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THE UNIVERSAL HOMOGENEOUS BINARY TREE

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Abstract. A partial order is called *semilinear* iff the upper bounds of each element are linearly ordered and any two elements have a common upper bound. There exists, up to isomorphism, a unique countable existentially closed semilinear order, which we denote by $(\mathbb{S}_2;\leq)$. We study the *reducts* of $(\mathbb{S}_2;\leq)$, that is, the relational structures with domain $\mathbb{S}_2$, all of whose relations are first-order definable in $(\mathbb{S}_2;\leq)$. Our main result is a classification of the model-complete cores of the reducts of $\mathbb{S}_2$. From this, we also obtain a classification of reducts up to first-order interdefinability, which is equivalent to a classification of all closed permutation groups that contain the automorphism group of $(\mathbb{S}_2;\leq)$.

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1. Introduction

A partial order $(P;\leq)$ is called *semilinear* iff for all $a, b \in P$ there exists $c \in P$ such that $a \leq c$ and $b \leq c$, and for every $a \in P$ the set $\{b \in P : a \leq b\}$ is linearly ordered, that is, contains no incomparable pair of elements. Finite semilinear orders are closely related to rooted trees: the transitive closure of a tree (viewed as a directed graph with the edges oriented towards the root) is a semilinear order, and the transitive reduction of any finite semilinear order is a rooted tree.

It follows from basic facts in model theory (e.g. Theorem 8.2.3. in [Hod97]) that there exists a countable semilinear order $(\mathbb{S}_2;\leq)$ which is *existentially closed* in the class of all countable semilinear orders, that is, for every embedding e of $(\mathbb{S}_2;\leq)$ into a countable semilinear order $(P;\leq)$, every existential formula $\phi(x_1,\dots,x_n)$, and all $p_1,\dots,p_n \in \mathbb{S}_2$ such that $\phi(e(p_1),\dots,e(p_n))$ holds in $(P;\leq)$ we have that $\phi(p_1,\dots,p_n)$ holds in $(\mathbb{S}_2;\leq)$. We write $x < y$ for $(x \leq y \wedge x \neq y)$ and $x \perp y$ for $\neg(x \leq y) \wedge \neg(y \leq x)$, that is, for incomparability with respect to $\leq$. Clearly, $(\mathbb{S}_2;\leq)$ is

- *dense*: for all $x, y \in \mathbb{S}_2$ such that $x < y$ there exists $z \in \mathbb{S}_2$ such that $x < z < y$;
- *unbounded*: for every $x \in \mathbb{S}_2$ there are $y, z \in \mathbb{S}_2$ such that $y < x < z$;
- *binary branching*: (a) for all $x, y \in \mathbb{S}_2$ such that $x < y$ there exists $u \in \mathbb{S}_2$ such that $u < y$ and $u \perp x$, and (b) for any three incomparable elements of $\mathbb{S}_2$ there is an element in $\mathbb{S}_2$ that is larger than two out of the three, and incomparable to the third;

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- *nice* (following terminology from [DHM91]): for every $x, y \in \mathbb{S}_2$ such that $x \perp y$ there exists $z \in \mathbb{S}_2$ such that $z > x$ and $z \perp y$.
- *without joins*: for all $x, y, z \in \mathbb{S}_2$ with $x, y \leq z$ and x, y incomparable, there exists a $u \in \mathbb{S}_2$ such that $x, y \leq u$ and $u < z$.

It can be shown by a back-and-forth argument (Proposition 3.2) that all countable, dense, unbounded, nice, and binary branching semilinear orders without joins are isomorphic to $(\mathbb{S}_2; \leq)$. Since all these properties of $(\mathbb{S}_2; \leq)$ can be expressed by first-order sentences, it follows that $(\mathbb{S}_2; \leq)$ is *ω-categorical*: it is, up to isomorphism, the unique countable model of its first-order theory. It also follows from general principles that the first-order theory T of $(\mathbb{S}_2; \leq)$ is *model-complete*, that is, embeddings between models of T preserve all first-order formulas, and that T is the *model-companion* of the theory of semilinear orders; again, we refer to [Hod97] (Theorem 8.3.6).

For $k \in \mathbb{N}$, a relational structure Δ is *k set–homogeneous* if whenever A and B are isomorphic k–element substructures of Δ, there is an automorphism g of Δ such that $g[A] = B$. In [Dro85], Droste studies 2 and 3 set–homogeneous semilinear orders. Of particular relevance here, Droste proved that $(\mathbb{S}_2; \leq)$ is the unique countably infinite, non-linear, 3 set–homogeneous semilinear order (see Theorem 6.22 of [Dro85]).

The structure $(\mathbb{S}_2; \leq)$ plays an important role in the study of a natural class of *constraint satisfaction problems (CSPs)* in theoretical computer science. CSPs from this class have been studied in artificial intelligence for qualitative reasoning about branching time [DÖ5, Hir96, BJ03], and, independently, in computational linguistics [Cor94, BK02] under the name *tree description* or *dominance* constraints.

A *reduct* of a relational structure Δ is a relational structure Γ with the same domain as Δ such that every relation of Γ has a first-order definition over Δ without parameters. All reducts of a countable ω-categorical structure are again ω-categorical [Hod93]. In this article we study the reducts of $(\mathbb{S}_2; \leq)$. Two structures Γ and Γ' with the same domain are called *(first-order) interdefinable* when Γ is a reduct of Γ', and Γ' is a reduct of Γ. We show that the reducts Γ of $(\mathbb{S}_2; \leq)$ fall into three equivalence classes with respect to interdefinability: either Γ is interdefinable with $(\mathbb{S}_2; =)$, with $(\mathbb{S}_2; \leq)$, or with $(\mathbb{S}_2; B)$, where B is the ternary *Betweenness relation.* The latter relation is defined by

$$B(x, y, z) \;\Leftrightarrow\; (x < y < z) \vee (z < y < x) \vee (x < y \wedge y \perp z) \vee (z < y \wedge y \perp x)\ .$$

We also classify the *model-complete cores* of the reducts of $(\mathbb{S}_2; \leq)$. A structure Γ is called *model-complete* iff its first-order theory is model-complete. A structure Δ is a *core* iff all endomorphisms of Δ are embeddings. It is known that every ω-categorical structure is *homomorphically equivalent* to a model-complete core Δ (that is, there is a homomorphism from Γ to Δ and vice versa; see [Bod07, BHM10]). The structure Δ is unique up to isomorphism, ω-categorical, and called the *model-complete core* of Γ. We show that for every reduct Γ of $(\mathbb{S}_2; \leq)$, the model-complete core of Γ is interdefinable with precisely one out of a list of ten structures (Corollary 2.2). The concept of model-complete cores is important for the aforementioned applications in constraint satisfaction, and implicitly used in complete complexity classifications for the CSPs of reducts of $(\mathbb{Q}; <)$ and the CSPs of reducts of the random graph [BK09, BP15]; also see [Bod12]. Our results have applications in this context which will be described in Section 5.

There are alternative formulations of our results in the language of permutation groups and transformation monoids, which also plays an important role in the proofs. By the theorem of Ryll-Nardzewski, two ω-categorical structures are first-order interdefinable if and only if

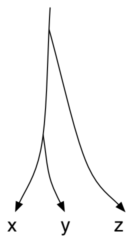

FIGURE 1. Illustration of $C(z, xy)$.

they have the same automorphisms. Our result about the reducts of $(\mathbb{S}_2; \leq)$ up to first-order interdefinability is equivalent to the statement that there are precisely three permutation groups that contain the automorphism group of $(\mathbb{S}_2; \leq)$ and that are closed in the full symmetric group $\mathrm{Sym}(\mathbb{S}_2)$ with respect to the *topology of pointwise convergence*, i.e., the product topology on $(\mathbb{S}_2)^{\mathbb{S}_2}$ where $\mathbb{S}_2$ is taken to be discrete. The link to transformation monoids comes from the fact that a countable ω-categorical structure Γ is model-complete if and only if $\mathrm{Aut}(\Gamma)$ is dense in the monoid $\mathrm{Emb}(\Gamma)$ of self-embeddings of Γ, i.e., the closure $\overline{\mathrm{Aut}(\Gamma)}$ of $\mathrm{Aut}(\Gamma)$ in $(\mathbb{S}_2)^{\mathbb{S}_2}$ equals $\mathrm{Emb}(\Gamma)$ [BP14]. Consequently, Γ is a model-complete core if and only if $\mathrm{Aut}(\Gamma)$ is dense in the endomorphism monoid $\mathrm{End}(\Gamma)$ of Γ, i.e., $\overline{\mathrm{Aut}(\Gamma)} = \mathrm{End}(\Gamma)$.

The proof method for showing our results relies on an analysis of the endomorphism monoids of reducts of $(\mathbb{S}_2; \leq)$. For that, we use a Ramsey-type statement for semilattices, due to Leeb [Lee73] (cf. also [GR74]). By results from [BP11, BPT13], that statement implies that if a reduct of $(\mathbb{S}_2; \leq)$ has an endomorphism that does not preserve a relation R, then it also has an endomorphism that does not preserve R and that behaves *canonically* in a formal sense defined in Section 3. Canonicity allows us to break the argument into finitely many cases.

We also mention a conjecture of Thomas, which states that every countable homogeneous structure Δ with a finite relational signature has only finitely many reducts up to interdefinability [Tho91]. By *homogeneous* we mean here that every isomorphism between finite substructures of Δ can be extended to an automorphism of Δ. Thomas' conjecture has been confirmed for various fundamental homogeneous structures, with particular activity in recent years [Cam76, Tho91, Tho96, Ben97, JZ08, Pon15, PPP$^+$14, BPP15, LP15]. The structure $(\mathbb{S}_2; \leq)$ is not homogeneous, but interdefinable with a homogeneous structure with a finite relational signature, so it falls into the scope of Thomas' conjecture.

