# CONGRUENCE BASED PROOFS OF THE RECOGNIZABILITY THEOREMS FOR FREE MANY-SORTED ALGEBRAS 

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#### Abstract

We generalize several recognizability theorems for free single-sorted algebras to the field of many-sorted algebras and provide, in a uniform way and without using neither regular tree grammars nor tree automata, purely algebraic proofs of them based on the concept of congruence.


## 1. Introduction

The definition of recognizable language set down by Rabin and Scott in 34, Definition 2, on p. 116, originated in the characterizations of regular languages given by Myhill 32, in terms of congruence relations of finite index, and Nerode 33, in terms of right invariant equivalence relations of finite index, and was formulated for sets of elements of an arbitrary single-sorted abstract algebra by Mezei and Wright 31. In this setting, a subset of an algebra is recognizable if there exists an homomorphism from it into a finite one for which the language coincides with the inverse image by the homomorphism of a subset of the finite algebra. Such a recognizability notion, as is well known, is equivalent to providing a congruence of finite index on the algebra for which the language in question is saturated.

This paper is devoted to the study of several recognizability results for free manysorted algebras. Concretely, we generalize to the field of many-sorted algebras the results presented by Gécseg and Steinby in Section 2.4 of [14], in the field of single-sorted algebras, on the preservation and, when appropriate, reflection of the recognizability under the action of the operators of substitution, iteration, quotient, inverse image by a tree homomorphism, and direct image by linear tree homomorphism on terms ( $\equiv$ trees) of free single-sorted algebras.

The device used by Gécseg and Steinby in [14 for proving the aforementioned results concerning recognizability was that of regular tree grammars. As in the case of formal languages, there exists another natural device intended for the same purpose: Tree automata. We recall that tree automata were defined, among others, by Doner [11, 12] and Thatcher and Wright [41, 42] and that the primary goal of the theory of tree automata was to apply it to decision problems of second order logic. On the other hand, regular tree grammars were defined by Brainerd 4] and the main result of 4, Theorem 4.9, on p. 230, is: "The sets of trees generated by regular systems [ $\equiv$ regular tree grammars, we add] are exactly those accepted by tree automata."

As an alternative to the above devices, regular tree grammars and tree automata, in this paper we supply congruence based proofs of the different recognizability theorems for the strictly more general setting of free many-sorted algebras. All results stated in this paper follow the same methodology: Starting from a congruence of finite index saturating input languages, which is ultimately based on the different syntactic congruences determined by the input languages, we provide a finite index refinement for it recognizing the transformed language which is obtained by means

[^0]of the aforementioned operators. This proof strategy, which, ultimately, leads to a proof schema, has proven very effective in providing a uniform approach for each of the cases in question, and, in addition, it has served to confirm, once again, the fundamental role played by the notion of congruence cogenerated by an equivalence relation on the underlying many-sorted set of a many-sorted algebra in the area of theoretical computer science.

In this regard it is worthwhile quoting (1) a part of Atiyah's answer in 35], p. 24, to the question, formulated by the interviewers about the underlying motivation for providing different proofs with different strategies for the Atiyah-Singer Index Theorem: "Any good theorem should have several proofs, the more the better. For two reasons: usually, different proofs have different strengths and weaknesses, and they generalise in different directions - they are not just repetitions of each other. [...] the more perspectives, the better!", and (2) what Lang, referring to his own review of the historical development of class field theory, wrote in the preface of [26]: "As I stated in the preface to my Algebraic Number Theory, there are several approaches to class field theory. None of them makes any other obsolete, and each gives a different insight from the others."

We next proceed to summarize the contents of the subsequent sections of this paper.

In Section 2, for the convenience of the reader, we recall those notions and facts, these latter mostly without proofs, on many-sorted sets, many-sorted algebras, and recognizability for many-sorted algebras, that we will need.

In Section 3 we provide congruence based proofs of the recognizability theorems for free many-sorted algebras. As a matter of fact, in order to deal with the different cases of recognizability, classified according to the type of operator under consideration, we have divided this section into several subsections. In Subsection 1, entitled Basic terms, we prove that the final sets containing a variable, a constant or an operation symbol applied to a suitable family of variables are recognizable. In Subsection 2, entitled Substitutions, after defining several substitution operators associated to a free many-sorted algebra and investigating the relationships between them, we state the main result of this subsection: If all input languages for a given substitution are recognizable, then the output language is recognizable as well. In Subsection 3, entitled Iterations, we introduce the notion of iteration of a language with respect to a variable and we prove that, if the input language is recognizable then its iteration with respect to a variable is also recognizable. In Subsection 4, entitled Quotients, we introduce the notion of quotient of a language by another, not necessarily recognizable, language with respect to a variable and we prove that if the input language is recognizable then its quotient by another language with respect to a variable is also recognizable. In Subsection 5, entitled Tree Homomorphisms, we define the notion of hyperderivor from a many-sorted signature labeled with a suitable many-sorted set (the domain of the hyperderivor) to another (the codomain of the hyperderivor) as consisting of two components: One transforming many-sorted operation symbols from the underlying many-sorted signature of the domain into terms (with variables in the coproduct of the underlying many-sorted set of the codomain and an "initial segment", of a fixed standard many-sorted set of variables, which depends on the many-sorted operation symbol) relative to the many-sorted signature of the codomain, and the other associating terms (with variables in the underlying many-sorted set of the codomain) relative to the many-sorted signature of the codomain, to variables from the underlying many-sorted set of the domain. In the particular case that, for each many-sorted operation symbol, it is fulfilled that, for every variable, the number of its occurrences in the term associated to the many-sorted operation symbol is at most one,
the hyperderivor is called linear. Then we show that each hyperderivor determines, in a canonical way, a tree homomorphism, which is, in fact, a homomorphism from a free many-sorted algebra obtained from the domain of the hyperderivor to another many-sorted algebra of the same many-sorted signature, itself derived from a many-sorted algebra obtained from the codomain of the hyperderivor. After this we prove that the inverse image of a recognizable language under a tree homomorphism is recognizable and that the direct image of a recognizable language under a linear tree homomorphism, which is a tree homomorphism determined by a linear hyperderivor, is also recognizable. Finally, in Subsection 6, entitled Derivors and recognizability, after defining the many-sorted finitary variety of Hall algebras, we define, by means of the homomorphisms between Hall algebras, the category of many-sorted signatures and derivors. Then we construct a category whose objects are ordered pairs consisting of a many-sorted signature and a many-sorted algebra of such a signature, and morphisms between them ordered pairs consisting of a derivor between the underlying many-sorted signatures and a homomorphism from the underlying many-sorted algebra of the source to a derived many-sorted algebra of the sink. Following this, after showing that every derivor, together with some additional data, gives rise to a hyperderivor, we state that the inverse image under a convenient morphism, of the just mentioned category, labeled by a derivor, and the direct image under a morphism, labeled by a linear derivor, of a recognizable language is also recognizable.

Our underlying set theory is ZFSk, Zermelo-Fraenkel-Skolem set theory (also known as ZFC, i.e., Zermelo-Fraenkel set theory with the axiom of choice) plus the existence of a Grothendieck universe $\mathcal{U}$, fixed once and for all (see [27], pp. 21-24). We recall that the elements of $\mathcal{U}$ are called $\mathcal{U}$-small sets and the subsets of $\mathcal{U}$ are called $\mathcal{U}$-large sets or classes. Moreover, from now on Set stands for the category of sets, i.e., the category whose set of objects is $\mathcal{U}$ and whose set of morphisms is the set of all mappings between $\mathcal{U}$-small sets.

In all that follows we use standard concepts and constructions from category theory, see e.g., [19], 21] and [27], universal algebra, see e.g., [2], 6], 18], and 43], and set theory, see e.g., 3. Nevertheless, regarding set theory, we have adopted the following conventions. An ordinal $\alpha$ is a transitive set that is well-ordered by $\in$, thus $\alpha=\{\beta \mid \beta \in \alpha\}$. The first transfinite ordinal $\omega_{0}$ will be denoted by $\mathbb{N}$, which is the set of all natural numbers, and, from what we have just said about the ordinals, for every $n \in \mathbb{N}, n=\{0, \ldots, n-1\}$. If $\Phi$ and $\Psi$ are (binary) relations in a set $A$, then we will say that $\Psi$ is a refinement of $\Phi$ if $\Psi \subseteq \Phi$. We will denote by $\operatorname{Fnc}(A, B)$ the set of all functions from $A$ to $B$. We recall that a function from $A$ to $B$ is a subset $F$ of $A \times B$ satisfying the functional condition, i.e., such that, for every $x \in A$, there exists a unique $y \in B$ such that $(x, y) \in F$. A function $F$ from $A$ to $B$ is usually denoted by $\left(F_{x}\right)_{x \in A}$. We will denote by $\operatorname{Hom}(A, B)$ (and, sometimes, also by $B^{A}$ ) the set of all mappings from $A$ to $B$. We recall that a mapping from $A$ to $B$ is an ordered triple $f=(A, F, B)$, denoted by $f: A \longrightarrow B$, in which $F$ is a function from $A$ to $B$. Therefore $\operatorname{Hom}(A, B)=\{A\} \times \operatorname{Fnc}(A, B) \times\{B\}$. We will denote by $\operatorname{Sub}(A)$ the set of all sets $X$ such that $X \subseteq A$ and if $X \in \operatorname{Sub}(A)$, then we will denote by $\mathrm{C}_{A} X$ or $A-X$ the complement of $X$ in $A$. Moreover, if $f$ is a mapping from $A$ to $B$, then the mapping $f[\cdot]$ from $\operatorname{Sub}(A)$ to $\operatorname{Sub}(B)$, of $f$-direct image formation, sends $X$ in $\operatorname{Sub}(A)$ to $f[X]=\{y \in B \mid \exists x \in X(y=f(x))\}$ in $\operatorname{Sub}(B)$, and the mapping $f^{-1}[\cdot]$ from $\operatorname{Sub}(B)$ to $\operatorname{Sub}(A)$, of $f$-inverse image formation, sends $Y$ in $\operatorname{Sub}(B)$ to $f^{-1}[Y]=\{x \in A \mid f(x) \in Y\}$ in $\operatorname{Sub}(A)$. In the sequel, for a mapping $f$ from $A$ to $B$ and a subset $X$ of $A$, we will write $\operatorname{Ker}(f)$ for the kernel of $f, \operatorname{Im}(f)$ to mean $f[A]$, and the restriction of $f$ to $X$ will be denoted by $f \upharpoonright_{X}$.

## 2. Preliminaries

In this section we collect the basic facts, mostly without proofs, about manysorted sets, many-sorted algebras, and recognizability for arbitrary many-sorted algebras, that we will need.

Assumption. From now on $S$ stands for a set of sorts in $\mathcal{U}$, fixed once and for all.
Definition 2.1. An $S$-sorted set is a mapping $A=\left(A_{s}\right)_{s \in S}$ from $S$ to $\mathcal{U}$. If $A$ and $B$ are $S$-sorted sets, an $S$-sorted mapping from $A$ to $B$ is an $S$-indexed family $f=\left(f_{s}\right)_{s \in S}$, where, for every $s$ in $S, f_{s}$ is a mapping from $A_{s}$ to $B_{s}$. Thus, an $S$-sorted mapping from $A$ to $B$ is an element of $\prod_{s \in S} \operatorname{Hom}\left(A_{s}, B_{s}\right)$. We will denote by $\operatorname{Hom}(A, B)$ the set of all $S$-sorted mappings from $A$ to $B$. $S$-sorted sets and $S$-sorted mappings form a category which we will denote henceforth by $\boldsymbol{S e t}^{S}$.

Definition 2.2. Let $I$ be a set in $\mathcal{U}$ and $\left(A^{i}\right)_{i \in I}$ an $I$-indexed family of $S$-sorted sets. Then the product of $\left(A^{i}\right)_{i \in I}$, denoted by $\prod_{i \in I} A^{i}$, is the $S$-sorted set defined, for every $s \in S$, as $\left(\prod_{i \in I} A^{i}\right)_{s}=\prod_{i \in I} A_{s}^{i}$, where

$$
\prod_{i \in I} A_{s}^{i}=\left\{\left(a_{i}\right)_{i \in I} \in \operatorname{Fnc}\left(I, \bigcup_{i \in I} A_{s}^{i}\right) \mid \forall i \in I\left(a_{i} \in A_{s}^{i}\right)\right\}
$$

For every $i \in I$, the $i$-th canonical projection, $\operatorname{pr}^{i}=\left(\operatorname{pr}_{s}^{i}\right)_{s \in S}$, is the $S$-sorted mapping from $\prod_{i \in I} A^{i}$ to $A^{i}$ that, for every $s \in S$, sends $\left(a_{i}\right)_{i \in I}$ in $\prod_{i \in I} A_{s}^{i}$ to $a_{i}$ in $A_{s}^{i}$. The ordered pair $\left(\prod_{i \in I} A^{i},\left(\operatorname{pr}^{i}\right)_{i \in I}\right)$ has the following universal property: For every $S$-sorted set $B$ and every $I$-indexed family of $S$-sorted mappings $\left(f^{i}\right)_{i \in I}$, where, for every $i \in I, f^{i}$ is an $S$-sorted mapping from $B$ to $A^{i}$, there exists a unique $S$-sorted mapping $\left\langle f^{i}\right\rangle_{i \in I}$ from $B$ to $\prod_{i \in I} A^{i}$ such that, for every $i \in I$, $\operatorname{pr}^{i} \circ\left\langle f^{i}\right\rangle_{i \in I}=f^{i}$.

The coproduct of $\left(A^{i}\right)_{i \in I}$, denoted by $\coprod_{i \in I} A^{i}$, is the $S$-sorted set defined, for every $s \in S$, as $\left(\coprod_{i \in I} A^{i}\right)_{s}=\coprod_{i \in I} A_{s}^{i}$, where

$$
\coprod_{i \in I} A_{s}^{i}=\bigcup_{i \in I}\left(A_{s}^{i} \times\{i\}\right) .
$$

For every $i \in I$, the $i$-th canonical injection, in ${ }^{i}$, is the $S$-sorted mapping from $A^{i}$ to $\coprod_{i \in I} A^{i}$ that, for every $s \in S$, sends $a$ in $A_{s}^{i}$ to $(a, i)$ in $\coprod_{i \in I} A_{s}^{i}$. The ordered pair $\left(\coprod_{i \in I} A^{i},\left(\mathrm{in}^{i}\right)_{i \in I}\right)$ has the following universal property: For every $S$-sorted set $B$ and every $I$-indexed family of $S$-sorted mappings $\left(f^{i}\right)_{i \in I}$, where, for every $i \in I$, $f^{i}$ is an $S$-sorted mapping from $A^{i}$ to $B$, there exists a unique $S$-sorted mapping $\left[f^{i}\right]_{i \in I}$ from $\coprod_{i \in I} A^{i}$ to $B$ such that, for every $i \in I,\left[f^{i}\right]_{i \in I} \circ \mathrm{in}^{i}=f^{i}$.

The remaining set-theoretic operations on $S$-sorted sets: $\times$ (binary product), $\amalg$ (binary coproduct), $\cup$ (union), $\cup$ (binary union), $\cap$ (intersection), $\cap$ (binary intersection), - (difference), and $\complement_{A}$ (complement of an $S$-sorted set in a fixed $S$-sorted $A$ ), are defined in a similar way, i.e., componentwise.

Definition 2.3. We will denote by $1^{S}$ the (standard) final $S$-sorted set of $\boldsymbol{\operatorname { S e t }}^{S}$, which is $1^{S}=(1)_{s \in S}$, and by $\varnothing^{S}$ the initial $S$-sorted set, which is $\varnothing^{S}=(\varnothing)_{s \in S}$. We shall abbreviate $1^{S}$ to 1 and $\varnothing^{S}$ to $\varnothing$ when this is unlikely to cause confusion.
Definition 2.4. If $A$ and $X$ are $S$-sorted sets, then we will say that $X$ is a subset of $A$, denoted by $X \subseteq A$, if, for every $s \in S, X_{s} \subseteq A_{s}$. We will denote by $\operatorname{Sub}(A)$ the set of all $S$-sorted sets $X$ such that $X \subseteq A$.

Remark. For every $S$-sorted set $A$, the ordered pairs ( $B, f$ ), where $B$ is an $S$-sorted set and $f$ a monomorphisms from $B$ to $A$, are classified by letting $(B, f) \equiv(C, g)$ if and only if there exists an isomorphism $h$ from $B$ to $C$ such that $f=g \circ h$, and the corresponding equivalence classes are called subobjects of $A$. Then $\operatorname{Sub}(A)$ is isomorphic to the set of all subobjects $A$.

Definition 2.5. Let $\delta$ be the mapping from $S \times \mathcal{U}$ to $\mathcal{U}^{S}$ that sends $(t, X)$ in $S \times \mathcal{U}$ to the $S$-sorted set $\delta^{t, X}=\left(\delta_{s}^{t, X}\right)_{s \in S}$ defined, for every $s \in S$, as follows: $\delta_{s}^{t, X}=X$, if $s=t ; \delta_{s}^{t, X}=\varnothing$, otherwise. We will call the value of $\delta$ at $(t, X)$ the delta of Kronecker associated to $(t, X)$. If $X=\{x\}$, then, for simplicity of notation, we will write $\delta^{t, x}$ instead of $\delta^{t,\{x\}}$. Moreover, for a sort $t$ in $S, \delta^{t, 1}$, the delta of Kronecker associated to $(t, 1)$, will be denoted by $\delta^{t}$ and called delta of Kronecker.

Remark. For a sort $t \in S$ and a set $X$, the $S$-sorted set $\delta^{t, X}$ is isomorphic to the $S$-sorted set $\coprod_{x \in X} \delta^{t}$, i.e., to the coproduct of the family $\left(\delta^{t}\right)_{x \in X}$.

For every sort $t \in S$ we have a functor $\delta^{t,}$ from Set to Set $^{S}$. In fact, for every set $X, \delta^{t, \cdot}(X)=\delta^{t, X}$, and, for every mapping $f: X \longrightarrow Y, \delta^{t, \cdot}(f)=\delta^{t, f}$, where, for $s \in S, \delta_{s}^{t, f}=\operatorname{id}_{\varnothing}$, if $s \neq t$, and $\delta_{t}^{t, f}=f$. Moreover, for every $t \in S$, the object mapping of the functor $\delta^{t, \cdot}$ is injective and $\delta^{t, \cdot}$ is full and faithful. Hence, for every $t \in S, \delta^{t, \cdot}$ is a full embedding from Set to $\mathbf{S e t}^{S}$.

The final object $1^{S}$ does not generate ( $\equiv$ separate) the category Set ${ }^{S}$. However, the set $\left\{\delta^{s} \mid s \in S\right\}$, of the deltas of Kronecker, is a generating ( $\equiv$ separating) set for the category $\mathbf{S e t}^{S}$. Therefore, every $S$-sorted set $A$ can be represented as a coproduct of copowers of deltas of Kronecker, i.e., $A$ is naturally isomorphic to $\coprod_{s \in S} \operatorname{card}\left(A_{s}\right) \cdot \delta^{s}$, where, for every $s \in S, \operatorname{card}\left(A_{s}\right) \cdot \delta^{s}$ is the copower of the family $\left(\delta^{s}\right)_{\alpha \in \operatorname{card}\left(A_{s}\right)}$, i.e., the coproduct of $\left(\delta^{s}\right)_{\alpha \in \operatorname{card}\left(A_{s}\right)}$. To this we add the following facts: (1) $\left\{\delta^{s} \mid s \in S\right\}$ is the set of atoms of the Boolean algebra $\operatorname{Sub}\left(1^{S}\right)$, of subobjects of $1^{S}$; (2) the Boolean algebras $\operatorname{Sub}\left(1^{S}\right)$ and $\operatorname{Sub}(S)$ are isomorphic; (3) for every $s \in S, \delta^{s}$ is a projective object; and (4) for every $s \in S$, every $S$-sorted mapping from $\delta^{s}$ to another $S$-sorted set is a monomorphism.

In view of the above, it must be concluded that the deltas of Kronecker are of crucial importance for many-sorted sets and associated fields.

Before proceeding any further, let us point out that it is no longer unusual to find in the works devoted to investigate both many-sorted algebras and many-sorted algebraic systems the following. (1) That an $S$-sorted set $A$ is defined in such a way that $\operatorname{Hom}\left(1^{S}, A\right) \neq \varnothing$, or, what is equivalent, requiring that, for every $s \in S$, $A_{s} \neq \varnothing$. This has as an immediate consequence that the corresponding category is not even finite cocomplete. Since cocompleteness (and completeness) are desirable properties for a category, we exclude such a convention in our work (the admission of $\varnothing^{S}$ is crucial in many applications). And (2) that an $S$-sorted set $A$ must be such that, for every $s, t \in S$, if $s \neq t$, then $A_{s} \cap A_{t}=\varnothing$. We also exclude such a requirement (the possibility of a common underlying set for the different sorts is very important in many applications). The above conventions are possibly based on the untrue widespread belief that many-sorted equational logic and many-sorted first-order logic are inessential variations of equational logic and first-order logic, respectively. One can find a definitive refutation to the just mentioned belief in [17] and 30], regarding many-sorted equational logic, and in [22], with respect to manysorted first-order logic.

We next define for an $S$-sorted mapping the associated mappings of direct and inverse image formation, its kernel, its image, as well as its restriction to a subset of its domain.

Definition 2.6. Let $f: A \longrightarrow B$ be an $S$-sorted mapping. Then the mapping $f[\cdot]: \operatorname{Sub}(A) \longrightarrow \operatorname{Sub}(B)$, of $f$-direct image formation, sends $X \in \operatorname{Sub}(A)$ to $f[X]=$ $\left(f_{s}\left[X_{s}\right]\right)_{s \in S} \in \operatorname{Sub}(B)$, and the mapping $f^{-1}[\cdot]: \operatorname{Sub}(B) \longrightarrow \operatorname{Sub}(A)$, of $f$-inverse image formation, sends $Y \in \operatorname{Sub}(B)$ to $f^{-1}[Y]=\left(f_{s}^{-1}\left[Y_{s}\right]\right)_{s \in S} \in \operatorname{Sub}(A)$. The kernel of $f$, denoted by $\operatorname{Ker}(f)$, is $\left(\operatorname{Ker}\left(f_{s}\right)\right)_{s \in S}$ and the image of $f$, denoted by $\operatorname{Im}(f)$, is $f[A]$. Moreover, if $X \subseteq A$, then the restriction of $f$ to $X$, denoted by
$f \upharpoonright_{X}$, is $f \circ \operatorname{in}_{X, A}$, where $\operatorname{in}_{X, A}=\left(\operatorname{in}_{X_{s}, A_{s}}\right)_{s \in S}$ is the canonical embedding of $X$ into $A$.

Before stating the following proposition, we recall that, for an $S$-sorted set $A$, the $S$-sorted set $\left(\operatorname{Sub}\left(A_{s}\right)\right)_{s \in S}$, usually denoted by $A^{\wp}$, is the power object of $A$ in the topos $\mathbf{S e t}^{S}$. Therefore, one should take great care not to confuse $\operatorname{Sub}(A)$, which is a set, i.e., an object of Set-naturally isomorphic to the set $\prod_{s \in S} 2^{A_{s}}$-, and $A^{\wp}$, which is an $S$-sorted set, i.e., an object of $\boldsymbol{S e t}^{S}$-naturally isomorphic to the $S$-sorted set $\left(2^{A_{s}}\right)_{s \in S}$. On the other hand, it is clear that there exists a natural isomorphism between $\operatorname{Sub}(A)$ and $\prod A^{\wp}$.

Proposition 2.7. Let $X$ and $Y$ be $S$-sorted sets and $f$ an $S$-sorted mapping from $X$ to $Y^{\wp}$. Then there exists a unique $S$-sorted mapping $f^{\mathfrak{p}}$ from $\left(\operatorname{Sub}\left(X_{s}\right)\right)_{s \in S}$ to $Y^{\wp}$ such that $f^{\mathfrak{p}}$ is completely additive, i.e., for every $s \in S$ and every $\mathcal{L} \subseteq \operatorname{Sub}\left(X_{s}\right)$, $f_{s}^{\mathfrak{p}}(\bigcup \mathcal{L})=\bigcup_{L \in \mathcal{L}} f_{s}^{\mathfrak{p}}(L)$, and $f^{\mathfrak{p}} \circ\{\cdot\}_{X}=f$, where $\{\cdot\}_{X}$ is the $S$-sorted mapping from $X$ to $X^{\wp}$ that, for every $s \in S$, sends $x \in X_{s}$ to $\{x\} \in \operatorname{Sub}\left(X_{s}\right)$.

Proof. The $S$-sorted mapping $f^{\mathfrak{p}}$ from $X^{\wp}$ to $Y^{\wp}$ that, for every $s \in S$, assigns to $L \subseteq X_{s}$ the set $\bigcup_{x \in L} f_{s}(x) \subseteq Y_{s}$ is completely additive and $f^{\mathfrak{p}} \circ\{\cdot\}_{X}=f$. Moreover, $f^{\mathfrak{p}}$ is clearly the unique $S$-sorted mapping from $X^{\wp}$ to $Y^{\wp}$ satisfying the aforementioned conditions.

Corollary 2.8. Let $\operatorname{Set}_{\wp, \mathrm{ca}}^{S}$ be the category whose objects are the $S$-sorted sets $X^{\wp}$, where $X \in \mathcal{U}^{S}$, and in which the set of morphisms from $X^{\wp}$ to $Y^{\wp}$ is the set of the completely additive $S$-sorted mappings from $X^{\wp}$ to $Y^{\wp}$. Then from $\mathbf{S e t}_{\wp, \mathrm{ca}}^{S}$ to $\boldsymbol{S e t}^{S}$ we have a canonical inclusion, denoted by $\operatorname{In}_{\text {Set }_{\phi, \text { a }}^{S}}^{S}$, Set $^{s}$, and, for every $S$-sorted set

Definition 2.9. We will denote by $(\cdot)^{\wp}$ the functor from $\boldsymbol{S e t}^{S}$ to $\boldsymbol{S e t}_{\ell, \text { ca }}^{S}$ that sends an $S$-sorted set $X$ to $X^{\wp}$ and an $S$-sorted mapping $f$ from $X$ to $Y$ to the $S$-sorted mapping $f^{\wp}=\left(\{\cdot\}_{Y} \circ f\right)^{\mathfrak{p}}$ from $X^{\wp}$ to $Y^{\wp}$ (the $S$-sorted mappings $\{\cdot\}_{X}$, for $X$ in $\mathcal{U}^{S}$, are the components of the unit of the adjunction: $\operatorname{Hom}_{\text {Set }_{p, \mathrm{ca}}^{S}}\left(X^{\wp}, Y^{\wp}\right) \cong$ $\left.\operatorname{Hom}_{\text {Set }^{S}}\left(X, Y^{\wp}\right)\right)$. Thus, $(\cdot)^{\wp}$ is a left adjoint of $\operatorname{In}_{\text {Set }_{\beta, \text { ca }}^{S}}^{S}$,Set ${ }^{S}$.
Remark. For an $S$-sorted mapping $f$ from $X$ to $Y$ the $S$-sorted mapping $f^{\wp}$ from $X^{\wp}$ to $Y^{\wp}$ sends, for every $s \in S, L \subseteq X_{s}$ to $f_{s}[L] \subseteq Y_{s}$, i.e., $f^{\wp}=\left(f_{s}[\cdot]\right)_{s \in S}$. One should be careful not to confuse $f[\cdot]$, which is a mapping from the set $\operatorname{Sub}(X)$ to the set $\operatorname{Sub}(Y)$, and $f^{6}$, which is an $S$-sorted mapping from the $S$-sorted set $X^{\wp}=\left(\operatorname{Sub}\left(X_{s}\right)\right)_{s \in S}$ to the $S$-sorted set $Y^{\wp}=\left(\operatorname{Sub}\left(Y_{s}\right)\right)_{s \in S}$. On the other hand, it is evident that $f[\cdot]$ and $\prod f^{\wp}$ are essentially the same mapping.

Definition 2.10. Let $A$ be an $S$-sorted set. Then the cardinal of $A$, denoted by $\operatorname{card}(A)$, is $\operatorname{card}(\amalg A)$, i.e., the cardinal of the set $\amalg A=\bigcup_{s \in S}\left(A_{s} \times\{s\}\right)$. An $S$-sorted set $A$ is finite if $\operatorname{card}(A)<\aleph_{0}$. We will say that an $S$-sorted set $X$ is a finite subset of $A$ if $X$ is finite and $X \subseteq A$. We will denote by $\operatorname{Sub}_{\mathrm{f}}(A)$ the set of all $S$-sorted sets $X$ in $\operatorname{Sub}(A)$ which are finite.

Remark. For an object $A$ in the topos $\mathbf{S e t}^{S}$ the following assertions are equivalent: (1) $A$ is finite, (2) $A$ is a finitary object of $\operatorname{Set}^{S}$, and (3) $A$ is a strongly finitary object of $\boldsymbol{S e t}^{S}$ (for the notions of finitary and strongly finitary object of a category see [21], Exercise 22E, on p. 155).

In Set ${ }^{S}$ there is another notion of finiteness: An $S$-sorted set $A$ is called $S$-finite or locally finite, abbreviated as lf, if and only if, for every $s \in S, A_{s}$ is finite. We will denote by $\operatorname{Sub}_{\mathrm{lf}}(A)$ the set of all $S$-sorted sets $X$ in $\operatorname{Sub}(A)$ which are locally finite. Although, unless $S$ is finite, this notion of finiteness is not categorial in
nature, however, it plays a relevant role in the field of many-sorted algebra and in computer science (see below, after the definition of many-sorted algebra and when we deal with the congruences of locally finite index, respectively).
Definition 2.11. Let $A$ be an $S$-sorted set. Then the support of $A$, denoted by $\operatorname{supp}_{S}(A)$, is the set $\left\{s \in S \mid A_{s} \neq \varnothing\right\}$.
Remark. An $S$-sorted set $A$ is finite if and only if $\operatorname{supp}_{S}(A)$ is finite and, for every $s \in \operatorname{supp}_{S}(A), A_{s}$ is finite.

We next recall, after fixing some notation with regard to an equivalence relation $\Phi$ on an $S$-sorted set $A$, the universal property of $\left(A / \Phi, \operatorname{pr}^{\Phi}\right)$, where $A / \Phi$ is the quotient $S$-sorted set of $A$ by $\Phi$ and $\mathrm{pr}^{\Phi}$ the canonical projection from $A$ to $A / \Phi$; the notion of transversal of $A / \Phi$ in $A$; the notion of $\Phi$-saturation of a subset $X$ of $A$; and those properties of this last notion which will be used afterwards.
Definition 2.12. An $S$-sorted equivalence relation on (or, to abbreviate, an $S$ sorted equivalence on) an $S$-sorted set $A$ is an $S$-sorted relation $\Phi$ on $A$, i.e., a subset $\Phi=\left(\Phi_{s}\right)_{s \in S}$ of the cartesian product $A \times A=\left(A_{s} \times A_{s}\right)_{s \in S}$ such that, for every $s \in S, \Phi_{s}$ is an equivalence relation on $A_{s}$. We will denote by $\operatorname{Eqv}(A)$ the set of all $S$-sorted equivalences on $A$ (which is an algebraic closure system on $A \times A)$, by $\operatorname{Eqv}(A)$ the algebraic lattice $(\operatorname{Eqv}(A), \subseteq)$, by $\nabla_{A}$ the greatest element of $\operatorname{Eqv}(A)$, and by $\Delta_{A}$ the least element of $\operatorname{Eqv}(A)$.

For an $S$-sorted equivalence relation $\Phi$ on $A$, the $S$-sorted quotient set of $A$ by $\Phi$, denoted by $A / \Phi$, is $\left(A_{s} / \Phi_{s}\right)_{s \in S}=\left(\left\{[x]_{\Phi_{s}} \mid x \in A_{s}\right\}\right)_{s \in S}\left(\subseteq A^{\wp}\right)$, where, for every $s \in S$ and every $x \in A_{s},[x]_{\Phi_{s}}$, the equivalence class of $x$ with respect to $\Phi_{s}$ (or, the $\Phi$-equivalence class of $x)$ is $\left\{y \in A_{s} \mid(x, y) \in \Phi_{s}\right\}$, and $\operatorname{pr}_{\Phi}: A \longrightarrow A / \Phi$, the canonical projection from $A$ to $A / \Phi$, is the $S$-sorted mapping $\left(\operatorname{pr}_{\Phi_{s}}\right)_{s \in S}$, where, for every $s \in S$, $\operatorname{pr}_{\Phi_{s}}$ is the canonical projection from $A_{s}$ to $A_{s} / \Phi_{s}$ (which sends $x$ in $A_{s}$ to $\operatorname{pr}_{\Phi_{s}}(x)=[x]_{\Phi_{s}}$, the $\Phi_{s}$-equivalence class of $x$, in $\left.A_{s} / \Phi_{s}\right)$.

The ordered pair $\left(A / \Phi, \operatorname{pr}_{\Phi}\right)$ has the following universal property: $\operatorname{Ker}\left(\operatorname{pr}_{\Phi}\right)$ is $\Phi$ and, for every $S$-sorted set $B$ and every $S$-sorted mapping $f$ from $A$ to $B$, if $\operatorname{Ker}(f) \supseteq \Phi$, then there exists a unique $S$-sorted mapping $h$ from $A / \Phi$ to $B$ such that $h \circ \operatorname{pr}_{\Phi}=f$. In particular, if $\Psi$ is an $S$-sorted equivalence relation on $A$ such that $\Phi \subseteq \Psi$, then we will denote by $\mathrm{p}_{\Phi, \Psi}$ the unique $S$-sorted mapping from $A / \Phi$ to $A / \Psi$ such that $\mathrm{p}_{\Phi, \Psi} \circ \mathrm{pr}_{\Phi}=\mathrm{pr}_{\Psi}$.

Remark. Let ClfdSet ${ }^{S}$ be the category whose objects are the classified $S$-sorted sets, i.e, the ordered pairs $(A, \Phi)$ where $A$ is an $S$-sorted set and $\Phi$ an $S$-sorted equivalence relation on $A$, and in which the set of morphisms from $(A, \Phi)$ to $(B, \Psi)$ is the set of all $S$-sorted mappings $f$ from $A$ to $B$ such that, for every $s \in S$ and every $(x, y) \in A_{s}^{2}$, if $(x, y) \in \Phi_{s}$, then $\left(f_{s}(x), f_{s}(y)\right) \in \Psi_{s}$. Let $G$ be the functor from Set ${ }^{S}$ to ClfdSet ${ }^{S}$ whose object mapping sends $A$ to $\left(A, \Delta_{A}\right)$ and whose morphism mapping sends $f: A \longrightarrow B$ to $f:\left(A, \Delta_{A}\right) \longrightarrow\left(B, \Delta_{B}\right)$. Then, for every classified $S$-sorted set $(A, \Phi)$, there exists a universal mapping from $(A, \Phi)$ to $G$, which is precisely the ordered pair $\left(A / \Phi, \mathrm{pr}_{\Phi}\right)$ with $\mathrm{pr}_{\Phi}:(A, \Phi) \longrightarrow\left(A / \Phi, \Delta_{A / \Phi}\right)$.
Definition 2.13. Let $A$ be an $S$-sorted set and $\Phi \in \operatorname{Eqv}(A)$. Then a transversal of $A / \Phi$ in $A$ is a subset $X$ of $A$ such that, for every $s \in S$ and every $a \in A_{s}$, $\operatorname{card}\left(X_{s} \cap[a]_{\Phi_{s}}\right)=1$.
Remark. For an $S$-sorted equivalence relation $\Phi$ on $A$, the set of all transversals of $A / \Phi$ in $A$ is isomorphic to the set of all cross-sections of $\mathrm{pr}_{\Phi}$, where an $S$-sorted mappings $f$ from $A / \Phi$ to $A$ is a cross-section of $\operatorname{pr}_{\Phi}$ if $\mathrm{pr}_{\Phi} \circ f=\mathrm{id}_{A / \Phi}$. Moreover, if $\Psi$ is another equivalence relation on $A, \Psi$ is a refinement of $\Phi$, i.e., $\Psi \subseteq \Phi$, and $X^{\Phi}$ is a transversal of $A / \Phi$ in $A$, then, for every $s \in S$ and every $a \in A_{s}$, there exists a unique $x \in X_{s}^{\Phi}$ such that $[a]_{\Psi_{s}} \subseteq[x]_{\Phi_{s}}$.

