# Faulty sets of Boolean formulas and Łukasiewicz logic



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# Faulty sets of Boolean formulas and Łukasiewicz logic[1]

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## Abstract

Suppose we are given a set $\Phi$ of $m$ Boolean formulas with the information that $e$ of these formulas are unconfirmed, while the actual set of unconfirmed formulas is not disclosed to us. Let us denote by $\mathrm{Rest}(\Phi, e)$ the family of all subsets of $\Phi$ having $m-e$ elements. We are interested in the problem whether a Boolean formula $\omega$ is a consequence of $\Psi$ for each $\Psi \in \mathrm{Rest}(\Phi, e)$. More generally, given for each $i = 1, \ldots, h$ a set $\Phi_i$ of $m_i$ Boolean formulas and an integer $0 \le e_i < m_i$, will $\omega$ be a consequence of $\Psi_1 \wedge \ldots \wedge \Psi_h$ for every choice of $\Psi_i \in \mathrm{Rest}(\Phi_i, e_i)$? We construct a quadratic reduction of this problem to the consequence problem in infinite-valued Łukasiewicz propositional logic $Ł_\infty$. Our reduction shows the usefulness of $Ł_\infty$ for the formal handling of unreliable Boolean information.

*Keywords*: Reasoning under uncertainty, Łukasiewicz calculus, Boolean logic, approximate reasoning, stable consequence, unreliable premises, polynomial time reduction, NP-complete, Rényi–Ulam games, Twenty Questions with Lies.

## 1 Foreword

Throughout, Boolean formulas are strings on the alphabet $\{X, |, \neg, \wedge, \vee, ), (\}$ as given by the usual syntax of propositional logic. Strings of the form $X|, X||, \ldots$ are called *variables*.

The *Stable Consequence problem* is defined as follows:

INSTANCE: A list $\Phi_1, \ldots, \Phi_k$ together with integers $e_1, \ldots, e_k$, where for each $i = 1, \ldots, k$, $\Phi_i$ is a set of $m_i$ Boolean formulas, and $0 \le e_i < m_i$.

QUESTION: Is the conjunction $\Psi_1 \wedge \ldots \wedge \Psi_k$ unsatisfiable for every possible choice of $\Psi_i \in \mathrm{Rest}(\Phi_i, e_i)$?

Again, $\mathrm{Rest}(\Phi_i, e_i)$ denotes the family of all subsets of $\Phi_i$ having $m_i - e_i$ elements.

The problem introduced in the abstract is the special case of the Stable Consequence problem with $\Phi_h = \{\neg\omega\}$ and $e_h = 0$.

A moment's reflection shows that the Stable Consequence problem is coNP-complete: for, it contains the Boolean unsatisfiability problem UNSAT, and is trivially in coNP.

In Theorem 5.2 and Corollary 5.3 we will construct a polytime reduction $\rho$ of the Stable Consequence problem to the consequence problem $\theta \vdash_\infty \phi$ in Łukasiewicz infinite-valued logic $Ł_\infty$.

[1]Dedicated to Alexander Leitsch.

Of course, other reductions can be extracted from the existing proofs of coNP-completeness of the consequence problem in $Ł_\infty$. However, since all these proofs (see e.g. [5, 18.3] and [2, 4.13(ii)]) are quite complex, so are the resulting reductions. By contrast, for any instance $I=(\Phi_1,\ldots,\Phi_k;e_1,\ldots,e_k)$ of the Stable Consequence problem, letting $v_I$ be the number of distinct variables in $I$, and $|I|$ its *length* (i.e. the number of occurrences of symbols in $I$), Corollary 5.3(ii) shows

$$|\rho(I)|<c\cdot v_I\cdot|I|<c\cdot|I|^2,$$

for some constant $c$ independent of $I$. $\rho(I)$ is a pair $(\theta_I,\phi_I)$ of $Ł_\infty$-formulas such that $I$ belongs to the Stable Consequence problem iff $\theta_I\vdash_\infty\phi_I$. Further, $I$ and $\rho(I)$ have the same variables. If $Ł_\infty$-formulas were also equipped with the operation of $n$-fold disjunction $n.\phi$, $(n=1,2,\ldots)$, then $|\rho(I)|<c|I|$.