2. Main results

To state our classification result, we need to introduce some homogeneous structures that appear in it. We have mentioned that $(\mathbb{S}_2; \leq)$ is not homogeneous, but interdefinable with a homogeneous structure with finite relational signature. Indeed, to obtain a homogeneous structure we can add a single first-order definable ternary relation C to $(\mathbb{S}_2; \leq)$, defined as

$$C(z, xy) \quad :\Leftrightarrow \quad x \perp y \ \wedge \ \exists u (x < u \wedge y < u \wedge u \perp z) \ . \tag{1}$$

See Figure 1.

We omit the comma between the last two arguments of C on purpose, since it increases readability, pointing out the symmetry $\forall x, y, z\ (C(z, xy) \Leftrightarrow C(z, yx))$. By a back-and-forth argument (Proposition 3.2) one can show that $(\mathbb{S}_2; \leq, C)$ is homogeneous, and clearly $(\mathbb{S}_2; \leq)$ and $(\mathbb{S}_2; \leq, C)$ are interdefinable.

We write $(\mathbb{L}_2; C)$ for the structure induced in $(\mathbb{S}_2; C)$ by any maximal antichain of $(\mathbb{S}_2; \leq)$; the reducts of $(\mathbb{L}_2; C)$, the *homogeneous binary branching C-relation on leaves*, were classified in [BJP14]. We mention in passing that the structure $(\mathbb{L}_2; C')$, where $C'(x, y, z) \Leftrightarrow \big(C(x, yz) \vee (y = z \wedge x \neq y)\big)$, is a so-called *C-relation*; we refer to [AN98] for the definition since we will not make further use of it.

It is known that two ω-categorical structures have the same endomorphisms if and only if they are existentially positively interdefinable, that is, if and only if each relation in one of the structures can be defined by an existential positive formula in the other structure [BP14]. We can now state one of our main results.

Theorem 2.1. *Let Γ be a reduct of $(\mathbb{S}_2; \leq)$. Then at least one of the following cases applies.*

(1) $\mathrm{End}(\Gamma)$ *contains a function whose range induces a chain in $(\mathbb{S}_2; \leq)$, and Γ is homomorphically equivalent to a reduct of the order of the rationals $(\mathbb{Q}; <)$.*
(2) $\mathrm{End}(\Gamma)$ *contains a function whose range induces an antichain in $(\mathbb{S}_2; \leq)$, and Γ is homomorphically equivalent to a reduct of $(\mathbb{L}_2; C)$.*
(3) $\mathrm{End}(\Gamma)$ *equals $\overline{\mathrm{Aut}(\mathbb{S}_2; B)}$; equivalently, Γ is existentially positively interdefinable with $(\mathbb{S}_2; B)$.*
(4) $\mathrm{End}(\Gamma)$ *equals $\overline{\mathrm{Aut}(\mathbb{S}_2; \leq)}$; equivalently, Γ is existentially positively interdefinable with $(\mathbb{S}_2; <, \perp)$.*

The reducts of $(\mathbb{L}_2; C)$ have been classified in [BJP14]. Each reduct of $(\mathbb{L}_2; C)$ is interdefinable with either

- $(\mathbb{L}_2; C)$ itself,
- $(\mathbb{L}_2; D)$ where $D(x, y, u, v)$ has the first-order definition $(C(u, xy) \wedge C(v, xy)) \vee (C(x, uv) \wedge C(y, uv))$ over $(\mathbb{L}_2; C)$, or
- $(\mathbb{L}_2; =)$.

.

The reducts of $(\mathbb{Q}; <)$ have been classified in [Cam76]. To describe them, it is convenient to write $\overrightarrow{x_1 \cdots x_n}$ whenever $x_1, \ldots, x_n \in \mathbb{Q}$ are such that $x_1 < \cdots < x_n$. Each reduct of $(\mathbb{Q}; <)$ is interdefinable with either

- the dense linear order $(\mathbb{Q}; <)$ itself,
- the structure $(\mathbb{Q}; \mathrm{Betw})$, where Betw is the ternary relation
$$\left\{(x, y, z) \in \mathbb{Q}^3 : \overrightarrow{xyz} \vee \overrightarrow{zyx}\right\},$$
- the structure $(\mathbb{Q}; \mathrm{Cyc})$, where Cyc is the ternary relation
$$\left\{(x, y, z) : \overrightarrow{xyz} \vee \overrightarrow{yzx} \vee \overrightarrow{zxy}\right\},$$
- the structure $(\mathbb{Q}; \mathrm{Sep})$, where Sep is the 4-ary relation
$$\left\{(x_1, y_1, x_2, y_2) : \overrightarrow{x_1x_2y_1y_2} \vee \overrightarrow{x_1y_2y_1x_2} \vee \overrightarrow{y_1x_2x_1y_2} \vee \overrightarrow{y_1y_2x_1x_2} \vee \overrightarrow{x_2x_1y_2y_1} \vee \overrightarrow{x_2y_1y_2x_1} \vee \overrightarrow{y_2x_1x_2y_1} \vee \overrightarrow{y_2y_1x_2x_1}\right\}, \text{ or}$$
- the structure $(\mathbb{Q}; =)$.

Corollary 2.2. *Let Γ be a reduct of $(\mathbb{S}_2; \leq)$. Then its model-complete core has only one element, or is isomorphic to a structure which is interdefinable with either $(\mathbb{S}_2; <, \perp)$, $(\mathbb{S}_2; B)$, $(\mathbb{L}_2; C)$, $(\mathbb{L}_2; D)$, $(\mathbb{Q}; <)$, $(\mathbb{Q}; \mathrm{Betw})$, $(\mathbb{Q}; \mathrm{Cyc})$, $(\mathbb{Q}; \mathrm{Sep})$, or $(\mathbb{Q}; \neq)$.*

Theorem 2.3. *Let Γ be a reduct of $(\mathbb{S}_2;\leq)$. Then Γ is first-order interdefinable with either $(\mathbb{S}_2;\leq)$, $(\mathbb{S}_2;B)$, or $(\mathbb{S}_2;=)$. Equivalently,* $\mathrm{Aut}(\Gamma)$ *equals either* $\mathrm{Aut}(\mathbb{S}_2;\leq)$, $\mathrm{Aut}(\mathbb{S}_2;B)$, *or* $\mathrm{Aut}(\mathbb{S}_2;=)$.

The permutation groups on $\mathbb{S}_2$ that are closed within $\mathrm{Sym}(\mathbb{S}_2)$ are precisely the automorphism groups of structures with domain $\mathbb{S}_2$. Moreover, the closed permutation groups on $\mathbb{S}_2$ that contain $\mathrm{Aut}(\mathbb{S}_2;\leq)$ are precisely the automorphism groups of reducts of $(\mathbb{S}_2;\leq)$. Therefore, the following is an immediate consequence of Theorem 2.3.

Corollary 2.4. *The closed subgroups of* $\mathrm{Sym}(\mathbb{S}_2)$ *containing* $\mathrm{Aut}(\mathbb{S}_2;\leq)$ *are precisely the permutation groups* $\mathrm{Aut}(\mathbb{S}_2;\leq)$, $\mathrm{Aut}(\mathbb{S}_2;B)$, *and* $\mathrm{Aut}(\mathbb{S}_2;=)$.

3. Preliminaries

In lack of a reference for the first-order axiomatization of $(\mathbb{S}_2;\leq)$ that we mentioned in the introduction, we prove it here for the convenience of the reader. We also prove the claim about the homogeneity of $(\mathbb{S}_2;\leq,C)$ made in Section 2. We then review the Ramsey properties of $(\mathbb{S}_2;\leq)$ after the expansion with a suitable linear order in Section 3.2. The Ramsey property will be used in our proof via the concept of *canonical functions*; they will be introduced in Section 3.3.

3.1. Homogeneity of $(\mathbb{S}_2;\leq,C)$.

We show that all countable semilinear orders that are unbounded, binary branching, nice, and without joins are isomorphic. As a by-product, we establish the homogeneity of $(\mathbb{S}_2;\leq,C)$.

We write $U<V$ for $U,V\subseteq\mathbb{S}_2$ when $u<v$ holds for all $u\in U$ and $v\in V$. The notation $U\leq V$ and $U\perp V$ is defined analogously. We also write $u<V$ for $\{u\}<V$ and $u\perp V$ for $\{u\}\perp V$.

Lemma 3.1. *Let $(P;\leq)$ be a dense, nice, and binary branching semilinear order without joins. Let $U,V,W\subset P$ be finite subsets such that U is non-empty, $U<V$, and $C(w,u_1u_2)$ for all $w\in W$ and $u_1,u_2\in U$. Then there exists an $x\in P$ such that $U<x$, $x<V$, and $x\perp W$.*

Proof. For $p,q\in V\cup W$, define $p\lhd q$ if

- $p<q$,
- $p\perp q$ and $u<p$ for all $u\in U$, or
- $C(q,pu)$ for all $u\in U$.

Note that $\lhd$ is transitive and irreflexive, a preorder on $U\cup V$. Let $m\in V\cup W$ be a minimal element with respect to $\lhd$.

We prove the statement by induction on the number of elements in U that are maximal with respect to $\leq$. Since U is non-empty and finite, there exists such a maximal element u_0. If there is just one such element, we distinguish whether $m\in V$ or $m\in W$. If $m\in V$ then we choose $x\in P$ such that $u_0<x<m$; such an x exists by density of $(P;\leq)$. If $m\in W$ then we choose $x\in P$ such that $u_0<x$ and $x\perp m$; such an x exists since $(P;\leq)$ is nice.

Now consider the case that there are two maximal elements $u_0,u_1\in U$. Again we distinguish two cases. If $m\in V$ then there exists an element $q\in P$ such that $u_0,u_1<x$ and $x<m$, since $(P;\leq)$ is without joins. Otherwise, $m\in W$. Since we have $C(m,u_0u_1)$ by assumption, there exists an element $x\in P$ such that $x\geq u_1,u_2$ and $x\perp m$, and this element x satisfies the required conditions.

