time used in reverse because it is in an antecedent of an implication.

The above conditions allow solving the problem in two different ways, leading to membership to the following inclusion:

```
C\exists \forall P \subseteq \exists (\forall \exists \forall P \cup C\exists P) \cap \forall (\exists \forall \exists P \cup C\exists P)
```

Some results that hold for these classes are: every class $\mathsf{C}K$ is closed under complementation [Tor91], and therefore $\mathsf{C}\exists K = \mathsf{C}\forall K$; both $\exists K$ and $\forall K$ are included into $\mathsf{C}K$ [Tor91]; and $PH \subseteq \mathsf{P^{\mathsf{CP}}}$ [Tod89]. Applying these results to the first class above we get:

```
\begin{array}{lll} \exists (\forall \exists \forall P \cup \mathsf{C}\exists P) & \subseteq & \exists (P^{PP} \cup \mathsf{C}\forall P) & \operatorname{Toda} \\ &=& \exists (NP^{PP} \cup \mathsf{C}\forall P) & \operatorname{because} P \subseteq NP \\ &=& \exists (NP^{CP} \cup \mathsf{C}\forall P) & \operatorname{because} PP = PP^P \text{ and } PP^K = \mathsf{C}K \\ &=& \exists (\exists \mathsf{C}P \cup \mathsf{C}\forall P) & \operatorname{since} NP^{\mathsf{C}K} = \exists \mathsf{C}K \\ &\subseteq& \exists (\exists \mathsf{C}\forall P \cup \exists \mathsf{C}\forall P) & \operatorname{adding} \text{ quantifier can only enlarge classes} \\ &=& \exists \exists \mathsf{C}\forall P & \operatorname{classes} \text{ with } \exists \text{ in front are closed under union} \\ &=& \exists \mathsf{C}\forall P & \operatorname{two} \text{ quantifiers of the same type} \end{array}
```

For the second class, we obtain a similar result:

```
\begin{array}{ll} \forall (\exists \forall \exists P \cup \mathsf{C} \exists P) & \subseteq & \forall (P^{\mathrm{PP}} \cup \mathsf{C} \exists P) \\ & \subseteq & \forall (\forall \mathsf{C} P \cup \mathsf{C} \exists P) \\ & \subseteq & \forall (\forall \mathsf{C} \exists P \cup \forall \mathsf{C} \exists P) \\ & = & \forall \mathsf{C} \exists P \end{array}
```

Therefore, the assumption of existence of a bijective faithful translation would imply that $C\exists \forall P$ is contained in both $\exists C\forall P$ and $\forall C\exists P$. This condition can be restated as: a counting quantifier can be swapped with either an existential or a universal one.

If $C\exists \forall P\subseteq \exists C\forall P$ then $\exists C\exists \forall P\subseteq \exists \exists C\forall P=\exists C\forall P\subseteq \exists C\exists \forall P$, and therefore $\exists C\exists \forall P=\exists C\forall P$. With a similar proof one can conclude that $\forall C\exists \forall P=\forall C\exists P$.

6 Conclusions

This article reports some results about the existence of bijective-faithful translations among variants of default logics. Translations between such variants have already been investigated in the literature; some of such translations are faithful: each extension of the original theory corresponds to an equivalent extension of the translated theory. This article makes the assumption that the translations can introduce new variables; that implies that faithful translations might not be bijective: each extension of the original theory may correspond to many extensions of the translated theory. We therefore considered translations that are not only faithful but also create a bijection between extensions.

The rationale of requiring such a bijection is that the translated theory provides a more close simulation of the original one. As an example, if one translates an instance of the planning problem into Reiter default logic in such a way each plan corresponds to an extension [Tur97], then translating this theory in another variant breaks this correspondence if the translation is not bijective. If one wants to enumerate all plans, and a non-bijective translation has been applied, enumerating the extensions of the translated theory does not automatically generate an enumeration of all possible plans, because some plans may be

generated more than once. As an extreme example, a planning instance having two plans P_1 and P_2 can be expressed into a Reiter default theory having two extensions E_1 and E_2 . If one then converts this theory into constrained default logic using a non-bijective translation, what may result is a theory having a large number of extensions corresponding to E_1 and a single one corresponding to E_2 . That means that enumerating all extensions of this theory is likely to find a large number of extensions corresponding to P_1 before finding the one corresponding to P_2 .