The succinct pair $(\theta_I,\phi_I)$ of $[0,1]$-valued $Ł_\infty$-formulas yields an interpretation of consequence in many-valued logic $Ł_\infty$ as an extension of the Stable Consequence problem: as above, suppose $\Phi$ is a set of $m$ Boolean formulas, but we are kept unaware of the number of unconfirmed formulas in $\Phi$. For definiteness let us further assume $\Phi\vdash\omega$ and $\omega$ is not a tautology. For each $0\le e<m$ we have an instance $I_e=(\Phi,\{\neg\omega\};e,0)$ of the Stable Consequence problem; writing for short $(\theta_e,\phi_e)$ instead of $(\theta_{I_e},\phi_{I_e})$, the pair of $Ł_\infty$-formulas $\rho(I_e)=(\theta_e,\phi_e)$ has the following property:

$$\boxed{\theta_e\vdash_\infty\phi_e \ \text{ iff } \ \text{in Boolean logic } \Psi\vdash\omega \text{ for each } \Psi\in\mathrm{Rest}(\Phi,e).}$$

Intuitively, $\theta_e\vdash_\infty\phi_e$ iff the deduction $\Phi\vdash\omega$ tolerates up to $e$ unconfirmed premises. Let $0\le e_{\max}$ = largest integer $e$ such that $\theta_e\vdash_\infty\phi_e$. Binary search yields $e_{\max}$ after checking $\theta_e\vdash_\infty\phi_e$ for only logarithmically few different values of $e$. Then a large $e_{\max}$ signifies that $\omega$, almost like a tautology, is largely independent of $\Phi$. At the other extreme, if $e_{\max}$ is small, the reliability of $\omega$ too critically depends on the unconfirmed formulas in $\Phi$.

Generalizing the familiar 'Guess a Number' game, in the Rényi–Ulam game [1, Section 5] one has the problem of guessing an unknown number $x$ in a search space $S=\{0,\ldots,2^n-1\}$ by asking (a minimum number of adaptive) yes–no questions $Q_1,\ldots,Q_t$ in such a way that $x$ can be uniquely recovered from the answers $A_1,\ldots,A_t$, even if up to $e$ of them may be wrong/inaccurate/mendacious. By a 'question' we mean a subset of $S$. By an 'answer' $A_j$ we mean a bit $A_j\in\{0,1\}=\{no,yes\}$. Identifying each number $y\in S$ with its binary notation as an $n$-bit string $\alpha_y$ (i.e. a Boolean valuation $\alpha_y$ over the $n$ variables $X_1,\ldots,X_n$), each question $Q_j$ can be written down as a Boolean formula $\chi_j(X_1,\ldots,X_n)$, in such a way that $y\in Q_j$ iff $\alpha_y$ satisfies $\chi_j$. Then for each $i=1,\ldots,t$, the information given by the pair $(Q_i,A_i)$ is represented by the Boolean formula $\theta_i$, where $\theta_i=\chi_i$ (if $A_i=1$) and $\theta_i=\neg\chi_i$ (if $A_i=0$). Given now a Boolean formula $\omega(X_1,\ldots,X_n)$, the problem whether '$\omega$ follows from $\theta_1,\ldots,\theta_t$ in the Rényi–Ulam game with $e$ lies' is immediately seen to be a special case of the Stable Consequence problem.

Within the fault-tolerant framework of the Rényi–Ulam game with lies one may perhaps give a reasonable justification of the adjective 'stable' in the Stable Consequence problem: here, from the premises $\theta_1,\ldots,\theta_t$ one wishes to infallibly draw consequences $\omega$, no matter the instability (uncertainty, unpredictability, dubiety, unsureness) caused by the fact that some of the $\theta_i$ may be false/wrong.

## 2 Consequence in infinite-valued Łukasiewicz logic

We refer to [1, Section 4] for background on Łukasiewicz propositional logic $Ł_\infty$, and to [4, Section 7] for (always polynomial time) reductions and NP-completeness.

To efficiently write down $Ł_\infty$-formulas it will be convenient to use the richer alphabet $\{X, |, \neg, \odot, \oplus, \wedge, \vee, ), (\}$. The symbols $\neg, \odot, \oplus$ are called the *negation, conjunction* and *disjunction* connective, respectively. We call $\wedge$ and $\vee$ the *idempotent* conjunction and disjunction. As shown in [1, (1.2), 1.1.5], the connective $\odot$, as well as the idempotent connectives are definable in terms of $\neg$ and $\oplus$. Following [1, (4.1)], we write $\alpha \to \beta$ as an abbreviation of $\beta \oplus \neg\alpha$. Further, $\alpha \leftrightarrow \beta$ stands for $(\alpha \to \beta) \odot (\beta \to \alpha)$.

To increase readability we assume that the negation connective $\neg$ is more binding than $\odot$, and the latter is more binding than $\oplus$; the idempotent connectives $\vee$ and $\wedge$ are less binding than any other connective.

For each $n = 1, 2, \ldots,$ we let $\mathsf{FORM}_n$ denote the set of formulas $\psi(X_1, \ldots, X_n)$ whose variables are contained in the set $\{X_1, \ldots, X_n\}$. More generally, for any set $\mathcal{X}$ of variables, $\mathsf{FORM}_{\mathcal{X}}$ denotes the set of formulas whose variables are contained in $\mathcal{X}$. For each formula $\phi$ we let $\mathrm{var}(\phi)$ be the set of variables occurring in $\phi$.