Now suppose that there are at least three maximal elements u_0, u_1, u_2 in U. Since $(P; \leq)$ is binary branching, there is an $s \in P$ larger than two out of u_0, u_1, u_2 and incomparable to the third; without loss of generality say that $s > u_0, u > u_1$, and $s \perp u_2$. Then we apply the inductive assumption for the set $U' := U \cup \{s\} \setminus \{u_0, u_1\}$ instead of U, which has one maximal element less. The element $x \in P$ that we obtain for U' also satisfies the requirements that we have for U. ■

Proposition 3.2. *All countable semilinear orders that are dense, unbounded, binary branching, nice, and without joins are isomorphic to* $(\mathbb{S}_2; \leq)$. *The structure* $(\mathbb{S}; \leq, C)$ *is homogeneous.*

Proof. Let $(P; \leq)$ and $(Q; \leq)$ be two semilinear orders with the properties given in the statement, and let Γ and Δ be the expansions of those structures with the signature $\{\leq, C\}$ where C denotes the relation as defined in (1) at the beginning of Section 2.

We fix enumerations $(p_i)_{i \in \omega}$ and $(q_j)_{j \in \omega}$ of P and Q, respectively. Assume that $D \subset P$ is a finite subset of P and that $\rho \colon D \to E$ is an isomorphism between the substructure induced by D in Γ and the substructure induced by E in Δ. Let $k \in \omega$ be smallest such that $p_k \in P \setminus D$. To go forth we need to extend the domain of the partial isomorphism ρ to $D \cup \{p_k\}$. Let $D_> := \{a \in D : a > p_k\}$ and $D_< := \{a \in D : a < p_k\}$ and $D_\perp := \{a \in D : a \perp p_k\}$. In each case we describe the element $q \in Q$ such that $\rho(p_k) := q$ defines an extension of ρ which is a partial isomorphism between $(P; \leq, C)$ and $(Q; \leq, C)$.

Case 1: $D_<$ is empty. Suppose first that there is an element $v \in D_>$ such that $v \perp w$ for all $w \in D_\perp$. In this case we can choose $q \in Q$ such that $q < \rho(v)$ by the unboundedness of $(P; \leq)$. Otherwise, there exists an element $w_0 \in D_\perp$ such that $w_0 < v$ for all $v \in D_>$. From all those, choose w_0 such that $C(w, p_k w_0)$ or $C(p_k, w_0 w)$ for all other $w \in D_\perp$. By Lemma 3.1 applied to $U := \{\rho(u) \mid u \in D_\perp \mid C(p_k, u w_0)\}$, $V := \rho[D_>]$, and $W := \rho[D_\perp] \setminus U$ we obtain an element $x \in Q$ such that $U < x$, $x < V$, and $x \perp W$. Another application of this lemma gives us an element $x' \in Q$ with the same properties and $x' < x$. Since $(P; \leq)$ is binary branching there exists an element $q \in P$ with $q < x$ and $q \perp x'$, and this element has the desired properties.

Case 2: $D_<$ is non-empty. We apply Lemma 3.1 to $U := \rho[D_<]$, $V := \rho[D_>]$, and $W := \rho[D_\perp]$. The element x from the statement of Lemma 3.1 has the properties that we require for q.

This allows us to take the step going forth. To take the step going back, we need to extend the range of ρ to $D' \cup \{q_k\}$ where k is the first such that $q_k \in Q \setminus D'$. The argument is analogous to the argument given above for going forth. This concludes the back-and-forth and the result follows. ■

3.2. **The convex linear Ramsey extension.** Let $(S; \leq)$ be a semilinear order. A linear order $\prec$ on S is called a *convex linear extension of* $\leq$ iff the following three conditions hold; here, the relations $<$, B, and C are defined over $(S; \leq)$ as they were defined over $(\mathbb{S}_2; \leq)$.

- $\prec$ is an extension, i.e., $x < y$ implies $x \prec y$ for all $x, y \in S$;
- for all $x, y, z \in S$, if $B(x, y, z)$, then y also lies between x and z with respect to $\prec$, i.e., $(x \prec y \prec z) \vee (z \prec y \prec x)$;
- for all $x, y, z \in S$ we have that $C(x, yz)$ implies that x cannot lie between y and z with respect to $\prec$, i.e., $(x \prec y \wedge x \prec z) \vee (y \prec x \wedge z \prec x)$.

For finite semilinear orders $(S; \leq)$, the convex linear extensions are precisely those linear orders obtained by first defining $\prec$ arbitrarily on the second-largest elements of $(S; \leq)$, then ordering the elements just below those elements, and so on. Subject to these choices, $\prec$ is uniquely determined by the above convexity extension rules.

Using Fraïssé's theorem [Hod93] one can show that in the case of $(\mathbb{S}_2; \leq)$, there exists a convex linear extension $\prec$ of $\leq$ such that $(\mathbb{S}_2; \leq, C, \prec)$ is homogeneous and such that $(\mathbb{S}_2; \leq, \prec)$ is *universal* in the sense that it contains all isomorphism types of convex linear extensions of finite semilinear orders; this extension is unique in the sense that all expansions of $(\mathbb{S}_2; \leq, C)$ by a convex linear extension with the above properties are isomorphic. We henceforth fix any such extension $\prec$. It follows from Lindström's Test ([Hod93] Theorem 8.3.4) that $(\mathbb{S}_2; \leq, C, \prec)$ is model complete. The structure $(\mathbb{S}_2; \leq, C, \prec)$ is combinatorially well-behaved in the following sense. For structures Σ, Π in the same language, we write $\binom{\Sigma}{\Pi}$ for the set of all embeddings of Π into Σ.

Definition 3.3. A countable homogeneous relational structure Δ is called a *Ramsey structure* iff for all finite substructures Ω of Δ, all substructures Γ of Ω, and all $\chi\colon \binom{\Delta}{\Gamma} \to 2$ there exists an $e_1 \in \binom{\Delta}{\Omega}$ such that χ is constant on $e_1 \circ \binom{\Omega}{\Gamma}$. An ω-categorical structure is called *Ramsey* if its (homogeneous) expansion by all first-order definable relations is Ramsey.

The following theorem is a special case of a Ramsey-type statement for semilinearly ordered semilattices due to Leeb [Lee73] (also see [GR74], page 276). A *semilinearly ordered semilattice* $(S; \vee, \leq)$ is a semilinear order $(S; \leq)$ which is closed under the binary function $\vee$, the *join* function, satisfying for all x and y, that $x \vee y$ is the least upper bound of $\{x, y\}$ with respect to $\leq$. If $\prec$ is a convex linear extension of $\leq$, then $(S; \vee, \leq, \prec)$ is a convex linear extension of the semilinearly ordered semilattice $(S; \vee, \leq)$. By Fraïssé's Theorem [Hod93] and a back and forth argument, there is a countably infinite homogeneous structure $(\mathbb{T}; \vee, \leq, \prec)$ which is the Fraïssé limit of the class of finite, semilinearly ordered semilattices with a convex linear extension.

Theorem 3.4 (Leeb)**.** *$(\mathbb{T}; \vee, \leq, \prec)$ is a Ramsey structure.*

Corollary 3.5. *$(\mathbb{S}_2; \leq, C, \prec)$ is a Ramsey structure.*

Proof. Let $(\mathbb{T}; \leq, C, \prec)$ be the structure obtained from $(\mathbb{T}; \vee, \leq, \prec)$ by restricting to the relations $(\leq, \prec)$ and making a definitional expansion with the ternary relation C (with formal definition as in Section 2 (1)). Note that $\mathrm{Aut}(\mathbb{T}; \vee, \leq, \prec) = \mathrm{Aut}(\mathbb{T}; \leq, C, \prec)$. Theorem 3.4 above then implies that $(\mathbb{T}; \leq, C, \prec)$ is a Ramsey structure. Every finite substructure of $(\mathbb{S}_2; \leq, C, \prec)$ is isomorphic to a substructure of $(\mathbb{T}; \leq, C, \prec)$ and vice versa, so they have the same age. As $(\mathbb{S}_2; \leq, C, \prec)$ is model complete, it is the model companion of $(\mathbb{T}; \leq, C, \prec)$. Using Theorem 3.15 of [Bod15] we conclude that $(\mathbb{S}_2; \leq, C, \prec)$ is a Ramsey structure. ∎

3.3. **Canonical functions.** The fact that $(\mathbb{S}_2; \leq, C, \prec)$ is a relational homogeneous Ramsey structure implies that endomorphism monoids of reducts of this structure, and hence also of $(\mathbb{S}_2; \leq, C)$, can be distinguished by so-called *canonical functions*.

Definition 3.6. Let Δ be a structure, and let a be an n-tuple of elements in Δ. The *type* of a in Δ is the set of first-order formulas with free variables $x_1, \ldots, x_n$ that hold for a in Δ.

Definition 3.7. Let Δ and Γ be structures. A *type condition* between Δ and Γ is a pair (t, s), such that t is the type on an n-tuple in Δ and s is the type of an n-tuple in Γ, for some $n \geq 1$. A function $f\colon \Delta \to \Gamma$ *satisfies* a type condition (t, s) iff the type of $(f(a_1), \ldots, f(a_n))$ in Γ equals s for all n-tuples $(a_1, \ldots, a_n)$ in Δ of type t.

A *behaviour* is a set of type conditions between Δ and Γ. We say that a function $f\colon \Delta \to \Gamma$ has a given behaviour iff it satisfies all of its type conditions.

Definition 3.8. Let Δ and Γ be structures. A function $f : \Delta \to \Gamma$ is *canonical* iff for every type t of an n-tuple in Δ there is a type s of an n-tuple in Γ such that f satisfies the type condition (t, s). That is, canonical functions send n-tuples of the same type to n-tuples of the same type, for all $n \geq 1$.

Note that any canonical function induces a function from the types over Δ to the types over Γ.

Definition 3.9. Let $\mathcal{F} \subseteq (\mathbb{S}_2)^{\mathbb{S}_2}$. We say that $\mathcal{F}$ *generates* a function $g\colon \mathbb{S}_2 \to \mathbb{S}_2$ iff g is contained in the smallest closed submonoid of $(\mathbb{S}_2)^{\mathbb{S}_2}$ which contains $\mathcal{F}$. This is the case iff for every finite subset $A \subset \mathbb{S}_2$ there exists an $n \geq 1$ and $f_1, \ldots, f_n \in \mathcal{F}$ such that $f_1 \circ \cdots \circ f_n$ agrees with g on A.