Definition 2.14. Let $A$ be an $S$-sorted set, $X$ a subset of $A$, and $\Phi \in \operatorname{Eqv}(A)$. Then the $\Phi$-saturation of $X$ (or, the saturation of $X$ with respect to $\Phi$ ), denoted by $[X]^{\Phi}$, is the $S$-sorted set defined, for every $s \in S$, as follows:

$$
[X]_{s}^{\Phi}=\left\{a \in A_{s} \mid X_{s} \cap[a]_{\Phi_{s}} \neq \varnothing\right\}=\bigcup_{x \in X_{s}}[x]_{\Phi_{s}}=\left[X_{s}\right]^{\Phi_{s}} .
$$

Let $X$ be a subset of $A$ and $\Phi \in \operatorname{Eqv}(A)$. Then we will say that $X$ is $\Phi$-saturated if and only if $X=[X]^{\Phi}$. We will denote by $\Phi-\operatorname{Sat}(A)$ the subset of $\operatorname{Sub}(A)$ defined as $\Phi-\operatorname{Sat}(A)=\left\{X \in \operatorname{Sub}(A) \mid X=[X]^{\Phi}\right\}$.
Remark. Let $A$ be an $S$-sorted set and $\Phi \in \operatorname{Eqv}(A)$. Then, for a subset $X$ of $A$, we have that the $\Phi$-saturation of $X$ is $\left(\operatorname{pr}_{\Phi}\right)^{-1}\left[\operatorname{pr}_{\Phi}[X]\right]$. Therefore, $X$ is $\Phi$-saturated if and only if $X \supseteq[X]^{\Phi}$. Besides, $X$ is $\Phi$-saturated if and only if there exists a $\mathcal{Y} \subseteq A / \Phi$ such that $X=\left(\mathrm{pr}^{\Phi}\right)^{-1}[\mathcal{Y}]$.

Proposition 2.15. Let $A$ be an $S$-sorted set and $\Phi, \Psi \in \operatorname{Eqv}(A)$. Then

$$
\Phi \subseteq \Psi \text { if and only if } \forall X \subseteq A\left(\left[[X]^{\Psi}\right]^{\Phi}=[X]^{\Psi}\right)
$$

Moreover, for a sort $s \in S$, we have that

$$
\Phi_{s} \subseteq \Psi_{s} \text { if and only if } \forall X \subseteq A_{s}\left(\left[[X]^{\Psi_{s}}\right]^{\Phi_{s}}=[X]^{\Psi_{s}}\right)
$$

Proof. We restrict ourselves to proving the first assertion. Let us assume that $\Phi \subseteq \Psi$ and let $X$ be a subset of $A$. In order to prove that $\left[[X]^{\Psi}\right]^{\Phi}=[X]^{\Psi}$ it suffices to verify that $\left[[X]^{\Psi}\right]^{\Phi} \subseteq[X]^{\Psi}$. Let $s$ be an element of $S$. Then, by definition, $a \in\left[[X]^{\Psi}\right]_{s}^{\Phi}$ if and only if there exists some $b \in[X]_{s}^{\Psi}$ such that $a \in[b]_{\Phi_{s}}$. Since $\Phi \subseteq \Psi$, we have that $a \in[b]_{\Psi_{s}}$, therefore $a \in[X]_{s}^{\Psi}$.

To prove the converse, let us assume that $\Phi \nsubseteq \Psi$. Then there exists some sort $s \in S$ and elements $a, b$ in $A_{s}$ such that $(a, b) \in \Phi_{s}$ and $(a, b) \notin \Psi_{s}$. Hence $b$ does not belong to $\left[\delta^{s,[a]_{s}}\right]_{s}^{\Psi}$, whereas it does belong to $\left[\left[\delta^{s,[a]_{\Psi_{s}}}\right]^{\Psi}\right]_{s}^{\Phi}$. It follows that $\left[\delta^{s,[a]_{s}}\right]^{\Psi} \neq\left[\left[\delta^{s,[a]_{\Psi_{s}}}\right]^{\Psi}\right]^{\Phi}$.
Corollary 2.16. Let $A$ be an $S$-sorted set and $\Phi, \Psi \in \operatorname{Eqv}(A)$. If $\Phi \subseteq \Psi$, then $\Psi-\operatorname{Sat}(A) \subseteq \Phi-\operatorname{Sat}(A)$. Moreover, for $s \in S$ and $L \subseteq A_{s}$, if $\Phi_{s} \subseteq \Psi_{s}$ and $\bar{L}=[L]^{\Psi s}$, then $L=[L]^{\Phi_{s}}$.

Remark. If, for an $S$-sorted set $A$, we denote by $(\cdot)$ - $\operatorname{Sat}(A)$ the mapping from $\operatorname{Eqv}(A)$ to $\operatorname{Sub}(\operatorname{Sub}(A))$ which sends $\Phi$ to $\Phi-\operatorname{Sat}(A)$, then the above corollary means that $(\cdot)-\operatorname{Sat}(A)$ is an antitone ( $\equiv$ order-reversing) mapping from the ordered set $(\operatorname{Eqv}(A), \subseteq)$ to the ordered set $(\operatorname{Sub}(\operatorname{Sub}(A)), \subseteq)$.

Proposition 2.17. Let $A$ be an $S$-sorted set and $X \subseteq A$. Then $X \in \nabla_{A}$ - $\operatorname{Sat}(A)$ if and only if, for every $s \in S$, if $s \in \operatorname{supp}_{S}(X)$, then $X_{s}=A_{s}$.
Proof. Let us suppose that there exists a $t \in S$ such that $X_{t} \neq \varnothing$ and $X_{t} \neq A_{t}$. Then, since $[X]_{t}^{\nabla_{A}}=\bigcup_{x \in X_{t}}[x]_{\nabla_{A_{t}}}$ and $X_{t} \neq \varnothing$, we have that, for some $y \in X_{t}$, $[y]_{\nabla_{A_{t}}}=A_{t}$. But $X_{t} \subset A_{t}$. Hence $[X]_{t}^{\nabla_{A}} \neq X_{t}$. Therefore $X \notin \nabla_{A}$-Sat $(A)$.

The converse implication is straightforward.
Remark. Let $A$ be an $S$-sorted set. Then, from the above proposition, it follows that $\varnothing^{S}, A \in \nabla_{A}$ - $\operatorname{Sat}(A)$. Moreover, for every subset $T$ of $S$, we have that $\bigcup_{t \in T} \delta^{t, A_{t}} \in \nabla_{A}-\operatorname{Sat}(A)$.
Proposition 2.18. Let $A$ be an $S$-sorted set, $X \subseteq A$, and $\Phi, \Psi \in \operatorname{Eqv}(A)$. Then $[X]^{\Phi \cap \Psi} \subseteq[X]^{\Phi} \cap[X]^{\Psi}$.
Proof. Let $s$ be a sort in $S$ and $b \in[X]_{s}^{\Phi \cap \Psi}$. Then, by definition, there exists an $a \in X_{s}$ such that $(a, b) \in(\Phi \cap \Psi)_{s}=\Phi_{s} \cap \Psi_{s}$. Hence, $(a, b) \in \Phi_{s}$ and $(a, b) \in \Psi_{s}$. Therefore $b \in[X]_{s}^{\Phi}$ and $b \in[X]_{s}^{\Psi}$. Consequently, $b \in\left([X]^{\Phi} \cap[X]^{\Psi}\right)_{s}$. Thus $[X]^{\Phi \cap \Psi} \subseteq[X]^{\Phi} \cap[X]^{\Psi}$.

Corollary 2.19. Let $A$ be an $S$-sorted set and $\Phi, \Psi \in \operatorname{Eqv}(A)$. Then we have that $\Phi-\operatorname{Sat}(A) \cap \Psi-\operatorname{Sat}(A) \subseteq(\Phi \cap \Psi)-\operatorname{Sat}(A)$.

We next state that the set $\Phi-\operatorname{Sat}(A)$ is the set of all fixed points of a suitable operator on $A$, i.e., of an endomapping of $\operatorname{Sub}(A)$.

Proposition 2.20. Let $A$ be an $S$-sorted set and $\Phi \in \operatorname{Eqv}(A)$. Then the mapping $[\cdot]^{\Phi}$ from $\operatorname{Sub}(A)$ to $\operatorname{Sub}(A)$ that sends $X$ in $\operatorname{Sub}(A)$ to $[\cdot]^{\Phi}(X)=[X]^{\Phi}$ in $\operatorname{Sub}(A)$ is a completely additive closure operator on $A$. Moreover, for every nonempty set $I$ in $\mathcal{U}$ and every $I$-indexed family $\left(X^{i}\right)_{i \in I}$ in $\operatorname{Sub}(A),\left[\bigcap_{i \in I} X^{i}\right]^{\Phi} \subseteq \bigcap_{i \in I}\left[X^{i}\right]^{\Phi}$ (and, obviously, $[A]^{\Phi}=A$ ), and, for every $X \subseteq A$, if $X=[X]^{\Phi}$, then $\complement_{A} X=\left[\complement_{A} X\right]^{\Phi}$. Besides, $[\cdot]^{\Phi}$ is uniform, i.e., is such that, for every $X, Y \subseteq A$, if $\operatorname{supp}_{S}(X)=$ $\operatorname{supp}_{S}(Y)$, then $\operatorname{supp}_{S}\left([X]^{\Phi}\right)=\operatorname{supp}_{S}\left([Y]^{\Phi}\right)$-hence, in particular, $[\cdot]^{\Phi}$ is a uniform algebraic closure operator on $A$. And $\Phi-\operatorname{Sat}(A)=\operatorname{Fix}\left([\cdot]^{\Phi}\right)$, where $\operatorname{Fix}\left([\cdot]^{\Phi}\right)$ is the set of all fixed point of the operator $[\cdot]^{\Phi}$.

Proposition 2.21. Let $A$ be an $S$-sorted set and $\Phi \in \operatorname{Eqv}(A)$. Then the ordered pair $\Phi-\operatorname{Sat}(A)=(\Phi-\operatorname{Sat}(A), \subseteq)$ is a complete atomic Boolean algebra.

Proof. The proof is straightforward and we leave it to the reader. We only point out that the atoms of $\Phi-\operatorname{Sat}(A)$ are precisely the deltas of Kronecker $\delta^{t,[x] \Phi_{t}}$, for some $t \in S$ and some $x \in A_{t}$, and that, obviously, every $\Phi$-saturated subset $X$ of $A$ is the join ( $\equiv$ union) of all atoms smaller than $X$.

We next define the concept of free monoid on a set and several notions associated to it that will be used afterwards to construct the free algebra on an $S$-sorted set and to define, in Section 3, diverse substitution operators.

Definition 2.22. Let $A$ be a set. The free monoid on $A$, denoted by $\mathbf{A}^{\star}$, is $\left(A^{\star}, \curlywedge, \lambda\right)$, where $A^{\star}$, the set of all words on $A$, is $\bigcup_{n \in \mathbb{N}} \operatorname{Hom}(n, A)$, the set of all mappings $w: n \longrightarrow A$ from some $n \in \mathbb{N}$ to $A$, $\lambda$, the concatenation of words on $A$, is the binary operation on $A^{\star}$ which sends a pair of words $(w, v)$ on $A$ to the mapping $w \curlywedge v$ from $|w|+|v|$ to $A$, where $|w|$ and $|v|$ are the lengths ( $\equiv$ domains) of the mappings $w$ and $v$, respectively, defined as follows:

$$
w \curlywedge v\left\{\begin{aligned}
|w|+|v| & \longrightarrow S \\
i & \longmapsto \begin{cases}w_{i}, & \text { if } 0 \leq i<|w| ; \\
v_{i-|w|}, & \text { if }|w| \leq i<|w|+|v| .\end{cases}
\end{aligned}\right.
$$

and $\lambda$, the empty word on $A$, is the unique mapping from $\varnothing$ to $A$. A word $w \in A^{\star}$ is usually denoted as a sequence $\left(a_{i}\right)_{i \in|w|}$, where, for $i \in|w|, a_{i}$ is the letter in $A$ satisfying $w(i)=a_{i}$. We will denote by $\eta_{A}$ the mapping from $A$ to $A^{\star}$ that sends $a \in A$ to $(a) \in A^{\star}$, i.e., to the mapping $(a): 1 \longrightarrow A$ that sends 0 to $a$. The ordered pair $\left(\mathbf{A}^{\star}, \eta_{A}\right)$ is a universal morphism from $A$ to the forgetful functor from the category Mon, of monoids, to Set.

Remark. For a word $w \in A^{\star},|w|$ is the value at $w$ of the unique homomorphism $|\cdot|$ from $\mathbf{A}^{\star}$ to $(\mathbb{N},+, 0)$, the additive monoid of the natural numbers, such that $|\cdot| \circ \eta_{A}=\kappa_{1}$, where $\kappa_{1}$ is the mapping from $A$ to $\mathbb{N}$ constantly 1 . Note that, for every $n \in \mathbb{N},|\cdot|$ sends $w \in \operatorname{Hom}(n, A)$ to $n$. Thus, for the family of mappings $\left(\kappa_{n}\right)_{n \in \mathbb{N}}$, where, for every $n \in \mathbb{N}$, $\kappa_{n}$ is the mapping from $\operatorname{Hom}(n, A)$ to $\mathbb{N}$ constantly $n$, and by applying the universal property of the coproduct, we have that $|\cdot|=\left[\kappa_{n}\right]_{n \in \mathbb{N}}$.

Definition 2.23. Let $w$ and $w^{\prime}$ be words in $A^{\star}$. We will say that $w^{\prime}$ is a subword of $w$ if there are words $u$ and $v$ such that $w=u \curlywedge w^{\prime} \curlywedge v$. A word $w$ may have several subwords equal to $w^{\prime}$. In that case, the equation $w=u \curlywedge w^{\prime} \curlywedge v$ has several solutions $(u, v)$. If the pairs $\left(u_{i}, v_{i}\right)(i \in n)$ are all solutions of $w=u \curlywedge w^{\prime} \curlywedge v$ and
if $\left|u_{0}\right|<\left|u_{1}\right|<\cdots<\left|u_{n-1}\right|$, then $\left(u_{i}, v_{i}\right)$ determines the $i$-th occurrence of $w^{\prime}$ in $w$. The solution $(u, v)$ in which either $u$ or $v$ is $\lambda$ is not excluded. Let $w$ be a word in $A^{\star}$ and $a \in A$. We will say that $a$ occurs in $w$ if there are words $u$, $v$ in $A^{\star}$ such that $w=u \curlywedge(a) \curlywedge v$. Note that $a$ occurs in $w$ if and only if there exists an $i \in|w|$ such that $w(i)=a$. We will denote by $|w|_{a}$ the natural number $\operatorname{card}(\{i \in|w| \mid w(i)=a\})=\operatorname{card}\left(w^{-1}[\{a\}]\right)$, i.e., the number of occurrences of $a$ in $w$. Moreover, we let $\left(i_{\alpha}\right)_{\alpha \in|w|_{a}}$ stand for the enumeration in ascending order of the occurrences of $a$ in $w$. Thus $\left(i_{\alpha}\right)_{\alpha \in|w|_{a}}$ is the order embedding of $\left(|w|_{a},<\right)$ into $(|w|,<)$ defined recursively as follows:

$$
i_{0}=\min \{i \in|w| \mid w(i)=a\} \text { and }
$$

$$
\text { for } \alpha \in|w|_{a}-\left\{|w|_{a}-1\right\}, i_{\alpha+1}=\min \left\{i \in|w|-\left\{i_{0}, \ldots, i_{\alpha}\right\} \mid w(i)=a\right\}
$$

If the pairs $\left(u_{i_{\alpha}}, v_{i_{\alpha}}\right)\left(\alpha \in|w|_{a}\right)$ are all solutions of $w=u \curlywedge(a) \curlywedge v$ and if $\left|u_{i_{0}}\right|<$ $\left|u_{i_{1}}\right|<\cdots<\left|u_{i_{|w| a-1} \mid}\right|$, then $\left(u_{i_{\alpha}}, v_{i_{\alpha}}\right)$ determines the $i_{\alpha}$-th occurrence of (a) in $w$ and we will say that $a$ occurs at the $i_{\alpha}$-th place of $w$.
Remark. Let $a$ be an element of $A$. Then, for $w \in A^{\star},|w|_{a}$, the number of occurrences of $a$ in $w$, is the value at $w$ of the unique homomorphism $|\cdot|_{a}$ from $\mathbf{A}^{\star}$ to $(\mathbb{N},+, 0)$ such that $|\cdot|_{a} \circ \eta_{A}=\delta_{a}$, where $\delta_{a}$ is the mapping from $A$ to $\mathbb{N}$ that sends $a \in A$ to $1 \in \mathbb{N}$ and $b \in A-\{a\}$ to $0 \in \mathbb{N}$. Therefore, since, for every $w \in A^{\star}$, the $A$-indexed family $\left(|w|_{a}\right)_{a \in A}$ in $\mathbb{N}$ is such that $|w|_{a}=0$ for all but a finite number of elements $a$ in $A$, i.e., is such that $\operatorname{card}\left(\left\{a \in A\left||w|_{a} \neq 0\right\}\right)<\aleph_{0}\right.$, we have that $\sum_{a \in A}|w|_{a} \in \mathbb{N}$ and, obviously, for every $w \in A^{\star},|w|=\sum_{a \in A}|w|_{a}$, i.e., $|\cdot|=\left.\sum_{a \in A}|\cdot|\right|_{a}$.

Definition 2.24. Let $w$ be a word in $A^{\star}$. Then we will denote by $\left({ }^{( }\right)_{a \in A}(w)$ the mapping from $\prod_{a \in A}\left(A^{\star}\right)^{|w|_{a}}$ to $A^{\star}$ that assigns to $\left(\left(q_{\alpha}^{a}\right)_{\alpha \in|w|_{a}}\right)_{a \in A}$ in $\prod_{a \in A}\left(A^{\star}\right)^{|w|_{a}}$ the word $\left(\left(q_{\alpha}^{a}\right)_{\alpha \in|w| a}^{a}\right)_{a \in A}(w)$ in $A^{\star}$ obtained by substituting in $w$, for every $a \in A$ and every $\alpha \in|w|_{a}, q_{\alpha}^{a}$ for the $i_{\alpha}$-th occurrence of $a$ in $w$. We call $\left(\left(q_{\alpha}^{a}\right)_{\alpha \in|w|_{a}}^{a}\right)_{a \in A}(w)$ the substitution of $\left(q_{\alpha}\right)_{\alpha \in|w|_{a}}$ for $a$ in $w$ for every $a$ in $A$, and $\left(\stackrel{a}{{ }^{a}}\right)_{a \in A}(w)$ the substitution operator for $w$. Let $c$ be an element of $A$. Then we will denote by $\left.{ }^{c}{ }^{c}\right)(w)$ the mapping from $\left(A^{\star}\right)^{|w|_{c}}$ to $A^{\star}$ that assigns to $\left(q_{\alpha}^{c}\right)_{\alpha \in|w|_{c}} \in\left(A^{\star}\right)^{|w|_{c}}$ the term $\left(\left(q_{\alpha}^{c}\right)_{\alpha \in|w|_{c}}^{c}\right)(w)$ in $A^{\star}$ obtained by substituting in $w$, for every $\alpha \in|w|_{c}, q_{\alpha}^{c}$ for the $i_{\alpha}$-th occurrence of $c$ in $w$. We call $\left(\left(q_{\alpha}^{c}\right)_{\alpha \in|w|_{c}}^{c}\right)(w)$ the substitution of $\left(q_{\alpha}^{c}\right)_{\alpha \in|w|_{c}}$ for $c$ in $w$.
Remark. Therefore, for every set $A$, we have obtained the family of substitution operators:

$$
\left(\left({ }^{( }\right)_{a \in A}(w)\right)_{w \in A^{\star}} \in \prod_{w \in A^{\star}} \operatorname{Hom}\left(\prod_{a \in A}\left(A^{\star}\right)^{|w|_{a}}, A^{\star}\right)
$$

Our next aim is to provide those notions from the field of many-sorted universal algebra that will be used afterwards.

Definition 2.25. An $S$-sorted signature is a function $\Sigma$ from $S^{\star} \times S$ to $\mathcal{U}$ which sends a pair $(w, s) \in S^{\star} \times S$ to the set $\Sigma_{w, s}$ of the formal operations of arity $w$, sort (or coarity) $s$, and rank (or biarity) $(w, s)$. If $\Sigma$ and $\Lambda$ are $S$-sorted signatures, then a morphism from $\Sigma$ to $\Lambda$ is an $S^{\star} \times S$-indexed family $d=\left(d_{w, s}\right)_{(w, s) \in S^{\star} \times S}$, where, for every $(w, s) \in S^{\star} \times S, d_{w, s}$ is a mapping from $\Sigma_{w, s}$ to $\Lambda_{w, s} . S$-sorted signatures and morphisms between $S$-sorted signatures form a category which we will denote henceforth by $\operatorname{Sig}(S)$.
Remark. For every set of sorts $S$, the category $\mathbf{S i g}(S)$ is $\boldsymbol{\operatorname { S e t }}^{S^{\star} \times S}$.
Assumption. From now on $\Sigma$ stands for an $S$-sorted signature, fixed once and for all.

We shall now give precise definitions of the concepts of many-sorted algebra and of homomorphism between many-sorted algebras.
Definition 2.26. The $S^{\star} \times S$-sorted set of the finitary operations on an $S$-sorted set $A$ is $\left(\operatorname{Hom}\left(A_{w}, A_{s}\right)\right)_{(w, s) \in S^{\star} \times S}$, where, for every $w \in S^{\star}, A_{w}=\prod_{i \in|w|} A_{w_{i}}$, with $|w|$ denoting the length of the word $w$ (if $w=\lambda$, then $A_{\lambda}$ is a final set). A structure of $\Sigma$-algebra on an $S$-sorted set $A$ is a family $\left(F_{w, s}\right)_{(w, s) \in S^{\star} \times S}$, denoted by $F$, where, for $(w, s) \in S^{\star} \times S, F_{w, s}$ is a mapping from $\Sigma_{w, s}$ to $\operatorname{Hom}\left(A_{w}, A_{s}\right)$ (if $(w, s)=(\lambda, s)$ and $\sigma \in \Sigma_{\lambda, s}$, then $F_{w, s}(\sigma)$ picks out an element of $\left.A_{s}\right)$. For a pair $(w, s) \in S^{\star} \times S$ and a formal operation $\sigma \in \Sigma_{w, s}$, in order to simplify the notation, the operation $F_{w, s}(\sigma)$ from $A_{w}$ to $A_{s}$ will be written as $F_{\sigma}$. A $\Sigma$-algebra is a pair $(A, F)$, abbreviated to $\mathbf{A}$, where $A$ is an $S$-sorted set and $F$ a structure of $\Sigma$-algebra on $A$. A $\Sigma$-homomorphism from $\mathbf{A}$ to $\mathbf{B}$, where $\mathbf{B}=(B, G)$, is a triple $(\mathbf{A}, f, \mathbf{B})$, abbreviated to $f: \mathbf{A} \longrightarrow \mathbf{B}$, where $f$ is an $S$-sorted mapping from $A$ to $B$ such that, for every $(w, s) \in S^{\star} \times S$, every $\sigma \in \Sigma_{w, s}$, and every $\left(a_{i}\right)_{i \in|w|} \in A_{w}$, we have that $f_{s}\left(F_{\sigma}\left(\left(a_{i}\right)_{i \in|w|}\right)\right)=G_{\sigma}\left(f_{w}\left(\left(a_{i}\right)_{i \in|w|}\right)\right)$, where $f_{w}$ is the mapping $\prod_{i \in|w|} f_{w_{i}}$ from $A_{w}$ to $B_{w}$ that sends $\left(a_{i}\right)_{i \in|w|}$ in $A_{w}$ to $\left(f_{w_{i}}\left(a_{i}\right)\right)_{i \in|w|}$ in $B_{w}$. We will denote by $\operatorname{Alg}(\Sigma)$ the category of $\Sigma$-algebras and $\Sigma$-homomorphisms (or, to abbreviate, homomorphisms) and by $\operatorname{Alg}(\Sigma)$ the set of objects of $\operatorname{Alg}(\Sigma)$.

In some cases, to avoid mistakes, we will denote by $F^{\mathbf{A}}$ the structure of $\Sigma$-algebra on $A$, and, for $(w, s) \in S^{\star} \times S$ and $\sigma \in \Sigma_{w, s}$, by $F_{\sigma}^{\mathbf{A}}$ the corresponding operation. Moreover, for $s \in S$ and $\sigma \in \Sigma_{\lambda, s}$, we will, usually, denote by $\sigma^{\mathbf{A}}$ the value of the mapping $F_{\sigma}^{\mathbf{A}}: 1 \longrightarrow A_{s}$ at the unique element in 1 .

We will denote by $\mathbf{1}^{S}$ or, to abbreviate, by $\mathbf{1}$, the (standard) final $\Sigma$-algebra.
Definition 2.27. Let $\mathbf{A}$ be a $\Sigma$-algebra. Then the support of $\mathbf{A}$, denoted by $\operatorname{supp}_{S}(\mathbf{A})$, is $\operatorname{supp}_{S}(A)$, i.e., the support of the underlying $S$-sorted set $A$ of $\mathbf{A}$.
Remark. The set $\left\{\operatorname{supp}_{S}(\mathbf{A}) \mid \mathbf{A} \in \operatorname{Alg}(\Sigma)\right\}$ is a closure system on $S$.
Definition 2.28. Let $\mathbf{A}$ be a $\Sigma$-algebra. We will say that $\mathbf{A}$ is finite if $A$, the underlying $S$-sorted set of $\mathbf{A}$, is finite.
Remark. In $\operatorname{Alg}(\Sigma)$, as was the case with $\operatorname{Set}^{S}$, there is another notion of finiteness: A $\Sigma$-algebra $\mathbf{A}$ is called $S$-finite or locally finite, abbreviated as lf, if and only if the underlying $S$-sorted set of $\mathbf{A}$ is $S$-finite. As was noted above this notion of finiteness plays a relevant role in the field of many-sorted algebra, e.g., to define $S$-finite, also called locally finite, terms and to distinguish, on the one hand, between many-sorted varieties and finitary many-sorted varieties, and, on the other hand, between many-sorted quasivarieties and finitary many-sorted quasivarieties (see, e.g., 17] and [30]).

We next define the subset many-sorted algebra associated to a many-sorted algebra and prove, in particular, that this is the object mapping of an endofunctor of $\operatorname{Alg}(\Sigma)$. We note that, in Section 3, the subset many-sorted algebra associated to a free many-sorted algebra will appear as the codomain of a substitution operator which will be used to prove recognizability results.

Proposition 2.29. Let $(\cdot)^{\wp}$ be the mapping that sends (1) a $\Sigma$-algebra $\mathbf{A}$ to the $\Sigma$-algebra $\mathbf{A}^{\wp}=\left(A^{\wp}, F^{\wp}\right)$ where $A^{\wp}$ is $\left(\operatorname{Sub}\left(A_{s}\right)\right)_{s \in S}$ and where, for every $(w, s) \in$ $S^{\star} \times S$ and every $\sigma \in \Sigma_{w, s}, F_{\sigma}^{\wp}$ is the mapping from $A_{w}^{\wp}=\prod_{i \in|w|} \operatorname{Sub}\left(A_{w_{i}}\right)$ to $\operatorname{Sub}\left(A_{s}\right)$ that sends $\left(L_{i}\right)_{i \in|w|}$ in $A_{w}^{\wp}$ to $\left\{F_{\sigma}\left(\left(x_{i}\right)_{i \in|w|}\right) \mid\left(x_{i}\right)_{i \in|w|} \in \prod_{i \in|w|} L_{i}\right\}$ in $\operatorname{Sub}\left(A_{s}\right)$, and (2) a homomorphism $f$ from $\mathbf{A}$ to $\mathbf{B}$ to the $S$-sorted mapping $f^{\wp}=$ $\left(f_{s}[\cdot]\right)_{s \in S}$ from $A^{\wp}$ to $B^{\wp}$. Then $(\cdot)^{\wp}$ is an endofunctor of $\mathbf{A l g}(\Sigma)$. The $\Sigma$-algebra $\mathbf{A}^{6}$ is called the subset algebra associated to $\mathbf{A}$ (this notion, but for single-sorted algebras, is due to Mezei and Wright, cf. 31, Definition 2.2). Moreover, there
exists a pointwise monomorphic natural transformation $\{\cdot\}^{\Sigma}$ from $\operatorname{Id}_{\mathbf{A l g}(\Sigma)}$ to $(\cdot)^{反}$, displayed as:


Proof. To show that $(\cdot)^{\wp}$ is an endofunctor of $\operatorname{Alg}(\Sigma)$ it suffices to verify that, for every homomorphism $f: \mathbf{A} \longrightarrow \mathbf{B}, f^{\wp}$ is, actually, a homomorphism from $\mathbf{A}^{\wp}$ to $\mathbf{B}^{\wp}$. Let $f$ be a homomorphism from $\mathbf{A}$ to $\mathbf{B},(w, s) \in S^{\star} \times S$, and $\sigma \in \Sigma_{w, s}$. Then the following diagram commutes

that is, for every sequence $\left(L_{i}\right)_{i \in|w|}$ in $A_{w}^{\wp}$, we have that

$$
f_{s}\left[F_{\sigma}^{\wp}\left(\left(L_{i}\right)_{i \in|w|}\right)\right]=G_{\sigma}^{\wp}\left(\left(f_{w_{i}}\left[L_{i}\right]\right)_{i \in|w|}\right)
$$

Now let $\{\cdot\}^{\Sigma}$ be the mapping from $\operatorname{Alg}(\Sigma)$ to $\operatorname{Mor}(\mathbf{A l g}(\Sigma))$ that assigns to a $\Sigma$-algebra $\mathbf{A}$ the $S$-sorted mapping $\{\cdot\}_{\mathbf{A}}^{\Sigma}$ from $A$ to $A^{\wp}$ that, for every $s \in S$, sends $a$ in $A_{s}$ to $\{a\}$ in $A_{s}^{\wp}$. It is easily seen that, for every $\Sigma$-algebra $\mathbf{A},\{\cdot\}_{\mathbf{A}}^{\Sigma}$ is an injective homomorphism from $\mathbf{A}$ to $\mathbf{A}^{\wp}$ and that $\{\cdot\}^{\Sigma}=\left(\{\cdot\}_{\mathbf{A}}^{\Sigma}\right)_{\mathbf{A} \in \operatorname{Alg}(\Sigma)}$ is, in fact, a natural transformation from $\operatorname{Id}_{\mathbf{A l g}(\Sigma)}$ to $(\cdot)^{反}$

We shall now go on to define the notion of subalgebra of a $\Sigma$-algebra $\mathbf{A}$ and the subalgebra generating operator for $\mathbf{A}$.

Definition 2.30. Let A be a $\Sigma$-algebra and $X \subseteq A$. Given $(w, s) \in S^{\star} \times S$ and $\sigma \in \Sigma_{w, s}$, we will say that $X$ is closed under the operation $F_{\sigma}: A_{w} \longrightarrow A_{s}$ if, for every $\left(a_{i}\right)_{i \in|w|} \in X_{w}, F_{\sigma}\left(\left(a_{i}\right)_{i \in|w|}\right) \in X_{s}$. We will say that $X$ is a subalgebra of $\mathbf{A}$ if $X$ is closed under the operations of $\mathbf{A}$. We will denote by $\operatorname{Sub}(\mathbf{A})$ the set of all subalgebras of $\mathbf{A}$ (which is an algebraic closure system on $A$ ) and by $\boldsymbol{\operatorname { S u b }}(\mathbf{A})$ the algebraic lattice $(\operatorname{Sub}(\mathbf{A}), \subseteq)$. We also say, equivalently, that a $\Sigma$-algebra $\mathbf{B}$ is a subalgebra of $\mathbf{A}$ if $B \subseteq A$ and the canonical embedding of $B$ into $A$ determines an embedding of $\mathbf{B}$ into $\mathbf{A}$.

Definition 2.31. Let $\mathbf{A}$ be a $\Sigma$-algebra. Then we will denote by $\mathrm{Sg}_{\mathbf{A}}$ the algebraic closure operator canonically associated to the algebraic closure system $\operatorname{Sub}(\mathbf{A})$ on $A$ and we call it the subalgebra generating operator for $\mathbf{A}$. Moreover, if $X \subseteq A$, then we call $\operatorname{Sg}_{\mathbf{A}}(X)$ the subalgebra of $\mathbf{A}$ generated by $X$, and if $X$ is such that $\operatorname{Sg}_{\mathbf{A}}(X)=A$, then we will say that $X$ is a generating subset of $\mathbf{A}$. Besides, $\mathbf{S g}_{\mathbf{A}}(X)$ denotes the algebra determined by $\operatorname{Sg}_{\mathbf{A}}(X)$.

Remark. Let $\mathbf{A}$ be a $\Sigma$-algebra. Then the algebraic closure operator $\mathrm{Sg}_{\mathbf{A}}$ is uniform, i.e., for every $X, Y \subseteq A$, if $\operatorname{supp}_{S}(X)=\operatorname{supp}_{S}(Y)$, then we have that $\operatorname{supp}_{S}\left(\operatorname{Sg}_{\mathbf{A}}(X)\right)=\operatorname{supp}_{S}\left(\operatorname{Sg}_{\mathbf{A}}(Y)\right)$.

We next recall the Principle of Proof by Algebraic Induction. This principle will be used in Section 3.

Proposition 2.32 (Principle of Proof by Algebraic Induction). Let A be a $\Sigma$ algebra generated by $X$. Then to prove that a subset $Y$ of $A$ is equal to $A$ it suffices to show: (1) $X \subseteq Y$ (algebraic induction basis) and (2) $Y$ is a subalgebra of $\mathbf{A}$ (algebraic induction step).

We next state that the forgetful functor $\mathrm{G}_{\Sigma}$ from $\boldsymbol{\operatorname { A l g }}(\Sigma)$ to $\boldsymbol{S e t}^{S}$ has a left adjoint $\mathbf{T}_{\Sigma}$ which assigns to an $S$-sorted set $X$ the free $\Sigma$-algebra $\mathbf{T}_{\Sigma}(X)$ on $X$. Let us note that in what follows, to construct the algebra of $\Sigma$-rows in $X$, and the free $\Sigma$-algebra on $X$, since neither the $S$-sorted signature $\Sigma$ nor the $S$-sorted set $X$ are subject to any constraint, coproducts must necessarily be used.