The same argument can be applied in general for the problem of generating all extensions of a default theory, finding the number of extensions, finding whether a theory has a unique extension [ZL02], etc. All these problems cannot be solved by first translating the theory into a different semantics and then solving the problem in that semantics, unless the translation is guaranteed to translate every extension into a single extension.

References

- [Ant99] G. Antoniou. A tutorial on default logics. *ACM Computing Surveys*, 31(4):337–359, 1999.
- [AS94] G. Antoniou and V. Sperschneider. Operational concepts of nonmonotonic logics, part 1: Default logic. *Artificial Intelligence Review*, 8(1):3–16, 1994.
- [Bes89] P. Besnard. An introduction to Default Logic. Springer, Berlin, 1989.
- [DS03] J. Delgrande and T. Schaub. On the relation between Reiter's default logic and its (major) variants. In Seventh European Conference on Symbolic and Quantitative Approaches to Reasoning with Uncertainty (ECSQARU 2003), pages 452–463, 2003.
- [DS05] J. Delgrande and T. Schaub. Expressing default logic variants in default logic. Journal of Logic and Computation, 2005. To appear.
- [ET93] J. Engelfriet and J. Treur. A temporal model theory for default logic. In *Proceedings* of the European Conference on Symbolic and Quantitative Approaches to Reasoning and Uncertainty (ECSQARU'93), pages 91–96, 1993.
- [FM92] C. Froidevaux and J. Mengin. A framework for default logics. In *European Workshop on Logics in AI (JELIA'92)*, pages 154–173, 1992.
- [FM94] C. Froidevaux and J. Mengin. Default logics: A unified view. *Computational Intelligence*, 10:331–369, 1994.
- [Got95] G. Gottlob. Translating default logic into standard autoepistemic logic. *Journal of the ACM*, 42:711–740, 1995.
- [Imi87] T. Imielinski. Results on translating defaults to circumscription. Artificial Intelligence, 32:131–146, 1987.
- [Jan98] T. Janhunen. On the intertranslatability of autoepistemic, default and priority logics, and parallel circumscription. In *Proceedings of the Sixth European Workshop on Logics in Artificial Intelligence (JELIA'98)*, pages 216–232, 1998.

- [Jan01] T. Janhunen. On the effect of default negation on the expressiveness of disjunctive rules. In *Proceedings of the Sixth International Conference on Logic Programming and Nonmonotonic Reasoning (LPNMR'01)*, pages 93–106, 2001.
- [Jan03] T. Janhunen. Evaluating the effect of semi-normality on the expressiveness of defaults. *Artificial Intelligence*, 144:233–250, 2003.
- [Kon88] K. Konolige. On the relationship between default and autoepistemic logic. *Artificial Intelligence*, 35:343–382, 1988.
- [Lib05] P. Liberatore. Representability in default logic. Journal of the Interest Group in Pure and Applied Logic, 13(3), 2005.
- [Lib06] P. Liberatore. Where fail-safe default logics fail. ACM Transactions on Computational Logic, 8(2), 2006.
- [LLM03] J. Lang, P. Liberatore, and P. Marquis. Propositional independence: Formulavariable independence and forgetting. *Journal of Artificial Intelligence Research*, 18:391–443, 2003.
- [Rei80] R. Reiter. A logic for default reasoning. Artificial Intelligence, 13:81–132, 1980.
- [Ros99] R. Rosati. Model checking for nonmonotonic logics. In *Proceedings of the Sixteenth International Joint Conference on Artificial Intelligence (IJCAI'99)*, pages 76–83, 1999.
- [Tod89] S. Toda. On the computational power of PP and \oplus P. In Proceedings of the Thirtieth Annual Symposium on the Foundations of Computer Science (FOCS'89), pages 514–519, 1989.
- [Tor91] J. Torán. Complexity classes defined by counting quantifiers. *Journal of the ACM*, 38:753–774, 1991.
- [Tur97] H. Turner. Representing actions in logic programs and default theories: a situation calculus approach. *Journal of Logic Programming*, 31(1–3):245–298, 1997.
- [Wag86] K. Wagner. The complexity of combinatorial problems with succinct input representation. *Acta Informatica*, 23:325–356, 1986.
- [ZL02] X. Zhao and P. Liberatore. Complexity of the unique extension problem in default logic. Fundamenta Informaticae, 53(1):79–104, 2002.