Our proof relies on the following proposition which is a consequence of [BP11, BPT13] and the fact that $(\mathbb{S}_2; \leq, C, \prec)$ is a homogeneous Ramsey structure. For a structure Δ and elements $c_1, \ldots, c_n$ in that structure, let $(\Delta, c_1, \ldots, c_n)$ denote the structure obtained from Δ by adding the constants $c_1, \ldots, c_n$ to the language.

Proposition 3.10. *Let $f\colon \mathbb{S}_2 \to \mathbb{S}_2$ be any injective function, and let $c_1, \ldots, c_n \in \mathbb{S}_2$. Then $\{f\} \cup \mathrm{Aut}(\mathbb{S}_2; \leq, \prec)$ generates an injective function $g\colon \mathbb{S}_2 \to \mathbb{S}_2$ such that*

- *g agrees with f on $\{c_1, \ldots, c_n\}$;*
- *g is canonical as a function from $(\mathbb{S}_2; \leq, C, \prec, c_1, \ldots, c_n)$ to $(\mathbb{S}_2; \leq, C, \prec)$.*

4. The Proof

4.1. **Rerootings and betweenness.** We start by examining what the self-embeddings, automorphisms, and endomorphisms of $(\mathbb{S}_2; B)$ look like.

Definition 4.1. A *rerooting* of $(\mathbb{S}_2; <)$ is an injective function $f\colon \mathbb{S}_2 \to \mathbb{S}_2$ for which there exists a set $S \subseteq \mathbb{S}_2$ such that

- S contains no incomparable elements and is upward closed with respect to $<$;
- f reverses the order $<$ on S;
- f preserves $<$ and $\perp$ on $\mathbb{S}_2 \setminus S$;
- whenever $x \in \mathbb{S}_2 \setminus S$ and $y \in S$, then $x < y$ implies $f(x) \perp f(y)$ and $x \perp y$ implies $f(x) < f(y)$.

We then say that f is a *rerooting with respect to S*.

It is not hard to see that whenever $S \subseteq \mathbb{S}_2$ is as above, then there is a rerooting with respect to S. A rerooting with respect to S is a self-embedding of $(\mathbb{S}_2; <)$ if and only if S is empty, and the image of any rerooting with respect to S is isomorphic to $(\mathbb{S}_2; <)$ if and only if S is a maximal chain or empty. In particular, there exist rerootings which are permutations of $\mathbb{S}_2$ and which are not self-embeddings of $(\mathbb{S}_2; <)$.

Proposition 4.2. $\mathrm{Emb}(\mathbb{S}_2; B)$ *consists precisely of the rerootings of* $(\mathbb{S}_2; <)$.

Proof. It is easy to check that rerootings preserve B and its negation. Let $f \in \mathrm{Emb}(\mathbb{S}_2; B)$. We first claim that either $f \in \mathrm{Emb}(\mathbb{S}_2; <)$, or there exist $x, y \in \mathbb{S}_2$ such that $x < y$ and $f(x) > f(y)$. To see this, suppose first that f violates $\perp$. Pick $a, b \in \mathbb{S}_2$ with $a \perp b$ and such that $f(a) < f(b)$. There exists $c \in \mathbb{S}_2$ such that $c > b$ and such that $B(a, c, b)$. Since f

preserves B we then must have $f(c) < f(b)$, and our claim follows. Now suppose f violates $<$, and pick $a, b \in \mathbb{S}_2$ with $a < b$ witnessing this. Then for any $c \in \mathbb{S}_2$ with $c > b$ we have $f(c) < f(b)$, proving the claim.

Let $S := \{x \in \mathbb{S}_2 \mid \exists y \in \mathbb{S}_2(x < y \wedge f(y) < f(x))\}$. By the above, S is non-empty. Since f preserves B, it follows easily that whenever $x \in S$, $y \in \mathbb{S}_2$ and $x < y$, then $f(y) > f(x)$. From this and again because f preserves B it follows that S is upward closed, i.e., if $x \in S$ and $y \in \mathbb{S}_2$ satisfy $y > x$, then $y \in S$. Hence, S cannot contain incomparable elements x, y, as otherwise for any $z \in S$ with $x < z$ and $y < z$ we would have $f(x) > f(z)$ and $f(y) > f(z)$, and so $f(x)$ and $f(y)$ would have to be comparable. But then f would violate $\neg B$ on $\{x, y, z\}$.

Consider $a \in \mathbb{S}_2 \setminus S$ and $b \in S$ with $a < b$. Pick $c \in S$ with $c > b$. Then $f(c) < f(b)$ and $B(a, b, c)$ imply that $f(a) > f(b)$ or $f(a) \perp f(b)$. The first case is impossible by the definition of S, and so $f(a) \perp f(b)$.

Next consider $a \in \mathbb{S}_2 \setminus S$ and $b \in S$ with $a \perp b$. Picking $c \in S$ with $B(a, c, b)$, we derive that $f(a) < f(b)$.

Let $x, y \in \mathbb{S}_2 \setminus S$ with $x < y$. Pick $z \in S$ such that $y < z$. Then $B(f(x), f(y), f(z))$, $f(x) \perp f(z)$ and $f(y) \perp f(z)$ imply that $f(x) < f(y)$.

Finally, given $x, y \in \mathbb{S}_2 \setminus S$ with $x \perp y$, we can pick $z \in S$ such that $x < z$ and $y < z$. Then $f(x) \perp f(z)$, $f(y) \perp f(z)$, $\neg B(f(x), f(y), f(z))$, and $\neg B(f(y), f(x), f(z))$ together imply $f(x) \perp f(y)$. ■

Corollary 4.3. $\mathrm{Aut}(\mathbb{S}_2; B)$ *consists precisely of the surjective rerootings with respect to a maximal chain or with respect to the empty set.*

Corollary 4.4. $\mathrm{Emb}(\mathbb{S}_2; B)$ *is generated by any of its functions which do not preserve* $<$.

Proof. By homogeneity of $(\mathbb{S}_2; \leq, C)$ and topological closure. ■

Proposition 4.5. *Any function in* $(\mathbb{S}_2)^{\mathbb{S}_2}$ *that preserves* B *is injective and preserves* $\neg B$. *Consequently,* $\mathrm{End}(\mathbb{S}_2; B) = \mathrm{Emb}(\mathbb{S}_2; B) = \overline{\mathrm{Aut}(\mathbb{S}_2; B)}$.

Proof. The existential positive formula

$$(a = b) \vee (b = c) \vee (c = a) \vee \exists x(B(a, x, b) \wedge B(b, x, c))$$

is equivalent to $\neg B(a, b, c)$. Therefore B and $\neg B$ are existentially positively interdefinable, and hence preserved by the same unary functions on $\mathbb{S}_2$ (cf. the discussion in the introduction). Moreover, for all $a, b \in \mathbb{S}_2$ we have that $a \neq b$ iff there exists $c \in \mathbb{S}_2$ such that $B(a, b, c)$, so inequality has an existential positive definition from B, and functions preserving B must be injective. Hence, every endomorphism of $(\mathbb{S}_2; B)$ is an embedding.

From Proposition 4.2 and 4.3 it follows that the restriction of any self-embedding of $(\mathbb{S}_2; B)$ to a finite subset of $\mathbb{S}_2$ extends to an automorphism, and hence $\mathrm{Emb}(\mathbb{S}_2; B) = \overline{\mathrm{Aut}(\mathbb{S}_2; B)}$. ■

4.2. **Ramsey-theoretic analysis.**

4.2.1. *Canonical functions without constants.* Every canonical function $f\colon (\mathbb{S}_2; \leq, C, \prec) \to (\mathbb{S}_2; \leq, C, \prec)$ induces a function on the 3-types of $(\mathbb{S}_2; \leq, C, \prec)$. Our first lemma shows that only few functions on those 3-types are induced by canonical functions, i.e., there are only few behaviors of canonical functions.

Definition 4.6. We call a function $f\colon \mathbb{S}_2 \to \mathbb{S}_2$

- *flat* iff its image induces an antichain in $(\mathbb{S}_2; \leq)$;
- *thin* iff its image induces a chain in $(\mathbb{S}_2; \leq)$.

Lemma 4.7. *Let $f\colon (\mathbb{S}_2; \leq, C, \prec) \to (\mathbb{S}_2; \leq, C, \prec)$ be an injective canonical function. Then either f is flat, or f is thin, or $f \in \operatorname{End}(\mathbb{S}_2; <, \perp)$.*

Proof. Let $u_1, u_2, v_1, v_2 \in \mathbb{S}_2$ be so that $u_1 < u_2$, $v_1 \perp v_2$, and $v_1 \prec v_2$. If $f(u_1) \perp f(u_2)$ and $f(v_1) \perp f(v_2)$, then f is flat by canonicity. If $f(u_1) \not\perp f(u_2)$ and $f(v_1) \not\perp f(v_2)$, then f is thin. It remains to check the following cases.

Case 1: $f(u_1) \perp f(u_2)$ and $f(v_1) < f(v_2)$. Let $x, y, z \in \mathbb{S}_2$ be such that $x < y$, $x \perp z$, $y \perp z$, $z \prec x$, and $z \prec y$. Then $f(x) \perp f(y)$, $f(x) > f(z)$, and $f(y) > f(z)$, in contradiction with the axioms of the semilinear order.

Case 2: $f(u_1) \perp f(u_2)$ and $f(v_1) > f(v_2)$. Let $x, y, z \in \mathbb{S}_2$ be such that $x < y$, $x \perp z$, $y \perp z$, $x \prec z$, and $y \prec z$. Then $f(x) \perp f(y)$, $f(x) > f(z)$, and $f(y) > f(z)$, in contradiction with the axioms of the semilinear order.