Definition 2.33. Let $X$ be an $S$-sorted set. The algebra of $\Sigma$-rows in $X$, denoted by $\mathbf{W}_{\Sigma}(X)$, is defined as follows:
(1) The underlying $S$-sorted set of $\mathbf{W}_{\Sigma}(X)$, written as $\mathrm{W}_{\Sigma}(X)$, is precisely the $S$-sorted set $\left((\amalg \Sigma \amalg \coprod X)^{\star}\right)_{s \in S}$, i.e., the mapping from $S$ to $\mathcal{U}$ constantly $(\amalg \Sigma \amalg \coprod X)^{\star}$, where $(\amalg \Sigma \amalg \coprod X)^{\star}$ is the set of all words on the set $\coprod \Sigma \amalg \coprod X$, i.e., on the set $\left[\left(\bigcup_{(w, s) \in S^{\star} \times S}\left(\Sigma_{w, s} \times\{(w, s)\}\right)\right) \times\{0\}\right] \cup\left[\left(\bigcup_{s \in S}\left(X_{s} \times\{s\}\right)\right) \times\{1\}\right]$.
(2) For every $(w, s) \in S^{\star} \times S$, and every $\sigma \in \Sigma_{w, s}$, the structural operation $F_{\sigma}$ associated to $\sigma$ is the mapping from $\mathrm{W}_{\Sigma}(X)_{w}$ to $\mathrm{W}_{\Sigma}(X)_{s}$ which sends $\left(P_{i}\right)_{i \in|w|} \in \mathrm{W}_{\Sigma}(X)_{w}$ to $(\sigma) \curlywedge \curlywedge_{i \in|w|} P_{i} \in \mathrm{~W}_{\Sigma}(X)_{s}$, where, for every $(w, s) \in$ $S^{\star} \times S$, and every $\sigma \in \Sigma_{w, s},(\sigma)$ stands for $(((\sigma,(w, s)), 0))$, which is the value at $\sigma$ of the canonical mapping from $\Sigma_{w, s}$ to $(\amalg \Sigma \amalg \coprod X)^{\star}$.
Definition 2.34. The free $\Sigma$-algebra on an $S$-sorted set $X$, denoted by $\mathbf{T}_{\Sigma}(X)$, is the $\Sigma$-algebra determined by $\operatorname{Sg}_{\mathbf{W}_{\Sigma}(X)}\left(\left(\left\{(x) \mid x \in X_{s}\right\}\right)_{s \in S}\right)$, the subalgebra of $\mathbf{W}_{\Sigma}(X)$ generated by $\left(\left\{(x) \mid x \in X_{s}\right\}\right)_{s \in S}$, where, for every $s \in S$ and every $x \in X_{s}$, $(x)$ stands for $(((x, s), 1))$, which is the value at $x$ of the canonical mapping from $X_{s}$ to $(\amalg \Sigma \amalg \coprod X)^{\star}$. We will denote by $\mathrm{T}_{\Sigma}(X)$ the underlying $S$-sorted of $\mathbf{T}_{\Sigma}(X)$ and, for $s \in S$, we will call the elements of $\mathrm{T}_{\Sigma}(X)_{s}$ terms of type $s$ with variables in $X$ or $(X, s)$-terms.
Remark. Since $\left(\left\{(x) \mid x \in X_{s}\right\}\right)_{s \in S}$ is a generating subset of $\mathbf{T}_{\Sigma}(X)$, to prove that a subset $\mathcal{T}$ of $\mathrm{T}_{\Sigma}(X)$ is equal to $\mathrm{T}_{\Sigma}(X)$ it suffices, by Proposition 2.32, to show: (1) $\left(\left\{(x) \mid x \in X_{s}\right\}\right)_{s \in S} \subseteq \mathcal{T}$ (algebraic induction basis) and (2) $\mathcal{T}$ is a subalgebra of $\mathbf{T}_{\Sigma}(X)$ (algebraic induction step).

In the many-sorted case we have, as in the single-sorted case, the following characterization of the elements of $\mathrm{T}_{\Sigma}(X)_{s}$, for $s \in S$.

Proposition 2.35. Let $X$ be an $S$-sorted set. Then, for every $s \in S$ and every $P \in \mathrm{~W}_{\Sigma}(X)_{s}$, we have that $P$ is a term of type $s$ with variables in $X$ if and only if $P=(x)$, for a unique $x \in X_{s}$, or $P=(\sigma)$, for a unique $\sigma \in \Sigma_{\lambda, s}$, or $P=(\sigma) \curlywedge \curlywedge\left(P_{i}\right)_{i \in|w|}$, for a unique $w \in S^{\star}-\{\lambda\}$, a unique $\sigma \in \Sigma_{w, s}$, and a unique family $\left(P_{i}\right)_{i \in|w|} \in \mathrm{T}_{\Sigma}(X)_{w}$. Moreover, the three possibilities are mutually exclusive.

From now on, for simplicity of notation, we will write $x, \sigma$, and $\sigma\left(P_{0}, \ldots, P_{|w|-1}\right)$ or $\sigma\left(\left(P_{i}\right)_{i \in|w|}\right)$ instead of $(x),(\sigma)$, and $(\sigma) \curlywedge \curlywedge\left(P_{i}\right)_{i \in|w|}$, respectively.

From the above proposition it follows, immediately, the universal property of the free $\Sigma$-algebra on an $S$-sorted set $X$, as stated in the subsequent proposition.

Proposition 2.36. For every $S$-sorted set $X$, the pair $\left(\eta_{X}, \mathbf{T}_{\Sigma}(X)\right)$, where $\eta_{X}$, the insertion of (the $S$-sorted set of generators) $X$ into $\mathrm{T}_{\Sigma}(X)$, is the co-restriction to $\mathrm{T}_{\Sigma}(X)$ of the canonical embedding of $X$ into $\mathrm{W}_{\Sigma}(X)$, has the following universal property: for every $\Sigma$-algebra $\mathbf{A}$ and every $S$-sorted mapping $f: X \longrightarrow A$, there exists a unique homomorphism $f^{\sharp}: \mathbf{T}_{\Sigma}(X) \longrightarrow \mathbf{A}$ such that $f^{\sharp} \circ \eta_{X}=f$.

Proof. For every $s \in S$ and every $(X, s)$-term $P$, the $s$-th coordinate $f_{s}^{\sharp}$ of $f^{\sharp}$ is defined recursively as follows: $f_{s}^{\sharp}(x)=f_{s}(x)$, if $P=x ; f_{s}^{\sharp}(\sigma)=\sigma$, if $P=$ $\sigma$; and, finally, $f_{s}^{\sharp}\left(\sigma\left(P_{0}, \ldots, P_{|w|-1}\right)\right)=F_{\sigma}\left(f_{w_{0}}^{\sharp}\left(P_{0}\right), \ldots, f_{w_{|w|-1}}^{\sharp}\left(P_{|w|-1}\right)\right)$, if $P=$ $\sigma\left(P_{0}, \ldots, P_{|w|-1}\right)$.

The just stated proposition allows us to carry out definitions by algebraic recursion on a free many-sorted algebra as indeed we will be doing throughout this paper.

Corollary 2.37. The functor $\mathbf{T}_{\Sigma}$, which sends an $S$-sorted set $X$ to $\mathbf{T}_{\Sigma}(X)$ and an $S$-sorted mapping $f$ from $X$ to $Y$ to $f^{@}\left(=\left(\eta_{Y} \circ f\right)^{\sharp}\right)$, the unique homomorphism from $\mathbf{T}_{\Sigma}(X)$ to $\mathbf{T}_{\Sigma}(Y)$ such that $f^{@} \circ \eta_{X}=\eta_{Y} \circ f$, is left adjoint for the forgetful functor $\mathrm{G}_{\Sigma}$ from $\operatorname{Alg}(\Sigma)$ to $\boldsymbol{S e t}^{S}$.

For every $\Sigma$-algebra it is possible to define a preorder on the coproduct of its underlying $S$-sorted set. Moreover, for the case of free algebras, such a preorder is, in fact, an order and this allows us to define the notion of subterm of a given term.

Definition 2.38. Let $\mathbf{A}=(A, F)$ be a $\Sigma$-algebra. Then $<_{\mathbf{A}}$ denotes the binary relation on $\coprod A$ consisting of the ordered pairs $((a, s),(b, t)) \in(\coprod A)^{2}$ for which there exists a $w \in S^{\star}-\{\lambda\}$, a $\sigma \in \Sigma_{w, t}$, and an $x \in A_{w}$ such that $F_{\sigma}(x)=b$ and, for some $i \in|w|, w_{i}=s$ and $x_{i}=a$. We will denote by $\leq_{\mathbf{A}}$ the reflexive and transitive closure of $<_{\mathbf{A}}$, i.e., the preorder on $\coprod A$ generated by $<_{\mathbf{A}}$.

Remark. The preorder $\leq_{\mathbf{A}}$ on $\coprod A$ is defined by letting $((a, s),(b, t)) \in \leq_{\mathbf{A}}$ mean that $s=t$ and $a=b$ or there exists an $n \in \mathbb{N}-\{0\}$, a $u \in S^{\star}$, and a family $\left(c_{i}\right)_{i \in|u|} \in A_{u}$ such that $|u|=n+1, u_{0}=s, u_{n}=t, c_{0}=a, c_{n}=b$, and, for every $i \in n,\left(\left(c_{i}, u_{i}\right),\left(c_{i+1}, u_{i+1}\right)\right) \in<_{\mathbf{A}}$.

Proposition 2.39. Let $X$ be an $S$-sorted set. Then $\leq_{\mathbf{T}_{\Sigma}(X)}$ is antisymmetric and does not have strictly descending $\omega_{0}$-chains, i.e., is an Artinian order.

Definition 2.40. Let $X$ be an $S$-sorted set, $t \in S$, and $P \in \mathrm{~T}_{\Sigma}(X)_{t}$. Then the $S$-sorted set of all subterms of $P$, denoted by $\operatorname{Subt}(P)$, is defined as follows:

$$
\operatorname{Subt}(P)=\left(\left\{Q \in \mathrm{~T}_{\Sigma}(X)_{s} \mid(Q, s) \leq_{\mathbf{T}_{\Sigma}(X)}(P, t)\right\}\right)_{s \in S}
$$

Remark. $\operatorname{Subt}(P) \in \operatorname{Sub}_{\mathfrak{f}}\left(\mathrm{T}_{\Sigma}(X)\right)$. Moreover, $\operatorname{Subt}(P)$ can also be characterized as the smallest subset $\mathcal{L}$ of $\mathrm{T}_{\Sigma}(X)$ which satisfies the following conditions: (1) $P \in \mathcal{L}_{t}$ and (2) for every $(w, s) \in S^{\star} \times S$, every $\xi \in \Sigma_{w, s}$, and every $\left(Q_{i}\right)_{i \in|w|} \in$ $\mathrm{T}_{\Sigma}(X)_{w}$, if $\xi\left(\left(Q_{i}\right)_{i \in|w|}\right) \in \mathcal{L}_{s}$, then, for every $i \in|w|, Q_{i} \in \mathcal{L}_{w_{i}}$. Note that the second condition is exactly the converse of the defining condition of the concept of subalgebra of a $\Sigma$-algebra.

Following this we associate to every term for $\Sigma$ of type ( $X, s$ ) its $S$-sorted set of variables. This will be used, in Section 3, in the proof of diverse propositions.

Definition 2.41. Let $X$ be an $S$-sorted set. Then $\operatorname{Fin}(X)$ is the $\Sigma$-algebra which has as underlying $S$-sorted set $\left(\operatorname{Sub}_{\mathrm{f}}(X)\right)_{s \in S}$, i.e., the $S$-sorted set constantly $\operatorname{Sub}_{\mathrm{f}}(X)$, and, for every $(w, s) \in S^{\star} \times S$ and every $\sigma \in \Sigma_{w, s}$, as operation $F_{\sigma}: \operatorname{Sub}_{\mathrm{f}}(X)^{|w|} \longrightarrow \operatorname{Sub}_{\mathrm{f}}(X)$ that one defined as $F_{\sigma}\left(\left(K^{i}\right)_{i \in|w|}\right)=\bigcup_{i \in|w|} K^{i}$. Let $\delta^{X}=\left(\delta_{s}^{X}\right)_{s \in S}$ be the $S$-sorted mapping defined, for every $s \in S$, as

$$
\delta_{s}^{X}\left\{\begin{aligned}
X_{s} & \longrightarrow \operatorname{Sub}_{\mathrm{f}}(X) \\
x & \longmapsto \delta^{s, x}
\end{aligned}\right.
$$

Then we will denote by $\operatorname{Var}^{X}$ the unique homomorphism $\left(\delta^{X}\right)^{\#}$ from $\mathbf{T}_{\Sigma}(X)$ to Fin $(X)$ such that the following diagram commutes


For a sort $s \in S$ and a term $P \in \mathrm{~T}_{\Sigma}(X)_{s}$ we will call $\operatorname{Var}_{s}^{X}(P)\left(\in \operatorname{Sub}_{\mathrm{f}}(X)\right)$ the $S$-sorted set of variables of $P$. Moreover, when this is unlikely to cause confusion, we will write $\operatorname{Var}^{X}(P)$ or, simply, $\operatorname{Var}(P)$ for $\operatorname{Var}_{s}^{X}(P)$.

Our next goal is to define the concepts of congruence on a $\Sigma$-algebra and of quotient of a $\Sigma$-algebra by a congruence on it. Moreover, we recall the notion of kernel of a homomorphism between $\Sigma$-algebras and the universal property of the quotient of a $\Sigma$-algebra by a congruence on it.

Definition 2.42. Let A be a $\Sigma$-algebra and $\Phi$ an $S$-sorted equivalence on $A$. We will say that $\Phi$ is an $S$-sorted congruence on (or, to abbreviate, a congruence on) $\mathbf{A}$ if, for every $(w, s) \in\left(S^{\star}-\{\lambda\}\right) \times S$, every $\sigma \in \Sigma_{w, s}$, and every $\left(a_{i}\right)_{i \in|w|},\left(b_{i}\right)_{i \in|w|} \in$ $A_{w}$, if, for every $i \in|w|,\left(a_{i}, b_{i}\right) \in \Phi_{w_{i}}$, then $\left(F_{\sigma}\left(\left(a_{i}\right)_{i \in|w|}\right), F_{\sigma}\left(\left(b_{i}\right)_{i \in|w|}\right)\right) \in \Phi_{s}$. We will denote by $\operatorname{Cgr}(\mathbf{A})$ the set of all $S$-sorted congruences on $\mathbf{A}$ (which is an algebraic closure system on $A \times A$ ), by $\mathbf{C g r}(\mathbf{A})$ the algebraic lattice $(\operatorname{Cgr}(\mathbf{A}), \subseteq)$, by $\nabla_{\mathbf{A}}$ the greatest element of $\operatorname{Cgr}(\mathbf{A})$, and by $\Delta_{\mathbf{A}}$ the least element of $\operatorname{Cgr}(\mathbf{A})$.

For a congruence $\Phi$ on $\mathbf{A}$, the quotient $\Sigma$-algebra of $\mathbf{A}$ by $\Phi$, denoted by $\mathbf{A} / \Phi$, is the $\Sigma$-algebra $\left(A / \Phi, F^{\mathbf{A} / \Phi}\right)$, where, for every $(w, s) \in S^{\star} \times S$ and every $\sigma \in \Sigma_{w, s}$, the operation $F_{\sigma}^{\mathbf{A} / \Phi}$ from $(A / \Phi)_{w}$ to $A_{s} / \Phi_{s}$, also denoted, to simplify, by $F_{\sigma}$, sends $\left(\left[a_{i}\right]_{\Phi_{w_{i}}}\right)_{i \in|w|}$ in $(A / \Phi)_{w}$ to $\left[F_{\sigma}\left(\left(a_{i}\right)_{i \in|w|}\right)\right]_{\Phi_{s}}$ in $A_{s} / \Phi_{s}$, and the canonical projection from $\mathbf{A}$ to $\mathbf{A} / \Phi$, denoted by $\operatorname{pr}_{\Phi}: \mathbf{A} \longrightarrow \mathbf{A} / \Phi$, is the homomorphism determined by the projection from $A$ to $A / \Phi$. The ordered pair $\left(\mathbf{A} / \Phi, \operatorname{pr}_{\Phi}\right)$ has the following universal property: $\operatorname{Ker}\left(\operatorname{pr}_{\Phi}\right)$ is $\Phi$ and, for every $\Sigma$-algebra $\mathbf{B}$ and every homomorphism $f$ from $\mathbf{A}$ to $\mathbf{B}$, if $\operatorname{Ker}(f) \supseteq \Phi$, then there exists a unique homomorphism $h$ from $\mathbf{A} / \Phi$ to $\mathbf{B}$ such that $h \circ \operatorname{pr}_{\Phi}=f$. In particular, if $\Psi$ is a congruence on $A$ such that $\Phi \subseteq \Psi$, then we will denote by $\mathrm{p}_{\Phi, \Psi}$ the unique homomorphism from $\mathbf{A} / \Phi$ to $\mathbf{A} / \Psi$ such that $\mathrm{p}_{\Phi, \Psi} \circ \mathrm{pr}_{\Phi}=\operatorname{pr}_{\Psi}$.

Remark. Let $\operatorname{Clfd} \operatorname{Alg}(\Sigma)$ be the category whose objects are the classified $\Sigma$ algebras, i.e, the ordered pairs $(\mathbf{A}, \Phi)$ where $\mathbf{A}$ is a $\Sigma$-algebra and $\Phi$ a congruence on $\mathbf{A}$, and in which the set of morphisms from $(\mathbf{A}, \Phi)$ to $(\mathbf{B}, \Psi)$ is the set of all homomorphisms $f$ from $\mathbf{A}$ to $\mathbf{B}$ such that, for every $s \in S$ and every $(x, y) \in$ $A_{s}^{2}$, if $(x, y) \in \Phi_{s}$, then $\left(f_{s}(x), f_{s}(y)\right) \in \Psi_{s}$. Let $G$ be the functor from $\operatorname{Alg}(\Sigma)$ to $\operatorname{Clfd} \operatorname{Alg}(\Sigma)$ whose object mapping sends $\mathbf{A}$ to $\left(\mathbf{A}, \Delta_{\mathbf{A}}\right)$ and whose morphism mapping sends $f: \mathbf{A} \longrightarrow \mathbf{B}$ to $f:\left(\mathbf{A}, \Delta_{\mathbf{A}}\right) \longrightarrow\left(\mathbf{B}, \Delta_{\mathbf{B}}\right)$. Then, for every classified $\Sigma$-algebra $(\mathbf{A}, \Phi)$, there exists a universal mapping from $(\mathbf{A}, \Phi)$ to $G$, which is precisely the ordered pair $\left(\mathbf{A} / \Phi, \mathrm{pr}_{\Phi}\right)$ with $\mathrm{pr}_{\Phi}:(\mathbf{A}, \Phi) \longrightarrow\left(\mathbf{A} / \Phi, \Delta_{\mathbf{A} / \Phi}\right)$.

We next define for a $\Sigma$-algebra the concepts of elementary translation and of translation with respect to it, and provide, by using the just mentioned concepts, two characterizations of the congruences on a $\Sigma$-algebra. To this we add that the concept of translation will allow us to define the concept of congruence cogenerated by an $S$-sorted subset of the underlying $S$-sorted set of a $\Sigma$-algebra, which will be all-important in our congruence based proofs of the recognizability theorems for free many-sorted algebras, in Section 3.

Definition 2.43. Let $\mathbf{A}$ be a $\Sigma$-algebra and $t \in S$. Then we will denote by $\operatorname{Etl}_{t}(\mathbf{A})$ the subset $\left(\operatorname{Etl}_{t}(\mathbf{A})_{s}\right)_{s \in S}$ of $\left(\operatorname{Hom}\left(A_{t}, A_{s}\right)\right)_{s \in S}$ defined, for every $s \in S$, as follows: For every mapping $T \in \operatorname{Hom}\left(A_{t}, A_{s}\right), T \in \operatorname{Etl}_{t}(\mathbf{A})_{s}$ if and only if there is a word $w \in S^{\star}-\{\lambda\}$, an $i \in|w|$, a $\sigma \in \Sigma_{w, s}$, a family $\left(a_{j}\right)_{j \in i} \in \prod_{j \in i} A_{w_{j}}$, and a family $\left(a_{k}\right)_{k \in|w|-(i+1)} \in \prod_{k \in|w|-(i+1)} A_{w_{k}}$ (recall that $i+1=\{0,1, \ldots, i\}$ and that $|w|-(i+1)=\{i+1, \ldots,|w|-1\})$ such that $w_{i}=t$ and, for every $x \in A_{t}$, $T(x)=F_{\sigma}\left(a_{0}, \ldots, a_{i-1}, x, a_{i+1}, \ldots, a_{|w|-1}\right)$. We call the elements of $\operatorname{Etl}_{t}(\mathbf{A})_{s}$ the $t$-elementary translations of sort $s$ for $\mathbf{A}$.

Definition 2.44. Let $\mathbf{A}$ be a $\Sigma$-algebra and $t \in S$. Then we will denote by $\mathrm{Tl}_{t}(\mathbf{A})$ the subset $\left(\mathrm{Tl}_{t}(\mathbf{A})_{s}\right)_{s \in S}$ of $\left(\operatorname{Hom}\left(A_{t}, A_{s}\right)\right)_{s \in S}$ defined, for every $s \in S$, as follows: For every mapping $T \in \operatorname{Hom}\left(A_{t}, A_{s}\right), T \in \mathrm{Tl}_{t}(\mathbf{A})_{s}$ if and only if there is an $n \in \mathbb{N}-1$, a word $\left(s_{j}\right)_{j \in n+1} \in S^{n+1}$, and a family $\left(T_{j}\right)_{j \in n}$ such that $s_{0}=t, s_{n}=s$, $T_{0} \in \operatorname{Etl}_{t}(\mathbf{A})_{s_{1}}, T_{1} \in \operatorname{Etl}_{s_{1}}(\mathbf{A})_{s_{2}}, \ldots, T_{n-1} \in \operatorname{Etl}_{s_{n-1}}(\mathbf{A})_{s}$ and $T=T_{n-1} \circ \cdots \circ T_{0}$. We call the elements of $\mathrm{Tl}_{t}(\mathbf{A})_{s}$ the $t$-translations of sort $s$ for $\mathbf{A}$. Besides, for every $t \in S$, the mapping $\operatorname{id}_{A_{t}}$ will be viewed as an element of $\operatorname{Tl}_{t}(\mathbf{A})_{t}$.
Remark. The $S \times S$-sorted set $\left(\mathrm{Tl}_{t}(\mathbf{A})_{s}\right)_{(t, s) \in S \times S}$ determines a category $\mathbf{T l}(\mathbf{A})$ whose object set is $S$ and in which, for every $(t, s) \in S \times S, \operatorname{Hom}_{\mathbf{T l}(\mathbf{A})}(t, s)$, the hom-set from $t$ to $s$, is $\operatorname{Tl}_{t}(\mathbf{A})_{s}$. Therefore, for every $t \in S, \operatorname{End}_{\mathbf{T l}(\mathbf{A})}(t)$ is equipped with a structure of monoid.

Given a $\Sigma$-algebra $\mathbf{A}$ and a translation $T \in \mathrm{Tl}_{t}(\mathbf{A})_{s}$ we next define, associated to the mappings $T[\cdot]: \operatorname{Sub}\left(A_{t}\right) \longrightarrow \operatorname{Sub}\left(A_{s}\right)$ and $T^{-1}[\cdot]: \operatorname{Sub}\left(A_{s}\right) \longrightarrow \operatorname{Sub}\left(A_{t}\right)$, operators: $T[\cdot]$ and $T^{-1}[\cdot]$ from $\operatorname{Sub}(A)$ to $\operatorname{Sub}(A), T[\cdot]$ from $\operatorname{Sub}\left(A_{t}\right)$ to $\operatorname{Sub}(A)$, and $T^{-1}[\cdot]$ from $\operatorname{Sub}\left(A_{s}\right)$ to $\operatorname{Sub}(A)$. This will be used below in the characterization of the recognizable languages.

Definition 2.45. Let $\mathbf{A}$ be a $\Sigma$-algebra, $L \subseteq A, s, t \in S, M \subseteq A_{t}, N \subseteq A_{s}$, and $T \in \mathrm{Tl}_{t}(\mathbf{A})_{s}$. Then
(1) $T[L]$ is the subset of $A$ defined as follows: $T[L]_{s}=T\left[L_{t}\right]$ and $T[L]_{u}=\varnothing$, if $u \neq s$. Therefore, $T[L]=\delta^{s, T\left[L_{t}\right]}$;
(2) $T^{-1}[L]$ is the subset of $A$ defined as follows: $T^{-1}[L]_{t}=T^{-1}\left[L_{s}\right]$ and $T^{-1}[L]_{u}=\varnothing$, if $u \neq t$. Therefore, $T^{-1}[L]=\delta^{t, T^{-1}\left[L_{s}\right]} ;$
(3) $T[M]$ is $\delta^{s, T[M]}\left(=T\left[\delta^{t, M}\right]\right)$; and
(4) $T^{-1}[N]$ is $\delta^{t, T^{-1}[N]}\left(=T^{-1}\left[\delta^{s, N}\right]\right)$.

Remark. Let A be a $\Sigma$-algebra, $K, L \subseteq A, s, t \in S, M \subseteq A_{t}, N \subseteq A_{s}$, and $T \in$ $\mathrm{Tl}_{t}(\mathbf{A})_{s}$. Then $T[M] \subseteq \delta^{s, N}$ if and only if $\delta^{t, M} \subseteq T^{-1}[N]$. Moreover, $T[K] \subseteq \delta^{s, L_{s}}$ if and only if $\delta^{t, K_{t}} \subseteq T^{-1}[L]$. Had the operators $T[\cdot]$ and $T^{-1}[\cdot]$ been defined (for $u \in S-\{s, t\})$ differently, then the suitably modified counterparts of the just stated connections would not be met.

As announced above, we next provide, by using the notions of elementary translation and of translation, two characterizations of the congruences on a $\Sigma$-algebra. This shows, in particular, the significance of the notions of elementary translation and of translation. We note that in [30, on p. 199, it was announced without proof a proposition similar to that set out below.
Proposition 2.46. Let A be a $\Sigma$-algebra and $\Phi$ an $S$-sorted equivalence on $A$. Then the following conditions are equivalent:
(1) $\Phi$ is a congruence on $\mathbf{A}$.
(2) $\Phi$ is closed under the elementary translations on $\mathbf{A}$, i.e., for every every $t$, $s \in S$, every $x, y \in A_{t}$, and every $T \in \operatorname{Etl}_{t}(\mathbf{A})_{s}$, if $(x, y) \in \Phi_{t}$, then $(T(x), T(y)) \in \Phi_{s}$.
(3) $\Phi$ is closed under the translations on $\mathbf{A}$, i.e., for every every $t, s \in S$, every $x, y \in A_{t}$, and every $T \in \mathrm{Tl}_{t}(\mathbf{A})_{s}$, if $(x, y) \in \Phi_{t}$, then $(T(x), T(y)) \in \Phi_{s}$.

Proof. Let us first prove that (1) and (2) are equivalent.
Let us suppose that $\Phi$ is a congruence on $\mathbf{A}$. We want to show that $\Phi$ is closed under the elementary translations on A. Let $t$ and $s$ be elements of $S$ and $T$ a $t$-elementary translation of sort $s$ for $\mathbf{A}$. Then $T: A_{t} \longrightarrow A_{s}$ and there is a word $w \in S^{\star}-\{\lambda\}$, an $i \in|w|$, a $\sigma \in \Sigma_{w, s}$, a family $\left(a_{j}\right)_{j \in i} \in \prod_{j \in i} A_{w_{j}}$, and a family $\left(a_{k}\right)_{k \in|w|-(i+1)} \in \prod_{k \in|w|-(i+1)} A_{w_{k}}$ such that $w_{i}=t$ and, for every $z \in A_{t}$, $T(z)=F_{\sigma}\left(a_{0}, \ldots, a_{i-1}, z, a_{i+1}, \ldots, a_{|w|-1}\right)$. Let $x$ and $y$ be elements of $A_{t}$ such that $(x, y) \in \Phi_{t}$. Since, for every $j \in i,\left(a_{j}, a_{j}\right) \in \Phi_{w_{j}}$, for every $k \in|w|-(i+1)$, $\left(a_{k}, a_{k}\right) \in \Phi_{w_{k}}$, and, in addition, $(x, y) \in \Phi_{t}=\Phi_{w_{i}}$, then $(T(x), T(y)) \in \Phi_{s}$.

Reciprocally, let us suppose that, for every $t, s \in S$, every $x, y \in A_{t}$, and every $T \in \operatorname{Etl}_{t}(\mathbf{A})_{s}$, if $(x, y) \in \Phi_{t}$, then $(T(x), T(y)) \in \Phi_{s}$. We want to show that $\Phi$ is a congruence on $\mathbf{A}$. Let $(w, u) \in\left(S^{\star}-\{\lambda\}\right) \times S, \sigma: w \longrightarrow u$, and $a=\left(a_{i}\right)_{i \in|w|}, b=\left(b_{i}\right)_{i \in|w|} \in A_{w}$ such that, for every $i \in|w|$ we have that $\left(a_{i}, b_{i}\right) \in \Phi_{w_{i}}$. We now define, for every $i \in|w|, T_{i}$, the $w_{i}$-elementary translation of sort $u$ for $\mathbf{A}$, as the mapping from $A_{w_{i}}$ to $A_{u}$ which sends $x \in A_{w_{i}}$ to $F_{\sigma}\left(b_{0}, \ldots, b_{i-1}, x, a_{i+1}, \ldots, a_{|w|-1}\right) \in A_{u}$. Then $F_{\sigma}\left(a_{0}, \ldots, a_{|w|-1}\right)=T_{0}\left(a_{0}\right)$ and $\left(T_{0}\left(a_{0}\right), T_{0}\left(b_{0}\right)\right) \in \Phi_{w_{0}}$. But $T_{0}\left(b_{0}\right)=T_{1}\left(a_{1}\right)$ and $\left(T_{1}\left(a_{1}\right), T_{1}\left(b_{1}\right)\right) \in \Phi_{w_{1}}$. By proceeding in the same way we, finally, come to $T_{|w|-2}\left(b_{|w|-2}\right)=T_{|w|-1}\left(a_{|w|-1}\right)$, $\left(T_{|w|-1}\left(a_{|w|-1}\right), T_{|w|-1}\left(b_{|w|-1}\right)\right) \in \Phi_{w_{|w|-1}}$, and $T_{|w|-1}\left(b_{|w|-1}\right)=F_{\sigma}\left(b_{0}, \ldots, b_{|w|-1}\right)$. Therefore $\left(F_{\sigma}(a), F_{\sigma}(b)\right) \in \Phi_{u}$.

We shall now proceed to verify that (2) and (3) are equivalent.
Since every elementary translations on $\mathbf{A}$ is a translation on $\mathbf{A}$, it is obvious that if $\Phi$ is closed under the translations on $\mathbf{A}$, then $\Phi$ is closed under the elementary translations on $\mathbf{A}$.

Reciprocally, let us suppose that $\Phi$ is closed under the elementary translations on $\mathbf{A}$. We want to show that $\Phi$ is closed under the translations on A. Let $t$ and $s$ be elements of $S, x, y$ elements of $A_{t}, T \in \mathrm{Tl}_{t}(\mathbf{A})_{s}$, and let us suppose that $(x, y) \in \Phi_{t}$. Then there is an $n \in \mathbb{N}-1$, a word $\left(s_{j}\right)_{j \in n+1} \in S^{n+1}$, and a family $\left(T_{j}\right)_{j \in n}$ such that $s_{0}=t, s_{n}=s, T_{0} \in \operatorname{Etl}_{t}(\mathbf{A})_{s_{1}}, T_{1} \in \operatorname{Etl}_{s_{1}}(\mathbf{A})_{s_{2}}, \ldots$, $T_{n-1} \in \operatorname{Etl}_{s_{n-1}}(\mathbf{A})_{s}$ and $T=T_{n-1} \circ \cdots \circ T_{0}$. Then, from $(x, y) \in \Phi_{t}=\Phi_{s_{0}}$, we infer that $\left(T_{0}(x), T_{0}(y)\right) \in \Phi_{s_{1}}$. By proceeding in the same way we, finally, come to $\left(T_{n-1}\left(\ldots\left(T_{0}(x)\right) \ldots\right), T_{n-1}\left(\ldots\left(T_{0}(y)\right) \ldots\right)\right) \in \Phi_{s}=\Phi_{s_{n}}$, i.e., to $(T(x), T(y)) \in$ $\Phi_{s}$.

Our next aim is to assign, in a functional way, to every subset $L$ of the underlying $S$-sorted set $A$ of a $\Sigma$-algebra $\mathbf{A}$ the so-called congruence on $\mathbf{A}$ cogenerated by $L$, and to investigate its properties. But before doing it we need to recall the following result. For every $S$-sorted set $A$, the mapping from $\operatorname{Sub}(A)$ to $\operatorname{Hom}\left(A,(2)_{s \in S}\right)$ that assigns to $L \in \operatorname{Sub}(A)$ precisely $\mathrm{ch}_{L}$, the character of $L$, i.e., the $S$-sorted mapping from $A$ to $(2)_{s \in S}$ whose $s$-th coordinate, for $s \in S$, is $\operatorname{ch}_{L_{s}}$, the characteristic mapping of $L_{s}$, is a natural isomorphism, in other words, $(2)_{s \in S}$ together with $\top^{S}=(\top)_{s \in S}: 1 \longrightarrow(2)_{s \in S}$, where, for every $s \in S, \top$ is the mapping from 1 to 2 that sends 0 to 1 , is a subobject classifier for $\mathbf{S e t}^{S}$. Then, given a $\Sigma$-algebra $\mathbf{A}$ and a subset $L$ of $A$, we associate to $L$ the $S$-sorted equivalence $\operatorname{Ker}\left(\operatorname{ch}_{L}\right)$ on $A$ determined by $\mathrm{ch}_{L}$. So, for every $s \in S$, we have that:

$$
\operatorname{Ker}\left(\operatorname{ch}_{L}\right)_{s}=\operatorname{Ker}\left(\operatorname{ch}_{L_{s}}\right)=\left\{(x, y) \in A_{s}^{2} \mid x \in L_{s} \leftrightarrow y \in L_{s}\right\} .
$$

We next prove that there exists a congruence $\Omega^{\mathbf{A}}(L)$ on $\mathbf{A}$ that saturates $L$, i.e., which is such that $\Omega^{\mathbf{A}}(L) \subseteq \operatorname{Ker}\left(\operatorname{ch}_{L}\right)$, and is the largest congruence on $\mathbf{A}$ having such a property.

Definition 2.47. Let A be a $\Sigma$-algebra and $L \subseteq A$. Then $\Omega^{\mathbf{A}}(L)$ is the binary relation on $A$ defined, for every $t \in S$, as follows:

$$
\Omega^{\mathbf{A}}(L)_{t}=\left\{\begin{array}{l|l}
(x, y) \in A_{t}^{2} & \begin{array}{c}
\forall s \in S \forall T \in \mathrm{Tl}_{t}(\mathbf{A})_{s} \\
\left(T(x) \in L_{s} \leftrightarrow T(y) \in L_{s}\right)
\end{array}
\end{array}\right\} .
$$

Proposition 2.48. Let $\mathbf{A}$ be a $\Sigma$-algebra and $L \subseteq A$. Then
(1) $\Omega^{\mathbf{A}}(L)$ is a congruence on $\mathbf{A}$.
(2) $\Omega^{\mathbf{A}}(L) \subseteq \operatorname{Ker}\left(\operatorname{ch}_{L}\right)$.
(3) For every congruence $\Phi$ on $\mathbf{A}$, if $\Phi \subseteq \operatorname{Ker}\left(\operatorname{ch}_{L}\right)$, then $\Phi \subseteq \Omega^{\mathbf{A}}(L)$.