For any formula $\phi \in \mathsf{FORM}_n$ and integer $k = 1, 2, \ldots,$ the iterated conjunction $\phi^k$ is defined by

$$\phi^1 = \phi,\ \phi^2 = \phi \odot \phi,\ \phi^3 = \phi \odot \phi \odot \phi, \ldots. \tag{1}$$

The iterated disjunction $k.\phi$ is defined by

$$1.\phi = \phi,\ 2.\phi = \phi \oplus \phi,\ 3.\phi = \phi \oplus \phi \oplus \phi, \ldots. \tag{2}$$

DEFINITION 2.1
A *valuation* (of $\mathsf{FORM}_n$ in $Ł_\infty$) is a function $V\colon \mathsf{FORM}_n \to [0,1]$ such that

$$V(\neg\phi) = 1 - V(\phi),\ V(\phi \oplus \psi) = \min(1, V(\phi) + V(\psi))$$

and, for the derived connectives $\odot, \vee, \wedge$,

$$V(\phi \odot \psi) = \max(0, V(\phi) + V(\psi) - 1) = V(\neg(\neg\phi \oplus \neg\psi))$$

$$V(\phi \vee \psi) = \max(V(\phi), V(\psi)) = V(\neg(\neg\phi \oplus \psi) \oplus \psi)$$

$$V(\phi \wedge \psi) = \min(V(\phi), V(\psi)) = V(\neg(\neg\phi \vee \neg\psi)).$$

We denote by $\mathsf{VAL}_n$ the set of valuations of $\mathsf{FORM}_n$. More generally, for any set $\mathcal{X}$ of variables, $\mathsf{VAL}_{\mathcal{X}}$ denotes the set of valuations $V\colon \mathsf{FORM}_{\mathcal{X}} \to [0,1]$.

Since Łukasiewicz logic $Ł_\infty$ is truth-functional, each $V \in \mathsf{VAL}_n$ is uniquely determined by its restriction to $\{X_1, \ldots, X_n\}$. Thus, for every point $x = (x_1, \ldots, x_n) \in [0,1]^n$ there is a uniquely determined valuation $V_x \in \mathsf{VAL}_n$ such that

$$V_x(X_i) = x_i \text{ for all } i = 1, \ldots, n. \tag{3}$$

Conversely, upon identifying the two sets $[0,1]^n$ and $[0,1]^{\{X_1,\ldots,X_n\}}$, we can write $x = V_x \restriction \{X_1, \ldots, X_n\}$.

For any set $\Phi \subseteq \mathsf{FORM}_{\mathcal{X}}$ and $V \in \mathsf{VAL}_{\mathcal{X}}$ we say that $V$ *satisfies* $\Phi$ if $V(\psi) = 1$ for all $\psi \in \Phi$. A formula $\phi$ is a *tautology* if it is satisfied by all valuations $V \in \mathsf{VAL}_{\mathrm{var}(\phi)}$.

PROPOSITION 2.2 (Hay–Wójcicki theorem, [3, 5, 6])
For all $n = 1, 2, \ldots$ and $\theta, \phi \in \mathsf{FORM}_n$ the following conditions are equivalent:

(i) Every valuation $V \in \mathsf{VAL}_n$ satisfying $\theta$ also satisfies $\phi$. In other words, $\phi$ is a semantic $Ł_\infty$-consequence of $\theta$;

(ii) For some integer $k>0$ the formula $\theta^k \to \phi$ is a tautology. (Notation of (1)).

(iii) For some integer $k>0$ the formula

$$\underbrace{\theta \to (\theta \to (\theta \to \cdots \to (\theta \to (\theta \to \phi))\cdots))}_{k \text{ occurrences of } \theta} \tag{4}$$

is a tautology.

(iv) For some integer $k>0$ there is a sequence of formulas $\chi_0,\ldots,\chi_{k+1}$ such that $\chi_0=\theta$, $\chi_{k+1}=\phi$, and for each $i=1,\ldots,k+1$ either $\chi_i$ is a tautology, or there are $p,q\in\{0,\ldots,i-1\}$ such that $\chi_q$ is the formula $\chi_p\to\chi_i$.

(v) For some integer $k>0$ there is a sequence of formulas $\chi_0,\ldots,\chi_{k+1}$ such that $\chi_0=\theta$, $\chi_{k+1}=\phi$, and for each $i=1,\ldots,k+1$ either $\chi_i$ is a tautology in $\mathsf{FORM}_n$, or there are $p,q\in\{0,\ldots,i-1\}$ such that $\chi_q$ is the formula $\chi_p\to\chi_i$. In other words, $\phi$ is a syntactic Ł$_\infty$-consequence of $\theta$.