Case 3: $f(u_1) < f(u_2)$ and $f(v_1) \perp f(v_2)$. Then f preserves $<$ and $\perp$.

Case 4: $f(u_1) > f(u_2)$ and $f(v_1) \perp f(v_2)$. Let $x, y, z \in \mathbb{S}_2$ such that $x \perp y$, $x \prec y$, $x < z$, and $y < z$. Then $f(x) \perp f(y)$, $f(x) > f(z)$, and $f(y) > f(z)$, in contradiction with the axioms of the semilinear order. ∎

4.2.2. *Canonical functions with constants.*

Lemma 4.8. *Let $f\colon \mathbb{S}_2 \to \mathbb{S}_2$ be a function. If f preserves incomparability but not comparability in $(\mathbb{S}; \leq)$, then $\{f\} \cup \operatorname{Aut}(\mathbb{S}_2; \leq)$ generates a flat function. If f preserves comparability but not incomparability in $(\mathbb{S}; \leq)$, then $\{f\} \cup \operatorname{Aut}(\mathbb{S}_2; \leq)$ generates a thin function.*

Proof. We show the first statement; the proof of the second statement is analogous. We first claim that for any finite set $A \subseteq \mathbb{S}_2$, f generates a function which sends A to an antichain. To see this, let A be given, and pick $a, b \in \mathbb{S}_2$ such that $a < b$ and $f(a) \perp f(b)$. If A contains elements u, v with $u < v$, then let $\alpha \in \operatorname{Aut}(\mathbb{S}_2; \leq)$ be so that $\alpha(u) = a$ and $\alpha(b) = v$. The function $f \circ \alpha$ sends A to a set which has less pairs (u, v) satisfying $u < v$ than A. Repeating this procedure on the image of A and so forth and composing functions we obtain a function which sends A to an antichain. Now let $\{s_0, s_1, \ldots\}$ be an enumeration of $\mathbb{S}_2$, and pick for every $n \geq 0$ a function g_n generated by $\{f\} \cup \operatorname{Aut}(\mathbb{S}_2; \leq)$ which sends $\{s_0, \ldots, s_n\}$ to an antichain. Since $(\mathbb{S}; \leq)$ is ω-categorical, by thinning out the sequence we may assume that for all $n \geq 0$ and all $i, j \geq n$ the type of the tuple $(g_i(s_0), \ldots, g_i(s_n))$ equals the type of $(g_j(s_0), \ldots, g_j(s_n))$ in $(\mathbb{S}; \leq)$. By composing with automorphisms of $(\mathbb{S}; \leq)$ from the left, we may even assume that these tuples are equal. But then the sequence $(g_n)_{n \in \omega}$ converges to a flat function. ∎

Definition 4.9. When $n \geq 1$ and $R \subseteq \mathbb{S}_2^n$ is an n-ary relation, then we say that $R(X_1, \ldots, X_n)$ holds for sets $X_1, \ldots, X_n \subseteq \mathbb{S}_2$ iff $R(x_1, \ldots, x_n)$ holds whenever $x_i \in X_i$ for all $1 \leq i \leq n$. We also use this notation when some of the X_i are elements of $\mathbb{S}_2$ rather than subsets, in which case we treat them as singleton subsets.

Definition 4.10. For $a \in \mathbb{S}_2$, we set

- $U^a_< := \{p \in \mathbb{S}_2 \mid p < a\}$;
- $U^a_> := \{p \in \mathbb{S}_2 \mid p > a\}$;
- $U^a_{\perp,\prec} := \{p \in \mathbb{S}_2 \mid p \perp a \wedge p \prec a\}$;
- $U^a_{\perp,\succ} := \{p \in \mathbb{S}_2 \mid p \perp a \wedge a \prec p\}$;
- $U^a_\perp := U^a_{\perp,\succ} \cup U^a_{\perp,\prec}$.

The first four sets defined above are precisely the infinite orbits of $\operatorname{Aut}(\mathbb{S}_2; \leq, \prec, a)$.

Lemma 4.11. *Let $a \in \mathbb{S}_2$, and let $f \colon (\mathbb{S}_2; \leq, C, \prec, a) \to (\mathbb{S}_2; \leq, C, \prec)$ be an injective canonical function. Then one of the following holds:*

1. *$\{f\} \cup \mathrm{Aut}(\mathbb{S}_2; \leq)$ generates a flat or a thin function;*
2. *$f \in \mathrm{End}(\mathbb{S}_2; <, \perp)$;*
3. *$f{\restriction}_{\mathbb{S}_2 \setminus \{a\}}$ behaves like a rerooting function with respect to $U^a_>$, and $f(a) \not< f[U^a_>]$.*

Moreover, if $f(a) \not> f[U^a_<]$ and $f(a) \not> f[U^a_>]$, then $\{f\} \cup \mathrm{Aut}(\mathbb{S}_2; \leq)$ generates a flat or a thin function.

Proof. The set $U^a_<$ induces an isomorphic copy of $(\mathbb{S}_2; \leq, C, \prec)$, and the restriction of f to this copy is canonical. By Lemma 4.7 we may assume that f preserves $<$ and $\perp$ on $U^a_<$ as otherwise $\{f\} \cup \mathrm{Aut}(\mathbb{S}_2; \leq)$ generates a flat or a thin function.

When $u, v \in U^a_{\perp,\prec}$ satisfy $u < v$, then there exists a subset of $U^a_{\perp,\prec}$ containing u and v which induces an isomorphic copy of $(\mathbb{S}_2; \leq, C, \prec)$. As above, we may assume that f preserves $<$ and $\perp$ on this subset, and hence $f(u) < f(v)$. If $u, v \in U^a_{\perp,\prec}$ satisfy $u \perp v$, then there exist subsets R, S of $U^a_{\perp,\prec}$ containing u and v, respectively, such that both R and S induce isomorphic copies of $(\mathbb{S}_2; \leq, C, \prec)$ and such that for all $r \in R$ and $s \in S$ the type of (r, s) equals the type of (u, v) in $(\mathbb{S}_2; \leq, C, \prec)$. Assuming as above that f preserves $<$ and $\perp$ on both copies, $f(u) < f(v)$ would imply $f[R] < f[S]$, which is in contradiction with the axioms of a semilinear order. Hence, we may assume that f preserves $<$ and $\perp$ on $U^a_{\perp,\prec}$, and by a similar argument also on $U^a_{\perp,\succ}$.

The sets $U^a_{\perp,\prec}$, $U^a_{\perp,\succ}$, and $U^a_<$ are pairwise incomparable, and the relation $\perp$ between them cannot be violated, as this would contradict the axioms of the semilinear order. Thus we may assume that f preserves $<$ and $\perp$ on $U^a_\perp \cup U^a_<$. Moreover, for no $p \in \{a\} \cup U^a_>$ we have $f(p) < f[U^a_{\perp,\prec}]$, $f(p) < f[U^a_{\perp,\succ}]$, or $f(p) < f[U^a_<]$, again by the properties of semilinear orders.

Assume that $U^a_>$ is mapped to an antichain by f. Then canonicity of f implies that $f[U^a_>] \perp f[U^a_\perp \cup U^a_<]$, as all other possibilities are in contradiction with the axioms of the semilinear order. In particular, f then preserves $\perp$ on $\mathbb{S}_2 \setminus \{a\}$. Given a finite $A \subseteq \mathbb{S}_2$ which is not an antichain, there exists $\alpha \in \mathrm{Aut}(\mathbb{S}_2; \leq)$ such that $\alpha[A] \subseteq \mathbb{S}_2 \setminus \{a\}$, and two comparable points are mapped into $U^a_>$ by α. Thus $f \circ \alpha$ preserves $\perp$ on A, and it maps at least one comparable pair in A to an incomparable one. As in Lemma 4.8, we see that $\{f\} \cup \mathrm{Aut}(\mathbb{S}_2; \leq)$ generates a flat function. So we may assume that the order on $U^a_>$ is either preserved or reversed by f. The rest of the proof is an analysis of the possible behaviours of f in these two cases. In order to talk about the behaviour of f, we choose elements $u_1 \in U^a_{\perp,\prec}$, $u_2 \in U^a_{\perp,\succ}$ and $z_1, z_2 \in U^a_>$ such that $z_1 < z_2$, $u_i \perp z_1$, and $u_i < z_2$ for $i \in \{1, 2\}$.

Case 1: f preserves the order on $U^a_>$. If $f(u_1) < f(z_1)$, then by transitivity of $<$ and canonicity of f we have that $f[U^a_{\perp,\prec}] < f[U^a_>]$. Given a finite $A \subseteq \mathbb{S}_2$ which is not a chain, there exists $\alpha \in \mathrm{Aut}(\mathbb{S}_2; \leq)$ such that $\alpha[A] \subseteq U^a_{\perp,\prec} \cup U^a_>$ and such that $\alpha(x) \in U^a_{\perp,\prec}$ and $\alpha(y) \in U^a_>$ for some elements $x, y \in A$ with $x \perp y$. Thus $f \circ \alpha$ preserves $<$ on A, and it maps at least one incomparable pair in A to a comparable one. As in Lemma 4.8, we conclude that $\{f\} \cup \mathrm{Aut}(\mathbb{S}_2; \leq)$ generates a thin function. We can argue similarly when $f(u_2) < f(z_1)$. Thus we may assume that $f(u_i) \perp f(z_1)$ for $i \in \{1, 2\}$. If $f(u_i) \perp f(z_2)$ for some $i \in \{1, 2\}$, then a similar argument shows that $\{f\} \cup \mathrm{Aut}(\mathbb{S}_2; \leq)$ generates a flat function. Hence, we may assume that $f(u_i) < f(z_2)$ for $i \in \{1, 2\}$, and so f preserves $<$ and $\perp$ on $U^a_\perp \cup U^a_>$.