In other words, $\Omega^{\mathbf{A}}(L)$ is the greatest congruence on $\mathbf{A}$ which saturates $L$.
Proof. To prove (1) it suffices to take into account Proposition 2.46 To prove (2), given $t \in S$ and $(x, y) \in \Omega^{\mathbf{A}}(L)_{t}$, it suffices to consider $\operatorname{id}_{A_{t}} \in \mathrm{Tl}_{t}(\mathbf{A})_{t}$, to conclude that $x \in L_{t}$ if and only if $y \in L_{t}$, i.e., that $(x, y) \in \operatorname{Ker}\left(\operatorname{ch}_{L}\right)_{t}$. We now proceed to prove (3). Let $\Phi$ be a congruence on $\mathbf{A}$ such that $\Phi \subseteq \operatorname{Ker}\left(\mathrm{ch}_{L}\right)$, i.e., such that, for every $s \in S$ and every $x, y \in A_{s}$, if $(x, y) \in \Phi_{s}$, then $x \in L_{s}$ if and only if $y \in L_{s}$. We want to show that, for every $t \in S, \Phi_{t} \subseteq \Omega^{\mathbf{A}}(L)_{t}$. Let $t$ be an element of $S$ and $(x, y) \in \Phi_{t}$. Then, since $\Phi$ is a congruence on $\mathbf{A}$, for every $s \in S$ and every $T \in \mathrm{Tl}_{t}(\mathbf{A})_{s}$, we have that $(T(x), T(y)) \in \Phi_{s}$. Hence, by the hypothesis on $\Phi, T(x) \in L_{s}$ if and only if $T(y) \in L_{s}$. Therefore $\Phi \subseteq \Omega^{\mathbf{A}}(L)$.
Definition 2.49. Let $\mathbf{A}$ be a $\Sigma$-algebra and $L \subseteq A$. Then we call $\Omega^{\mathbf{A}}(L)$ the congruence on A cogenerated by $L$, or the syntactic congruence on $\mathbf{A}$ determined by $L$ (which is shorthand for "the congruence on $\mathbf{A}$ cogenerated by the equivalence $\operatorname{Ker}\left(\mathrm{ch}_{L}\right)$ on $\mathbf{A}$ canonically associated to $\left.L "\right)$.
Remark. For every subset $L$ of the underlying $S$-sorted set of a $\Sigma$-algebra A, $\Omega^{\mathbf{A}}(L)$ can be viewed as the value at $L$ of a mapping $\Omega^{\mathbf{A}}$ from $\operatorname{Sub}(A)$ to $\operatorname{Cgr}(\mathbf{A})$. We will call $\Omega^{\mathbf{A}}$ the congruence cogenerating operator for $\mathbf{A}$ (with regard to subsets of $A$ ).

With regard to the congruence cogenerated by an equivalence it is worthwhile to quote what Büchi, in [5], on p. 113, wrote: "The notion of induced [ $\equiv$ cogenerated, we add] congruence therefore is clearly relevant to automata theory. For some reason it seems to have escaped the attention of algebraists. In contrast, its mate, the generated congruence, is widely used in algebra."

It may be worth reminding the reader that in the theory of formal languages, a congruence of the type $\Omega^{\mathbf{A}^{\star}}(L)$, where $L$ is a subset of the underlying set of a free monoid $\mathbf{A}^{\star}$ on an alphabet $A$, is called the syntactic congruence determined by $L$ (or the two-sided principal congruence of $L$ ). According to Lallement (in [25], on p. 175): "the fact that the principal equivalence of $L$ is the largest congruence for which $L$ is a union of equivalence classes is due to Teissier in [39] [for demi-groups, i.e., sets with an associative binary operation, we add]." To this we append that these congruences were defined by Schützenberger (in [36], on p. 10) for monoids (he speaks of: "demi-groupes contenant un élément neutre"). It should also be pointed out that, for a single-sorted algebra $\mathbf{A}$ and for an equivalence relation $\Phi$ on $A$, in [20], on p. 65, Green defined the congruence on A analysed ( $\equiv$ cogenerated, we add) by $\Phi$ as the join ( $\equiv$ supremum) of the set of all congruences $\Psi$ on $\mathbf{A}$ contained in $\Phi$ and asserted that it is the greatest congruence on $\mathbf{A}$ contained in $\Phi$; and in [37], on pp. 32-33, Słomiński proved that there exists the greatest congruence on A contained in $\Phi$. In this respect it should be noted that in [1], and for singlesorted algebras, Almeida addressed, in particular, the issue of the definition and basic properties of the syntactic congruence.

The theorem of Green and Słomiński is also valid for the many-sorted case, as shown below.

Proposition 2.50. Let A be a $\Sigma$-algebra and $\Phi$ an $S$-sorted equivalence on $A$. Then there exists a congruence $\widetilde{\Omega}^{\mathbf{A}}(\Phi)$ on $\mathbf{A}$ such that $\widetilde{\Omega}^{\mathbf{A}}(\Phi) \subseteq \Phi$ and, for every congruence $\Psi$ on $\mathbf{A}$, if $\Psi \subseteq \Phi$, then $\Psi \subseteq \widetilde{\Omega}^{\mathbf{A}}(\Phi)$. We will call $\widetilde{\Omega}^{\mathbf{A}}(\Phi)$ the congruence on A cogenerated by $\Phi$.

Proof. Since: (1) the join in $\mathbf{E q v}(A)$ of a family $\left(\Psi^{i}\right)_{i \in I}$ of congruences on $\mathbf{A}$ is the $S$-sorted equivalence $\Psi$ on $A$ whose $s$-th coordinate, $\Psi_{s}$, for $s \in S$, is:

$$
\Psi_{s}=\left\{\begin{array}{l|c}
(a, b) \in A_{s}^{2} & \begin{array}{c}
\exists n \geq 1 \exists x \in A_{s}^{n+1}\left(x_{0}=a \& x_{n}=b \quad \&\right. \\
\left.\forall p \in n\left(\left(x_{p}, x_{p+1}\right) \in \bigcup_{i \in I} \Psi_{s}^{i}\right)\right)
\end{array}
\end{array}\right\}
$$

(2) for the join $\Psi$ in $\operatorname{Eqv}(A)$ of a family $\left(\Psi^{i}\right)_{i \in I}$ of congruences on A, we have that, for every $(w, s) \in\left(S^{\star}-\{\lambda\}\right) \times S$, every $\sigma \in \Sigma_{w, s}$, every $j \in|w|$, every $a$, $b \in A_{w_{j}}$, every $\left(c_{i}\right)_{i \in j} \in \prod_{i \in j} A_{w_{i}}$, and every $\left(c_{k}\right)_{k \in|w|-(j+1)} \in \prod_{k \in|w|-(j+1)} A_{w_{k}}$, if $(a, b) \in \Psi_{w_{j}}$, then

$$
\left(F_{\sigma}\left(c_{0}, \ldots, c_{j-1}, a, c_{j+1}, \ldots, c_{|w|-1}\right), F_{\sigma}\left(c_{0}, \ldots, c_{j-1}, b, c_{j+1}, \ldots, c_{|w|-1}\right)\right) \in \Psi_{s}
$$

(3) the join in $\operatorname{Eqv}(A)$ of a family $\left(\Psi^{i}\right)_{i \in I}$ of congruences on $\mathbf{A}$ is a congruence on $\mathbf{A}$ (this follows from (2)); and (4) $\Delta_{\mathbf{A}}$ is a congruence on $\mathbf{A}$ contained in $\Phi$, it suffices to take as $\widetilde{\Omega}^{\mathbf{A}}(\Phi)$ the join of $\{\Psi \in \operatorname{Cgr}(\mathbf{A}) \mid \Psi \subseteq \Phi\}$ in $\mathbf{E q v}(A)$.

Remark. For every $S$-sorted equivalence $\Phi$ on the underlying $S$-sorted set of a $\Sigma$ algebra $\mathbf{A}, \widetilde{\Omega}^{\mathbf{A}}(\Phi)$ can be viewed as the value at $\Phi$ of a mapping $\widetilde{\Omega}^{\mathbf{A}}$ from $\operatorname{Eqv}(A)$ to $\operatorname{Cgr}(\mathbf{A})$. We will call $\widetilde{\Omega}^{\mathbf{A}}$ the congruence cogenerating operator for $\mathbf{A}$ (with regard to $S$-sorted equivalences on $A$ ).

Remark. For every subset $L$ of the underlying $S$-sorted set of a $\Sigma$-algebra A, $\Omega^{\mathbf{A}}(L)=\widetilde{\Omega}^{\mathbf{A}}\left(\operatorname{Ker}\left(\operatorname{ch}_{L}\right)\right)$. In other words, the mapping $\Omega^{\mathbf{A}}$ from $\operatorname{Sub}(A)$ to $\operatorname{Cgr}(\mathbf{A})$ is the composition of the mapping from $\operatorname{Sub}(A)$ to $\operatorname{Eqv}(A)$ that sends $L$ in $\operatorname{Sub}(A)$ to $\operatorname{Ker}\left(\operatorname{ch}_{L}\right)$ in $\operatorname{Eqv}(A)$ (which in general is neither injecive, nor surjective, nor isotone) and the mapping $\widetilde{\Omega}^{\mathbf{A}}$ from $\operatorname{Eqv}(A)$ to $\operatorname{Cgr}(\mathbf{A})$.

Remark. It is possible to provide a proof of Proposition 2.50 along the lines of Proposition 2.48 Let $\Phi$ be an $S$-sorted equivalence on the underlying $S$-sorted set of a $\Sigma$-algebra $\mathbf{A}$. Then the $S$-sorted binary relation $\widetilde{\Omega}^{\mathbf{A}}(\Phi)$ on $A$ defined, for every $t \in S$, as follows:

$$
\widetilde{\Omega}^{\mathbf{A}}(\Phi)_{t}=\left\{\begin{array}{l|l}
(x, y) \in A_{t}^{2} & \begin{array}{c}
\forall s \in S \forall T \in \mathrm{Tl}_{t}(\mathbf{A})_{s} \\
\left.(T(x), T(y)) \in \Phi_{s}\right)
\end{array}
\end{array}\right\}
$$

is the greatest congruence on $\mathbf{A}$ contained in $\Phi$.
We next provide, for a $\Sigma$-algebra $\mathbf{A}$, some basic properties of the congruence cogenerating operator $\widetilde{\Omega}^{\mathbf{A}}$.

Proposition 2.51. Let $\mathbf{A}$ be a $\Sigma$-algebra. Then $\widetilde{\Omega}^{\mathbf{A}}$, considered as an endomapping of $\operatorname{Eqv}(A)$, is a kernel ( $\equiv$ interior) operator, i.e., it is contractive ( $\equiv$ deflationary), isotone, and idempotent. Moreover, $\widetilde{\Omega}^{\mathbf{A}}\left(\Delta_{\mathbf{A}}\right)=\Delta_{\mathbf{A}}, \widetilde{\Omega}^{\mathbf{A}}\left(\nabla_{\mathbf{A}}\right)=\nabla_{\mathbf{A}}$ and, for every nonempty set $I$ in $\mathcal{U}$ and every $\left(\Phi^{i}\right)_{i \in I} \in \operatorname{Eqv}(A)^{I}, \widetilde{\Omega}^{\mathbf{A}}\left(\bigcap_{i \in I} \Phi^{i}\right)=\bigcap_{i \in I} \widetilde{\Omega}^{\mathbf{A}}\left(\Phi^{i}\right)$.
Proposition 2.52. Let A be a $\Sigma$-algebra, $\Phi$ an $S$-sorted equivalence on $A, t, s \in S$, and $T \in \mathrm{Tl}_{t}(\mathbf{A})_{s}$. Then $\Omega^{\mathbf{A}}(\Phi) \subseteq \Omega^{\mathbf{A}}\left((T \times T)^{-1}[\Phi]\right)$, where $(T \times T)^{-1}[\Phi]$ stands for the $S$-sorted equivalence on $A$ defined, for every $u \in S$, as follows:

$$
\Omega^{\mathbf{A}}\left((T \times T)^{-1}[\Phi]\right)_{u}=\left\{\begin{array}{l}
(T \times T)^{-1}\left[\Phi_{s}\right], \text { if } u=t \\
\nabla_{A_{u}}, \text { otherwise } .
\end{array}\right.
$$

Proposition 2.53. Let $f$ be a homomorphism from $\mathbf{A}$ to $\mathbf{B}$ and $\Upsilon$ an $S$-sorted equivalence on $B$. Then $(f \times f)^{-1}\left[\widetilde{\Omega}^{\mathbf{B}}(\Upsilon)\right] \subseteq \widetilde{\Omega}^{\mathbf{A}}\left((f \times f)^{-1}[\Upsilon]\right)$. Moreover, if $f$ is an epimorphism, then $(f \times f)^{-1}\left[\widetilde{\Omega}^{\mathbf{B}}(M)\right]=\widetilde{\Omega}^{\mathbf{A}}\left((f \times f)^{-1}[\Upsilon]\right)$.
Remark. For every $\Sigma$-algebra $\mathbf{A}, \widetilde{\Omega}^{\mathbf{A}}$ can be regarded as the component at $\mathbf{A}$ of a natural transformation $\widetilde{\Omega}$ between two contravariant functors from a suitable category of $\Sigma$-algebras to the category Set. In fact, let $\operatorname{Alg}(\Sigma)_{\text {epi }}$ be the category whose objects are the $\Sigma$-algebras and whose morphisms are the epimorphisms between $\Sigma$-algebras. Then we have, on the one hand, the functor Eqv from $\operatorname{Alg}(\Sigma)_{\mathrm{epi}}^{\mathrm{op}}$, the dual of $\operatorname{Alg}(\Sigma)_{\text {epi }}$, to Set which assigns to a $\Sigma$-algebra $\mathbf{A}$ the set $\operatorname{Eqv}(A)$, and to an epimorphism $f: \mathbf{A} \longrightarrow \mathbf{B}$ the mapping $(f \times f)^{-1}[\cdot]$ from $\operatorname{Eqv}(B)$ to $\operatorname{Eqv}(A)$, and, on the other hand, the functor Cgr from $\operatorname{Alg}(\Sigma)_{\mathrm{epi}}^{\mathrm{op}}$ to Set which assigns to a $\Sigma$-algebra $\mathbf{A}$ the set $\operatorname{Cgr}(\mathbf{A})$, and to an epimorphism $f: \mathbf{A} \longrightarrow \mathbf{B}$ the mapping $(f \times f)^{-1}[\cdot]$ from $\operatorname{Cgr}(\mathbf{B})$ to $\operatorname{Cgr}(\mathbf{A})$. Then the mapping $\widetilde{\Omega}$ from $\operatorname{Alg}(\Sigma)$, the set of objects of $\operatorname{Alg}(\Sigma)$, to $\operatorname{Mor}(\mathbf{S e t})$, the set of morphisms of Set, which assigns to a $\Sigma$-algebra $\mathbf{A}$ the mapping $\widetilde{\Omega}^{\mathbf{A}}$ from $\operatorname{Eqv}(A)$ to $\operatorname{Cgr}(\mathbf{A})$ is a natural transformation from Eqv to Cgr, because, for every epimorphism $f: \mathbf{A} \longrightarrow \mathbf{B}$, we have that $(f \times f)^{-1}[\cdot] \circ \widetilde{\Omega}^{\mathbf{B}}=\widetilde{\Omega}^{\mathbf{A}} \circ(f \times f)^{-1}[\cdot]$, i.e., for every $S$-sorted equivalence $\Upsilon$ on $B$, $(f \times f)^{-1}\left[\widetilde{\Omega}^{\mathbf{B}}(\Upsilon)\right]=\widetilde{\Omega}^{\mathbf{A}}\left((f \times f)^{-1}[\Upsilon]\right)$.

We next gather together, for a $\Sigma$-algebra $\mathbf{A}$, some basic properties of the congruence cogenerating operator $\Omega^{\mathbf{A}}$.
Proposition 2.54. Let $\mathbf{A}$ be a $\Sigma$-algebra, $L$ a subset of $A$, and $\Phi \in \operatorname{Cgr}(\mathbf{A})$. Then $L \in \Phi-\operatorname{Sat}(A)$, i.e., $L=[L]^{\Phi}$, if and only if $\Phi \subseteq \Omega^{\mathbf{A}}(L)$. Moreover, for $s \in S$ and $L \subseteq A_{s}$, we have that $L=[L]^{\Phi_{s}}$ if and only if $\Phi_{s} \subseteq \Omega^{\mathbf{A}}\left(\delta^{s, L}\right)_{s}$.
Proof. Let us suppose that $L=[L]^{\Phi}$. Then, since $\Omega^{\mathbf{A}}(L)$ is the greatest congruence on $\mathbf{A}$ such that $L=[L]^{\Omega^{\mathbf{A}}(L)}$, we have that $\Phi \subseteq \Omega^{\mathbf{A}}(L)$.

Reciprocally, let us suppose that $\Phi \subseteq \Omega^{\mathbf{A}}(\bar{L})$ then, by Corollary 2.16, we have that $\Omega^{\mathbf{A}}(L)$-Sat $(A) \subseteq \Phi-\operatorname{Sat}(A)$. Since $L$ belongs to $\Omega^{\mathbf{A}}(L)$-Sat $(A)$, we conclude that $L \in \Phi-\operatorname{Sat}(A)$.

Proposition 2.55. Let $\mathbf{A}$ be a $\Sigma$-algebra and $L$ a subset of $A$. Then it happens that $\Omega^{\mathbf{A}}(L)=\Omega^{\mathbf{A}}\left(\complement_{A} L\right)$.
Proposition 2.56. Let $\mathbf{A}$ be a $\Sigma$-algebra, $I$ a nonempty set in $\mathcal{U}$, and $\left(L^{i}\right)_{i \in I}$ an I-indexed family of subsets of $A$. Then $\bigcap_{i \in I} \Omega^{\mathbf{A}}\left(L^{i}\right) \subseteq \Omega^{\mathbf{A}}\left(\bigcap_{i \in I} L^{i}\right)$, i.e., $\bigcap_{i \in I} \widetilde{\Omega}^{\mathbf{A}}\left(\operatorname{Ker}\left(\operatorname{ch}_{L^{i}}\right)\right) \subseteq \widetilde{\Omega}^{\mathbf{A}}\left(\operatorname{Ker}\left(\operatorname{ch}_{\bigcap_{i \in I} L^{i}}\right)\right)$.
Proof. To prove that $\bigcap_{i \in I} \Omega^{\mathbf{A}}\left(L^{i}\right) \subseteq \Omega^{\mathbf{A}}\left(\bigcap_{i \in I} L^{i}\right)$ it suffices, by part (3) of Proposition 2.48, to verify that $\bigcap_{i \in I} \Omega^{\mathbf{A}}\left(L^{i}\right) \subseteq \operatorname{Ker}\left(\operatorname{ch}_{\bigcap_{i \in I} L^{i}}\right)$. But $\bigcap_{i \in I} \Omega^{\mathbf{A}}\left(L^{i}\right) \subseteq$ $\bigcap_{i \in I} \operatorname{Ker}\left(\mathrm{ch}_{L^{i}}\right)$ and, in addition, we have that $\bigcap_{i \in I} \operatorname{Ker}\left(\operatorname{ch}_{L^{i}}\right) \subseteq \operatorname{Ker}\left(\mathrm{ch}_{\bigcap_{i \in I} L^{i}}\right)$. Therefore $\bigcap_{i \in I} \Omega^{\mathbf{A}}\left(L^{i}\right) \subseteq \operatorname{Ker}\left(\operatorname{ch}_{\bigcap_{i \in I} L^{i}}\right)$.

With regard to the following proposition, it must be borne in mind that, for every set $C, \operatorname{Ker}\left(\operatorname{ch}_{\varnothing}\right)=\operatorname{Ker}\left(\operatorname{ch}_{C}\right)=\nabla_{C}$.
Proposition 2.57. Let A be a $\Sigma$-algebra, $L$ a subset of $A$, $t, s \in S$, and $T \in$ $\mathrm{Tl}_{t}(\mathbf{A})_{s}$. Then $\Omega^{\mathbf{A}}(L) \subseteq \Omega^{\mathbf{A}}\left(T^{-1}[L]\right)$, i.e., $\widetilde{\Omega}^{\mathbf{A}}\left(\operatorname{Ker}\left(\operatorname{ch}_{L}\right)\right) \subseteq \widetilde{\Omega}^{\mathbf{A}}\left(\operatorname{Ker}\left(\operatorname{ch}_{T^{-1}[L]}\right)\right)$, where, for every $u \in S-\{t\}, \operatorname{Ker}\left(\operatorname{ch}_{T^{-1}[L]}\right)_{u}=\operatorname{Ker}\left(\operatorname{ch}_{\varnothing}\right)=\nabla_{A_{u}}$ and

$$
\operatorname{Ker}\left(\operatorname{ch}_{T^{-1}[L]}\right)_{t}=\operatorname{Ker}\left(\operatorname{ch}_{T^{-1}\left[L_{s}\right]}\right)=\left\{(x, y) \in A_{t}^{2} \mid T(x) \in L_{s} \leftrightarrow T(y) \in L_{s}\right\} .
$$

Proposition 2.58. Let $f$ be a homomorphism from $\mathbf{A}$ to $\mathbf{B}$ and $M$ a subset of $B$. Then $(f \times f)^{-1}\left[\Omega^{\mathbf{B}}(M)\right] \subseteq \Omega^{\mathbf{A}}\left(f^{-1}[M]\right)$. Moreover, if $f$ is an epimorphism, then $(f \times f)^{-1}\left[\Omega^{\mathbf{B}}(M)\right]=\Omega^{\mathbf{A}}\left(f^{-1}[M]\right)$.

Remark. Let $\mathrm{P}^{-}$be the functor from $\operatorname{Alg}(\Sigma)_{\text {epi }}^{\mathrm{op}}$ to Set which assigns to a $\Sigma$ algebra $\mathbf{A}$ the set $\operatorname{Sub}(A)$, and to an epimorphism $f: \mathbf{A} \longrightarrow \mathbf{B}$ the mapping $f^{-1}[\cdot]$ from $\operatorname{Sub}(B)$ to $\operatorname{Sub}(A)$. Then the mapping $\Omega$ from $\operatorname{Alg}(\Sigma)$ to $\operatorname{Mor}(\operatorname{Set})$ which assigns to a $\Sigma$-algebra $\mathbf{A}$ the mapping $\Omega^{\mathbf{A}}$ from $\operatorname{Sub}(A)$ to $\operatorname{Cgr}(\mathbf{A})$ is a natural transformation from $\mathrm{P}^{-}$to Cgr , since, for every epimorphism $f: \mathbf{A} \longrightarrow \mathbf{B}$, we have that $(f \times f)^{-1}[\cdot] \circ \Omega^{\mathbf{B}}=\Omega^{\mathbf{A}} \circ f^{-1}[\cdot]$, i.e., for every $M \subseteq B,(f \times f)^{-1}\left[\Omega^{\mathbf{B}}(M)\right]=$ $\Omega^{\mathbf{A}}\left(f^{-1}[M]\right)$.

We finish this section by reviewing a few aspects of recognizability for subsets of the underlying many-sorted set of an arbitrary many-sorted algebra. But before going any further it is worth noting what Gécseg and Steinby, in [14], at the beginning of Chapter 2, wrote: "... one should note that there are often many ways to generalize from languages [sets of words of the underlying set of a free monoid on a set, we add] to forest [sets of terms of the underlying set of a free algebra on a set, we add], and a right choice among the alternatives is essential if one wants to generalize the corresponding results, too." In this regard, concerning the congruences on a many-sorted algebra-on which, ultimately, the notion of recognizability will be founded-we have two nonequivalent ways of defining the concept of congruence of "finite" index on it, depending on the notion of finiteness we choose: The categorial or the non-categorial notion of finiteness.

We shall now go on to define both notions of congruence of finite index on a many-sorted algebra.
Definition 2.59. Let $\mathbf{A}$ be a $\Sigma$-algebra and $\Phi \in \operatorname{Cgr}(\mathbf{A})$. We will say that $\Phi$ is of finite index, abbreviated as fi, if $A / \Phi \in \operatorname{Sub}_{\mathrm{f}}\left(A^{\wp}\right)$, i.e., if $\operatorname{card}\left(\operatorname{supp}_{S}(A / \Phi)\right)$ is finite and, for every $s \in \operatorname{supp}_{S}(A / \Phi), A_{s} / \Phi_{s}$ is finite. We will denote by $\operatorname{Cgr}_{\mathrm{fi}}(\mathbf{A})$ the set of all congruences on $\mathbf{A}$ of finite index. Moreover, we will say that $\Phi$ is of $S$-finite index or of locally finite index, abbreviated as lfi, if $A / \Phi \in \operatorname{Sub}_{\mathrm{lf}}\left(A^{\wp}\right)$ i.e., if, for every $s \in S, \operatorname{card}\left(A_{s} / \Phi_{s}\right)<\aleph_{0}$. We will denote by $\operatorname{Cgr}_{1 \mathrm{lfi}}(\mathbf{A})$ the set of all congruences on $\mathbf{A}$ of locally finite index.

Let us note that in 69, on p. 99, and in [10, on p. 30, and for a set of sorts $S$, eventually infinite, and a many-sorted algebra $\mathbf{A}$ such that, for every $s \in S$, $A_{s} \neq \varnothing$, Courcelle says that a congruence on $\mathbf{A}$ is locally finite if it has finitely many classes of each sort. To this we add that he uses such a type of congruence to investigate, for a many-sorted algebra $\mathbf{A}$, subject to satisfy the above condition, and a sort $s \in S$, the recognizable subsets of $A_{s}$. So, if we disregard the condition imposed by Courcelle on the many-sorted algebras, our notion of congruence of locally finite index coincides with Courcelle's notion of locally finite congruence.
Proposition 2.60. Let $\mathbf{A}$ be a $\Sigma$-algebra. Then $\operatorname{Cgr}_{\mathrm{fi}}(\mathbf{A}) \neq \varnothing$ if and only if $\operatorname{supp}_{S}(\mathbf{A})$ is finite. Moreover, $\operatorname{card}(S)<\aleph_{0}$ if and only if, for every $\Sigma$-algebra $\mathbf{A}$, $\operatorname{Cgr}_{\mathrm{f}}(\mathbf{A}) \neq \varnothing$. Therefore, the following conditions are equivalent: (1) for every $\Sigma$-algebra $\mathbf{A}, \operatorname{supp}_{S}(\mathbf{A})$ is finite, (2) for every $\Sigma$-algebra $\mathbf{A}, \operatorname{Cgr}_{\mathrm{fi}}(\mathbf{A}) \neq \varnothing$, and (3) $S$ is finite.
Proof. Since the first assertion is straightforward, we restrict ourselves to verify the second one. If the set of sorts $S$ is finite, then, for every $\Sigma$-algebra $\mathbf{A}, \nabla_{\mathbf{A}} \in$ $\operatorname{Cgr}_{\mathrm{fi}}(\mathbf{A})$. If $S$ is infinite, then the final $\Sigma$-algebra $\mathbf{1}$ is such that $\operatorname{Cgr}_{\mathrm{fi}}(\mathbf{1})=\varnothing$.
Remark. If $S$ is finite, then, obviously, for every $X \in \mathcal{U}^{S}, \operatorname{supp}_{S}\left(\mathbf{T}_{\Sigma}(X)\right)$ is finite. If $S$ is infinite, then there exists an $X \in \mathcal{U}^{S}$ such that $\operatorname{supp}_{S}\left(\mathbf{T}_{\Sigma}(X)\right)$ is infinite, e.g., for $X=1=(1)_{s \in S}$, we have that $\operatorname{supp}_{S}\left(\mathbf{T}_{\Sigma}(1)\right)=S$, thus $\operatorname{supp}_{S}\left(\mathbf{T}_{\Sigma}(1)\right)$ is infinite. Hence, if, for every $X \in \mathcal{U}^{S}, \operatorname{supp}_{S}\left(\mathbf{T}_{\Sigma}(X)\right)$ is finite, then $S$ is finite. Therefore, the following conditions are equivalent: (1) for every $X \in \mathcal{U}^{S}$, $\operatorname{supp}_{S}\left(\mathbf{T}_{\Sigma}(X)\right)$ is finite, (2) for every $X \in \mathcal{U}^{S}, \operatorname{Cgr}_{\mathrm{f}}\left(\mathbf{T}_{\Sigma}(X)\right) \neq \varnothing$, and (3) $S$ is finite.

Proposition 2.61. Let A be a $\Sigma$-algebra. Then
(1) for every $n \in \mathbb{N}-\{0\}$ and every $\left(\Phi^{i}\right)_{i \in n} \in \operatorname{Cgr}_{\mathrm{fi}}(\mathbf{A})^{n}, \bigcap_{i \in n} \Phi^{i} \in \operatorname{Cgr}_{\mathrm{fi}}(\mathbf{A})$;
(2) $\operatorname{Cgr}_{\mathrm{fi}}(\mathbf{A})$ is an upward closed set of the lattice $\mathbf{C g r}(\mathbf{A})$, i.e., for every $\Phi, \Psi \in$ $\operatorname{Cgr}(\mathbf{A})$, if $\Phi \subseteq \Psi$ and $\Phi \in \operatorname{Cgr}_{\mathrm{fi}}(\mathbf{A})$, then $\Psi \in \operatorname{Cgr}_{\mathrm{fi}}(\mathbf{A})$.
Therefore, if $\operatorname{supp}_{S}(\mathbf{A})$ is finite, then $\operatorname{Cgr}_{\mathrm{f}}(\mathbf{A})$ is a filter of the lattice $\operatorname{Cgr}(\mathbf{A})$.
Proof. Let $n$ be a non-zero natural number and $\left(\Phi^{i}\right)_{i \in n} \in \operatorname{Cgr}_{\mathrm{fi}}(\mathbf{A})^{n}$. Then the congruence $\bigcap_{i \in n} \Phi^{i}$ on $\mathbf{A}$ is of finite index because $\prod_{i \in n} \mathbf{A} / \Phi^{i}$ is finite and the homomorphism $\mathrm{p}^{\left(\Phi^{i}\right)_{i \in n}}$ from $\mathbf{A} / \bigcap_{i \in n} \Phi^{i}$ to $\prod_{i \in n} \mathbf{A} / \Phi^{i}$, defined, for every $s \in S$, as follows:

$$
\mathrm{p}_{s}^{\left(\Phi^{i}\right)_{i \in n}}\left\{\begin{aligned}
A_{s} / \bigcap_{i \in n} \Phi_{s}^{i} & \longrightarrow \prod_{i \in n} A_{s} / \Phi_{s}^{i} \\
{[a]_{\bigcap_{i \in n} \Phi_{s}^{i}} } & \longmapsto\left([a]_{\Phi_{s}^{i}}\right)_{i \in n}
\end{aligned}\right.
$$

is a subdirect embedding of $\mathbf{A} / \bigcap_{i \in n} \Phi^{i}$ into $\prod_{i \in n} \mathbf{A} / \Phi^{i}$. Let us note that $\mathrm{p}^{\left(\Phi^{i}\right)_{i \in n}}$ is $\left\langle\mathrm{p}_{i \in n} \Phi^{i}, \Phi^{i}\right\rangle_{i \in n}$ (the unique homomorphism from $\mathbf{A} / \bigcap_{i \in n} \Phi^{i}$ to $\prod_{i \in n} \mathbf{A} / \Phi^{i}$ such that, for every $i \in n, \operatorname{pr}_{\Phi^{i}} \circ\left\langle\mathrm{p}_{\bigcap_{i \in n} \Phi^{i}, \Phi^{i}}\right\rangle_{i \in n}=\mathrm{p}_{\bigcap_{i \in n} \Phi^{i}, \Phi^{i}}$, where $\mathrm{p}_{\bigcap_{i \in n} \Phi^{i}, \Phi^{i}}$ and $\mathrm{pr}_{\Phi^{i}}$ are the canonical homomorphisms from $\mathbf{A} / \bigcap_{i \in n} \Phi^{i}$ and $\prod_{i \in n} \mathbf{A} / \Phi^{i}$, respectively, to $\mathbf{A} / \Phi^{i}$ ).

Now let $\Phi$ be a congruence on $\mathbf{A}$ of finite index and $\Psi$ a congruence on $\mathbf{A}$ such that $\Phi \subseteq \Psi$. Then, since $\mathbf{A} / \Phi$ is finite and $\operatorname{pr}_{\Phi, \Psi}$ (the unique homomorphism from $\mathbf{A} / \Phi$ to $\mathbf{A} / \Psi$ such that $\operatorname{pr}_{\Phi, \Psi} \circ \operatorname{pr}_{\Phi}=\operatorname{pr}_{\Psi}$ ) is surjective, $\mathbf{A} / \Psi$ is finite. Hence $\Psi \in \operatorname{Cgr}_{\mathrm{fi}}(\mathbf{A})$.

Proposition 2.62. Let $\mathbf{A}$ be a $\Sigma$-algebra. Then $\operatorname{Cgr}_{1 \mathrm{lf}}(\mathbf{A})$ is a filter of the lattice $\mathbf{C g r}(\mathrm{A})$.

As for congruences of finite index on a many-sorted algebra, for many-sorted languages we have, on the one hand, those whose definition is based on the categorial notion of finiteness and which will be called recognizable, and, on the other, those whose definition is founded on the notion of local finiteness and which will be called locally finitely recognizable, abbreviated to lf recognizable.

Definition 2.63. Let $\mathbf{A}$ be a $\Sigma$-algebra, $T \subseteq S$, and $L \subseteq A \upharpoonright_{T}=\left(A_{t}\right)_{t \in T}$.
(1) We will say that $L$ is $T$-recognizable if there exists a finite $\Sigma$-algebra $\mathbf{B}$, a homomorphism $f: \mathbf{A} \longrightarrow \mathbf{B}$, and a subset $M$ of $B \upharpoonright_{T}$ such that $\left(f \upharpoonright_{T}\right)^{-1}[M]=$ $L$, where $\left(f \upharpoonright_{T}\right)^{-1}[M]=\left(f_{t}^{-1}\left[M_{t}\right]\right)_{t \in T}$. We will denote by $\operatorname{Rec}_{T}(\mathbf{A})$ the set of all subsets of $A \upharpoonright_{T}$ which are $T$-recognizable.
(2) We will say that $L$ is (lf, $T$ )-recognizable if there exists a locally finite ( $\equiv$ $S$-finite) $\Sigma$-algebra $\mathbf{B}$, a homomorphism $f: \mathbf{A} \longrightarrow \mathbf{B}$, and a subset $M$ of $B \upharpoonright_{T}$ such that $L=\left(\left(f \upharpoonright_{T}\right)^{-1}[M]\right.$. We will denote by $\operatorname{Rec}_{\mathrm{lf}, T}(\mathbf{A})$ the set of all subsets of $A \upharpoonright_{T}$ which are (lf, $T$ )-recognizable.
For $T=S$ we will denote by $\operatorname{Rec}(\mathbf{A})$ and $\operatorname{Rec}_{\mathrm{lf}}(\mathbf{A})$ the sets $\operatorname{Rec}_{S}(\mathbf{A})$ and $\operatorname{Rec}_{\text {lf }, S}(\mathbf{A})$, respectively. We will call the elements of $\operatorname{Rec}(\mathbf{A})$ and $\operatorname{Rec}_{\text {lf }}(\mathbf{A})$ recognizable and lf-recognizable, respectively.