PROOF. (ii)⇔(iii) is promptly verified, because the two formulas (4) and $\theta^k\to\phi$ are equivalent in Ł$_\infty$. (iv)⇔(i) follows from [1, 4.5.2, 4.6.7]. (iv)⇔(iii) follows from [1, 4.6.4]. (v)⇒(iv) is trivial. Finally, to prove (iii)⇒(v), arguing by induction on $k$, one verifies that $\phi$ can be obtained as the final formula $\chi_{k+1}$ of a sequence $\chi_0,\ldots,\chi_{k+1}$ as in (v), which only requires the assumed tautology (4). Also see [5, 1.7]. ∎

We write $\theta\vdash_\infty\phi$ if $\theta$ and $\phi$ satisfy the equivalent conditions above, and we say that $\phi$ is an *Ł$_\infty$-consequence* of $\theta$ without fear of ambiguity.

An instance of the Ł$_\infty$-*consequence problem* is a pair of formulas $(\theta,\phi)$. The problem asks if $\phi$ is an Ł$_\infty$-consequence of $\theta$.

## 3 The function $\hat{\phi}$ associated with an Ł$_\infty$-formula $\phi$

PROPOSITION 3.1
With every formula $\phi=\phi(X_1,\ldots,X_n)\in\mathsf{FORM}_n$ let us associate a function, denoted $\widehat{\phi}\colon [0,1]^n\to[0,1]$, via the following inductive procedure: for all $x=(x_1,\ldots,x_n)\in[0,1]^n$,

$$\begin{aligned}
\widehat{X_i}(x) &= x_i \ (i=1,\ldots,n),\\
\widehat{\neg\psi}(x) &= 1-\widehat{\psi(x)},\\
\widehat{\psi\oplus\chi}(x) &= \min(1,\widehat{\psi}(x)+\widehat{\chi}(x)),\\
\widehat{\psi\odot\chi}(x) &= \max(0,\widehat{\psi}(x)+\widehat{\chi}(x)-1),\\
\widehat{\psi\wedge\chi}(x) &= \min(\widehat{\psi}(x),\widehat{\chi}(x)),\\
\widehat{\psi\vee\chi}(x) &= \max(\widehat{\psi}(x),\widehat{\chi}(x)).
\end{aligned}$$

Then generalizing (3) we have the identity

$$\hat{\phi}(x)=V_x(\phi) \ \text{ for all } x\in[0,1]^n. \tag{5}$$

PROOF. Immediate by Definition 2.1, arguing by induction on the number of connectives in $\phi$. ∎

PROPOSITION 3.2
For each $n=1,2,\ldots,$ $e=2,3,\ldots,$ and valuation $V\colon \mathsf{FORM}_n \to [0,1]$, the following conditions are equivalent:

(i) $V$ satisfies $\bigwedge_{i=1}^{n}(X_i^e \leftrightarrow \neg X_i)\vee(X_i \leftrightarrow \neg\, e.X_i)$. (Notation of (1)–(2)).

(ii) For each $i=1,\ldots,n,$ $V(X_i)\in\left\{\frac{1}{e+1},\frac{e}{e+1}\right\}$.

PROOF. Let $\xi_e$ be the Ł$_\infty$-formula $X^e\leftrightarrow\neg X$, and $\widehat{\xi_e}\colon [0,1]\to[0,1]$ its associated function. Recalling (5) and the definition of the $\leftrightarrow$ connective, for every $y\in[0,1]$, we can write $\widehat{\xi_e}(y)=1$ iff $\widehat{X^e}(y)=1-y$. Further, by induction on $e$,

$$\widehat{X^e}(y)=\underbrace{y\odot\cdots\odot y}_{e\text{ times}}=\max(0,ey-e+1)=\begin{cases}0 & \text{if } 0\le y<\frac{e-1}{e}\\ ey-e+1 & \text{if } \frac{e-1}{e}\le y\le 1.\end{cases}$$

Thus, $\widehat{\xi_e}(y)=1$ iff $ey-e+1=1-y$ iff $y=\frac{e}{e+1}$. In other words, a valuation satisfies $X^e\leftrightarrow\neg X$ iff it evaluates $X$ to $\frac{e}{e+1}$.

Similarly, letting $\chi_e$ be the formula $X\leftrightarrow\neg\, e.X$ we obtain $\widehat{\chi_e}(y)=\widehat{\xi_e}(1-y)$, whence $\widehat{\chi_e}(y)=1$ iff $\widehat{\xi_e}(1-y)=1$ iff $1-y=\frac{e}{e+1}$ iff $y=\frac{1}{e+1}$. Thus, a valuation satisfies $X\leftrightarrow\neg\, e.X$ iff it evaluates $X$ to $\frac{1}{e+1}$.