Assume that $f[U^a_<] \perp f[U^a_>]$. Given a finite $A \subseteq \mathbb{S}_2$ which is not an antichain, there exists $\alpha \in \mathrm{Aut}(\mathbb{S}_2; \leq)$ such that $\alpha[A] \subseteq \mathbb{S}_2 \setminus \{a\}$ and such that $\alpha(x) \in U^a_<$ and $\alpha(y) \in U^a_>$ for some $x, y \in A$ with $x < y$. Thus $f \circ \alpha$ preserves $\perp$ on A, and it maps at least one comparable pair in A to an incomparable one. The proof of Lemma 4.8 shows that $\{f\} \cup \mathrm{Aut}(\mathbb{S}_2; \leq)$ generates a flat function. So we may assume that $f[U^a_<] < f[U^a_>]$, and consequently, f preserves $<$ and $\perp$ on $\mathbb{S}_2 \setminus \{a\}$.

If $f(a) > f[U^a_>]$, then by transitivity of $<$ we have $f(a) > f[\mathbb{S}_2 \setminus \{a\}]$, and we can easily show that $\{f\} \cup \mathrm{Aut}(\mathbb{S}_2; \leq)$ generates a thin function. Similarly, if $f(a) \perp f[U^a_>]$, then by the axioms of the semilinear order we have $f(a) \perp f[\mathbb{S}_2 \setminus \{a\}]$, and $\{f\} \cup \mathrm{Aut}(\mathbb{S}_2; \leq)$ generates a flat function. Thus we may assume that $f(a) < f[U^a_>]$. If $f(a) > f[U^a_{\perp,\prec}]$ or $f(a) > f[U^a_{\perp,\succ}]$, then by transitivity of $<$ we have $f[U^a_{\perp,\prec}] < f[U^a_>]$ or $f[U^a_{\perp,\succ}] < f[U^a_>]$, a contradiction. Hence, $f(a) \perp f[U^a_\perp]$. Finally, if $f(a) \perp f[U^a_<]$, then $\{f\} \cup \mathrm{Aut}(\mathbb{S}_2; \leq)$ generates a flat function. Thus we may assume that $f(a) > f[U^a_<]$, and so f preserves $<$ and $\perp$, proving the lemma.

Case 2: f reverses the order on $U^a_>$. If $f(u_1) \perp f(z_1)$, then by $f(z_2) < f(z_1)$ and the axioms of the semilinear order we have that $f(u_1) \perp f(z_2)$. Moreover, $f{\restriction}_{U^a_{\perp,\prec} \cup U^a_>}$ preserves $\perp$. Since the comparable elements u_1, z_2 are sent to incomparable ones, the standard iterative argument shows that $\{f\} \cup \mathrm{Aut}(\mathbb{S}_2; \leq)$ generates a flat function. An analogous argument works if $f(u_2) \perp f(z_1)$. Thus we may assume that $f(u_i) < f(z_1)$ for $i \in \{1, 2\}$. If $f(u_i) < f(z_2)$ for some $i \in \{1, 2\}$, then a similar argument shows that $\{f\} \cup \mathrm{Aut}(\mathbb{S}_2; \leq)$ generates a thin function. Thus we may assume that $f(u_i) \perp f(z_2)$ for $i \in \{1, 2\}$, and $f{\restriction}_{U^a_\perp \cup U^a_>}$ behaves like a rerooting.

Assume that $f[U^a_<] < f[U^a_>]$. Let $A \subseteq \mathbb{S}_2$ be finite. Pick a minimal element $b \in A$, and let $C \subseteq A$ be those elements $c \in A$ with $b \leq c$. Let $\alpha \in \mathrm{Aut}(\mathbb{S}_2; \leq)$ be such that $\alpha(b) \in U^a_<$, $\alpha[C \setminus \{b\}] \subseteq U^a_>$ and $\alpha[A \setminus C] \subseteq U^a_\perp$. Then there exists $\beta \in \mathrm{Aut}(\mathbb{S}_2; \leq)$ such that $\beta \circ f \circ \alpha[C] \subseteq U^a_>$ and $\beta \circ f \circ \alpha[A \setminus C] \subseteq U^a_\perp$. Let $g := f \circ \beta \circ f \circ \alpha$. Then $g{\restriction}_{A \setminus \{b\}}$ preserves $<$ and $\perp$, and $g(b) \geq g[A]$. By iterating such steps, A can be mapped to a chain. Hence, as in Lemma 4.8, $\{f\} \cup \mathrm{Aut}(\mathbb{S}_2; \leq)$ generates a thin function. Thus we may assume that $f[U^a_<] \perp f[U^a_>]$. By replacing $U^a_<$ with $\{a\}$ in this argument, one can show that if $f(a) < f[U^a_>]$, then $\{f\} \cup \mathrm{Aut}(\mathbb{S}_2; \leq)$ generates a thin function. Thus we may assume that $f(a) \not< f[U^a_>]$, and so Item (3) applies.

To show the second part of the lemma, suppose that $f(a) \not> f[U^a_<]$ and $f(a) \not> f[U^a_>]$. Then f violates $<$, thus Item (2) cannot hold for f. Hence, either $\{f\} \cup \mathrm{Aut}(\mathbb{S}_2; \leq)$ generates a flat or a thin function, or the conditions in Item (3) hold for f. We assume the latter. In particular, $f(a) \perp f[U^a_\perp]$, by the axioms of the semilinear order, and hence $f(a) \perp f[U^a_>]$.

Let $A \subseteq \mathbb{S}_2$ be finite such that A is not an antichain. Pick some $x \in A$ with is maximal in A with respect to $\leq$ and such that there exists $y \in A$ with $y < x$. Let $\alpha \in \mathrm{Aut}(\mathbb{S}_2; \leq)$ be such that $\alpha(x) = a$. Then $f \circ \alpha$ preserves $\perp$ on A, and $f(y) \perp f(x)$. Hence, iterating such steps A can be mapped to an antichain, and $\{f\} \cup \mathrm{Aut}(\mathbb{S}_2; \leq)$ generates a flat function. ■

4.2.3. *Applying canonicity.*

Lemma 4.12. *Let $f\colon \mathbb{S}_2 \to \mathbb{S}_2$ be an injective function that violates $<$. Then either $\{f\} \cup \mathrm{Aut}(\mathbb{S}_2; \leq)$ generates a flat or a thin function, or $\{f\} \cup \mathrm{Aut}(\mathbb{S}_2; \leq)$ generates* $\mathrm{End}(\mathbb{S}_2; B)$.

Proof. If f preserves comparability and incomparability, then f cannot violate $<$. If f preserves comparability and violates incomparability, then $\{f\} \cup \mathrm{Aut}(\mathbb{S}_2; \leq)$ generates a thin function by Lemma 4.8. Thus we may assume that f violates comparability. Let $a, b \in \mathbb{S}_2$ such that $a < b$ and $f(a) \perp f(b)$. According to Proposition 3.10, there exists a canonical function $g\colon (\mathbb{S}_2; \leq, C, \prec, a, b) \to (\mathbb{S}_2, \leq, C, \prec)$ that is generated by $\{f\} \cup \mathrm{Aut}(\mathbb{S}_2; \leq)$ such that $g(a) \perp g(b)$. The set $U^b_<$ induces in $(\mathbb{S}_2; \leq, C, \prec, a)$ a structure isomorphic to $(\mathbb{S}_2; \leq, C, \prec, a)$, and the restriction of g to this set is canonical. By Lemma 4.11 either $\{g\} \cup \mathrm{Aut}(\mathbb{S}_2; \leq)$

generates a thin or a flat function, or a rerooting, or g preserves $<$ and $\perp$ on $U^b_<$. We may assume the latter. By a similar argument, either $\{g\} \cup \mathrm{Aut}(\mathbb{S}_2; \leq)$ generates a thin or a flat function, or a rerooting, or g preserves $<$ and $\perp$ on $U^a_< \cup U^b_\perp \cup U^b_> \cup \{b\}$. However, the latter is impossible as it would imply that $g(t) < g(a)$ and $g(t) < g(b)$ for all $t \in U^a_<$ while $g(a) \perp g(b)$, which is in contradiction with the axioms of the semilinear order. ■

Lemma 4.13. *Let $f : \mathbb{S}_2 \to \mathbb{S}_2$ be an injective function that violates B. Then $\{f\} \cup \mathrm{Aut}(\mathbb{S}_2; B)$ generates a flat or a thin function.*

Proof. Let $a, b, c \in \mathbb{S}_2$ be such that $B(a, b, c)$ and $\neg B(f(a), f(b), f(c))$. Then it follows from Corollary 4.4 that there exist $\alpha, \beta \in \mathrm{Aut}(\mathbb{S}_2; B)$ such that $\alpha(a) < \alpha(b) < \alpha(c)$ and such that $\{\beta(f(a)), \beta(f(b)), \beta(f(c))\}$ induces an antichain. Replacing f by $\beta \circ f \circ \alpha^{-1}$, we may assume that there are $a, b, c \in \mathbb{S}_2$ such that $a < b < c$ and such that $\{f(a), f(b), f(c)\}$ induce an antichain. By Proposition 3.10, there exists a canonical function $g : (\mathbb{S}_2; \leq, C, \prec, a, b, c) \to (\mathbb{S}_2, \leq, C)$ that is generated by $\{f\} \cup \mathrm{Aut}(\mathbb{S}_2; \leq)$ such that $\{g(a), g(b), g(c)\}$ induces an antichain.

By the axioms of the semilinear order, at most one $y \in \{g(a), g(b), g(c)\}$ can satisfy $y > g[U^a_<]$ and at most one such element can satisfy $y > g[U^c_>]$. Hence, there exists an $x \in \{a, b, c\}$ such that $g(x) \not> g[U^a_<]$ and $g(x) \not> g[U^c_>]$. The set $X := U^a_< \cup \{x\} \cup U^c_> \cup U^c_\perp$ induces in $(\mathbb{S}_2; \leq, C, \prec)$ a structure isomorphic to $(\mathbb{S}_2; \leq, C, \prec)$, and $g{\restriction}_X$ is canonical as a function from $(\mathbb{S}_2; \leq, C, \prec, x)$ to $(\mathbb{S}_2; \leq, C, \prec)$. According to the second part of Lemma 4.11, $\{g\} \cup \mathrm{Aut}(\mathbb{S}_2; \leq)$ generates a flat or a thin function. ■

4.3. **Endomorphisms and the proof of Theorem 2.1.**

Proposition 4.14. *Let Γ be a reduct of $(\mathbb{S}_2; \leq)$. Then one of the following holds.*

(1) $\mathrm{End}(\Gamma)$ *contains a flat or a thin function.*
(2) $\mathrm{End}(\Gamma) = \overline{\mathrm{Aut}(\mathbb{S}_2; \leq)}$.
(3) $\mathrm{End}(\Gamma) = \overline{\mathrm{Aut}(\mathbb{S}_2; B)}$.