For $s \in S$ we will denote by $\operatorname{Rec}_{s}(\mathbf{A})$ and $\operatorname{Rec}_{\text {lf }, s}(\mathbf{A})$ the sets $\operatorname{Rec}_{\{s\}}(\mathbf{A})$ and $\operatorname{Rec}_{l f,\{s\}}(\mathbf{A})$, respectively. We will call the elements of $\operatorname{Rec}_{s}(\mathbf{A})$ and $\operatorname{Rec}_{\mathrm{lf}, s}(\mathbf{A})$ $s$-recognizable and (lf, s)-recognizable, respectively.

Remark. Let A be a $\Sigma$-algebra, $T \subseteq S$, and $s \in S$. Then $\operatorname{Rec}_{T}(\mathbf{A}) \subseteq \operatorname{Rec}_{\text {lf }, T}(\mathbf{A})$, $\operatorname{Rec}(\mathbf{A}) \subseteq \operatorname{Rec}_{\mathrm{lf}}(\mathbf{A})$, and $\operatorname{Rec}_{s}(\mathbf{A}) \subseteq \operatorname{Rec}_{\mathrm{lf}, s}(\mathbf{A})$. If $S$ is infinite, then the converse inclusions are not valid. Let $T$ be an infinite subset of $S$ and $\mathbf{A}$ a $\Sigma$-algebra such that $T \subseteq \operatorname{supp}_{S}(\mathbf{A})$. Then $A \upharpoonright_{T}$ is a language in $\operatorname{Rec}_{l . f, T}(\mathbf{A})$, but not in $\operatorname{Rec}_{T}(\mathbf{A})$.

Remark. If $S$ is finite, $T \subseteq S, s \in S$, and $\mathbf{A}$ is a $\Sigma$-algebra, then $\operatorname{Rec}_{T}(\mathbf{A})=$ $\operatorname{Rec}_{\mathrm{lf}, T}(\mathbf{A}), \operatorname{Rec}(\mathbf{A})=\operatorname{Rec}_{\mathrm{lf}}(\mathbf{A})$, and $\operatorname{Rec}_{s}(\mathbf{A})=\operatorname{Rec}_{\mathrm{lf}, s}(\mathbf{A})$.

In what follows, among other things, we investigate, for a many-sorted algebra and a language of it, the relationships between the different notions of recognizability for the language and the two notions of congruence of finite index on the many-sorted algebra saturating the language. In particular, we investigate such a relationship for the case of the congruence cogenerated by a many-sorted language.

Proposition 2.64. Let $\mathbf{A}$ be a $\Sigma$-algebra, $T \subseteq S$, and $L \subseteq A \upharpoonright_{T}$. Then the following assertions are equivalent:
(1) $L$ is $T$-recognizable.
(2) There exists a congruence $\Phi$ on $\mathbf{A}$ of finite index such that $L=[L]^{\Phi \dagger_{T}}$, where $\left.\Phi\right|_{T}=\left(\Phi_{t}\right)_{t \in T}$ and, for every $t \in T,[L]_{t}^{\Phi_{T} T}=\left[L_{t}\right]^{\Phi_{t}}$.
(3) $\Omega^{\mathbf{A}}\left(\left[L, \varnothing^{S-T}\right]\right)$ is of finite index, where $\varnothing^{S-T}{ }^{\text {is }}$ is the mapping from $S-T$ to $\mathcal{U}$ constantly $\varnothing$ and $\left[L, \varnothing^{S-T}\right]$ the unique mapping from $S$ to $\mathcal{U}$ such that $\left[L, \varnothing^{S-T}\right] \upharpoonright_{T}=L$ and $\left[L, \varnothing^{S-T}\right] \Gamma_{S-T}=\varnothing^{S-T}$.

Proposition 2.65. Let A be a $\Sigma$-algebra and $L \subseteq A$. Then the following assertions are equivalent:
(1) $L$ is recognizable.
(2) There exists a congruence $\Phi$ on $\mathbf{A}$ of finite index such that $L=[L]^{\Phi}$.
(3) $\Omega^{\mathbf{A}}(L)$ is of finite index.

Proof. We first prove that $(1) \Rightarrow(2)$.
Let us suppose that $L$ is recognizable, i.e., that there exists a finite $\Sigma$-algebra $\mathbf{B}$, a homomorphism $f: \mathbf{A} \longrightarrow \mathbf{B}$, and a subset $M$ of $B$ such that $L=f^{-1}[M]$. Then, since $\operatorname{Im}(f)$ is a subalgebra of a finite algebra, $\mathbf{\operatorname { I m }}(f)$ is finite. Therefore $\mathbf{A} / \operatorname{Ker}(f)$ is finite, because it is canonically isomorphic to $\operatorname{Im}(f)$. Hence $\operatorname{Ker}(f)$ is a congruence on $\mathbf{A}$ of finite index. Let us check that $L=[L]^{\operatorname{Ker}(f)}$. The inclusion $L \subseteq[L]^{\operatorname{Ker}(f)}$ is always true. To show that $[L]^{\operatorname{Ker}(f)} \subseteq L$, let $s$ be a sort in $S, a \in A_{s}$, $x \in L_{s}$, and let us suppose that $(a, x) \in \operatorname{Ker}\left(f_{s}\right)$, i.e., that $f_{s}(a)=f_{s}(x)$. Then, since $L=f^{-1}[M]$, we have that $L_{s}=f_{s}^{-1}\left[M_{s}\right]$, thus $f_{s}(x) \in M_{s}$, but $f_{s}(a)=f_{s}(x)$, therefore $f_{s}(a) \in M_{s}$, consequently $a \in L_{s}$. From this it follows that $[L]^{\operatorname{Ker}(f)} \subseteq L$.

We next prove that $(2) \Rightarrow(3)$.
Let us suppose that there exists a congruence $\Phi$ on $\mathbf{A}$ such that $\Phi$ is of finite index and $L=[L]^{\Phi}$. Then, by Proposition [2.54, we have that $\Phi \subseteq \Omega^{\mathbf{A}}(L)$. Thus, by Proposition 2.61, since, by hypothesis, $\Phi \in \operatorname{Cgr}_{\text {fi }}(\mathbf{A}), \Omega^{\mathbf{A}}(L) \in \operatorname{Cgr}_{\mathrm{fi}}(\mathbf{A})$.

Finally, we prove that $(3) \Rightarrow(1)$.
Let us suppose that $\Omega^{\mathbf{A}}(L) \in \operatorname{Cgr}_{\mathrm{f}}(\mathbf{A})$. Then $\mathbf{A} / \Omega^{\mathbf{A}}(L)$ is finite. Let $\mathcal{M}$ be the subset $\operatorname{pr}_{\Omega^{\mathbf{A}}(L)}[L]$ of $A / \Omega^{\mathbf{A}}(L)$, where $\operatorname{pr}_{\Omega^{\mathbf{A}}(L)}$ is the canonical projection from $\mathbf{A}$ to $\mathbf{A} / \Omega^{\mathbf{A}}(L)$. To verify that $L=\left(\operatorname{pr}_{\Omega^{\mathbf{A}}(L)}\right)^{-1}[\mathcal{M}]$ it suffices to check the inclusion from right to left. Let $s$ be an element of $S$ and $b \in \operatorname{pr}_{\Omega^{\mathbf{A}}(L)_{s}}^{-1}\left[\mathcal{M}_{s}\right]$ (recall that $\left(\operatorname{pr}_{\Omega^{\mathbf{A}}(L)}\right)^{-1}[\mathcal{M}]=\left(\operatorname{pr}_{\Omega^{\mathbf{A}}(L)_{s}}^{-1}\left[\mathcal{M}_{s}\right)_{s \in S}\right)$. Then $\operatorname{pr}_{\Omega^{\mathbf{A}}(L)_{s}}[b] \in \mathcal{M}_{s}$. Hence, there exists an $a \in L_{s}$ such that $[a]_{\Omega^{\mathbf{A}}(L)_{s}}=[b]_{\Omega^{\mathbf{A}}(L)_{s}}$. But, by Proposition $2.54 L$ is $\Omega^{\mathbf{A}}(L)$ saturated and, since $a \in L_{s}$, we have that $b \in L_{s}$. Therefore $\left(\operatorname{pr}_{\Omega^{\mathbf{A}}(L)}\right)^{-1}[\mathcal{M}] \subseteq$ $L$.

Proposition 2.66. Let $\mathbf{A}$ be a $\Sigma$-algebra, $s \in S$, and $L \subseteq A_{s}$. Then the following assertions are equivalent:
(1) $L$ is s-recognizable.
(2) There exists a congruence $\Phi$ on $\mathbf{A}$ of finite index such that $L=[L]^{\Phi_{s}}$.
(3) $\Omega^{\mathbf{A}}\left(\delta^{s, L}\right)$ is of finite index.

Proposition 2.67. Let $\mathbf{A}$ and $\mathbf{B}$ be $\Sigma$-algebras. Then
(1) $\varnothing^{S}, A \in \operatorname{Rec}(\mathbf{A})$.
(2) If $K, L \in \operatorname{Rec}(\mathbf{A})$, then $K \cup L, K \cap L, K-L \in \operatorname{Rec}(\mathbf{A})$.
(3) If $T \in \mathrm{Tl}_{t}(\mathbf{A})_{s}$, and $L \in \operatorname{Rec}(\mathbf{A})$, then $T^{-1}[L]=\delta^{t, T^{-1}}\left[L_{s}\right] \in \operatorname{Rec}(\mathbf{A})$.
(4) If $f: \mathbf{A} \longrightarrow \mathbf{B}$ and $M \in \operatorname{Rec}(\mathbf{B})$, then $f^{-1}[M] \in \operatorname{Rec}(\mathbf{A})$.

Proposition 2.68. Let $\mathbf{A}$ be a $\Sigma$-algebra, $s \in S$, and $L \subseteq A_{s}$. Then $L \in \operatorname{Rec}_{s}(\mathbf{A})$ if and only if $\delta^{s, L} \in \operatorname{Rec}(\mathbf{A})$. Thus $\operatorname{Rec}_{s}(\mathbf{A})$ is isomorphic to a subset of $\operatorname{Rec}(\mathbf{A})$.

Assumption. To prove the following proposition we will assume that the set of sorts $S$ is finite.

Proposition 2.69. Let $\mathbf{A}$ be a $\Sigma$-algebra and $L \subseteq A$. Then $L \in \operatorname{Rec}(\mathbf{A})$ if and only if, for every $s \in S, L_{s} \in \operatorname{Rec}_{s}(\mathbf{A})$. Thus there exists an embedding from $\operatorname{Rec}(\mathbf{A})$ into $\prod_{s \in S} \operatorname{Rec}_{s}(\mathbf{A})$ and $\operatorname{Rec}(\mathbf{A})$ is a subdirect product of $\left(\operatorname{Rec}_{s}(\mathbf{A})\right)_{s \in S}$.

Proof. If $L \in \operatorname{Rec}(\mathbf{A})$, then, from the definitions, it follows that, for every $s \in S$, $L_{s} \in \operatorname{Rec}_{s}(\mathbf{A})$.

Conversely, let us suppose that, for every $s \in S, L_{s} \in \operatorname{Rec}_{s}(\mathbf{A})$. Then, for every $s \in S$, there exists a finite $\Sigma$-algebra $\mathbf{B}^{s}$, a homomorphism $f^{s}: \mathbf{A} \longrightarrow \mathbf{B}^{s}$ and $M_{s} \subseteq B_{s}^{s}$ such that $\left(f_{s}^{s}\right)^{-1}\left[M_{s}\right]=L_{s}$. Let $\prod_{s \in S} \mathbf{B}^{s}$ be the product of $\left(\mathbf{B}^{\mathbf{s}}\right)_{s \in S}$, which is finite-since, for every $s \in S, \mathbf{B}^{s}$ is finite and, by hypothesis, $S$ is finite-, and $\left\langle f^{s}\right\rangle_{s \in S}$ the canonical homomorphism from $\mathbf{A}$ to $\prod_{s \in S} \mathbf{B}^{s}$. Let $N=\left(N_{t}\right)_{t \in S}$ be the subset of $\prod_{s \in S} B^{s}=\left(\prod_{s \in S} B_{t}^{s}\right)_{t \in S}$ defined, for every $t \in S$, up to isomorphism, as: $N_{t}=M_{t} \times \prod_{s \in S-\{t\}} B_{t}^{s}$. Then $\left(\left\langle f^{s}\right\rangle_{s \in S}\right)^{-1}[N]=L$. Consequently, $L$ is recognizable.

Before proceeding any further, we provide another proof of the just proved result. If, for every $s \in S, L_{s} \in \operatorname{Rec}_{s}(\mathbf{A})$, then, for every $s \in S$, by Proposition 2.68, $\delta^{s, L_{s}} \in \operatorname{Rec}(\mathbf{A})$. Therefore, by part (2) of Proposition 2.67 and since, by hypothesis, $S$ is finite, $\bigcup_{s \in S} \delta^{s, L_{s}}=L \in \operatorname{Rec}(\mathbf{A})$.

The mapping from $\operatorname{Rec}(\mathbf{A})$ to $\prod_{s \in S} \operatorname{Rec}_{s}(\mathbf{A})$ that sends $L$ in $\operatorname{Rec}(\mathbf{A})$ to $L$ in $\prod_{s \in S} \operatorname{Rec}_{s}(\mathbf{A})$ is well-defined and injective. Moreover, for every $s \in S$ and every $K \in \operatorname{Rec}_{s}(\mathbf{A})$, the $S$-sorted set $\delta^{s, K}$ is in $\operatorname{Rec}(\mathbf{A})$ and its $s$-th projection is $K$.

From the just stated proposition (which, we recall, has been obtained under the assumption that $S$ is finite) and since, for a $\Sigma$-algebra $\mathbf{A}$ and an $L \subseteq A$, $L=\bigcup_{s \in S} \delta^{s, L_{s}}$, it follows that in order to investigate the languages in $\operatorname{Rec}(\mathbf{A})$ it suffices to investigate, for every $s \in S$, the languages in $\operatorname{Rec}_{s}(\mathbf{A})$.
Proposition 2.70. Let $\mathbf{A}$ and $\mathbf{B}$ be $\Sigma$-algebras and $s, t \in S$. Then
(1) $\varnothing, A_{s} \in \operatorname{Rec}_{s}(\mathbf{A})$.
(2) If $K, L \in \operatorname{Rec}_{s}(\mathbf{A})$, then $K \cup L, K \cap L, K-L \in \operatorname{Rec}_{s}(\mathbf{A})$.
(3) If $T \in \operatorname{Tl}_{t}(\mathbf{A})_{s}$, and $L \in \operatorname{Rec}_{s}(\mathbf{A})$, then $T^{-1}[L] \in \operatorname{Rec}_{t}(\mathbf{A})$.
(4) If $f: \mathbf{A} \longrightarrow \mathbf{B}$ and $M \in \operatorname{Rec}_{s}(\mathbf{B})$, then $f_{s}^{-1}[M] \in \operatorname{Rec}_{s}(\mathbf{A})$.

For the notions of lf-recognizability and congruence of locally finite index there are results similar to those we have just stated for recognizability and congruence of finite index. By way of illustration, here are some examples of it.
Proposition 2.71. Let $\mathbf{A}$ be a $\Sigma$-algebra, $T \subseteq S$, and $L \subseteq A \upharpoonright_{T}$. Then the following assertions are equivalent:
(1) $L$ is If-recognizable.
(2) There exists a congruence $\Phi$ on $\mathbf{A}$ of $S$-finite index such that $L=[L]^{\Phi \upharpoonright_{T}}$, where $\Phi \upharpoonright_{T}=\left(\Phi_{t}\right)_{t \in T}$ and, for every $t \in T,[L]_{t}^{\Phi \upharpoonright_{T}}=\left[L_{t}\right]^{\Phi_{t}}$.
(3) $\Omega^{\mathbf{A}}\left(\left[L, \varnothing^{S-T}\right]\right)$ is of $S$-finite index.

Proposition 2.72. Let $\mathbf{A}$ be a $\Sigma$-algebra, $s \in S$, and $L \subseteq A_{s}$. Then $L$ is (lf, $s$ )recognizable if and only if $\delta^{s, L}$ is lf-recognizable, i.e., $L \in \operatorname{Rec}_{\mathrm{lf}, s}(\mathbf{A})$ if and only if $\delta^{s, L} \in \operatorname{Rec}_{\mathrm{lf}}(\mathbf{A})$. Thus $\operatorname{Rec}_{\mathrm{lf}, s}(\mathbf{A})$ is isomorphic to a subset of $\operatorname{Rec}_{\mathrm{lf}}(\mathbf{A})$.

Proposition 2.73. Let $\mathbf{A}$ be a $\Sigma$-algebra and $L \subseteq A$. Then the following assertions are equivalent:
(1) $L \in \operatorname{Rec}_{\mathrm{lf}}(\mathbf{A})$.
(2) There exists a congruence $\Phi$ on $\mathbf{A}$ of $S$-finite index such that $L=[L]^{\Phi}$.
(3) $\Omega^{\mathbf{A}}(L) \in \operatorname{Cgr}_{l \mathrm{fi}}(\mathbf{A})$.

## 3. Recognizable subsets of a free many-sorted algebra

This section is devoted to provide congruence based proofs of the recognizability theorems for free many-sorted algebras. Actually, in order to deal with the different cases of recognizability, classified according to the type of operator under consideration, we have divided this section into several subsections. In Subsection 1, entitled Basic terms, we prove that the final sets containing a variable, a constant or an operation symbol applied to a suitable family of variables are recognizable. These results will be used later on to prove that the set of recognizable languages of a many-sorted term algebra forms an algebra of the same signature. In Subsection 2, entitled Substitutions, after defining several substitution operators associated to a free many-sorted algebra and investigating the relationships between them, we prove that, if all input languages for a given substitution are recognizable, then the output language is recognizable as well. In Subsection 3, entitled Iterations, we introduce the notion of iteration of a language with respect to a variable and we prove that, if the input language is recognizable then its iteration with respect to a variable is also recognizable. In Subsection 4, entitled Quotients, we introduce the notion of quotient of a language by another with respect to a variable and we prove that if the input language is recognizable then its quotient is also recognizable. In Subsection 5, entitled Tree Homomorphisms, we introduce the notion of hyperderivor as a way of transforming terms relative to a signature into terms relative to another signature. We show that tree homomorphisms are, really, homomorphisms from a free many-sorted algebra to another many-sorted algebra, itself derived from a many-sorted algebra relative to another signature. Then we prove that the inverse image of a recognizable language under a tree homomorphism is recognizable and that the direct image of a recognizable language by a linear tree homomorphism, which is a particular type of tree homomorphism, is also recognizable. Finally, Subsection 6 is devoted to the study of derivors. We first introduce the category of many-sorted signatures and derivors. Next, after defining a contravariant functor from this category to the category of $\mathcal{U}$-locally small categories and functors, we obtain by means of the Grothendieck contruction, the category of ordered pairs consisting of a many-sorted signature and an algebra of the same signature and the pairs of derivors and derived homomorphisms of algebras as morphisms. As a consequence, after showing that every derivor, together with some additional data, gives rise to a hyperderivor, the counterparts of the two main results in Subsection 5 are immediately obtained.

From now on, following a strongly rooted tradition in the fields of formal languages and automata, we agree to call languages the subsets of the underlying many-sorted set of the free many-sorted algebra on a many-sorted set.
3.1. Basic terms. In this subsection we prove some basic recognizability results relative to a free algebra on an $S$-sorted set. Concretely, we prove that the final sets consisting, respectively, of a variable, a constant, and the action of an operator symbol on a family of variables, are recognizable.

Assumption. In this subsection we will assume that $S$ is finite.
Proposition 3.1. Let $\Sigma$ be an $S$-sorted signature, $X$ an $S$-sorted set, $s \in S$, and $x$ and element in $X_{s}$. Then the language $\{x\} \subseteq \mathrm{T}_{\Sigma}(X)_{s}$ is recognizable.

Proof. We will denote by $2^{S}$ or, to abbreviate, by 2 , the $S$-sorted set (2) $)_{s \in S}$, where $2=\{0,1\}$. Let $\mathbf{2}=\left(2, G^{\mathbf{2}}\right)$ be the finite $\Sigma$-algebra defined as follows: For every $(w, t) \in S^{\star} \times S$ and every $\sigma \in \Sigma_{w, t}$, the structural operation associated to $\sigma$, $G_{\sigma}^{2}: 2_{w} \longrightarrow 2$, is the constant mapping with value 0 . Let $f: X \longrightarrow 2$ be the $S$ sorted mapping such that $f_{s}: X \longrightarrow 2$ is $\operatorname{ch}_{\{x\}}$, the characteristic mapping of $\{x\}$, and, for $t \in S-\{s\}, f_{t}$ is the mapping constantly 0 . Then, for $f^{\sharp}$, the unique homomorphism from $\mathbf{T}_{\Sigma}(X)$ to $\mathbf{2}$ such that $f^{\sharp} \circ \eta_{X}=f$, we have that, for every $P \in \mathrm{~T}_{\Sigma}(X)_{s}, f_{s}^{\sharp}(P)=\operatorname{ch}_{\{x\}}(P)=1$ if and only if $P=x$. Hence, since $\{x\}=$ $\left(f_{s}^{\sharp}\right)^{-1}[\{1\}]$, the language $\{x\}$ is recognizable.

Proposition 3.2. Let $\Sigma$ be an $S$-sorted signature, $X$ an $S$-sorted set, and $\sigma \in \Sigma_{\lambda, s}$. Then the language $\{\sigma\} \subseteq \mathrm{T}_{\Sigma}(X)_{s}$ is recognizable.

Proof. Let $\mathbf{2}=\left(2, H^{\mathbf{2}}\right)$ be the finite $\Sigma$-algebra defined as follows: $H_{\sigma}^{\mathbf{2}}$ is the mapping from $2_{\lambda}$ to 2 that sends the unique member of $2_{\lambda}$ to 1 , and, for every $(w, t) \in S^{\star} \times S$ and every $\tau \in \Sigma_{w, t}, H_{\tau}^{2}: 2_{w} \longrightarrow 2$ is the constant mapping to 0 . Let $\kappa: X \longrightarrow 2$ be the $S$-sorted mapping constantly zero at each sort. Then, for $\kappa^{\sharp}$, the unique homomorphism from $\mathbf{T}_{\Sigma}(X)$ to $\mathbf{2}$ such that $\kappa^{\sharp} \circ \eta_{X}=\kappa$, we have that, for every $P \in \mathrm{~T}_{\Sigma}(X)_{s}, \kappa_{s}^{\sharp}(P)=1$ if and only if $P=\sigma$. Hence, since $\{\sigma\}=\left(\kappa_{s}^{\sharp}\right)^{-1}[\{1\}]$, the language $\{\sigma\}$ is recognizable.

Proposition 3.3. Let $\Sigma$ be an $S$-sorted signature, $X$ an $S$-sorted set, $(w, s) \in$ $\left(S^{\star}-\{\lambda\}\right) \times S, \sigma \in \Sigma_{w, s}$, and $\left(x_{i}\right)_{i \in|w|}$ a family of variables in $X_{w}$. Then the language $\left\{\sigma\left(\left(x_{i}\right)_{i \in|w|}\right)\right\} \subseteq \mathrm{T}_{\Sigma}(X)_{\text {s }}$ is recognizable.

Proof. For $w \in S^{\star}-\{\lambda\}$, let $\downarrow w$ be the $S$-sorted set defined, for every $t \in S$ as follows: $(\downarrow w)_{t}=\left\{i \in|w| \mid w_{i}=t\right\}$, i.e., $(\downarrow w)_{t}=w^{-1}[\{t\}]$. Therefore, for every $t \in S-\operatorname{Im}(w),(\downarrow w)_{t}=\varnothing$, while, for every $t \in \operatorname{Im}(w),(\downarrow w)_{t} \neq \varnothing$. For every $S$-sorted set $X$, the sets $\operatorname{Hom}(\downarrow w, X)$ and $X_{w}$ are naturally isomorphic. If $x=\left(x_{i}\right)_{i \in|w|} \in X_{w}$, i.e., if $x$ is a mapping from $|w|$ to $\bigcup_{i \in|w|} X_{w_{i}}$ such that, for every $i \in|w|, x_{i} \in X_{w_{i}}$, then we will denote by $\bar{x}$ the $S$-sorted mapping from $\downarrow w$ to $X$ which is associated, in virtue of the natural isomorphism, to $x$ and is defined as follows: For every $t \in S$ and every $i \in(\downarrow w)_{t}$, then $\bar{x}_{t}(i)=x_{i}$. Let us note that, for every $t \in S-\operatorname{Im}(w), \bar{x}_{t}$ is the unique mapping from $\varnothing$ to $X_{t}$. Thus, given $w \in S^{\star}-\{\lambda\}$ and $x=\left(x_{i}\right)_{i \in|w|} \in X_{w}$, let $k=\left(k_{t}\right)_{t \in S}$ be the $S$-sorted set defined, for every $t \in S$, as $k_{t}=\operatorname{card}\left(\left(\operatorname{Im}\left(\bar{x}_{t}\right)\right)\right)=\operatorname{card}\left(\left\{x_{i} \mid i \in w^{-1}[\{t\}]\right\}\right)$. Then, for every $t \in S, k_{t}$ is the number of different variables of type $t$ in $x$.

Let $\varphi: \operatorname{Im}(\bar{x}) \longrightarrow k$ be a fixed $S$-sorted mapping such that, for every $t \in S$, $\varphi_{t}: \operatorname{Im}\left(\bar{x}_{t}\right) \longrightarrow k_{t}$ is a bijection. Let $K$ be the $S$-sorted set such that, for every $t \in S-\{s\}, K_{t}=k_{t}+1$ and $K_{s}=k_{s}+2$. Then let $\mathbf{K}=\left(K, I^{\mathbf{K}}\right)$ be the $\Sigma$-algebra defined as follows: For $(u, t) \in S^{\star} \times S$ and $\tau \in \Sigma_{u, t}$, if $(u, t) \neq(w, s)$ or $\tau \neq \sigma$, then $I^{\mathbf{K}}: K_{u} \longrightarrow K_{t}$ is the constant mapping with value $k_{t}$, and for $\tau=\sigma, I^{\mathbf{K}}$ is the mapping:

$$
I_{\sigma}^{\mathrm{K}}\left\{\begin{aligned}
K_{w} & \longrightarrow K_{s} \\
a & \longmapsto \begin{cases}k_{s}+1, & \text { if } a=\varphi(x) ; \\
k_{s}, & \text { otherwise } .\end{cases}
\end{aligned}\right.
$$

Let $f: X \longrightarrow K$ be the $S$-sorted mapping defined, for every $s \in S$, as follows:

$$
f_{s}\left\{\begin{aligned}
X_{s} & \longrightarrow K_{s} \\
z & \longmapsto \begin{cases}\varphi_{s}(z), & \text { if } z \text { is a variable in } \operatorname{Im}(\bar{x})_{s} ; \\
k_{s}, & \text { otherwise }\end{cases}
\end{aligned}\right.
$$

Then, for the unique homomorphism $f^{\sharp}$ from $\mathbf{T}_{\Sigma}(X)$ to $\mathbf{K}$ such that $f^{\sharp} \circ \eta_{X}=f$, we have that, for every $P \in \mathrm{~T}_{\Sigma}(X)_{s}, f^{\sharp}(P)=k_{s}+1$ if and only if $P=\sigma\left(\left(x_{i}\right)_{i \in|w|}\right)$. Hence, since $\left\{\sigma\left(\left(x_{i}\right)_{i \in|w|}\right)\right\}=\left(f^{\sharp}\right)_{s}^{-1}\left[\left\{k_{s}+1\right\}\right]$, the language $\left\{\sigma\left(\left(x_{i}\right)_{i \in n}\right)\right\}$ is recognizable.
3.2. Substitutions. In this subsection we introduce several substitution operators associated to a free algebra with the aim of proving that, if the languages under consideration are recognizable, then the language that results from the substitution operator applied to these languages is also recognizable.

Definition 3.4. Let $X$ be an $S$-sorted set, $s \in S$, and $P \in \mathrm{~T}_{\Sigma}(X)_{s}$. Then we will denote by $\left({ }^{(x, t)}\right)_{(x, t) \in \amalg_{X}}(P)$ the mapping from $\prod_{(x, t) \in \amalg_{X}} \mathrm{~T}_{\Sigma}(X)_{t}^{|P|_{x}}$ to $\mathrm{T}_{\Sigma}(X)_{s}$ that assigns to $\left(\left(Q_{\alpha}^{x, t}\right)_{\alpha \in|P|_{x}}\right)_{(x, t) \in \amalg X}$ in $\prod_{(x, t) \in \amalg X} \mathrm{~T}_{\Sigma}(X)_{t}^{|P|_{x}}$ the term $\left(\begin{array}{c}\left(Q_{\alpha}^{x, t}\right)_{\alpha \in|P| x}\end{array}\right)_{(x, t) \in \amalg X}(P)$ in $\mathrm{T}_{\Sigma}(X)_{s}$ obtained by substituting in $P$, for every $t \in S$, every $x \in X_{t}$, and every $\alpha \in|P|_{x}, Q_{\alpha}^{x, t}$ for the $i_{\alpha}$-th occurrence of $x$ in $P$. We will call $\binom{(x, t)}{\left(Q_{\alpha}^{x, t}\right)_{\alpha \in|P| x}}_{(x, t) \in \amalg_{X}}(P)$ the substitution of $\left(Q_{\alpha}^{x, t}\right)_{\alpha \in|P|_{x}}$ for $x$ in $P$ for every $t \in S$ and every $x$ in $X_{t}$, and $\left({ }^{(x, t)}\right)_{(x, t) \in \amalg X}(P)$ the global substitution operator for $P$.

Let $u$ be a sort in $S$ and $z \in X_{u}$. Then we will denote by $\left({ }^{( }{ }^{z}\right)(P)$ the mapping from $\mathrm{T}_{\Sigma}(X)_{u}^{|P|_{z}}$ to $\mathrm{T}_{\Sigma}(X)_{s}$ that assigns to $\left(Q_{\alpha}^{z}\right)_{\alpha \in|P|_{z}}$ in $\mathrm{T}_{\Sigma}(X)_{u}^{|P|_{z}}$ the term $\left(\left(Q_{\alpha}^{z}\right)_{\alpha \in|P|_{z}}^{z}\right)(P)$ in $\mathrm{T}_{\Sigma}(X)_{s}$ obtained by substituting in $P$, for every $\alpha \in|P|_{z}, Q_{\alpha}^{z}$ for the $i_{\alpha}$-th occurrence of $z$ in $P$. We will call $\left(\left(Q_{\alpha}^{z}\right)_{\alpha \in|P| z}^{z}\right)(P)$ the substitution of $\left(Q_{\alpha}^{z}\right)_{\alpha \in|P|_{z}}$ for $z$ in $P$.

Remark. For an $S$-sorted set $X$, a sort $s \in S$, and a term $P$ in $\mathrm{T}_{\Sigma}(X)_{s}$, the domain of $\left({ }^{(x, t)}\right)_{(x, t) \in \amalg X}(P)$ is the set of all choice functions for $\left(\mathrm{T}_{\Sigma}(X)_{t}^{|P|_{x}}\right)_{(x, t) \in \amalg X}$. Regarding the index set $\coprod X$, since $\operatorname{Var}^{X}(P)$, the $S$-sorted of the variables of $P$, is finite, we can, without loss of generality, replace it by $\bigcup_{t \in \operatorname{sort}\left(\operatorname{Var}^{x}{ }_{(P)}\right)}\left(X_{t} \times\{t\}\right)$, where $\operatorname{sort}\left(\operatorname{Var}^{X}(P)\right)$ is the finite set of the sorts of the variables which appear in $P$. However, for uniformity, we will continue using $\amalg X$.
Remark. Let $X$ be an $S$-sorted set, $s$ a sort in $S$, and $P$ a term in $\mathrm{T}_{\Sigma}(X)_{s}$. Then we will denote by $\left({ }^{( }\right)_{x \in X_{s}}(P)$ the mapping from $\prod_{x \in X_{s}} \mathrm{~T}_{\Sigma}(X)_{s}^{|P|_{x}}$ to $\mathrm{T}_{\Sigma}(X)_{s}$ that assigns to $\left(\left(Q_{\alpha}^{x}\right)_{\alpha \in|P|_{x}}\right)_{x \in X_{s}}$ in $\prod_{x \in X_{s}} \mathrm{~T}_{\Sigma}(X)_{s}^{|P|_{x}}$ the term $\left({\left(Q_{\alpha}^{x}\right)_{\alpha \in|P| x}^{x}}_{x}\right)_{x \in X_{s}}(P)$ in $\mathrm{T}_{\Sigma}(X)_{s}$ obtained by substituting in $P$, for every $x \in X_{s}$ and every $\alpha \in|P|_{x}, Q_{\alpha}^{x}$ for the $i_{\alpha}$-th occurrence of $x$ in $P$. We will call $\left(\left(Q_{\alpha}^{x}\right)_{\alpha \in|P|_{x}}^{x}\right)_{x \in X_{s}}(P)$ the substitution of $\left(Q_{\alpha}^{x}\right)_{\alpha \in|P|_{x}}$ for $x$ in $P$ for every $x$ in $X_{s}$, and $\left({ }^{(x}\right)_{x \in X_{s}}(P)$ the substitution operator for $P$.

Since (1) for every $s \in S, X_{s} \cong X_{s} \times\{s\}$, (2) by the associativity of the product, $\prod_{(x, t) \in \amalg X} \mathrm{~T}_{\Sigma}(X)_{t}^{|P|_{x}} \cong \prod_{t \in S}\left(\prod_{x \in X_{t}} \mathrm{~T}_{\Sigma}(X)_{t}^{|P|_{x}}\right)$, and (3) $\prod_{x \in X_{s}} \mathrm{~T}_{\Sigma}(X)_{s}^{|P|_{x}}$ is naturally isomorphic to a subset of $\prod_{t \in S}\left(\prod_{x \in X_{t}} \mathrm{~T}_{\Sigma}(X)_{t}^{|P|_{x}}\right)$, we have that $\left({ }^{x}\right)_{x \in X_{s}}(P)$ is, essentially, the restriction of $\left({ }^{(x, t)}\right)_{(x, t) \in \amalg X}(P)$ to $\prod_{x \in X_{s}} \mathrm{~T}_{\Sigma}(X)_{s}^{|P|_{x}}$. Similarly $\left.{ }^{(z}\right)(P)$ is, essentially, the restriction of $\left({ }^{(x, t)}\right)_{(x, t) \in \amalg X}(P)$ to $\mathrm{T}_{\Sigma}(X)_{u}^{|P|_{z}}$.

We next prove that, for every $s \in S$ and every $P \in \mathrm{~T}_{\Sigma}(X),\left({ }^{(x, t)}\right)_{(x, t) \in \amalg X}(P)$ is, actually, a mapping from $\prod_{(x, t) \in \amalg X} \mathrm{~T}_{\Sigma}(X)_{t}^{|P|_{x}}$ to $\mathrm{T}_{\Sigma}(X)_{s}$.