Summing up, a valuation satisfies $\bigwedge_{i=1}^{n}(X_i^e\leftrightarrow\neg X_i)\vee(X_i\leftrightarrow\neg\, e.X_i)$ iff it evaluates each $X_i$ either to $\frac{1}{e+1}$ or to $\frac{e}{e+1}$. ■

## 4 The ‡-transform of a Boolean formula

As the reader will recall, every Boolean formula $\psi$ in this article is constructed from the variables using the connectives $\neg,\vee,\wedge$. A Boolean formula is said to be in *negation normal form* if the negation symbol can only precede a variable. Any Boolean formula $\psi$ can be immediately reduced into an equivalent formula $\psi^\dagger$ in negation normal form by using De Morgan's laws to push negation inside all conjunctions and disjunctions, and eliminating double negations. The same variables occur in $\psi$ and $\psi^\dagger$. Further, the number of occurrences of variables in $\psi$ is the same as in $\psi^\dagger$.

DEFINITION 4.1
Let $\psi=\psi(X_1,\ldots,X_n)$ be a Boolean formula. We denote by $\psi^\ddagger$ the Ł$_\infty$-formula obtained from $\psi$ by the following procedure:

— write the negation normal form $\psi^\dagger$, and for each $i=1,\ldots,n$,
— replace every occurrence of $\neg X_i$ in $\psi^\dagger$ by the formula $X_i\vee\neg(X_i\odot X_i)$,
— and simultaneously replace every occurrence of the non-negated variable $X_i$ by the formula $\neg X_i\vee(X_i\oplus X_i)$, $\quad i=1,\ldots,n$.

In other words, the ‡*-transform* $\psi^\ddagger$ of $\psi$ is the Ł$_\infty$-formula defined by:

$$\begin{aligned}(\neg X_i)^\ddagger &= X_i\vee\neg(X_i\odot X_i),\\ X_i^\ddagger &= \neg X_i\vee(X_i\oplus X_i),\ \text{ if } X_i \text{ is not preceded by } \neg\end{aligned}$$

and by induction on the number of binary connectives in $\psi^\dagger$,

$$(\sigma\wedge\tau)^\ddagger = \sigma^\ddagger\wedge\tau^\ddagger$$
$$(\sigma\vee\tau)^\ddagger = \sigma^\ddagger\vee\tau^\ddagger.$$

DEFINITION 4.2
Fix $e=2,3,\ldots$. For each $y\in\{0,1\}$ we let $y^{\langle e\rangle}$ be the only point of $[0,1]$ lying at a distance $\frac{1}{e+1}$ from $y$. More generally, for any $x=(x_1,\ldots,x_m)\in\{0,1\}^m$, the point $x^{\langle e\rangle}\in[0,1]^m$ is defined by $x^{\langle e\rangle}=(x_1^{\langle e\rangle},\ldots,x_m^{\langle e\rangle})$.

PROPOSITION 4.3
For any Boolean valuation

$$W:\{\text{Boolean formulas in the variables } X_1,\ldots,X_n\}\to\{0,1\},$$

let $w\in\{0,1\}^{\{X_1,\ldots,X_n\}}=\{0,1\}^n$ be the restriction of W to the set $\{X_1,\ldots,X_n\}$. Then for every Boolean formula $\psi(X_1,\ldots,X_n)$ and $e=2,3,\ldots$ we have:

$$W \text{ satisfies } \psi \text{ iff } \widehat{\psi^\ddagger}(w^{\langle e\rangle})=1$$
$$W \text{ does not satisfy } \psi \text{ iff } \widehat{\psi^\ddagger}(w^{\langle e\rangle})=\frac{e}{e+1}.$$

PROOF. Our assumption about $e$ ensures that $0^{\langle e\rangle}<1^{\langle e\rangle}$. For each variable $X$ we first prove (see Figure 1):

(i) $\widehat{X^\ddagger}(\frac{1}{e+1})=\frac{e}{e+1}$,
(ii) $\widehat{X^\ddagger}(\frac{e}{e+1})=1$,
(iii) $\widehat{\neg X^\ddagger}(\frac{1}{e+1})=1$,
(iv) $\widehat{\neg X^\ddagger}(\frac{e}{e+1})=\frac{e}{e+1}$.