Proof. Assume that there exist $x, y \in \mathbb{S}_2$ with $x < y$ and $f \in \mathrm{End}(\Gamma)$ such that $f(x) = f(y)$. By collapsing comparable pairs one-by-one using f and automorphisms of $(\mathbb{S}_2; \leq)$, it is possible to generate a flat function. Similarly, if there exist a pair of elements $x \perp y$ and $f \in \mathrm{End}(\Gamma)$ such that $f(x) = f(y)$, then $\{f\} \cup \mathrm{Aut}(\mathbb{S}_2; \leq)$ generates a thin function. Hence, we may assume that every endomorphism of Γ is injective. If $\mathrm{End}(\Gamma)$ preserves $<$ and $\perp$, then $\mathrm{End}(\Gamma) = \mathrm{Emb}(\mathbb{S}_2; \leq) = \overline{\mathrm{Aut}(\mathbb{S}_2; \leq)}$. If $\mathrm{End}(\Gamma)$ preserves $<$ and violates $\perp$, then $\mathrm{End}(\Gamma)$ contains a thin function. Thus we may assume that some $f \in \mathrm{End}(\Gamma)$ violates $<$. By Lemma 4.12 either $\mathrm{End}(\Gamma)$ contains a flat or a thin function, or $\mathrm{Emb}(\mathbb{S}_2; B) \subseteq \mathrm{End}(\Gamma)$. Since $\mathrm{Emb}(\mathbb{S}_2; B) = \overline{\mathrm{Aut}(\mathbb{S}_2; B)}$, we may assume that $\mathrm{Emb}(\mathbb{S}_2; B) \subsetneq \mathrm{End}(\Gamma)$, as otherwise Item (1) or (3) holds. Hence, there exists a function $f \in \mathrm{End}(\Gamma)$ that violates either B or $\neg B$. By Proposition 4.5 f violates B, and then $\mathrm{End}(\Gamma)$ contains a flat or a thin function by Lemma 4.13. ■

Lemma 4.15. *Let Γ be a reduct of $(\mathbb{S}_2; \leq)$ which has a flat endomorphism. Then Γ is homomorphically equivalent to a reduct of $(\mathbb{L}_2; C)$.*

Proof. Let f be that endomorphism. By Zorn's lemma, there exists a maximal antichain M in $\mathbb{S}_2$ that contains the image of f. By definition M induces in $(\mathbb{S}_2; C)$ a structure Σ which is isomorphic to $(\mathbb{L}_2; C)$. The structure Δ with domain M and all relations that are restrictions of the relations of Γ to M is a reduct of Σ, as $(\mathbb{S}_2; \leq, C)$ has quantifier elimination.

The inclusion map of M into $\mathbb{S}_2$ is a homomorphism from Δ to Γ, and the function f is a homomorphism from Γ to Δ. ■

Lemma 4.16. *Let Γ be a reduct of $(\mathbb{S}_2; \leq)$ which has a thin endomorphism. Then Γ is homomorphically equivalent to a reduct of the dense linear order.*

Proof. Analogous to the proof of Lemma 4.15, using the obvious fact that maximal chains in $(\mathbb{S}_2; \leq)$ are isomorphic to $(\mathbb{Q}; \leq)$. ■

Proof of Theorem 2.1. Follows directly from Propositions 4.5 and 4.14, Lemmas 4.15 and 4.16, and the easily verifiable fact that $\mathrm{End}(\mathbb{S}_2; <, \perp) = \overline{\mathrm{Aut}(\mathbb{S}_2; \leq)}$. ■

4.4. **Embeddings and the proof of Theorem 2.3.**

Lemma 4.17. *Let Γ be a reduct of $(\mathbb{S}_2; \leq)$ with a thin self-embedding. Then Γ is isomorphic to a reduct of $(\mathbb{Q}; <)$.*

Proof. By Proposition 3.10 there is a thin canonical function $g\colon (\mathbb{S}_2; \leq, C, \prec) \to (\mathbb{S}_2; \leq, C, \prec)$ such that $g \in \mathrm{Emb}(\Gamma)$. There are four possible behaviours of g, as it can preserve or reverse $<$, and independently, it can preserve or reverse $\prec$ on incomparable pairs. In all four of these cases, the structure Σ induced by the image of f in $(\mathbb{S}_2; \leq)$ is isomorphic to $(\mathbb{Q}; \leq)$. The structure Δ on this image whose relations are the restrictions of the relations of Γ to $f[\mathbb{S}_2]$ is a reduct of Σ, as $(\mathbb{S}_2; \leq, C)$ has quantifier elimination. The claim follows as g is an isomorphism between Γ and Δ. ■

Lemma 4.18. *Let Γ be a reduct of $(\mathbb{S}_2; \leq)$ which is isomorphic to a reduct of $(\mathbb{Q}; <)$. Then Γ is existentially interdefinable with $(\mathbb{S}_2; =)$.*

Proof. Pick any pairwise incomparable elements $a_1, \ldots, a_5 \in \mathbb{S}_2$. Then there exist distinct $i, j \in \{1, \ldots, 5\}$ and an automorphism of $(\mathbb{S}_2; \leq)$ which flips a_i, a_j and fixes the other three elements. From Cameron's classification of the reducts of $(\mathbb{Q}; <)$ ([Cam76], cf. the description in Section 2) we know that the only automorphism group of such a reduct which can perform this is the full symmetric group, since all other groups fix at most one or all of five elements when they act on them. Hence, $\mathrm{Aut}(\Gamma)$ contains all permutations of $\mathbb{S}_2$. Thus, all injections of $\mathbb{S}_2$ are self-embeddings of Γ, and the lemma follows. ■

Definition 4.19. Let $R(x, y, z)$ be the ternary relation on $\mathbb{S}_2$ defined by the formula

$$C(z, xy) \vee (x < z \wedge y < z) \vee (x \perp z \wedge y \perp z \wedge (x < y \vee y < x)).$$

Proposition 4.20. *$(\mathbb{S}_2; R)$ and $(\mathbb{S}_2; \leq)$ are interdefinable. However, $(\mathbb{S}_2; R)$ is not model-complete, i.e., it has a self-embedding which is not an element of $\overline{\mathrm{Aut}(\mathbb{S}_2; R)}$.*

Proof. By definition, R has a first-order definition in $(\mathbb{S}_2; \leq)$. To see the converse, observe that for $a, b \in \mathbb{S}_2$ we have that $a \leq b$ if and only if there exists no $c \in \mathbb{S}_2$ such that $R(b, c, a)$. Hence, $(\mathbb{S}_2; R)$ and $(\mathbb{S}_2; \leq)$ are interdefinable, and in particular, $\mathrm{Aut}(\mathbb{S}_2; R) = \mathrm{Aut}(\mathbb{S}_2; \leq)$.

To show that $(\mathbb{S}_2; R)$ is not model-complete, let $f \in (\mathbb{S}_2)^{\mathbb{S}_2}$ map $\mathbb{S}_2$ to an antichain in $(\mathbb{S}_2; \leq)$ in such a way that $R(a, b, c)$ if and only if $C(f(c), f(a)f(b))$ for all $a, b, c \in \mathbb{S}_2$. It is an easy proof by induction that such a mapping exists. Clearly, f is not an element of $\overline{\mathrm{Aut}(\mathbb{S}_2; R)}$, since it does not preserve comparability. ■

The previous proposition is the reason for the special case concerning R in the following lemma.

Lemma 4.21. *Let Γ be a reduct of $(\mathbb{S}_2; \leq)$ with a flat self-embedding. Then Γ is isomorphic to a reduct of $(\mathbb{Q}; <)$, or it has a flat self-embedding that preserves R.*

Proof. Let f be the flat self-embedding. By Proposition 3.10 we may assume that f is canonical as a function from $(\mathbb{S}_2; \leq, C, \prec)$ to $(\mathbb{S}_2; \leq, C, \prec)$. By composing f, if necessary, from the right with an automorphism α of $(\mathbb{S}_2; \leq, C)$ which reverses the order $\prec$ on incomparable pairs, we may assume that f is canonical as a function from $(\mathbb{S}_2; \prec)$ to $(\mathbb{S}_2; \prec)$; that is, it either preserves or reverses the order $\prec$. In the latter case, $\alpha \circ f$ preserves $\prec$, so in any case we may assume that f preserves $\prec$. To simplify notation, we shall write x' instead of $f(x)$ for all $x \in \mathbb{S}_2$, and we write $xy|z$ or $z|xy$ instead of $C(z, xy)$ for all $x, y, z \in \mathbb{S}_2$.

Let $a_1, \ldots, a_5 \in \mathbb{S}_2$ be so that $a_1 \prec \cdots \prec a_5$ and so that $a_1 \perp a_2$, $a_1, a_2 < a_3$, $a_3 \perp a_4$, and $a_1, \ldots, a_4 < a_5$. We shall analyse the possible behaviours of f on these elements. Since f preserves $\prec$, we have that either $a_1'a_2'|a_3'$ or $a_1'|a_2'a_3'$.