Proposition 3.5. Let $X$ be an $S$-sorted set, $s \in S$, and $P$ a term in $\mathrm{T}_{\Sigma}(X)_{s}$. Then, for every $\left(\left(Q_{\alpha}^{x, t}\right)_{\alpha \in|P|_{x}}\right)_{(x, t) \in \amalg X}$ in $\prod_{(x, t) \in \amalg X} \mathrm{~T}_{\Sigma}(X)_{t}^{|P|_{x}}$, we have that $\left(\begin{array}{c}\left(Q_{\alpha}^{x, t}\right)_{\alpha \in|P| x}\end{array}\right)_{(x, t) \in \amalg X}(P)$ is a term in $\mathrm{T}_{\Sigma}(X)_{s}$. Therefore $\left({ }^{(x, t)}\right)_{(x, t) \in \amalg X}(P)$ is a mapping from $\prod_{(x, t) \in \amalg X} \mathrm{~T}_{\Sigma}(X)_{t}^{|P|_{x}}$ to $\mathrm{T}_{\Sigma}(X)_{s}$.
Proof. Let $\mathcal{T}$ be the subset of $\mathrm{T}_{\Sigma}(X)$ defined, for every $s \in S$, as follows:

$$
\mathcal{T}_{s}=\left\{P \in \mathrm{~T}_{\Sigma}(X)_{s} \mid \operatorname{Im}\left(((x, t))_{(x, t) \in \amalg X}(P)\right) \subseteq \mathrm{T}_{\Sigma}(X)_{s}\right\} .
$$

To prove that $\mathcal{T}=\mathrm{T}_{\Sigma}(X)$ it suffices to show, by Proposition 2.32, that $X \subseteq \mathcal{T}$ and that $\mathcal{T}$ is a subalgebra of $\mathbf{T}_{\Sigma}(X)$.

We first prove that $X \subseteq \mathcal{T}$.
Let $s$ be a sort in $S$ and $z \in X_{s}$. Then we have that $|z|_{z}=1$, whilst, for every $x \in X_{s}-\{z\},|z|_{x}=0$, and, for every $t \in S-\{s\}$ and every $x \in X_{t}|z|_{x}=0$. Therefore, for a term $Q^{z, s}$ in $\mathrm{T}_{\Sigma}(X)_{s}$ associated to $z$, the global substitution of $Q^{z, s}$ for $z$ in $z$ is equal to $Q^{z, s}$. Hence is a term in $\mathrm{T}_{\Sigma}(X)_{s}$. Consequently $X \subseteq \mathcal{T}$.

We next prove that $\mathcal{T}$ is a subalgebra of $\mathbf{T}_{\Sigma}(X)$.
Let $s$ be a sort in $S$ and $\sigma \in \Sigma_{\lambda, s}$. Then, for every $t \in S$ and every $x \in X_{t}$, we have that $|\sigma|_{x}=0$. Hence the global substitution operator leaves $\sigma$ invariant.

Let $(w, s) \in\left(S^{\star}-\{\lambda\}\right) \times S, \sigma \in \Sigma_{w, s}$, and let $\left(P_{i}\right)_{i \in|w|} \in \mathrm{T}_{\Sigma}(X)_{w}$ be such that, for every $i \in|w|, P_{i}$ satisfies the given requirement. Then, for every $t \in S$ and every $x \in X_{t}$, we have that

$$
\left|\sigma\left(\left(P_{i}\right)_{i \in|w|}\right)\right|_{x}=\sum_{i \in|w|}\left|P_{i}\right|_{x}
$$

But we have that

$$
\left.\begin{array}{rl}
\left(\left(Q_{\alpha}^{x, t}\right)_{\alpha \in\left|\sigma\left(\left(P_{i}\right)_{i \in|w|}\right)\right| x}^{(x, t)}\right. \tag{3.1}
\end{array}\right)_{(x, t) \in \amalg X}\left(\sigma\left(\left(P_{i}\right)_{i \in|w|}\right)\right)=0
$$

and, by induction hypothesis, for every $i \in|w|,\left(\begin{array}{c}\left(Q_{\alpha, t \sum_{k \in i}\left|P_{k}\right| x}^{x, t}\right)_{\alpha \in\left|P_{i}\right| x}\end{array}\right)_{(x, t) \in \amalg X}\left(P_{i}\right)$ is a term in $\mathrm{T}_{\Sigma}(X)_{w_{i}}$. Therefore the left side of the above equation is a term in $\mathrm{T}_{\Sigma}(X)_{s}$. Consequently $\mathcal{T}$ is a subalgebra of $\mathbf{T}_{\Sigma}(X)$. Thus $\mathcal{T}=\mathrm{T}_{\Sigma}(X)$. Hereby completing our proof.

Corollary 3.6. Let $X$ be an $S$-sorted set, $u \in S, z \in X_{u}$, and $P$ a term in $\mathrm{T}_{\Sigma}(X)_{s}$. Then, for every $\left(Q_{\alpha}^{z}\right)_{\alpha \in|P|_{z}}$ in $\mathrm{T}_{\Sigma}(X)_{u}^{|P|_{z}}$, we have that $\left(\left(Q_{\alpha}^{z}\right)_{\alpha \in|P| z}^{z}\right)(P)$ is a term in $\mathrm{T}_{\Sigma}(X)_{s}$. Therefore ( $\left.{ }_{(\%)}^{(.)}\right)(P)$ is a mapping from $\mathrm{T}_{\Sigma}(X)_{u}^{|P|_{z}}$ to $\mathrm{T}_{\Sigma}(X)_{s}$.

Remark. For every $S$-sorted set $X$ the family $\left(((x, t))_{(x, t) \in \amalg X}(P)\right)_{s \in S}$ of global substitution operators is in $\prod_{(P, s) \in \amalg \mathrm{T}_{\Sigma}(X)} \operatorname{Hom}\left(\prod_{(x, t) \in \amalg X} \mathrm{~T}_{\Sigma}(X)_{t}^{|P|_{x}}, \mathrm{~T}_{\Sigma}(X)_{s}\right)$.

We next prove that every homomorphism from $\mathbf{T}_{\Sigma}(X)$ is compatible with the substitution operator.

Lemma 3.7. Let $X$ be an $S$-sorted set, $s \in S$, $W \in \mathrm{~T}_{\Sigma}(X)_{s}$, A a $\Sigma$-algebra, and $g$ a homomorphism from $\mathbf{T}_{\Sigma}(X)$ to A. Then, for every $\left(\left(P_{\alpha}^{x, t}\right)_{\alpha \in|W|_{x}}\right)_{(x, t) \in \amalg X}$,
$\left(\left(Q_{\alpha}^{x, t}\right)_{\alpha \in|W|_{x}}\right)_{(x, t) \in \amalg X} \in \prod_{(x, t) \in \amalg X} \mathrm{~T}_{\Sigma}(X)_{t}^{|W|_{x}}$, if, for every $(x, t) \in \amalg X$ and every $\alpha \in|W|_{x}$, we have that $g_{t}\left(P_{\alpha}^{x, t}\right)=g_{t}\left(Q_{\alpha}^{x, t}\right)$, then

$$
g_{s}\left(\left({\left(P_{\alpha}^{x, t}\right)_{\alpha \in|W| x}^{(x, t)}}_{)_{(x, t) \in \amalg X}}(W)\right)=g_{s}\left(\left({\left(Q_{\alpha}^{x, t}\right)_{\alpha \in|W|_{x}}^{(x, t)}}^{(x, t) \in \amalg X}\right)_{(W)}\right) .\right.
$$

Proof. We prove it by algebraic induction on $W$. But for $W$, either (1) $W=z$, for a unique $z \in X_{s}$, or (2) $W=\sigma$, for a unique $\sigma \in \Sigma_{\lambda, s}$, or (3) $W=\sigma\left(\left(W_{i}\right)_{i \in|w|}\right)$, for a unique $w \in S^{\star}-\{\lambda\}$, a unique $\sigma \in \Sigma_{w, s}$, and a unique family $\left(W_{i}\right)_{i \in|w|} \in \mathrm{T}_{\Sigma}(X)_{w}$.

In case (1), $|W|_{z}=1$, whilst for every $x \in X_{s}-\{z\},|W|_{x}=0$, and for every $t \in S-\{s\}$ and every $x \in X_{t},|W|_{x}=0$. Therefore, for two terms $P^{z, s}$ and $Q^{z, s}$ in $\mathrm{T}_{\Sigma}(X)_{s}$ with $g_{s}\left(P^{z, s}\right)=g\left(Q^{z, s}\right)$ then it holds that

$$
g_{s}\left(P^{z, s}\right)=g_{s}\left(\left(P_{\alpha}^{z, s}\right)(z)\right)=g_{s}\left(\left(Q_{\alpha}^{z, s}\right)(z)\right)=g_{s}\left(Q^{z, s}\right) .
$$

In case (2), we note that no constant symbol $\sigma \in \Sigma_{\lambda, s}$ has variables in $X$ and, consequently no proper substitution is made. To prove the statement, we need to show that $g_{s}(\sigma)=g_{s}(\sigma)$, which trivially holds.

Finally, in case (3), we have the following equations

$$
\begin{align*}
& g_{s}\left({\left(P_{\alpha}^{x, t}\right)_{\alpha \in|W| x}^{(x, t)}}_{)_{(x, t) \in \amalg X}}(W)\right) \\
& =g_{s}\left(\left(\left(P_{\alpha}^{x, t}\right)_{\alpha \in\left|\sigma\left(\left(W_{i}\right)_{i \in|w|}\right)\right| x}^{(x, t)}\right)_{(x, t) \in \amalg X}\left(\sigma\left(\left(W_{i}\right)_{i \in|w|}\right)\right)\right) \\
& \left.=g_{s}\left(\sigma\left(\left({\left(P_{\alpha+\sum_{k \in i}^{x, t}\left|W_{k}\right| x}^{x, t}\right)_{\alpha \in\left|W_{i}\right| x}}^{(x, t)}\right)_{(x, t) \in \amalg X}\left(W_{i}\right)\right)_{i \in|w|}\right)\right) \tag{1}
\end{align*}
$$

$$
\begin{align*}
& =\sigma^{\mathbf{A}}\left(\left(g_{w_{i}}\left(\left({\left(Q_{\alpha+\sum_{k \in i}^{x, W_{k} \mid x}}^{x, t}\right)_{\alpha \in\left|W_{i}\right| x}}^{(x, t)}\right)_{(x, t) \in \amalg X}\left(W_{i}\right)\right)\right)_{i \in|w|}\right)\left(\dagger_{3}\right) \\
& =g_{s}\left(\sigma\left(\left({\left(Q_{\alpha+\sum_{k \in i}^{x}\left|W_{k}\right| x}^{x, t}\right)_{\alpha \in\left|W_{i}\right| x}}_{(x, t)}^{(x, t) \in \amalg X}{ }^{\left(W_{i}\right)}\right)_{i \in|w|}\right)\right)  \tag{2}\\
& =g_{s}\left(\left(\left(Q_{\alpha}^{x, t}\right)_{\alpha \in\left|\sigma\left(\left(W_{i}\right)_{i \in|w|}\right)\right| x}^{(x, t)}\right)_{(x, t) \in \amalg X}\left(\sigma\left(\left(W_{i}\right)_{i \in|w|}\right)\right)\right)  \tag{1}\\
& =g_{s}\left(\left({\left(Q_{\alpha}^{x, t}\right)_{\alpha \in|W| x}^{(x, t)}}_{)_{(x, t) \in \amalg X}}(W)\right),\right.
\end{align*}
$$

where, to shorten notation, we let $\left(\dagger_{1}\right)$, $\left(\dagger_{2}\right)$, and $\left(\dagger_{3}\right)$ stand for (by equation 3.1), (by definition of homomorphism), and (by inductive hypothesis), respectively.

As a consequence of the last lemma we have the following corollary, that states that every congruence on $\mathbf{T}_{\Sigma}(X)$ is compatible with the substitution operator.
Corollary 3.8. Let $X$ be an $S$-sorted set, $s \in S, W \in \mathrm{~T}_{\Sigma}(X)_{s}$, and $\Phi$ be a congruence on $\mathbf{T}_{\Sigma}(X)$. Then, for every $\left(\left(P_{\alpha}^{x, t}\right)_{\alpha \in|W|_{x}}\right)_{(x, t) \in \amalg X},\left(\left(Q_{\alpha}^{x, t}\right)_{\alpha \in|W|_{x}}\right)_{(x, t) \in \amalg X} \in$ $\prod_{(x, t) \in \amalg X} \mathrm{~T}_{\Sigma}(X)_{t}^{|W|_{x}}$, if, for every $(x, t) \in \coprod X$ and every $\alpha \in|W|_{x}$, we have that $\left[P_{\alpha}^{x, t}\right]_{\Phi_{t}}=\left[Q_{\alpha}^{x, t}\right]_{\Phi_{t}}$, then

$$
\left[\binom{(x, t)}{\left(P_{\alpha}^{x, t}\right)_{\alpha \in|W| x}}_{(x, t) \in \amalg X}(W)\right]_{\Phi_{s}}=\left[\binom{(x, t)}{\left(Q_{\alpha}^{x, t}\right)_{\alpha \in|W| x}}_{(x, t) \in \amalg X}(W)\right]_{\Phi_{s}}
$$

Since it will be used in the following definition we recall that, as a particular case of Proposition [2.29, for an $S$-sorted set $X$, we have the power $\Sigma$-algebra
$\mathbf{T}_{\Sigma}(X)^{\wp}$ associated to $\mathbf{T}_{\Sigma}(X)$, which has as underlying $S$-sorted set $\mathrm{T}_{\Sigma}(X)^{\wp}=$ $\left(\operatorname{Sub}\left(\mathrm{T}_{\Sigma}(X)_{s}\right)\right)_{s \in S}$ and, for every $(w, s) \in S^{\star} \times S$ and every $\sigma \in \Sigma_{w, s}$, as structural operation associated to $\sigma$, the mapping $\sigma^{\wp}$ from $\mathrm{T}_{\Sigma}(X)_{w}^{\wp}=\prod_{i \in|w|} \operatorname{Sub}\left(\mathrm{T}_{\Sigma}(X)_{w_{i}}\right)$ to $\mathrm{T}_{\Sigma}(X)_{s}^{\wp}=\operatorname{Sub}\left(\mathrm{T}_{\Sigma}(X)_{s}\right)$ that sends $\left(L_{i}\right)_{i \in|w|}$ in $\mathrm{T}_{\Sigma}(X)_{w}^{\wp}$ to

$$
\sigma^{\wp}\left(\left(L_{i}\right)_{i \in|w|}\right)=\left\{\sigma\left(\left(P_{i}\right)_{i \in|w|}\right) \mid\left(P_{i}\right)_{i \in|w|} \in \prod_{i \in|w|} L_{i}\right\}
$$

in $\mathrm{T}_{\Sigma}(X)_{s}^{\wp}$. We remind the reader that in accordance with what is established in Proposition 2.35, we have let, for abbreviation, for every $(w, s) \in S^{\star} \times S$ and every $\sigma \in \Sigma_{w, s}, \sigma$ stand for $F_{\sigma}^{\mathbf{T}_{\Sigma}(X)}$, the structural operation of $\mathbf{T}_{\Sigma}(X)$ associated to $\sigma$.
Definition 3.9. For every $t \in S$, let $\left(L_{x}\right)_{x \in X_{t}}$ be a mapping from $X_{t}$ to $\mathrm{T}_{\Sigma}(X)_{t}^{\wp}=$ $\operatorname{Sub}\left(\mathrm{T}_{\Sigma}(X)_{t}\right)$, also written, in this context, as $\binom{x}{L_{x}}_{x \in X_{t}}$. Then, for the $S$-sorted mapping $\left(\left({ }_{L_{x}}^{x}\right)_{x \in X_{t}}\right)_{t \in S}$ from $X$ to $\mathrm{T}_{\Sigma}(X)^{\wp}$, we will denote by $\left(\left(\binom{x}{L_{x}}_{x \in X_{t}}\right)_{t \in S}\right)^{\sharp}$ the unique homomorphism from $\mathbf{T}_{\Sigma}(X)$ to $\mathbf{T}_{\Sigma}(X)^{反}$ such that

$$
\left(\left(\binom{x}{L_{x}}_{x \in X_{t}}\right)_{t \in S}\right)^{\sharp} \circ \eta_{X}=\left(\binom{x}{L_{x}}_{x \in X_{t}}\right)_{t \in S} .
$$

Let $u$ be a sort in $S, z$ an element of $X_{u}$, and $L \in \mathrm{~T}_{\Sigma}(X)_{u}^{\wp}$. Then, for the $S$ sorted mapping $(\underset{L}{z})$ from $X$ to $\mathrm{T}_{\Sigma}(X)^{\wp}$ defined as: $(\underset{L}{z})_{u}$ is the mapping from $X_{u}$ to $\operatorname{Sub}\left(\mathrm{T}_{\Sigma}(X)_{u}\right)$ that sends $z$ to $L$ and $y \in X_{u}-\{z\}$ to $\{y\}$, while, for $t \in S-\{u\},\binom{z}{L}_{t}$ is the mapping from $X_{t}$ to $\operatorname{Sub}\left(\mathrm{T}_{\Sigma}(X)_{t}\right)$ that sends $x \in X_{t}$ to $\{x\}$, we will denote by $\binom{z}{L}^{\sharp}$ the unique homomorphism from $\mathbf{T}_{\Sigma}(X)$ to $\mathbf{T}_{\Sigma}(X)^{\wp}$ such that $\binom{z}{L}^{\sharp} \circ \eta_{X}=\binom{z}{L}$.

We recall that the homomorphism $\left(\left(\binom{x}{L_{x}}_{x \in X_{t}}\right)_{t \in S}\right)^{\sharp}$ from $\mathbf{T}_{\Sigma}(X)$ to $\mathbf{T}_{\Sigma}(X)^{\wp}$ is defined, by algebraic recursion, as follows. Let $s$ be a sort in $S$ and $P \in \mathrm{~T}_{\Sigma}(X)_{s}$. Then we know that $P$ either has the form (1) $z$, for a unique $z \in X_{s}$, or (2), $\sigma$, for a unique $\sigma \in \Sigma_{\lambda, s}$, or (3) $\sigma\left(\left(P_{i}\right)_{i \in|w|}\right)$, for a unique $w \in S^{\star}-\{\lambda\}$, a unique $\sigma \in \Sigma_{w, s}$, and a unique family $\left(P_{i}\right)_{i \in|w|} \in \mathrm{T}_{\Sigma}(X)_{w}$.

In case (1), we have that $\left(\left(\binom{x}{L_{x}}_{x \in X_{t}}\right)_{t \in S}\right)_{s}^{\#}(z)=L_{z}$.
In case (2), we have that $\left(\left(\left(L_{L_{x}}^{x}\right)_{x \in X_{t}}\right)_{t \in S}\right)_{s}^{\#}(\sigma)=\{\sigma\}$.
Finally, in case (3), and under the hypothesis that, for every $i \in|w|$, the subset $\left(\left(\binom{x}{L_{x}}_{x \in X_{t}}\right)_{t \in S}\right)_{w_{i}}^{\sharp}\left(P_{i}\right)$ of $\mathrm{T}_{\Sigma}(X)_{w_{i}}$ has been defined, we have that

$$
\begin{aligned}
&\left(\left(\binom{x}{L_{x}}_{x \in X_{t}}\right)_{t \in S}\right)_{s}^{\sharp}\left(\sigma\left(\left(P_{i}\right)_{i \in|w|}\right)\right)= \\
& \sigma^{\wp}\left(\left(\left(\left(\binom{x}{L_{x}}_{x \in X_{t}}\right)_{t \in S}\right)_{w_{i}}^{\sharp}\left(P_{i}\right)\right)_{i \in|w|}\right)= \\
&\left\{\sigma\left(\left(Q_{i}\right)_{i \in|w|}\right) \left\lvert\,\left(Q_{i}\right)_{i \in|w|} \in \prod_{i \in|w|}\left(\left(\binom{x}{L_{x}}_{x \in X_{t}}\right)_{t \in S}\right)_{w_{i}}^{\sharp}\left(P_{i}\right)\right.\right\} .
\end{aligned}
$$

We next prove that, for every $S$-sorted mapping $\left(\binom{x}{L_{x}}_{x \in X_{t}}\right)_{t \in S}$ from $X$ to $\mathrm{T}_{\Sigma}(X)^{\wp}$, every $s \in S$, and every $P \in \mathrm{~T}_{\Sigma}(X)_{s},\left(\left(\binom{x}{L_{x}}_{x \in X_{t}}\right)_{t \in S}\right)_{s}^{\sharp}(P)$ is given by collecting together the terms $\binom{(x, t)}{\left(Q_{\alpha}^{x, t}\right)_{\alpha \in|P| x}}_{(x, t) \in \amalg X}(P)$ with $\left(\left(Q_{\alpha}^{x, t}\right)_{\alpha \in|P|_{x}}\right)_{(x, t) \in \amalg X}$ varying in $\prod_{(x, t) \in \amalg X} L_{x}^{|P|_{x}}$. Let us note that, since, for every $x \in X_{t}, L_{x}^{|P|_{x}}$ is embedded into $\mathrm{T}_{\Sigma}(X)_{t}^{|P|_{x}}, \prod_{x \in X_{t}} L_{x}^{|P|_{x}}$ is also embedded into $\prod_{x \in X_{t}} \mathrm{~T}_{\Sigma}(X)_{t}^{|P|_{x}}$. Therefore $\prod_{(x, t) \in \amalg X} L_{x}^{|P|_{x}}$ is embedded into $\prod_{(x, t) \in \amalg X} \mathrm{~T}_{\Sigma}(X)_{t}^{|P|_{x}}$ and, consequently, the restriction of $\left({ }^{(x, t)}\right)_{(x, t) \in \amalg_{X}}(P)$ to $\prod_{(x, t) \in \amalg_{X}} L_{x}^{|P|_{x}}$ is available.

Proposition 3.10. Let $X$ be an $S$-sorted set, $s \in S$, and $P$ a term in $\mathrm{T}_{\Sigma}(X)_{s}$. Then, for every $S$-sorted mapping $\left(\left({ }_{L_{x}}^{x}\right)_{x \in X_{t}}\right)_{t \in S}$ from $X$ to $\mathrm{T}_{\Sigma}(X)^{\wp}$, we have that

$$
\left(\left(\binom{x}{L_{x}}_{x \in X_{t}}\right)_{t \in S}\right)_{s}^{\sharp}(P)=\operatorname{Im}\left(((x, t))_{(x, t) \in \amalg X}(P) \upharpoonright_{\Pi_{(x, t) \in \amalg X} L_{x}^{|P| x}}\right) .
$$

Proof. Let $\mathcal{T}$ be the subset of $\mathrm{T}_{\Sigma}(X)$ defined, for every $s \in S$, as follows: For every $P \in \mathrm{~T}_{\Sigma}(X)_{s}, P \in \mathcal{T}_{s}$ if and only if

$$
\left(\left(\binom{x}{L_{x}}_{x \in X_{t}}\right)_{t \in S}\right)_{s}^{\sharp}(P)=\operatorname{Im}\left(((x, t))_{(x, t) \in \amalg X}(P) \Gamma_{\Pi_{(x, t) \in \amalg X} L_{x}^{|P| x}}\right) .
$$

To prove that $\mathcal{T}=\mathrm{T}_{\Sigma}(X)$ it suffices to show, by Proposition 2.32, that $X \subseteq \mathcal{T}$ and that $\mathcal{T}$ is a subalgebra of $\mathbf{T}_{\Sigma}(X)$.

We first prove that $X \subseteq \mathcal{T}$.
Let $s$ be a sort in $S$ and $z \in X_{s}$. Then $\left(\left(\binom{x}{L_{x}}_{x \in X_{t}}\right)_{t \in S}\right)_{s}^{\sharp}(z)=L_{z}$. On the other hand, we have that $|z|_{z}=1$, whilst, for every $x \in X_{s}-\{z\},|z|_{x}=0$, and, for every $t \in S-\{s\}$ and every $x \in X_{t}|z|_{x}=0$. Therefore, since, for every $Q \in L_{z}$, it happens that $\left({ }_{Q}^{z}\right)(z)=Q$, we have that $\operatorname{Im}\left(((x, t))_{(x, t) \in \amalg X}(P)\right)=$ $L_{z}$. Thus $\operatorname{Im}\left(\left(\left(^{(x, t)}\right)_{(x, t) \in \amalg X}(P) \upharpoonright_{\prod_{(x, t) \in \amalg X} L_{x}^{|P| x}}\right)=L_{z}\right.$. Hence both sets coincide. Consequently $X \subseteq \mathcal{T}$.

We next prove that $\mathcal{T}$ is a subalgebra of $\mathbf{T}_{\Sigma}(X)$.
Let $s$ be a sort in $S$ and $\sigma \in \Sigma_{\lambda, s}$. Then $\left(\left(\binom{x}{L_{x}}_{x \in X_{t}}\right)_{t \in S}\right)_{s}^{\sharp}(\sigma)=\{\sigma\}$, which is the interpretation of the constant symbol $\sigma$ in the subset algebra. On the other hand, $\operatorname{Im}\left(\left({ }^{(x, t)}\right)_{(x, t) \in \amalg_{X}}(P) \upharpoonright_{\prod_{(x, t) \in \amalg X} L_{x}^{|P| x}}\right)$, which is the result of collecting together the terms $\binom{(x, t)}{\left(Q_{\alpha}^{x, t}\right)_{\alpha \in|P| x}}_{(x, t) \in \amalg X}(\sigma)$ when $\left(\left(Q_{\alpha}^{x, t}\right)_{\alpha \in|P|_{x}}\right)_{(x, t)}$ varies in $\prod_{(x, t) \in \amalg X} L_{x}^{|P|_{x}}$, is, simply, $\{\sigma\}$. Consequently

$$
\operatorname{Im}\left(((x, t))_{(x, t) \in \amalg_{X}}(P) \upharpoonright_{\prod_{(x, t) \in \amalg X} L_{x}^{|P| x}}\right)=\{\sigma\} .
$$

Let $(w, s) \in\left(S^{\star}-\{\lambda\}\right) \times S, \sigma \in \Sigma_{w, s}$, and let $\left(P_{i}\right)_{i \in|w|} \in \mathrm{T}_{\Sigma}(X)_{w}$ be such that, for every $i \in|w|, P_{i}$ satisfies the given requirement. Then, since $\left(\left(\binom{x}{L_{x}}_{x \in X_{t}}\right)_{t \in S}\right)^{\sharp}$ is a homomorphism from $\mathbf{T}_{\Sigma}(X)$ to $\mathbf{T}_{\Sigma}(X)^{\wp}$, we have that

$$
\left(\left(\binom{x}{L_{x}}_{x \in X_{t}}\right)_{t \in S}\right)_{s}^{\sharp}\left(\sigma\left(\left(P_{i}\right)_{i \in|w|}\right)\right)=\sigma^{\wp}\left(\left(\left(\left(\binom{x}{L_{x}}_{x \in X_{t}}\right)_{t \in S}\right)_{w_{i}}^{\sharp}\left(P_{i}\right)\right)_{i \in|w|}\right)
$$

On the other hand, by the induction hypothesis, bearing in mind that, for every $t \in$ $S$, every $x \in X_{t},\left|\sigma\left(\left(P_{i}\right)_{i \in n}\right)\right|_{x}=\sum_{i \in n}\left|P_{i}\right|_{x}$, and taking into account Equation [3.1, we have that
$\operatorname{Im}\left(((x, t))_{(x, t) \in \amalg X}(P) \upharpoonright_{\prod_{(x, t) \in \amalg X} L_{x}^{|P| x}}\right)=\sigma^{\wp}\left(\left(\left(\left(\left({ }_{L_{x}}^{x}\right)_{x \in X_{t}}\right)_{t \in S}\right)_{w_{i}}^{\sharp}\left(P_{i}\right)\right)_{i \in|w|}\right)$.
Consequently $\mathcal{T}$ is a subalgebra of $\mathbf{T}_{\Sigma}(X)$. Thus $\mathcal{T}=\mathrm{T}_{\Sigma}(X)$. Hereby completing our proof.

Remark. Given an $S$-sorted mapping $\left(\binom{x}{L_{x}}_{x \in X_{t}}\right)_{t \in S}$ from $X$ to $\mathrm{T}_{\Sigma}(X)^{\wp}$, and $P$ a term in $\mathrm{T}_{\Sigma}(X)_{s}$, for some $s \in S$, we have that $\left(\left(\binom{x}{L_{x}}_{x \in X_{t}}\right)_{t \in S}\right)_{s}^{\sharp}(P)$ is equal to

$$
\bigcup_{\left(\left(Q_{\alpha}^{x, t}\right)_{\alpha \in|P| x}\right)_{(x, t) \in \amalg X} \in \prod_{(x, t) \in \amalg X} L_{x}^{|P| x}\left\{\binom{(x, t)}{\left(Q_{\alpha}^{x, t}\right)_{\alpha \in|P|_{x}}}_{(x, t) \in \amalg X}(P)\right\} . . . ~} \text {. }
$$

In other words, the value of $\left(\left(\binom{x}{L_{x}}_{x \in X_{t}}\right)_{t \in S}\right)_{s}^{\sharp}$ at $P$ is the union of the image of the mapping $\{\cdot\}_{\mathrm{T}_{\Sigma}(X)_{s}} \circ\left(((x, t))_{(x, t) \in \amalg X}(P) \upharpoonright_{\prod_{(x, t) \in \amalg X} L_{x}^{|P| x}}\right)$ from $\prod_{(x, t) \in \amalg X} L_{x}^{|P|_{x}}$ to $\operatorname{Sub}\left(\mathrm{T}_{\Sigma}(X)\right)$, where $\left({ }^{(x, t)}\right)_{(x, t) \in \amalg_{X}}(P) \upharpoonright_{\prod_{(x, t) \in \amalg X} L_{x}^{|P| x}}$ is the restriction of the substitution mapping $\left({ }^{(x, t)}\right)_{(x, t) \in \amalg X}(P)$ to $\prod_{(x, t) \in \amalg X} L_{x}^{|P|_{x}}$, and $\{\cdot\}_{\mathrm{T}_{\Sigma}(X)_{s}}$ the canonical embedding of $\mathrm{T}_{\Sigma}(X)_{s}$ into $\mathrm{T}_{\Sigma}(X)_{s}^{\wp}$.

Corollary 3.11. Let $s$ and $u$ be sorts in $S, z \in X_{u}, L \in \mathrm{~T}_{\Sigma}(X)_{u}^{\wp}$, and $P$ a term in $\mathrm{T}_{\Sigma}(X)_{s}$. Then

$$
\binom{z}{L}_{s}^{\sharp}(P)=\operatorname{Im}\left((\stackrel{z}{.})(P) \upharpoonright_{L^{|P| z}}\right) .
$$

Definition 3.12. Let $\left(\binom{x}{L_{x}}_{x \in X_{t}}\right)_{t \in S}$ be an $S$-sorted mapping from $X$ to $\mathrm{T}_{\Sigma}(X)^{\wp}$. Then we will denote by $\left(\left(\left({ }_{L_{x}}^{x}\right)_{x \in X_{t}}\right)_{t \in S}\right)^{\sharp p}$ the $S$-sorted mapping from $\mathrm{T}_{\Sigma}(X)^{\wp}$ to $\mathrm{T}_{\Sigma}(X)^{\wp}$ that, for every $s \in S$, sends $K$ in $\mathrm{T}_{\Sigma}(X)_{s}^{\wp}$ to

$$
\left(\left(\binom{x}{L_{x}}_{x \in X_{t}}\right)_{t \in S}\right)_{s}^{\sharp \mathfrak{p}}(K)=\bigcup_{P \in K}\left(\left(\binom{x}{L_{x}}_{x \in X_{t}}\right)_{t \in S}\right)_{s}^{\sharp}(P)
$$

in $\mathrm{T}_{\Sigma}(X)_{s}^{\wp}$. Therefore $\left(\left(\binom{x}{L_{x}}_{x \in X_{t}}\right)_{t \in S}\right)^{\sharp \mathfrak{p}}$ is the canonical extension of the underlying $S$-sorted mapping of the homomorphism $\left(\left(\binom{x}{L_{x}}_{x \in X_{t}}\right)_{t \in S}\right)^{\sharp}$ from $\mathbf{T}_{\Sigma}(X)$ to $\mathbf{T}_{\Sigma}(X)^{6}$.

Let $u$ be a sort in $S, z \in X_{u}$, and $L \in \mathrm{~T}_{\Sigma}(X)_{u}^{\wp}$. Then we will denote by $\binom{z}{L}^{\sharp \mathfrak{p}}$ the canonical extension of the underlying mapping of the homomorphism $\binom{z}{L}^{\sharp}$ from $\mathbf{T}_{\Sigma}(X)$ to $\mathbf{T}_{\Sigma}(X)^{\wp}$.

Remark. Let $\left(\binom{x}{L_{x}}_{x \in X_{t}}\right)_{t \in S}$ be an $S$-sorted mapping from $X$ to $\mathrm{T}_{\Sigma}(X)^{\wp}, s \in S$, $K \subseteq \mathrm{~T}_{\Sigma}(X)_{s}$, and $W \in \mathrm{~T}_{\Sigma}(X)_{s}$. Then $W \in\left(\left(\binom{x}{L_{x}}_{x \in X_{t}}\right)_{t \in S}\right)_{s}^{\sharp \mathfrak{p}}(K)$ if and only if there are $P \in K$ and $\left(\left(Q_{\alpha}^{x, t}\right)_{\alpha \in|P|_{x}}\right)_{(x, t) \in \amalg X} \in \prod_{(x, t) \in \amalg X} L_{x}^{|P|_{x}}$ such that

$$
W=\binom{(x, t)}{\left(Q_{\alpha}^{x, t}\right)_{\alpha \in|P| x}}_{(x, t) \in \amalg X}(P) .
$$

In particular, for $z \in X_{s}$ and $L \in \operatorname{Sub}\left(\mathrm{~T}_{\Sigma}(X)_{s}\right)$, we have that $W \in\binom{z}{L}_{s}^{\sharp \mathfrak{p}}(K)$ if and


We next prove the many-sorted version of Theorem 4.6, on p. 74, in [14]. The proposition states that, for every sort $s \in S$, every $s$-recognizable language $K$, and every operator $\left(\binom{x}{L_{x}}_{x \in X_{t}}\right)_{t \in S}$ such that, for every $t \in S$ and every $x \in X_{t}$, $L_{x} \in \operatorname{Rec}_{t}\left(\mathbf{T}_{\Sigma}(X)\right),\left(\left(\binom{x}{L_{x}}_{x \in X_{t}}\right)_{t \in S}\right)_{s}^{\sharp \mathfrak{p}}(K)$ is $s$-recognizable.
Assumption. To prove the following proposition we will assume that $S$ and $X$ are finite.