(i)–(ii) By (5), for all $y\in[0,1]$ we can write $\widehat{X^\ddagger}(y)=\max(\widehat{\neg X}(y),\widehat{X\oplus X}(y)) = \max(1-y,\min(1,2y))$. Thus,

$$\widehat{X^\ddagger}\left(\frac{1}{e+1}\right)=\max\left(\frac{e}{e+1},\min(1,\frac{2}{e+1})\right)=\max\left(\frac{e}{e+1},\frac{2}{e+1}\right)=\frac{e}{e+1}$$

and

$$\widehat{X}^{\ddagger}\left(\frac{e}{e+1}\right)=\max\left(\frac{1}{e+1},\min(1,\frac{2e}{e+1})\right)=\max\left(\frac{1}{e+1},1\right)=1.$$

(iii)–(iv) Again by (5), we can write $\widehat{\neg X}^{\ddagger}(y)=\max(\widehat{X}(y),\neg\widehat{(X\odot X)}(y))=\max(y,1-\max(0,2y-1))=\max(y,\min(1,2-2y))$, whence

$$\widehat{\neg X}^{\ddagger}\left(\frac{1}{e+1}\right)=\max\left(\frac{1}{e+1},\min(1,2-\frac{2}{e+1})\right)=\max\left(\frac{1}{e+1},1\right)=1$$

and

$$\widehat{\neg X}^{\ddagger}\left(\frac{e}{e+1}\right)=\max\left(\frac{e}{e+1},\min(1,2-\frac{2e}{e+1})\right)=\max\left(\frac{e}{e+1},\frac{2}{e+1}\right)=\frac{e}{e+1}.$$

Having thus settled (i)–(iv), the proof now proceeds by induction on the number $b$ of binary connectives in $\psi^{\dagger}$, the equivalent counterpart of $\psi$ in negation normal form as in 4.1:

*Basis,* $b=0$. Then $\psi^{\dagger}\in\{X_i,\neg X_i\}$.

In case $\psi^{\dagger}=X_i$ we have

$$\begin{aligned}
&W \text{ satisfies } \psi\\
&\text{iff } W \text{ satisfies } X_i, \text{ (because } \psi^{\dagger} \text{ is equivalent to } \psi)\\
&\text{iff } w_i=1, \text{ by definition of } w\\
&\text{iff } w_i^{\langle e\rangle}=\frac{e}{e+1}, \text{ by definition of } w_i{}^{\langle e\rangle}\\
&\text{iff } \widehat{X_i}^{\ddagger}(w_i^{\langle e\rangle})=\widehat{\psi}^{\ddagger}(w_i^{\langle e\rangle})=1.
\end{aligned}$$

The ($\Downarrow$)-direction of the last bi-implication follows from (ii). Conversely, for the ($\Uparrow$)-direction, if $w_i^{\langle e\rangle}\neq\frac{e}{e+1}$ then $w_i^{\langle e\rangle}=\frac{1}{e+1}$, whence by (i), $\widehat{X_i}^{\ddagger}(w_i^{\langle e\rangle})=\frac{e}{e+1}\neq 1$.

The case $\psi^{\dagger}=\neg X_i$ is similarly proved using (iii)–(iv).

*Induction step.* Suppose $\psi^{\dagger}=\sigma\wedge\tau$. Then

$$\begin{aligned}
&W \text{ satisfies } \psi\\
&\text{iff } W \text{ satisfies } \psi^{\dagger}\\
&\text{iff } W \text{ satisfies both } \sigma^{\dagger} \text{ and } \tau^{\dagger}\\
&\text{iff } W \text{ satisfies both } \sigma \text{ and } \tau\\
&\text{iff } \widehat{\sigma}^{\ddagger}(w^{\langle e\rangle})=\widehat{\tau}^{\ddagger}(w^{\langle e\rangle})=1, \text{ by induction hypothesis.}
\end{aligned}$$

Thus, if $W$ satisfies $\psi$ then

$$\widehat{\psi}^{\ddagger}(w^{\langle e\rangle})=(\widehat{\sigma}^{\ddagger}\wedge\widehat{\tau}^{\ddagger})(w^{\langle e\rangle})=\min(1,1)=1.$$

Conversely,

$$
\begin{array}{ll}
 & W \text{ does not satisfy } \psi \\
\text{iff} & \text{either } \sigma \text{ or } \tau \text{ is not satisfied by } W \\
\text{iff} & \text{either } \widehat{\sigma^{\ddagger}}(w^{\langle e\rangle}) = \dfrac{e}{e+1} \text{ or } \widehat{\tau^{\ddagger}}(w^{\langle e\rangle}) = \dfrac{e}{e+1},
\end{array}
$$

$$
\text{whence } \widehat{\psi^{\ddagger}}(w^{\langle e\rangle}) = \min(\widehat{\sigma^{\ddagger}}(w^{\langle e\rangle}), \widehat{\tau^{\ddagger}}(w^{\langle e\rangle})) = \frac{e}{e+1}.
$$