We claim that in the first case, $a_2'a_3'|a_4'$. Otherwise, pick $x > a_2$ such that $a_1x|a_4$. Since $a_1'a_2'|a_3'$, we must have $a_1'a_2'|a_4'$ by the properties of $\prec$, and so $a_1'x'|a_4'$ by canonicity. But then $a_2'x'|a_4'$ since $a_1' \prec a_2' \prec x'$, and hence indeed $a_2'a_3'|a_4'$ by canonicity. This together with $a_1'a_2'|a_3'$ implies $a_1'a_3'|a_4'$. Since $a_1'a_2'|a_3'$, we have $a_1'a_4'|a_5'$ by canonicity, leaving us with the following possibility which uniquely determines the type of the tuple $(a_1', \ldots, a_5')$ in $(\mathbb{S}_2; \leq, C, \prec)$:

(A1) $a_1'a_2'|a_3'$, $a_1'a_3'|a_4'$, $a_1'a_4'|a_5'$.

Now assume $a_1'|a_2'a_3'$; then $a_1'|a_2'a_5'$ by canonicity. The latter implies $a_1'|a_3'a_4'$, and thus $a_2'|a_3'a_4'$ again by canonicity. Taking into account that $a_1'|a_2'a_3'$ and canonicity imply $a_3'|a_4'a_5'$, this leaves us with the following possibility:

(A2) $a_1'|a_2'a_5'$, $a_2'|a_3'a_5'$, $a_3'|a_4'a_5'$.

Next let $b_1, \ldots, b_5 \in \mathbb{S}_2$ be so that $b_1 \prec \cdots \prec b_5$ and so that $b_1 \perp b_4$, $b_2, b_3 < b_4$, $b_2 \perp b_3$, and $b_1, \ldots, b_4 < b_5$.

If $b_2'|b_3'b_4'$, then canonicity implies $b_1'|b_2'b_5'$ and $b_2'|b_3'b_5'$ leaving us with only two non-isomorphic possibilities, namely $b_3'|b_4'b_5'$ and $b_3'b_4'|b_5'$.

(B1) $b_1'|b_2'b_5'$, $b_2'|b_3'b_5'$, $b_3'|b_4'b_5'$;
(B2) $b_1'|b_2'b_5'$, $b_2'|b_3'b_5'$, $b_3'b_4'|b_5'$.

If on the other hand $b_2'b_3'|b_4'$, then canonicity tells us that $b_1'b_4'|b_5'$. One possibility here is that $b_1'b_2'|b_3'$, which together with $b_2'b_3'|b_4'$ implies $b_1'b_3'|b_4'$, and so we have:

(B3) $b_1'b_4'|b_5'$, $b_1'b_3'|b_4'$, $b_1'b_2'|b_3'$.

Finally, suppose that $b_2'b_3'|b_4'$ and $b_1'|b_2'b_3'$. Pick $x > b_3$ such that $b_2 \perp x$. Then $b_1'|b_2'x'$ by canonicity, and hence $b_2' \prec b_3' \prec x$ implies that we must have $b_1'|b_3'x'$. But then canonicity gives us $b_1'|b_2'b_4'$, and hence the following:

(B4) $b_1'b_4'|b_5'$, $b_1'|b_2'b_4'$, $b_2'b_3'|b_4'$.

We now consider all possible combinations of these situations. Assume first that (A1) holds; then neither (B1) nor (B2) hold because otherwise $a_1'a_4'|a_5'$ and $b_1'|b_4'b_5'$ together would contradict canonicity. If we have (B3), then for all a, b, c in the range of f we have that $ab|c$ iff $a, b \prec c$. Hence, the formula $a \prec c \wedge b \prec c$ defines the relation C on the image. It is clear that the structure induced by $f[\mathbb{S}_2]$ in $(\mathbb{S}_2; \prec)$ is isomorphic to $(\mathbb{Q}; <)$, since $(\mathbb{S}_2; \prec)$ is isomorphic to it and since f preserves $\prec$. Thus Γ is isomorphic to a reduct of $(\mathbb{Q}; <)$. If we have (B4), then f is a flat self-embedding of Γ that preserves R.

Now assume that (A2) holds. Then $a_1'|a_4'a_5'$ and canonicity imply that (B1) or (B2) is the case. However, (B2) is in fact impossible by virtue of $a_1'a_3'|a_5'$ and $b_2'|b_4'b_5'$, leaving us with

(B1). Here, we argue that Γ is isomorphic to a reduct of $(\mathbb{Q};<)$ precisely as in the case (A1)+(B3). ■

Lemma 4.22. *Let Γ be a reduct of $(\mathbb{S}_2;\leq)$. Assume that there is a flat function in $\overline{\mathrm{Aut}(\Gamma)}$ that preserves R. Then Γ is isomorphic to a reduct of $(\mathbb{Q};<)$.*

Proof. Let f be that function. We use induction to show that the action of $\mathrm{Aut}(\Gamma)$ is n-set transitive for all $n \geq 1$, i.e., if two subsets of $\mathbb{S}_2$ have the same finite cardinality n, then there exists an automorphism of Γ sending one set to the other. The statement is obvious for $n = 1, 2$. Assume that the claim holds for some $n \in \mathbb{N}$, and let A_1, A_2 be $(n+1)-$element subsets with $a_i \in A_i$ for $i \in \{1,2\}$. By the induction hypothesis, for all $i \in \{1,2\}$ there exists an $\alpha_i \in \mathrm{Aut}(\Gamma)$ such that $\alpha_i[A_i \setminus \{a_i\}]$ is a chain. Using the fact that f preserves R, we then get that $(f \circ \alpha_1)[A_1]$ and $(f \circ \alpha_2)[A_2]$ induce isomorphic substructures in $(\mathbb{S}_2;\leq,C)$: namely, for both $i \in \{1,2\}$ there exists a linear order $\sqsubseteq_i$ on $(f \circ \alpha_i)[A_i]$ such that for all pairwise distinct $a,b,c \in (f \circ \alpha_i)[A_i]$ the relation $C(c,ab)$ holds if and only if $a \sqsubseteq_i c$ and $b \sqsubseteq_i c$. Thus there exist $\beta_1, \beta_2, \gamma \in \mathrm{Aut}(\Gamma)$ such that $\beta_i{\restriction}_{A_i} = (f \circ \alpha_i){\restriction}_{A_i}$ for $i \in \{1,2\}$ and $\gamma[(f \circ \alpha_1)[A_1]] = (f \circ \alpha_2)[A_2]$. Hence, $\beta_2^{-1} \circ \gamma \circ \beta_1[A_1] = A_2$.

As Γ is n-set transitive for all $n \geq 1$, the assertion follows from Cameron's theorem in [Cam76]. ■

Proof of Theorem 2.3. Let Γ' be the structure that we obtain from Γ by adding all first-order definable relations in Γ. Then $\mathrm{Aut}(\Gamma) = \mathrm{Aut}(\Gamma')$ and $\overline{\mathrm{Aut}(\Gamma')} = \mathrm{Emb}(\Gamma') = \mathrm{End}(\Gamma')$. The theorem follows by applying Proposition 4.14 and Lemmas 4.17, 4.18, 4.21, and 4.22 to the structure Γ'. ■

5. Applications in Constraint Satisfaction

Let Γ be a structure with a finite relational signature τ. Then $\mathrm{CSP}(\Gamma)$, the *constraint satisfaction problem* for Γ, is the computational problem of deciding for a given finite τ-structure whether there exists a homomorphism to Γ. There are several computational problems in the literature that can be formulated as CSPs for reducts of $(\mathbb{S}_2;\leq)$.

When Γ_b is the reduct of $(\mathbb{S}_2;\leq)$ that contains precisely the *binary* relations with a first-order definition in $(\mathbb{S}_2;\leq)$, then $\mathrm{CSP}(\Gamma_b)$ has been studied under the name "network consistency problem for the branching-time relation algebra" by Hirsch [Hir96]; it is shown there that the problem can be solved in polynomial time. For concreteness, we mention that in particular the problem $\mathrm{CSP}(\mathbb{S}_2;<,\perp)$ can be solved in polynomial time, since it can be seen as a special case of $\mathrm{CSP}(\Gamma_b)$. Broxvall and Jonsson [BJ03] found a better algorithm for $\mathrm{CSP}(\Gamma_b)$ which improves the running time from $O(n^5)$ to $O(n^{3.326})$, where n is the number of variables in the input. Yet another algorithm with a running time that is quadratic in the input size has been described in [BK02]. The complexity of the CSP of *disjunctive reducts* of $(\mathbb{S}_2;\leq,\prec)$ has been determined in [BJ03]; a *disjunctive reduct* is a reduct each of whose relations can be defined by a disjunction of the basic relations in such a way that the disjuncts do not share common variables.

Independently from this line of research, motivated by research in computational linguistics, Cornell [Cor94] studied the reduct Γ_c of $(\mathbb{S}_2;\leq,\prec)$ containing all binary relations that are first-order definable over $(\mathbb{S}_2;\leq,\prec)$. Contrary to a conjecture of Cornell, it has been shown that $\mathrm{CSP}(\Gamma_c)$ (and in fact already $\mathrm{CSP}(\mathbb{S}_2;<,\perp)$) cannot be solved by establishing *path consistency* [BM11]. However, $\mathrm{CSP}(\Gamma_c)$ can be solved in polynomial time [BK07].

It is a natural but challenging research question to ask for a classification of the complexity of CSP(Γ) for all reducts of $(\mathbb{S}_2; \leq)$. In this context, we call the reducts of $(\mathbb{S}_2; \leq)$ *tree description constraint languages*. Such classifications have been obtained for the reducts of $(\mathbb{Q}; \leq)$ and the reducts of the random graph [BK09, BP15]. In both these previous classifications, the classification of the model-complete cores of the reducts played a central role. Our Theorem 2.1 shows that every tree description language belongs to at least one out of four cases; in cases one and two, the CSP has already been classified. It is easy to show (and this will appear in forthcoming work) that the CSP is NP-hard when case three of Theorem 2.1 applies. It is also easy to see (again we have to refer to forthcoming work) that in case four of Theorem 2.1, adding the relations $<$ and $\perp$ to Γ does not change the computational complexity of the CSP. The corresponding fact for the reducts of $(\mathbb{Q}; \leq)$ and the reducts of the random graph has been extremely useful in the subsequent classification. Therefore, the present paper and in particular Theorem 2.1 are highly relevant for the study of the CSP for tree description constraint languages.

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