Proposition 3.13. Let $s$ be a sort in $S, K \in \operatorname{Rec}_{s}\left(\mathbf{T}_{\Sigma}(X)\right)$, and $\left(\left({ }_{L_{x}}^{x}\right)_{x \in X_{t}}\right)_{t \in S}$ an $S$-sorted mapping from $X$ to $\left(\operatorname{Rec}_{t}\left(\mathbf{T}_{\Sigma}(X)\right)\right)_{t \in S}$. Then

$$
\left(\left(\binom{x}{L_{x}}_{x \in X_{t}}\right)_{t \in S}\right)_{s}^{\sharp \mathfrak{p}}(K) \in \operatorname{Rec}_{s}\left(\mathbf{T}_{\Sigma}(X)\right) .
$$

Proof. Let $\Phi$ be the congruence on $\mathbf{T}_{\Sigma}(X)$ defined as follows:

$$
\Phi=\bigcap\left(\left(\bigcup_{t \in S}\left\{\Omega^{\mathbf{T}_{\Sigma}(X)}\left(\delta^{t, L_{x}}\right) \mid x \in X_{t}\right\}\right) \cup\left\{\Omega^{\mathbf{T}_{\Sigma}(X)}\left(\delta^{s, K}\right)\right\}\right) .
$$

By the assumption and Proposition 2.61, $\Phi$ is of finite index. Moreover, for every $t \in S-\{s\}$ and every $x \in X_{t}, \Phi_{t}$ saturates $L_{x}$, and, for $t=s, \Phi_{s}$ saturates $K$.

From now on, for every $r \in S, k_{r}$ and $\mathcal{W}_{\Phi_{r}}=\left\{W_{r, l} \mid l \in k_{r}\right\}$ stand for the index of $\Phi_{r}$ and a fixed transversal of $\mathrm{T}_{\Sigma}(X)_{r} / \Phi_{r}$ in $\mathrm{T}_{\Sigma}(X)_{r}$, respectively. Moreover, $k$ and $\mathcal{W}_{\Phi}$ denote the $S$-sorted sets $\left(k_{r}\right)_{r \in S}$ and $\left(\mathcal{W}_{\Phi_{r}}\right)_{r \in S}$, respectively.

Let $\Psi=\left(\Psi_{r}\right)_{r \in S}$ be the binary relation on $\mathrm{T}_{\Sigma}(X)$ defined as follows: For every $r \in S, \Psi_{r}$ is the subset of $\mathrm{T}_{\Sigma}(X)_{r}^{2}$ consisting of all ordered pairs $(P, Q)$ in $\mathrm{T}_{\Sigma}(X)_{r}^{2}$ such that the following two conditions are satisfied:
(1) $(P, Q) \in \Phi_{r}$.
(2) For every $l \in k_{r}$

$$
\left(P \in\left(\left(\binom{x}{L_{x}}_{x \in X_{t}}\right)_{t \in S}\right)_{r}^{\sharp \mathfrak{p}}\left(\left[W_{r, l}\right]_{\Phi_{r}}\right) \leftrightarrow Q \in\left(\left(\binom{x}{L_{x}}_{x \in X_{t}}\right)_{t \in S}\right)_{r}^{\sharp \mathfrak{p}}\left(\left[W_{r, l}\right]_{\Phi_{r}}\right)\right) .
$$

By definition, for every $r \in S, \Psi_{r}$ is a refinement of $\Phi_{r}$ and an equivalence relation on $\mathrm{T}_{\Sigma}(X)_{r}$. Moreover, for every $r \in S$, the index of $\Psi_{r}$ on $\mathrm{T}_{\Sigma}(X)_{r}$ is bounded by $k_{r} 2^{k_{r}}$. Consequently, the $S$-sorted set $\mathrm{T}_{\Sigma}(X) / \Psi$ is finite.

Let us check that $\Psi$ is a congruence on $\mathbf{T}_{\Sigma}(X)$. Let $(w, u) \in\left(S^{\star}-\{\lambda\}\right) \times S$, $\sigma \in \Sigma_{w, u}$, and let $\left(P_{i}\right)_{i \in|w|}$ and $\left(Q_{i}\right)_{i n|w|}$ be sequences of terms in $\mathrm{T}_{\Sigma}(X)_{w}$ such that, for every $i \in|w|,\left(P_{i}, Q_{i}\right) \in \Psi_{w_{i}}$. We want to show that

$$
\left(\sigma\left(\left(P_{i}\right)_{i \in|w|}\right), \sigma\left(\left(Q_{i}\right)_{i \in|w|}\right)\right) \in \Psi_{u}
$$

Let us note that, by definition of $\Psi$, for every $i \in|w|$, we have that $\left(P_{i}, Q_{i}\right) \in \Phi_{w_{i}}$. Since $\Phi$ is a congruence on $\mathrm{T}_{\Sigma}(X)$, we conclude that $\left(\sigma\left(\left(P_{i}\right)_{i \in|w|}\right), \sigma\left(\left(Q_{i}\right)_{i \in|w|}\right)\right)$ is a pair in $\Phi_{u}$, so $\sigma\left(\left(P_{i}\right)_{i \in|w|}\right)$ and $\sigma\left(\left(Q_{i}\right)_{i \in|w|}\right)$ satisfy the first condition for being related under $\Psi$.

Regarding the second condition, let $l$ be an element of $k_{u}$. Assume that

$$
\sigma\left(\left(P_{i}\right)_{i \in|w|}\right) \in\left(\left(\binom{x}{L_{x}}_{x \in X_{t}}\right)_{t \in S}\right)_{u}^{\sharp \mathfrak{p}}\left(\left[W_{u, l}\right]_{\Phi_{u}}\right) .
$$

Then there are $W^{\dagger} \in\left[W_{u, l}\right]_{\Phi_{u}}$ and $\left(\left(U_{\alpha}^{x, t}\right)_{\alpha \in\left|W^{\dagger}\right|_{x}}\right)_{(x, t) \in \amalg X}$ in $\prod_{(x, t) \in \amalg X} L_{x}^{\left|W^{\dagger}\right|_{x}}$ such that

$$
\begin{equation*}
\sigma\left(\left(P_{i}\right)_{i \in|w|}\right)=\binom{(x, t)}{\left(U_{\alpha}^{x, t}\right)_{\alpha \in\left|W^{\dagger}\right| x}}_{(x, t) \in \amalg X}\left(W^{\dagger}\right) . \tag{3.2}
\end{equation*}
$$

But for $W^{\dagger}$, either (a) $\operatorname{Var}\left(W^{\dagger}\right) \cap X=\varnothing^{S}$ or (b) $\operatorname{Var}\left(W^{\dagger}\right) \cap X \neq \varnothing^{S}$.
In case (a), since for every $t \in S$ and $x \in X_{t},\left|W^{\dagger}\right|_{x}=0$, we derive from Equation 3.2 that $W^{\dagger}=\sigma\left(\left(P_{i}\right)_{i \in|w|}\right)$. It follows that, for every $i \in|w|$, for every $t \in S$ and $x \in X_{t},\left|P_{i}\right|_{x}=0$. Therefore, for every $i \in|w|$, the term $P_{i}$ is a term in $\left(\left(\binom{x}{L_{x}}_{x \in X_{t}}\right)_{t \in S}\right)_{w_{i}}^{\sharp \mathfrak{p}}\left(\left[P_{i}\right]_{\Phi_{w_{i}}}\right)$. Thus, by assumption, for every $i \in|w|$, the term $Q_{i}$ is in $\left(\left(\binom{x}{L_{x}}_{x \in X_{t}}\right)_{t \in S}\right)_{w_{i}}^{\sharp \mathfrak{p}}\left(\left[P_{i}\right]_{\Phi_{w_{i}}}\right)$. Hence, for every $i \in|w|$, there are $W^{\ddagger i} \in\left[P_{i}\right]_{\Phi_{w_{i}}}$ and $\left(\left(V_{\beta}^{x, t, i}\right)_{\beta \in\left|W^{\ddagger i}\right|_{x}}\right)_{(x, t) \in \amalg X}$ in $\prod_{(x, t) \in \amalg X} L_{x}^{\left|W^{\ddagger i}\right|_{x}}$ such that

$$
Q_{i}=\left({\left(V_{\alpha}^{x, t, i}\right)_{\alpha \in\left|W^{\ddagger i}\right|_{x}}^{(x, t)}}_{)_{(x, t) \in \amalg X}}\left(W^{\ddagger i}\right) .\right.
$$

Let us denote by $W^{\ddagger}$ the term $\sigma\left(\left(W^{\ddagger i}\right)_{i \in|w|}\right)$. Since, for every $i \in|w|, W^{\ddagger i}$ is a term in $\left[P_{i}\right]_{\Phi_{w_{i}}}$ and $\Phi$ is a congruence on $\mathrm{T}_{\Sigma}(X)$, we have that $\left(W^{\dagger}, W^{\ddagger}\right) \in \Phi_{w_{u}}$. Thus $W^{\ddagger} \in\left[W_{u, l}\right]_{\Phi_{u}}$. Moreover, let us note that, for every $t \in S$ and $x \in X_{t},\left|W^{\ddagger}\right|_{x}=$ $\sum_{i \in|w|}\left|W^{\ddagger}\right|_{x}$. Let $\left(\left(V_{\beta}^{x, t}\right)_{\beta \in\left|W^{\ddagger}\right|_{x}}\right)_{(x, t) \in \amalg X}$ be the element in $\prod_{(x, t) \in \amalg X} L_{x}^{\left|W^{\ddagger}\right|_{x}}$ obtained by joining, in order, the family $\left(\left(\left(V_{\beta}^{x, t, i}\right)_{\beta \in \mid W^{\ddagger} i^{i_{x}}}\right)_{(x, t) \in \amalg X}\right)_{i \in|w|}$. Then,
from Equation 3.1, it follows that

$$
\begin{aligned}
\sigma\left(\left(Q_{i}\right)_{i \in|w|}\right) & =\sigma\left(\left(\left(\begin{array}{c}
\left(V_{\alpha}^{x, t, i}\right)_{\alpha \in\left|W^{\ddagger}\right| x \mid x}
\end{array}\right)_{(x, t) \in \amalg X}\left(W^{\ddagger i}\right)\right)_{i \in|w|}\right) \\
& =\binom{(x, t)}{\left(V_{\alpha}^{x, t}\right)_{\alpha \in\left|W^{\ddagger}\right| x}}_{(x, t) \in \amalg X}\left(W^{\ddagger}\right) .
\end{aligned}
$$

Therefore, $\sigma\left(\left(Q_{i}\right)_{i \in|w|}\right) \in\left(\left(\binom{x}{L_{x}}_{x \in X_{t}}\right)_{t \in S}\right)_{u}^{\sharp \mathfrak{p}}\left(\left[W_{u, l}\right]_{\Phi_{u}}\right)$. The other implication follows by a similar reasoning, thus proving case (a).

In case (b), either (b.1) $W^{\dagger}$ is a variable or (b.2) $W^{\dagger}$ has the form of an operation symbol applied to a suitable family of terms. Let us note that the case in which $W^{\dagger}$ is a constant is excluded because a term of such type does not contain any variable.

In case (b.1), we have that $W^{\dagger}=z$, for some $z \in X_{u}$. Thus, $\left|W^{\dagger}\right|_{z}=1$, whilst, for every $x \in X_{u}-\{z\},\left|W^{\dagger}\right|_{x}=0$, and, for every $r \in S-\{u\}$ and every $x \in X_{r}$, $\left|W^{\dagger}\right|_{x}=0$. Therefore, from Equation 3.2 we obtain the following equation

$$
\left(\left(U_{\alpha}^{x, t}\right)_{\alpha \in\left|W^{\dagger}\right| x}^{(x, t)}\right)_{(x, t) \in \amalg X}\left(W^{\dagger}\right)=\binom{(x, t)}{\left(U_{\alpha}^{x, t}\right)_{\alpha \in\left|W^{\dagger}\right| x}}_{(x, t) \in \amalg X}(z)=U_{0}^{z, u}=\sigma\left(\left(P_{i}\right)_{i \in|w|}\right) .
$$

Hence $\sigma\left(\left(P_{i}\right)_{i \in|w|}\right)$ is a term in $L_{z}$. Let us consider the term $V_{0}^{z, u}=\sigma\left(\left(Q_{i}\right)_{i \in|w|}\right)$ in $L_{z}$. Since, for every $i \in|w|,\left(P_{i}, Q_{i}\right) \in \Phi_{w_{i}}$, we have that $\left(U_{0}^{z, u}, V_{0}^{z, u}\right) \in \Phi_{u}$, and, since $L_{z}$ is $\Phi_{u}$-saturated, we have that $V_{0}^{z, u}$ is a term in $L_{x}$. Moreover, $\sigma\left(\left(Q_{i}\right)_{i \in|w|}\right)=\left(V_{0}^{z}, u\right)(z)$. Therefore $\sigma\left(\left(Q_{i}\right)_{i \in|w|}\right) \in\left(\left(\binom{x}{L_{x}}_{x \in X_{t}}\right)_{t \in S}\right)_{u}^{\sharp \mathfrak{p}}\left(\left[W_{u, l}\right]_{\Phi_{u}}\right)$. The other implication follows from a similar reasoning, thus proving case (b.1).

In case (b.2), we conclude, from Equation 3.2, that the term $W^{\dagger}$ has the form $\sigma\left(\left(W^{\dagger_{i}}\right)_{i \in|w|}\right)$ for some sequence of terms $\left(W^{\dagger_{i}}\right)_{i \in|w|} \in \mathrm{T}_{\Sigma}(X)_{w}$, since no substitution replaces the operation symbol. From Equation 3.1, for every $i \in|w|$,

$$
P_{i}=\left(\left(\begin{array}{cc}
\left(U_{\alpha+\Sigma_{j \in i}\left|W^{\dagger}\right|_{\mid x}}\right)_{\alpha \in \mid W^{\left.\dagger_{i}\right|_{x}}}
\end{array}\right)_{(x, t) \in \amalg X}\left(W^{\dagger_{i}}\right)\right) .
$$

Hence, for every $i \in|w|, P_{i} \in\left(\left(\binom{x}{L_{x}}_{x \in X_{t}}\right)_{t \in S}\right)_{w_{i}}^{\sharp \mathfrak{p}}\left(\left[W^{\dagger}\right]_{\Phi_{w_{i}}}\right)$.
By definition of $\Psi$, for every $i \in|w|, Q_{i} \in\left(\left(\left(L_{x}^{x}\right)_{x \in X_{t}}\right)_{t \in S}\right)_{w_{i}}^{\sharp \mathfrak{p}}\left(\left[W^{\dagger_{i} i}\right]_{\Phi_{w_{i}}}\right)$. Therefore, for every $i \in|w|$, there are $W^{\ddagger_{i}} \in\left[W^{\dagger_{i}}\right]_{\Phi_{w_{i}}}$ and $\left(\left(V_{\beta}^{x, t, i}\right)_{\beta \in \mid W^{\ddagger} i_{x}}\right)_{(x, t) \in \amalg X}$ in $\prod_{(x, t) \in \amalg X} L_{x}^{\left|W^{\ddagger}\right|_{x}}$ such that

$$
Q_{i}=\left(\begin{array}{c}
\left(V_{\alpha}^{x, t, i}\right)_{\alpha \in\left|W^{\ddagger}\right|_{x}}
\end{array}\right)_{(x, t) \in \amalg X}\left(W^{\ddagger i}\right) .
$$

For $W^{\ddagger}=\sigma\left(\left(W^{\ddagger i}\right)_{i \in|w|}\right)$ and $\left(\left(V_{\beta}^{x, t}\right)_{\beta \in\left|W^{\ddagger}\right|_{x}}\right)_{(x, t) \in \amalg X}$ of $\prod_{(x, t) \in \amalg X} L_{x}^{\left|W^{\ddagger}\right|_{x}}$ obtained by joining, in order, the family $\left(\left(\left(V_{\beta}^{x, t, i}\right)_{\beta \in\left|W^{\ddagger i}\right|_{x}}\right)_{(x, t) \in \amalg X}\right)_{i \in|w|}$, we have that $\sigma\left(\left(Q_{i}\right)_{i \in|w|}\right)$ is in $\left(\left(\left(L_{x}^{x}\right)_{x \in X_{t}}\right)_{t \in S}\right)_{u}^{\sharp \mathfrak{p}}\left(\left[W_{u, l}\right]_{\Phi_{u}}\right)$. The other implication follows by a similar reasoning, thus proving case (b.2).

Therefore $\Psi$ is a congruence of finite index in $\operatorname{Cgr}_{\mathrm{fi}_{\mathrm{f}}}\left(\mathbf{T}_{\Sigma}(X)\right)$.
Finally, we prove that $\left(\left(\left(L_{x}^{x}\right)_{x \in X_{t}}\right)_{t \in S}\right)_{s}^{\sharp \mathfrak{p}}(K)$ is $\Psi_{s}$-saturated. Note that, by definition of $\Psi$, for every $r \in S$ and every $l \in k_{r},\left(\left(\binom{x}{L_{x}}_{x \in X_{t}}\right)_{t \in S}\right)_{r}^{\sharp \mathfrak{p}}\left(\left[W_{r, l}\right]_{\Phi_{r}}\right)$ is
$\Psi_{r}$-saturated. Moreover,

Hence, $\left(\left(\binom{x}{L_{x}}_{x \in X_{t}}\right)_{t \in S}\right)_{s}^{\sharp \mathfrak{p}}(K)$ is $\Psi_{s}$-saturated because it is a finite union of $\Psi_{s^{-}}$ saturated languages. The statement follows from Proposition 2.67.

Corollary 3.14. Let $u$ and $s$ be sorts in $S, z \in X_{u}, L \in \operatorname{Rec}_{u}\left(\mathbf{T}_{\Sigma}(X)\right)$, and $K \in \operatorname{Rec}_{s}\left(\mathbf{T}_{\Sigma}(X)\right)$. Then $\binom{z}{L}_{s}^{\sharp p}(K) \in \operatorname{Rec}_{s}\left(\mathbf{T}_{\Sigma}(X)\right)$.

We next state the many-sorted version of Corollary 4.12, on p. 78, in [14].
Corollary 3.15. Let $(w, s)$ be an element of $S^{\star} \times S, \sigma \in \Sigma_{w, s}$, and $\left(L_{i}\right)_{i \in|w|} \in$ $\prod_{i \in|w|} \operatorname{Rec}_{w_{i}}\left(\mathbf{T}_{\Sigma}(X)\right)$. Then the language $\sigma^{\wp}\left(\left(L_{i}\right)_{i \in|w|}\right) \in \operatorname{Rec}_{s}\left(\mathbf{T}_{\Sigma}(X)\right)$.
Proof. It follows from Proposition 3.3 and Proposition 3.13
Corollary 3.16. The $S$-sorted set $\left(\operatorname{Rec}_{s}\left(\mathbf{T}_{\Sigma}(X)\right)\right)_{s \in S}$ is a subalgebra of $\mathbf{T}_{\Sigma}(X)^{反}$. We will denote by $\operatorname{Rec} .\left(\mathbf{T}_{\Sigma}(X)\right)$ the $\Sigma$-algebra canonically associated to the subalgebra $\left(\operatorname{Rec}_{s}\left(\mathbf{T}_{\Sigma}(X)\right)\right)_{s \in S}$ of $\mathbf{T}_{\Sigma}(X)^{\wp}$.
3.3. Iterations. In this subsection we introduce the notion of iteration of a language with respect to a variable with the aim of proving that, when the considered language is recognizable, then its iteration with respect to a variable is also a recognizable language.

We begin by stating the many-sorted counterpart of the single-sorted notion of $z$-iteration as defined by Gécseg and Steinby in [14], Definition 4.7, on p. 76.
Definition 3.17. Let $s$ be a sort in $S, z \in X_{s}$, and $L \in \mathrm{~T}_{\Sigma}(X)_{s}^{\wp}$. Then the $z$-iteration of $L$ is the language

$$
L^{\star z}=\bigcup_{j \in \mathbb{N}} L^{j, z},
$$

where $\left(L^{j, z}\right)_{j \in \mathbb{N}}$ is the family of subsets of $\mathrm{T}_{\Sigma}(X)_{s}$ defined recursively as follows:

$$
L^{0, z}=\{z\}, \text { and, for } j \in \mathbb{N}, L^{j+1, z}=L^{j, z} \cup\left(L^{j, z}\right)_{s}^{\sharp \mathfrak{p}}(L) .
$$

Remark. The language $L^{\star z}$ is obtained as follows. First include $z$. New members of $L^{\star z}$ are obtained by substituting in some term of $L$, for every occurrence of $z$, some term already known to be in $L^{\star z}$. Let us note that $L^{1, z}=L \cup\{z\}$ and that $\left(L^{j, z}\right)_{j \in \mathbb{N}}$ is an ascending chain of subsets of $\mathrm{T}_{\Sigma}(X)_{s}$, i.e., that, for every $j \in \mathbb{N}$, $L^{j, z} \subseteq L^{j+1, z}$.

We next prove the many-sorted version of Theorem 4.8., on p. 76, in [14. The proposition states that, for every sort $s \in S$ and $z \in X_{s}$, if the input language $L \subseteq \mathrm{~T}_{\Sigma}(X)_{s}$ is recognizable, then its $z$-iteration is also recognizable.
Assumption. To prove the following proposition we will assume that $S$ is finite.
Proposition 3.18. Let $s$ be a sort in $S$ and $z \in X_{s}$. If $L \in \operatorname{Rec}_{s}\left(\mathbf{T}_{\Sigma}(X)\right)$, then $L^{\star z} \in \operatorname{Rec}_{s}\left(\mathbf{T}_{\Sigma}(X)\right)$.

Proof. Let $\Phi$ be the congruence on $\mathbf{T}_{\Sigma}(X)$ defined as follows:

$$
\Phi=\Omega^{\mathbf{T}_{\Sigma}(X)}\left(\delta^{s, L}\right) \cap \Omega^{\mathbf{T}_{\Sigma}(X)}\left(\delta^{s, z}\right)
$$

By Proposition 2.61, $\Phi$ is of finite index (recall that, by Proposition [3.1, $\{z\} \subseteq$ $\mathrm{T}_{\Sigma}(X)_{s}$ is recognizable). Moreover, for the sort $s \in S, \Phi_{s}$ saturates $L$ and $\{z\}$.

From now on, for every $r \in S, k_{r}$ and $\mathcal{W}_{\Phi_{r}}=\left\{W_{r, l} \mid l \in k_{r}\right\}$ stand for the index of $\Phi_{r}$ and a fixed transversal of $\mathrm{T}_{\Sigma}(X)_{r} / \Phi_{r}$ in $\mathrm{T}_{\Sigma}(X)_{r}$, respectively. Moreover, $k$ and $\mathcal{W}_{\Phi}$ denote the $S$-sorted sets $\left(k_{r}\right)_{r \in S}$ and $\left(\mathcal{W}_{\Phi_{r}}\right)_{r \in S}$, respectively.

Let $\Psi=\left(\Psi_{r}\right)_{r \in S}$ be the binary relation on $\mathrm{T}_{\Sigma}(X)$ that is defined as follows: For every $r \in S, \Psi_{r}$ is the binary relation on $\mathrm{T}_{\Sigma}(X)_{r}$ consisting of all ordered pairs $(P, Q) \in \mathrm{T}_{\Sigma}(X)_{r}^{2}$ such that the following two conditions are satisfied:
(1) $(P, Q) \in \Phi_{r}$.
(2) For every $l \in k_{r},\left(P \in\left(\underset{L^{z} z}{z}\right)_{r}^{\sharp \mathfrak{p}}\left(\left[W_{r, l}\right]_{\Phi_{r}}\right) \leftrightarrow Q \in\left(\frac{L^{\star z}}{z}\right)_{r}^{\sharp \mathfrak{p}}\left(\left[W_{r, l}\right]_{\Phi_{r}}\right)\right)$.

By definition, for every $r \in S, \Psi_{r}$ is a refinement of $\Phi_{r}$ and an equivalence relation on $\mathrm{T}_{\Sigma}(X)_{r}$. Moreover, for every $r \in S$, the index of $\Psi_{r}$ on $\mathrm{T}_{\Sigma}(X)_{r}$ is bounded by $k_{r} 2^{k_{r}}$. Consequently, the $S$-sorted $\mathrm{T}_{\Sigma}(X) / \Psi$ is finite.

Let us check that $\Psi$ is a congruence on $\mathbf{T}_{\Sigma}(X)$. Let $(w, u) \in\left(S^{\star}-\{\lambda\} \times S\right), \sigma \in$ $\Sigma_{w, u}$ and let $\left(P_{i}\right)_{i \in|w|}$ and $\left(Q_{i}\right)_{i \in|w|}$ be sequences of terms in $\mathrm{T}_{\Sigma}(X)_{w}$ such that for every $i \in|w|$, $\left(P_{i}, Q_{i}\right) \in \Psi_{w_{i}}$. We want to show that $\left(\sigma\left(\left(P_{i}\right)_{i \in|w|}\right), \sigma\left(\left(Q_{i}\right)_{i \in|w|}\right)\right) \in$ $\Psi_{u}$.

Let us note that, by definition of $\Psi$, for every $i \in|w|$, we have that $\left(P_{i}, Q_{i}\right) \in \Phi_{w_{i}}$. Since $\Phi$ is a congruence on $\mathrm{T}_{\Sigma}(X)$, we conclude that $\left(\sigma\left(\left(P_{i}\right)_{i \in|w|}\right), \sigma\left(\left(Q_{i}\right)_{i \in|w|}\right)\right)$ is a pair in $\Phi_{u}$, so $\sigma\left(\left(P_{i}\right)_{i \in|w|}\right)$ and $\sigma\left(\left(Q_{i}\right)_{i \in|w|}\right)$ satisfy the first condition for being related under $\Psi_{u}$.

Regarding the second condition, let $l$ be an element of $k_{u}$. Assume that

$$
\left.\sigma\left(\left(P_{i}\right)_{i \in|w|}\right)\right) \in\left(L_{L^{\star} z}^{z}\right)_{u}^{\sharp \mathfrak{p}}\left(\left[W_{u, l}\right]_{\Phi_{u}}\right) .
$$

Then there are $W^{\dagger} \in\left[W_{u, l}\right]_{\Phi_{s}}$ and $\left(U_{\alpha}^{z}\right)_{\alpha \in\left|W^{\dagger}\right|_{z}}$ in $\left(L^{\star z}\right)^{\left|W^{\dagger}\right|_{z}}$ such that

$$
\begin{equation*}
\sigma\left(\left(P_{i}\right)_{i \in|w|}\right)=\left(\underset{\left(U_{\alpha}^{z}\right)_{\alpha \in\left|W^{\dagger}\right| z}^{z}}{z}\right)\left(W^{\dagger}\right) \tag{3.3}
\end{equation*}
$$

Note that, for $W^{\dagger}$, either (a) $z \notin \operatorname{Var}\left(W^{\dagger}\right)_{s}$ or (b) $z \in \operatorname{Var}\left(W^{\dagger}\right)_{s}$.
Case (a) follows by a similar argument to that presented in case (a) of Proposition 3.13

In case (b), either (b.1) $W^{\dagger}$ is the variable $z$ (and $u=s$ ) or (b.2) $W^{\dagger}$ has the form of an operation symbol applied to a suitable family of terms.

In case (b.1), from Equation 3.3, we obtain the following equation:

$$
\left(\left(U_{\alpha}^{z}\right)_{\alpha \in\left|W^{\dagger}\right| z}^{z}\right)\left(W^{\dagger}\right)=\left(\left(U_{\alpha}^{z}\right)_{\alpha \in\left|W^{\dagger}\right| z}^{z}\right)(z)=U_{0}^{z}=\sigma\left(\left(P_{i}\right)_{i \in|w|}\right) .
$$

It follows that $\sigma\left(\left(P_{i}\right)_{i \in|w|}\right)$ is a term in $L^{\star z}$. Thus, there exists a $j \in \mathbb{N}$ such that $\sigma\left(\left(P_{i}\right)_{i \in|w|}\right) \in L^{j, z}$. Let $j \in \mathbb{N}$ be the smallest natural number satisfying the just mentioned property. Since $L^{0, z}=\{z\}$ it follows that $j \neq 0$. Therefore, we have that

$$
\sigma\left(\left(P_{i}\right)_{i \in|w|}\right) \in L^{j, z}=L^{j-1, z} \cup\left(L_{L^{j-1, z}}^{z}\right)_{s}^{\sharp \mathfrak{p}}(L) .
$$

By the minimality of $j$, we conclude that $\sigma\left(\left(P_{i}\right)_{i \in|w|}\right) \in\left(\frac{L^{j-1, z}}{z}\right)_{s}^{\sharp p}(L)$. Then there are $\bar{W}^{\dagger} \in L$ and $\left(\bar{U}_{\alpha}^{z}\right)_{\alpha \in\left|\bar{W}^{\dagger}\right|_{z}} \in\left(L^{j-1, z}\right)^{\left|\bar{W}^{\dagger}\right|_{z}}$ such that

$$
\begin{equation*}
\sigma\left(\left(P_{i}\right)_{i \in|w|}\right)=\left(\left(\bar{U}_{\alpha}^{z}\right)_{\alpha \in\left|\bar{W}^{\dagger}\right| z}^{z}\right)\left(\bar{W}^{\dagger}\right) \tag{3.4}
\end{equation*}
$$

For $\bar{W}^{\dagger}$, either (b.1.i) $z \notin \operatorname{Var}\left(\bar{W}^{\dagger}\right)_{s}$ or (b.1.ii) $z \in \operatorname{Var}\left(\bar{W}^{\dagger}\right)_{s}$.
In case (b.1.i). since we are assuming that $\left|\bar{W}^{\dagger}\right|_{z}=0$, the substitution leaves $\bar{W}^{\dagger}$ invariant. Hence $\bar{W}^{\dagger}=\sigma\left(\left(P_{i}\right)_{i \in|w|}\right)$. It follows that $\sigma\left(\left(P_{i}\right)_{i \in|w|}\right) \in L$. Let $V_{0}^{z}=\sigma\left(\left(Q_{i}\right)_{i \in|w|}\right)$. Let us note that, for every $i \in|w|,\left(P_{i}, Q_{i}\right) \in \Phi_{w_{i}}$. Therefore, $\left(U_{0}^{z}, V_{0}^{z}\right) \in \Phi_{s}$ and, since $L$ is $\Phi_{s}$-saturated, we conclude that $V_{0}^{z} \in L \subseteq L^{\star z}$. It follows that $\sigma\left(\left(Q_{i}\right)_{i \in|w|}\right)$ is a term in $\left(L_{L^{\star} z}^{z}\right)_{s}^{\sharp p}\left(\left[W_{s, l}\right]_{\Phi_{s}}\right)$, as desired.

In case (b.1.ii), where we are assuming that $\left|\bar{W}^{\dagger}\right|_{z} \neq 0$, we claim that $\bar{W}^{\dagger}$ cannot be the term $z$. Otherwise, from Equation [3.4 we can conclude that $\sigma\left(\left(P_{i}\right)_{i \in|w|}\right)$ is a term in $L^{j-1, z}$, contradicting the minimality of $j$. Thus, from Equation 3.4
we conclude that $\bar{W}^{\dagger}=\sigma\left(\left(\bar{W}^{\dagger}\right)_{i \in|w|}\right)$, for a unique $\left(\bar{W}^{\dagger}\right)_{i \in|w|} \in \mathrm{T}_{\Sigma}(X)_{w}$. Hence, from Equation 3.1, for every $i \in|w|$, we have that

$$
P_{i}=\left(\left(\bar{U}_{\alpha+\sum_{k \in i}^{z}\left|\bar{W}^{\dagger} k\right| z}^{z}\right)_{\alpha \in \mid \bar{W}^{\dagger} i_{z}}\right)\left(\bar{W}^{\dagger i}\right) \in\left(L_{L^{\star} z}^{z}\right)_{w_{i}}^{\sharp \mathfrak{p}}\left(\left[\bar{W}^{\dagger_{i}}\right]_{\Phi_{w_{i}}}\right) .
$$

By construction, for every $i \in|w|, Q_{i} \in\left(L^{z} z\right)_{w_{i}}^{\not \mathrm{p}_{i}}\left(\left[\bar{W}^{\dagger_{i}}\right]_{\Phi_{w_{i}}}\right)$. Therefore, for every $i \in|w|$, there are $\bar{W}^{\ddagger i} \in\left[\bar{W}^{\dagger i}\right]_{\Phi_{w_{i}}}$ and $\left(\bar{V}_{\beta}^{z, i}\right)_{\beta \in\left|\bar{W}^{\ddagger}\right|_{z}} \in\left(L^{\star z}\right)^{\left|\bar{W}^{\ddagger}\right|_{z}}$ such that

$$
Q_{i}=\left(\left(\bar{V}_{\beta}^{z, i}\right)_{\beta \in\left|\bar{W}^{\ddagger}\right|_{z}}^{z}\right)\left(\bar{W}^{\ddagger i}\right) .
$$

Let us note that the term $\bar{W}^{\ddagger}=\sigma\left(\left(\bar{W}^{\ddagger i}\right)_{i \in n}\right)$ is in $L$ because, for every $i \in|w|$, the pair $\left(\bar{W}^{\dagger i}, \bar{W}^{\ddagger i}\right)$ is in $\Phi_{s}, \bar{W}^{\dagger} \in L$, and $L$ is $\Phi_{s}$-saturated.

Let $\left(\bar{V}_{\beta}^{z}\right)_{\beta \in\left|\bar{W}^{\ddagger}\right|_{z}}$ be the element of $\left(L^{\star z}\right)^{\left|\bar{W}^{\ddagger}\right|_{z}}$ obtained by joining, in order, the $|w|$-indexed family $\left(\left(\bar{V}_{\beta}^{z, i}\right)_{\beta \in\left|\bar{W}^{\dagger}\right|_{z}}\right)_{i \in|w|}$. Since $\left(\bar{V}_{\beta}^{z}\right)_{\beta \in\left|\bar{W}^{\dagger}\right|_{z}}$ is finite, there exists a $t \in \mathbb{N}$ such that, for every $\beta \in\left|\bar{W}^{\ddagger}\right|_{x}$, the term $\bar{V}_{\beta}^{z}$ is in $L^{t, z}$. On the whole, we conclude that

$$
\sigma\left(\left(Q_{i}\right)_{i \in|w|}\right) \in\left(L_{L^{t, z}}^{z}\right)_{s}^{\sharp p}(L) \subseteq L^{\star z} .
$$

Therefore $\sigma\left(\left(Q_{i}\right)_{i \in|w|}\right)$ is a term in $\left(L^{z} z\right)_{s}^{\sharp \mathfrak{p}}\left(\left[W_{s, l}\right]_{\Phi_{s}}\right)$, as desired. The other implication follows by a similar reasoning, thus proving case (1.b.ii).

Case (b.2) follows by an argument similar to that used in case (b.2) of Proposition 3.13

Therefore $\Psi$ is a congruence of finite index in $\operatorname{Cgr}_{\mathrm{fi}}\left(\mathbf{T}_{\Sigma}(X)\right)$.
Finally, we prove that $L^{\star z}$ is $\Psi_{s^{-}}$-saturated. Note that, by definition of $\Psi$, for every $r \in S$ and $l \in k_{r}$, the language $\left(L^{z}\right)_{r}^{\sharp p}\left(\left[W_{r, l}\right]_{\Phi_{r}}\right)$ is $\Psi_{r}$-saturated. In particular, since $\Phi_{s}$ recognizes $\{z\}$, we conclude that $L^{\star z}=\left(\frac{L^{\star z}}{z}\right)_{s}^{\sharp p}(\{z\})$ is $\Psi_{s}$-saturated, thus proving the stamement.
3.4. Quotients. We next define the notion of quotient of a language by another with respect to a variable of a specified sort with the aim of proving that, when one of the languages is recognizable, then the resulting quotient is also recognizable.

We begin by stating the many-sorted counterpart of the single-sorted notion of $z$-quotient as defined by Gécseg and Steinby in [14], Definition 4.9., on p. 77.