The case $\psi^{\dagger} = \sigma \vee \tau$ is similar. ∎

## 5 Main results

The incorporation into Ł$_\infty$-formulas of the numerical parameters $e_i$ of the Stable Consequence problem relies on the following:

PROPOSITION 5.1
For $\Phi = \{\phi_1, \ldots, \phi_u\}$ a finite set of Boolean formulas in the variables $X_1 \ldots, X_n$, let the integers $d$ and $e$ satisfy the conditions $0 \le d < u$ and $e \ge \max(2, d)$. Then the following conditions are equivalent:

(i) Every subset $\Psi$ of $\Phi$ obtained by deleting $d$ elements of $\Phi$ is unsatisfiable.
(i') Every subset $\Psi$ of $\Phi$ obtained by deleting up to $d$ elements of $\Phi$ is unsatisfiable.
(ii) For each valuation $V \in \mathsf{VAL}_n$ such that $V(X_i) \in \left\{\frac{1}{e+1}, \frac{e}{e+1}\right\}$ for all $i = 1, \ldots, n$, we have $V\left(\left(\bigodot_{j=1}^{u} \phi_j^{\ddagger}\right) \to (X_1 \vee \neg X_1)^{d+1}\right) = 1$.

PROOF. (i)⇔(i') is trivial. (i') ⇒ (ii) Let $V$ be a counter-example to (ii). Since for all $i = 1, \ldots, n$, $V(X_i) \in \left\{\frac{1}{e+1}, \frac{e}{e+1}\right\}$, upon identifying the restriction $V \restriction \{X_1, \ldots, X_n\}$ with the point $(V(X_1), \ldots, V(X_n)) \in [0,1]^n$ we can write

$$
V \restriction \{X_1, \ldots, X_n\} = (W \restriction \{X_1, \ldots, X_n\})^{\langle e\rangle} \tag{6}
$$

for a unique Boolean valuation $W$ of the set of Boolean formulas in the variables $X_1, \ldots, X_n$. Since (ii) fails for $V$, by definition of the implication connective in Ł$_\infty$ we can write

$$
V\left(\bigodot_{j=1}^{u} \phi_j^{\ddagger}\right) > V((X_1 \vee \neg X_1)^{d+1}).
$$

From

$$
V(X_1 \vee \neg X_1) = \max\left(\frac{1}{e+1}, \frac{e}{e+1}\right) = \frac{e}{e+1}
$$

we obtain

$$
V((X_1 \vee \neg X_1)^{d+1}) = 1 - \frac{d+1}{e+1},
$$

whence

$$
V\left(\bigodot_{j=1}^{u} \phi_j^{\ddagger}\right) > 1 - \frac{d+1}{e+1}. \tag{7}
$$

Our assumption about $V$ is to the effect that $V\left(\bigodot_{j=1}^{u}\phi_j^{\ddagger}\right)$ is an integer multiple of $\frac{1}{e+1}$, whence by (7),

$$V\left(\bigodot_{j=1}^{u}\phi_j^{\ddagger}\right) \geq 1-\frac{d}{e+1}, \tag{8}$$

and by Definition 4.1,

$$V\left(\phi_j^{\ddagger}\right) \in \left\{\frac{e}{e+1}, 1\right\}, \text{ for all } j=1,\dots,u.$$

Thus by (8), at most $d$ among the formulas $\phi_1^{\ddagger},\dots,\phi_u^{\ddagger}$ are evaluated to $e/e+1$ by $V$. By (6) together with Propositions 3.1 and 4.3, at most $d$ among the formulas $\phi_1,\dots,\phi_u$ are evaluated to 0 by $W$. Thus, at least $u-d$ are satisfied by $W$, against assumption (i').

(ii) $\Rightarrow$ (i) If (i) fails then without loss of generality we can assume the set $\Psi=\{\phi_1,\dots,\phi_{u-d}\}$ to be satisfiable by some Boolean valuation $Y$. Let the point $z=(Y(X_1),\dots,Y(X_n))\in\{0,1\}^n$ be (identified with) the restriction of $Y$ to the set of variables $\{X_1,\dots,X_n\}$. Let $U\in\mathsf{VAL}_n$ be uniquely determined by the stipulation $U\lceil\{X_1,\dots,X_n\}=z^{\langle e\rangle}$. Then $U$ satisfies the hypothesis of (ii),

$$U(X_i)\in\left\{\frac{1}{e+1},\frac{e}{e+1}\right\} \text{ for all } i=1,\dots,n,$$

whence

$$U((X_1\vee\neg X_1)^{d+1})=1-\frac{d+1}{e+1}.$$

Since $Y$ satisfies $\Psi$, from Proposition 4.3 we get

$$U\left(\bigodot_{j=1}^{u}\phi_j^{\ddagger}\right)\geq 1-\frac{d}{e+1}.$$

Thus,

$$U\left(\bigodot_{j=1}^{u}\phi_j^{\ddagger}\right) > 1-\frac{d+1}{e+1}=U((X_1\vee\neg X_1)^{d+1}),$$

and, by definition of the $\to$ connective, (ii) fails. ■

THEOREM 5.2
Let $n$ and $k$ be integers $>0$. For each $i=1,\dots,k$ let $\Phi_i=\{\phi_{i1},\phi_{i2},\dots,\phi_{iu(i)}\}$ be a finite set of Boolean formulas in the variables $X_1,\dots,X_n$. Further, let the integer $e_i$ satisfy $0\leq e_i<u(i)$. Then the following conditions are equivalent:

(i) For each $i=1,\dots,k$ and $\Psi_i\in\mathrm{Rest}(\Phi_i,e_i)$, the Boolean formula $\bigwedge_{i=1}^{k}\Psi_i$ is unsatisfiable.
(ii) In infinite-valued Łukasiewicz logic $Ł_\infty$ we have

$$\bigwedge_{t=1}^{n}\left((X_t^e\leftrightarrow\neg X_t)\vee(X_t\leftrightarrow\neg\, e\,.X_t)\right)\vdash_\infty\bigwedge_{i=1}^{k}\left(\left(\bigodot_{j=1}^{u(i)}\phi_{ij}^{\ddagger}\right)\to(X_1\vee\neg X_1)^{e_i+1}\right),$$

where $e=\max(2,e_1,\dots,e_k)$.

PROOF. Immediate from Propositions 2.2 and 5.1, using the characterization (Proposition 3.2) of all valuations satisfying $\bigwedge_{t=1}^{n}\big((X_t^e \leftrightarrow \neg X_t) \vee (X_t \leftrightarrow \neg\, e \,\mathbf{.}\, X_t)\big)$. ■

COROLLARY 5.3
For any instance

$$I = \big(\{\phi_{11}, \ldots, \phi_{1u(1)}\}, \ldots, \{\phi_{k1}, \ldots, \phi_{ku(k)}\}; e_1, \ldots, e_k\big)$$

of the Stable Consequence problem for Boolean formulas in the variables $X_1, \ldots, X_n$, let $\rho(I)$ be the pair of $Ł_\infty$-formulas

$$\left( \bigwedge_{t=1}^{n} \big((X_t^e \leftrightarrow \neg X_t) \vee (X_t \leftrightarrow \neg\, e \,\mathbf{.}\, X_t)\big),\ \bigwedge_{i=1}^{k} \left( \left( \bigodot_{j=1}^{u(i)} \phi_{ij}^{\ddagger} \right) \to (X_1 \vee \neg X_1)^{e_i+1} \right) \right),$$

where $e = \max(2, e_1, \ldots, e_k)$.

(i) Then $\rho$ reduces in polynomial time the Stable Consequence problem to the $Ł_\infty$-consequence problem.
(ii) There is a constant $c$ such that

$$|\rho(I)| \leq c \cdot n \cdot |I| < c \cdot |I|^2 \tag{9}$$

for all $n$ and $I$.

PROOF. (i) By Theorem 5.2, $\rho(I)$ belongs to the $Ł_\infty$-consequence problem iff $I$ belongs to the Stable Consequence problem. Trivially, $\rho$ is computable in polynomial time. (ii) These inequalities immediately follow by direct inspection. ■

REMARK 5.4
With reference to the notational conventions (1)–(2), it should be noted that we do not have in $Ł_\infty$ an exponentiation connective for $\psi^e$, nor a multiplication connective for $e \,\mathbf{.}\, \psi$: these would further simplify $\rho(I)$, reducing (9) to $|\rho(I)| \leq d \cdot |I|$ for some fixed constant $d$.

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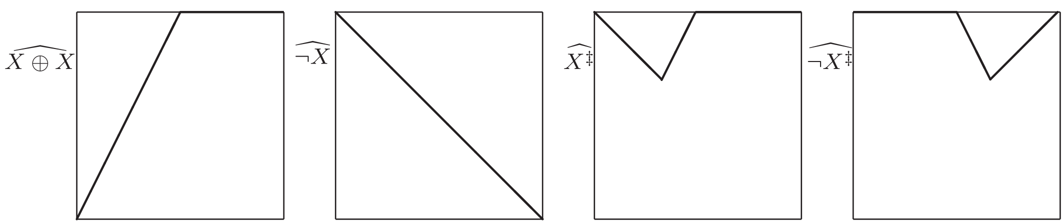


FIGURE 1. The graphs of the functions $\widehat{X \oplus X}$, $\widehat{\neg X}$, $\widehat{X^{\ddagger}}$ and $\widehat{\neg X^{\ddagger}}$.