Definition 3.19. Let $s$ be a sort in $S$ and $L \in \mathrm{~T}_{\Sigma}(X)_{s}^{\wp}$. Let $t$ be a sort in $S, z$ an element of $X_{t}$, and $K \in \mathrm{~T}_{\Sigma}(X)_{t}^{\wp}$. Then the $z$-quotient of $L$ by $K$ is the language

$$
K^{-z} L=\left\{U \in \mathrm{~T}_{\Sigma}(X)_{s} \mid(\underset{K}{z})_{s}^{\sharp}(U) \cap L \neq \varnothing\right\} .
$$

This operation may be seen as a converse of the $z$-substitution. If $K=\{x\}$ is a final set, then we will write $x^{-z} L$ instead of $K^{-z} L$.

We prove now the many-sorted version of Theorem 4.10., on p. 77, in 14. The proposition states that, for a sort $s$ in $S$, if $L \in \mathrm{~T}_{\Sigma}(X)_{s}^{\wp}$ is recognizable, then, for every $t \in S, z \in X_{t}$ and every $K \in \mathrm{~T}_{\Sigma}(X)_{t}^{\wp}$, the $z$-quotient of $L$ by $K$ is also recognizable.

Assumption. To prove the following proposition we will assume that $S$ is finite.
Proposition 3.20. Let $s$ be a sort in $S$. If $L \in \operatorname{Rec}_{s}\left(\mathbf{T}_{\Sigma}(X)\right)$, then, for every $t \in S$, every $z \in X_{t}$, and every $K \in \mathrm{~T}_{\Sigma}(X)_{t}^{\wp}, K^{-z} L \in \operatorname{Rec}_{s}\left(\mathbf{T}_{\Sigma}(X)\right)$. Moreover, the number of different $z$-quotients $K^{-z} L$ for any fixed $L \in \operatorname{Rec}_{s}\left(\mathbf{T}_{\Sigma}(X)\right)$ is finite.

Proof. Let $\Phi$ be the congruence on $\mathbf{T}_{\Sigma}(X)$ defined as follows:

$$
\Phi=\Omega^{\mathbf{T}_{\Sigma}(X)}\left(\delta^{s, L}\right)
$$

Then $\Phi$ is of finite index because it is the syntactic congruence of a recognizable language. Moreover, for the sort $s \in S, \Phi_{s}$ saturates $L$.

From now on, for every $r \in S$, let $k_{r}$ and $\mathcal{W}_{\Phi_{r}}=\left\{W_{r, l} \mid l \in k_{r}\right\}$ stand for the index of $\Phi_{r}$ and a fixed transversal of $\mathrm{T}_{\Sigma}(X)_{r} / \Phi_{r}$ in $\mathrm{T}_{\Sigma}(X)_{r}$, respectively. Moreover, $k$ and $\mathcal{W}_{\Phi}$ denote the $S$-sorted sets $\left(k_{r}\right)_{r \in S}$ and $\left(\mathcal{W}_{\Phi_{r}}\right)_{r \in S}$, respectively.

Let $\Psi=\left(\Psi_{r}\right)_{r \in S}$ be the binary relation on $\mathrm{T}_{\Sigma}(X)$ that is defined as follows: For every $r \in S, \Psi_{r}$ is the binary relation on $\mathrm{T}_{\Sigma}(X)_{r}$ consisting of all ordered pairs $(P, Q) \in \mathrm{T}_{\Sigma}(X)_{r}^{2}$ such that the following two conditions are satisfied:
(1) $(P, Q) \in \Phi_{r}$.
(2) for every $l \in k_{r}\left(P \in K^{-z}\left[W_{r, l}\right]_{\Phi_{r}} \leftrightarrow Q \in K^{-z}\left[W_{r, l}\right]_{\Phi_{r}}\right)$.

By definition, for every $r \in S, \Psi_{r}$ is a refinement of $\Phi_{r}$ and an equivalence relation on $\mathrm{T}_{\Sigma}(X)_{r}$. Moreover, for every $r \in S$, the index of $\Psi_{r}$ on $\mathrm{T}_{\Sigma}(X)_{r}$ is bounded by $k_{r} 2^{k_{r}}$. Consequently, the $S$-sorted set $\mathrm{T}_{\Sigma}(X) / \Psi$ is finite.

Analysis similar to that in the proof of Proposition 3.13 shows that $\Psi$ is a congruence on $\mathbf{T}_{\Sigma}(X)$.

Finally, we prove that $K^{-z} L$ is $\Psi_{s}$-saturated. Note that, by definition of $\Psi$, for every $r \in S$ and every $l \in k_{r}$, the language $K^{-z}\left[W_{r, l}\right]_{\Phi_{r}}$ is $\Psi_{r}$-saturated. Moreover

$$
K^{-z} L=\bigcup_{W_{s, l} \in \mathcal{W}_{\Phi_{s}} \&\left[W_{s, l}\right]_{\Phi_{s} \subseteq L} K^{-z}\left[W_{s, l}\right]_{\Phi_{s}} .}
$$

Hence, $K^{-z} L$ is $\Psi_{s}$-saturated because it is a finite union of $\Psi_{s}$-saturated languages. The statement follows from Proposition 2.67 .
3.5. Tree Homomorphisms. Tree automata and tree homomorphisms were defined for the first time by Thatcher in [40]. In the just cited paper Thatcher proved, among other things, that linear tree homomorphisms preserve recognizability. We shall now consider a class of many-sorted homomorphisms, the tree homomorphisms - which are the generalization to the many-sorted field of the tree homomorphism defined by Gécseg and Steinby in [14], Definition 4.13., on p. 78which go from a free many-sorted algebra (of a certain many-sorted signature ( $S, \Sigma$ ) ) to another many-sorted algebra (of the same many-sorted signature), itself derived from a free many-sorted algebras (of another many-sorted signature ( $T, \Xi$ )). These tree homomorphisms, as we will prove, have the property of reflecting suitable recognizable languages. Moreover, we will define a proper subset of the set of the tree homomorphisms, the linear tree homomorphisms, and we will prove that they have the remarkable property of preserving recognizable languages.

Definition 3.21. Let $\varphi: S \longrightarrow T$ be a mapping. Then we will denote by $\Delta_{\varphi}$ the functor from $\mathbf{S e t}^{T}$ to $\mathbf{S e t}^{S}$ that sends a $T$-sorted set $A$ the $S$-sorted set $A_{\varphi}=$ $\left(A_{\varphi(s)}\right)_{s \in S}$, i.e., $A \circ \varphi$, and a $T$-sorted mapping $f: A \longrightarrow B$ to the $S$-sorted mapping $f_{\varphi}=\left(f_{\varphi(s)}\right)_{s \in S}: A_{\varphi} \longrightarrow B_{\varphi}$.
Remark. The functor $\Delta_{\varphi}$ has a left and a right adjoint.
Let $\varphi: S \longrightarrow T$ be a mapping, $\varphi^{\star}$ the canonical homomorphism from $\mathbf{S}^{\star}$, the free monoid on $S$, to $\mathbf{T}^{\star}$, the free monoid on $T$, and $w \in S^{\star}$. Then, for a standard $T$-infinite countable $T$-sorted set of variables $V^{T}=\left(\left\{v_{n}^{t} \mid n \in \mathbb{N}\right\}\right)_{t \in T}$, which is assumed to be disjoint from all other alphabets, we will denote by $V_{\downarrow \varphi^{\star}(w)}^{T}=$ $\left(V_{\left(\downarrow \varphi^{\star}(w)\right)_{t}}^{T}\right)_{t \in T}$ the $T$-sorted set, where, for every $t \in T,\left(\downarrow \varphi^{\star}(w)\right)_{t}=\left(\varphi^{\star}(w)\right)^{-1}[\{t\}]$ and $V_{\left(\downarrow \varphi^{\star}(w)\right)_{t}}^{T}$ is the subset of $V_{t}^{T}$ defined as follows:

$$
V_{\left(\downarrow \varphi^{\star}(w)\right)_{t}}^{T}=\left\{v_{i}^{t} \mid i \in\left(\downarrow \varphi^{\star}(w)\right)_{t}\right\}=\left\{v_{i}^{t}|i \in| \varphi^{\star}(w) \mid \& \varphi\left(w_{i}\right)=t\right\} .
$$

Since $V_{\downarrow \varphi^{\star}(w)}^{T}$ is isomorphic to $\downarrow \varphi^{\star}(w)$, we abbreviate $V_{\downarrow \varphi^{\star}(w)}^{T}$ to $\downarrow \varphi^{\star}(w)$. Let us note that $\operatorname{card}\left(V_{\downarrow \varphi^{\star}(w)}^{T}\right)$, the total number of variables, is $|w|=\left|\varphi^{\star}(w)\right|$. Moreover, for every $i \in|w|$, the number of variables of type $\varphi\left(w_{i}\right)$ is $\left|\varphi^{\star}(w)\right|_{\varphi\left(w_{i}\right)}$ (while, for $\left.t \in T-\operatorname{Im}\left(\varphi^{\star}(w)\right), V_{\left(\downarrow \varphi^{\star}(w)\right)_{t}}^{T}=\varnothing\right)$.
Remark. For $V_{\downarrow \varphi^{\star}(w)}^{T}$, if we disregard the classification into types, then we have the following variables:

$$
v_{0}^{\varphi\left(w_{0}\right)}, \ldots, v_{i}^{\varphi\left(w_{i}\right)}, \ldots, v_{|w|-1}^{\varphi\left(w_{|w|-1}\right)}
$$

On the other hand, instead of $V_{\downarrow^{\star}(w)}^{T}$ we can, equivalently, use the $T$-sorted set $\left(\downarrow v_{\left|\varphi^{\star}(w)\right|_{t}}^{t}\right)_{t \in T}$, where, for every $t \in T, \downarrow v_{\left|\varphi^{\star}(w)\right|_{t}}^{t}$ is the set of the first $\left|\varphi^{\star}(w)\right|_{t}$ variables in $V_{t}^{T}$. Therefore, $\downarrow v_{\left|\varphi^{\star}(w)\right|_{t}}^{t}=\varnothing$, if $t \notin \operatorname{Im}\left(\varphi^{\star}(w)\right)$; while $\downarrow v_{\left|\varphi^{\star}(w)\right|_{t}}^{t}=$ $\left\{\left.v_{j}^{t}|j \in| \varphi^{\star}(w)\right|_{t}\right\}$, if $t \in \operatorname{Im}\left(\varphi^{\star}(w)\right)$.

Thus, preserving the classification into types of the variables, we have:

$$
\begin{array}{ccc}
v_{0}^{\varphi\left(w_{0}\right)} & \ldots & v_{\left|\varphi^{\star}(w)\right|_{\varphi\left(w_{0}\right)}-1}^{\varphi\left(w_{0}\right)} \\
\vdots & \ddots & \vdots \\
v_{0}^{\varphi\left(w_{|w|-1}\right)} & \ldots & v_{\left.\left|\varphi^{\star}(w)\right|_{\varphi(w|w|-1}\right)-1}^{\varphi\left(w_{|w|-1}\right)}
\end{array}
$$

Definition 3.22. A many-sorted signature is an ordered pair $\boldsymbol{\Sigma}=(S, \Sigma)$ where $S$ is a set (of sorts) and $\Sigma$ an $S$-sorted signature.

We next define the notion of hyperderivor from a pair $(\boldsymbol{\Sigma}, X)$, where $\boldsymbol{\Sigma}$ is a manysorted signature and $X$ an $S$-sorted set, to another pair $(\boldsymbol{\Xi}, Y)$, where $\boldsymbol{\Xi}=(T, \boldsymbol{\Xi})$ is a many-sorted signature and $Y$ a $T$-sorted set.

Definition 3.23. Let $\boldsymbol{\Sigma}$ and $\boldsymbol{\Xi}$ be many-sorted signatures, $X$ an $S$-sorted set, and $Y$ a $T$-sorted set. A hyperderivor from $(\boldsymbol{\Sigma}, X)$ to $(\boldsymbol{\Xi}, Y)$ is an ordered pair $((\varphi, c), f)$, denoted by $(\mathbf{c}, f)$, where $\varphi$ is a mapping from $S$ to $T, c=\left(c_{w, s}\right)_{(w, s) \in S^{\star} \times S}$ an $S^{\star} \times S$ sorted mapping from $\Sigma$ to $\left(\mathrm{T}_{\Xi}\left(Y \cup \downarrow \varphi^{\star}(w)\right)_{\varphi(s)}\right)_{(w, s) \in S^{\star} \times S}$, and $f$ an $S$-sorted mapping from $X$ to $\mathrm{T}_{\Xi}(Y)_{\varphi}$. We will say that a hyperderivor $(\mathbf{c}, f)$ from $(\boldsymbol{\Sigma}, X)$ to $(\boldsymbol{\Xi}, Y)$ is linear if, for every $(w, s) \in S^{\star} \times S$, every $\sigma \in \Sigma_{w, s}$, and every $i \in|w|$, no variable $v_{i}^{\varphi\left(w_{i}\right)}$ appears more than once in $c_{w, s}(\sigma)$, i.e., $\left|c_{w, s}(\sigma)\right|_{v_{i}^{\varphi\left(w_{i}\right)}} \leq 1$.
Remark. The reason for the terminology hyperderivor lies in the analogy with the notion of derivor defined by Goguen, Thatcher, and Wagner in [16], on p. 137.

We show now that, for every hyperderivor $(\mathbf{c}, f)$ from $(\boldsymbol{\Sigma}, X)$ to $(\boldsymbol{\Xi}, Y)$, the $S$ sorted set $\mathrm{T}_{\Xi}(Y)_{\varphi}$ is equipped, in a natural way, with a structure of $\Sigma$-algebra. But for this we need to show that, given a mapping $\varphi$ from a set of sorts $S$ to another $T$ and a $T$-sorted set $Y$, it happens that, for every $w \in S^{\star}$ and every $\left(P_{i}\right)_{i \in|w|} \in \mathrm{T}_{\Xi}(Y)_{\varphi^{\star}(w)}$, there exists a canonical homomorphism $\mathrm{S}_{\left(P_{i}\right)_{i \in|w|}}^{*}$ from $\mathbf{T}_{\Xi}\left(Y \cup \downarrow \varphi^{\star}(w)\right)$ to $\mathbf{T}_{\Xi}(Y)$. In what follows we will assume that, for every $w \in S^{\star}$, $Y \cap \downarrow \varphi^{\star}(w)=\varnothing^{T}$.

Remark. Let us note that the just stated assumption is not, in anyway, a loss in generality. Actually, given a $T$-sorted set $A$ and an $I$-indexed family $\left(B^{i}\right)_{i \in I}$ of $T$-sorted sets there exists a $T$-sorted set $C$ such that $A \cong C$ and, for every $i \in I$, $C \cap B^{i}=\varnothing^{T}$. In fact, it suffices, by the Axiom of Regularity, to take as $C$ the $T$-sorted set defined, for every $t \in T$, as $C_{t}=A_{t} \times\left\{\left\{\left(i, B_{t}^{i}\right) \mid i \in I\right\}\right\}$.

Let $w$ be an element of $S^{\star}$. Then the sets $\mathrm{T}_{\Xi}(Y)_{\varphi^{\star}(w)}$ and $\operatorname{Hom}\left(\downarrow \varphi^{\star}(w), \mathrm{T}_{\Xi}(Y)\right)$ are naturally isomorphic. On the basis of this isomorphism, we let $\binom{v_{i}^{\varphi\left(w_{i}\right)}}{P_{i}}_{i \in|w|}$
stand for the $T$-sorted mapping from $\downarrow \varphi^{\star}(w)$ to $\mathrm{T}_{\Xi}(Y)$ canonically associated to $\left(P_{i}\right)_{i \in|w|} \in \mathrm{T}_{\Xi}(Y)_{\varphi^{\star}(w)}$. And then we let $\left(\left(v_{i}^{\varphi\left(w_{i}\right)}\right)_{i \in|w|}\right)^{\sharp}$ stand for the unique homomorphism from $\mathbf{T}_{\Xi}\left(\downarrow \varphi^{\star}(w)\right)$ to $\mathbf{T}_{\Xi}(Y)$ such that

$$
\left(\binom{v_{i}^{\varphi\left(w_{i}\right)}}{P_{i}}_{i \in|w|}\right)^{\sharp} \circ \eta_{\downarrow \varphi^{\star}(w)}=\binom{v_{i}^{\varphi\left(w_{i}\right)}}{P_{i}}_{i \in|w|} .
$$

On the other hand, since $\mathbf{T}_{\Xi}$ has a right adjoint, the $\Xi$-algebras $\mathbf{T}_{\Xi}\left(Y \cup \downarrow \varphi^{\star}(w)\right)$ and $\mathbf{T}_{\Xi}(Y) \coprod \mathbf{T}_{\Xi}\left(\downarrow \varphi^{\star}(w)\right)$ are naturally isomorphic. Then, finally, we let $\mathrm{S}_{\left(P_{i}\right)_{i \in|w|}^{w}}^{w}$ stand for $\left[\operatorname{id}_{\mathbf{T}_{\Xi(Y)}},\left(\left(v_{i}^{v_{P_{i}}^{\varphi\left(w_{i}\right)}}\right)_{i \in|w|}\right)^{\sharp}\right]$, the unique homomorphism from $\mathbf{T}_{\Xi}(Y \cup \downarrow$ $\left.\varphi^{\star}(w)\right)$ to $\mathbf{T}_{\Xi}(Y)$ obtained, from $\operatorname{id}_{\mathbf{T}_{\Xi(Y)}}$ and $\left(\binom{v_{i}^{\varphi\left(w_{i}\right)}}{P_{i}}_{i \in|w|}\right)^{\sharp}$, by the universal property of the coproduct.
Proposition 3.24. Let $(\mathbf{c}, f)$ be a hyperderivor from $(\boldsymbol{\Sigma}, X)$ to $(\boldsymbol{\Xi}, Y)$. Then the $S$-sorted set $\mathrm{T}_{\Xi}(Y)_{\varphi}$ is equipped, in a natural way, with a structure of $\Sigma$-algebra.
Proof. Let $\mathbf{c}\left(\mathbf{T}_{\Xi}(Y)\right)$ be the $\Sigma$-algebra defined as follows: The underlying $S$-sorted set of $\mathbf{c}\left(\mathbf{T}_{\Xi}(Y)\right)$ is $\mathrm{T}_{\Xi}(Y)_{\varphi}$ while, for $(w, s) \in S^{\star} \times S$ and $\sigma \in \Sigma_{w, s}$, the operation $\sigma^{\mathbf{c}\left(\mathbf{T}_{\Xi}(Y)\right)}$ from $\mathbf{T}_{\Xi}(Y)_{\varphi^{\star}(w)}$ to $\mathbf{T}_{\Xi}(Y)_{\varphi(s)}$ associated to $\sigma$ is defined as:

$$
\sigma^{\mathbf{c}\left(\mathbf{T}_{\Xi(Y)}\right)}\left\{\begin{aligned}
\mathbf{T}_{\Xi}(Y)_{\varphi^{\star}(w)} & \longrightarrow \mathbf{T}_{\Xi}(Y)_{\varphi(s)} \\
\left(P_{i}\right)_{i \in|w|} & \longmapsto \mathrm{S}_{\left(P_{i}\right)_{i \in|w|}}^{w}\left(c_{w, s}(\sigma)\right)
\end{aligned}\right.
$$

Thus $\sigma^{\mathbf{c (}\left(\mathbf{T}_{\Xi}(Y)\right)}\left(\left(P_{i}\right)_{i \in|w|}\right)$ is the term obtained by substituting in $c_{w, s}(\sigma)$, for every $i \in|w|, P_{i}$ for $v_{i}^{\varphi\left(w_{i}\right)}$. Let us note that since $\left(\bigcup_{i \in|w|} \operatorname{Var}\left(P_{i}\right)\right) \cap V^{T}=\varnothing^{T}$, $\operatorname{Var}\left(\sigma^{\mathbf{c}\left(\mathbf{T}_{\Xi}(Y)\right)}\left(\left(P_{i}\right)_{i \in|w|}\right)\right) \cap V^{T}=\varnothing^{T}$. Therefore $\sigma^{\mathbf{c}\left(\mathbf{T}_{\Xi}(Y)\right)}\left(\left(P_{i}\right)_{i \in|w|}\right) \in \mathrm{T}_{\Xi}(Y)_{\varphi(s)}$. Consequently, the operation $\sigma^{\mathbf{c}\left(\mathbf{T}_{\Xi}(Y)\right)}$ is well-defined.

Definition 3.25. Let $(\mathbf{c}, f)$ be a hyperderivor from $(\boldsymbol{\Sigma}, X)$ to $(\boldsymbol{\Xi}, Y)$. Then the unique homomorphism $f^{\sharp}$ from $\mathbf{T}_{\Sigma}(X)$ to $\mathbf{c}\left(\mathbf{T}_{\Xi}(Y)\right)$ such that $f^{\sharp} \circ \eta_{X}=f$ will be called the tree homomorphism determined by the hyperderivor (c, $f$ ). Moreover, $f^{\sharp}$ will be called linear if $(\mathbf{c}, f)$ is linear.

The just defined tree homomorphisms are a generalization of those proposed by Gécseg and Steinby, in [14, Definition 4.13., on p. 78, for single-sorted algebras, which, in its turn, generalize those of Thatcher in 40].

Gécseg and Steinby in [14], on p. 70, wrote: "Tree homomorphisms are not really homomorphisms in the sense of algebra." Literally speaking the above assertion is true due, simply, to the fact that the signatures of the domain and codomain of a tree homomorphism are, in general, not identical. However, as we have seen $\mathrm{T}_{\Xi}(Y)_{\varphi}$ is equipped with a structure of $\Sigma$-algebra and $f^{\sharp}$ is a homomorphism of $\Sigma$-algebras from $\mathbf{T}_{\Sigma}(X)$ to $\mathbf{c}\left(\mathbf{T}_{\Xi}(Y)\right)$.

In 14, Theorem 4.18., on p. 82, Gécseg and Steinby proved, in the single-sorted case, that the inverse image of a recognizable language under a tree homomorphism is a recognizable language. We next extend Gécseg and Steinby's result to the many-sorted case.
Proposition 3.26. Let $(\mathbf{c}, f)$ be a hyperderivor from $(\boldsymbol{\Sigma}, X)$ to $(\boldsymbol{\Xi}, Y)$, $\mathbf{A}$ a $\boldsymbol{\Xi}$ algebra, and $g$ a surjective homomorphism from $\mathbf{T}_{\Xi}(Y)$ to $\mathbf{A}$. Then there exists a structure of $\Sigma$-algebra $F^{\mathbf{c}(\mathbf{A})}$ on $A_{\varphi}$ such that $g_{\varphi}$ is a homomorphism from $\mathbf{c}\left(\mathbf{T}_{\Lambda}(Y)\right)$ to $\mathbf{c}(\mathbf{A})=\left(A_{\varphi}, F^{\mathbf{c}(\mathbf{A})}\right)$.

Proof. Let $(w, s)$ be an element of $S^{\star} \times S$ and $\sigma \in \Sigma_{w, s}$. Then we denote by $F_{\sigma}^{\mathbf{c}(\mathbf{A})}$ the mapping from $A_{\varphi^{\star}(w)}$ to $A_{\varphi(s)}$ defined as follows:

$$
F_{\sigma}^{\mathbf{c}(\mathbf{A})}\left\{\begin{aligned}
A_{\varphi^{\star}(w)} & \longrightarrow A_{\varphi(s)} \\
\left(a_{i}\right)_{i \in|w|} & \longmapsto g_{\varphi(s)}\left(\sigma^{\mathbf{c}\left(\mathbf{T}_{\Xi}(Y)\right)}\left(\left(P_{i}\right)_{i \in|w|}\right)\right)
\end{aligned}\right.
$$

where, for every $i \in|w|, P_{i} \in \mathrm{~T}_{\Lambda}(Y)_{\varphi\left(w_{i}\right)}$ such that $g_{\varphi\left(w_{i}\right)}\left(P_{i}\right)=a_{i}$.
The operation $F_{\sigma}^{\mathbf{c}(\mathbf{A})}$ is well-defined. In fact, let $\left(a_{i}\right)_{i \in|w|}$ be an element of $A_{\varphi^{\star}(w)}$ and $\left(P_{i}\right)_{i \in|w|},\left(Q_{i}\right)_{i \in|w|}$ elements of $\mathrm{T}_{\Lambda}(Y)_{\varphi^{\star}(w)}$ such that, for every $i \in|w|$,

$$
g_{\varphi\left(w_{i}\right)}\left(P_{i}\right)=a_{i}=g_{\varphi\left(w_{i}\right)}\left(Q_{i}\right)
$$

But we have that:

$$
\begin{align*}
g_{\varphi(s)}\left(\sigma^{\left.\mathbf{T}_{\Lambda}(Y)_{\varphi}^{\Sigma}\left(\left(P_{i}^{\varphi\left(w_{i}\right)}\right)_{i \in|w|}\right)\right)}\right. & =g_{\varphi(s)}\left(\binom{v_{i}^{\varphi\left(w_{i}\right)}}{P_{i}^{\varphi\left(w_{i}\right)}}_{i \in|w|}\left(c_{w, s}(\sigma)\right)\right)  \tag{1}\\
& =g_{\varphi(s)}\left(\binom{v_{i}^{\varphi\left(w_{i}\right)}}{Q_{i}^{\varphi\left(w_{i}\right)}}_{i \in|w|}\left(c_{w, s}(\sigma)\right)\right)  \tag{2}\\
& =g_{\varphi(s)}\left(\sigma^{\mathbf{T}_{\Lambda}(Y)_{\varphi}^{\Sigma}}\left(\left(Q_{i}^{\varphi\left(w_{i}\right)}\right)_{i \in|w|}\right)\right) \tag{1}
\end{align*}
$$

where, to shorten notation, $\left(\dagger_{1}\right)$ and $\left(\dagger_{2}\right)$ stand for (by definition of $\mathbf{c}\left(\mathbf{T}_{\Xi}(Y)\right)$ ) and (by Lemma 3.7), respectively. Therefore

$$
g_{\varphi(s)}\left(\sigma^{\mathbf{c}\left(\mathbf{T}_{\Xi}(Y)\right)}\left(\left(P_{i}\right)_{i \in|w|}\right)\right)=g_{\varphi(s)}\left(\sigma^{\mathbf{c}\left(\mathbf{T}_{\Xi}(Y)\right)}\left(\left(Q_{i}\right)_{i \in|w|}\right)\right) .
$$

It is obvious that $g_{\varphi}$ is a homomorphism of $\Sigma$-algebras.
By construction $g_{\varphi}^{\sharp}$ is a homomorphism from $\mathbf{T}_{\Lambda}(Y)_{\varphi}^{\Sigma}$ to $g^{\sharp}\left[\mathbf{T}_{\Lambda}(Y)\right]_{\varphi}^{\Sigma}$. Indeed, for $\sigma \in \Sigma_{w, s}$ and a family $\left(P_{i}^{\varphi\left(w_{i}\right)}\right)_{i \in|w|} \in \mathrm{T}_{\Lambda}(Y)_{\varphi^{\star}(w)}$ we have that

$$
g_{\varphi(s)}^{\sharp}\left(\sigma^{\mathbf{T}_{\Lambda}(Y)_{\varphi}^{\Sigma}}\left(\left(P_{i}^{\varphi\left(w_{i}\right)}\right)_{i \in|w|}\right)\right)=\sigma^{g^{\sharp}\left[\mathbf{T}_{\Lambda}(Y)\right]_{\varphi}^{\Sigma}}\left(\left(g^{\sharp}\left(P_{i}^{\varphi\left(w_{i}\right)}\right)\right)_{i \in|w|}\right) .
$$

Proposition 3.27. Let $(\mathbf{c}, f)$ be a hyperderivor from $(\boldsymbol{\Sigma}, X)$ to $(\boldsymbol{\Xi}, Y), f^{\sharp}$ the tree homomorphism from $\mathbf{T}_{\Sigma}(X)$ to $\mathbf{c}\left(\mathbf{T}_{\Xi}(Y)\right)$ determined by $(\mathbf{c}, f)$, $s \in S$, and $L \in \operatorname{Rec}_{\varphi(s)}\left(\mathbf{T}_{\Xi}(Y)\right)$. Then $\left(f_{\varphi(s)}^{\sharp}\right)^{-1}[L] \in \operatorname{Rec}_{s}\left(\mathbf{T}_{\Sigma}(X)\right)$.
Proof. It follows from the definition of recognizability relative to a sort and from Proposition 3.26

Our immediate goal is to prove the many-sorted version of Theorem 4.16., on p. 80, in [14]. The proposition states that, for a sort $s \in S$, if the language $L$ is $s$-recognizable, then its direct image by the $s$-th coordinate of a linear tree homomorphism $f^{\sharp}$ is $\varphi(s)$-recognizable.

Assumption. To prove the following proposition we will assume that $S, T, \Sigma$ and $X$ are finite.

Proposition 3.28. Let $(\mathbf{c}, f)$ be a linear hyperderivor from $(\boldsymbol{\Sigma}, X)$ to $(\boldsymbol{\Xi}, Y), f^{\sharp}$ the linear tree homomorphism from $\mathbf{T}_{\Sigma}(X)$ to $\mathbf{c}\left(\mathbf{T}_{\Xi}(Y)\right)$ determined by $(\mathbf{c}, f), s \in S$, and $L \in \mathrm{~T}_{\Sigma}(X)_{s}^{\wp}$. If $L \in \operatorname{Rec}_{s}\left(\mathbf{T}_{\Sigma}(X)\right)$, then $f_{s}^{\sharp}[L] \in \operatorname{Rec}_{\varphi(s)}\left(\mathbf{T}_{\Xi}(Y)\right)$.

Proof. Let $\Phi$ be the congruence on $\mathbf{T}_{\Xi}(Y)$ defined as follows:

$$
\Phi=\bigcap_{(x, r) \in \amalg X} \Omega^{\mathbf{T} \Xi(Y)}\left(\delta^{\varphi(r),\left\{f_{r}(x)\right\}}\right)
$$

Since $f: X \longrightarrow \mathrm{~T}_{\Xi}(Y)_{\varphi}$ we have that, for every $r \in S$ and every $x \in X_{r}$, the language $\left\{f_{r}(x)\right\}$ is in $\operatorname{Rec}_{\varphi(r)}\left(\mathbf{T}_{\Xi}(Y)\right)$. Hence, $\Omega^{\mathbf{T}_{\Xi}(Y)}\left(\delta^{\varphi(r),\left\{f_{r}(x)\right\}}\right)$ is of finite index. Therefore, by the assumption and Proposition 2.61, $\Phi$ is of finite index.

Moreover, for every $r \in S$ and every $x \in X_{r}$, the language $\left\{f_{r}(x)\right\}$ is $\Phi_{\varphi(r)^{-}}$ saturated.

For abbreviation we let $\Theta$ stand for $\Omega^{\mathbf{T}_{\Sigma}(X)}\left(\delta^{s, L}\right)$, the syntactic congruence on $\mathbf{T}_{\Sigma}(X)$ determined by $L$. Since $L$ is a language in $\operatorname{Rec}_{s}\left(\mathbf{T}_{\Sigma}(X)\right), \Theta$ is of finite index. Moreover, for $s \in S, L$ is $\Theta_{s}$-saturated.

From now on, for every $r \in S, k_{r}$ and $\mathcal{W}_{\Theta_{r}}=\left\{W_{r, l} \mid l \in k_{r}\right\}$ stand for the index of $\Theta_{r}$ and a fixed transversal of $\mathrm{T}_{\Sigma}(X) / \Theta_{r}$ in $\mathrm{T}(X)_{r}$, respectively. Moreover, $k$ and $\mathcal{W}_{\Theta}$ denote the $S$-sorted sets $\left(k_{r}\right)_{r \in S}$ and $\left(\mathcal{W}_{\Theta_{r}}\right)_{r \in S}$, respectively.

Let $\Psi=\left(\Psi_{t}\right)_{t \in T}$ be the binary relation on $\mathrm{T}_{\Xi}(Y)$ defined as follows: For every $t \in T, \Psi_{t}$ is the binary relation on $\mathrm{T}_{\Xi}(Y)_{t}$ consisting of all ordered pairs $(M, N) \in$ $\mathrm{T}_{\Xi}(Y)_{t}^{2}$ such that the following two conditions are satisfied:
(1) $(M, N) \in \Phi_{t}$.
(2) $\forall(w, r) \in S^{\star} \times S, \forall \sigma \in \Sigma_{w, r}, \forall R \in \operatorname{Subt}\left(c_{w, r}(\sigma)\right)_{t}, \forall i \in|w|, \forall l_{i} \in k_{w_{i}}$

$$
\left.\left(M \in\binom{v_{i}^{\varphi\left(w_{i}\right)}}{f_{w_{i}}^{\sharp}\left[\left[W_{w_{i}}, l_{i}\right]_{\left.\Theta_{w_{i}}\right]}\right]}_{i \in|w|}\right)_{t}^{\sharp}(R) \leftrightarrow N \in\left(\binom{v_{i}^{\varphi\left(w_{i}\right)}}{f_{w_{i}}^{\sharp}\left[\left[W_{w_{i}}, l_{i}\right] \Theta_{w_{i}}\right.}_{i \in|w|}\right)_{t}^{\sharp}(R)\right) .
$$

By definition, for every $t \in T, \Psi_{t}$ is a refinement of $\Phi_{t}$ and an equivalence relation on $\mathrm{T}_{\Xi}(Y)_{t}$. Moreover, if $a=\operatorname{card}\left(\mathrm{T}_{\Xi}(Y) / \Phi\right), b=\operatorname{card}(\Sigma)$,

$$
d=\max \left\{\operatorname{card}\left(\operatorname{Subt}\left(c_{w, r}(\sigma)\right)_{t}\right) \mid(w, r) \in S^{\star} \times S, \sigma \in \Sigma_{w, r}, t \in T\right\}
$$

and $e=\max \left\{|w| \mid(w, r) \in S^{\star} \times S \& \Sigma_{w, r} \neq \varnothing\right\}$, then the index of $\Psi_{t}$ is bounded by $a 2^{b d \operatorname{card}(k)^{e}}$. In addition, all different possibilities defining $\Psi$ are bounded. Consequently, the $T$-sorted set $\mathrm{T}_{\Xi}(Y) / \Psi$ is finite.

Let us check that $\Psi$ is a congruence on $\mathbf{T}_{\Xi}(Y)$. Let $(u, t)$ be an element of $\left(T^{\star}-\{\lambda\}\right) \times T, \xi \in \Xi_{u, t}$, and let $\left(M_{j}\right)_{j \in|u|}$ and $\left(N_{j}\right)_{j \in|u|}$ be elements of $\mathrm{T}_{\Xi}(Y)_{u}$ such that, for every $j \in|u|,\left(M_{j}, N_{j}\right) \in \Psi_{u_{j}}$. We want to show that $\left(\xi\left(\left(M_{j}\right)_{j \in|u|}\right), \xi\left(\left(N_{j}\right)_{j \in|u|}\right)\right)$ is an element of $\Psi_{t}$. Let us note that, by definition of $\Psi$, for every $j \in|u|$, we have that $\left(M_{j}, N_{j}\right) \in \Phi_{u_{j}}$. Therefore, since $\Phi$ is a congruence on $\mathrm{T}_{\Xi}(Y),\left(\xi\left(\left(M_{j}\right)_{j \in|u|}\right), \xi\left(\left(N_{j}\right)_{j \in|u|}\right)\right)$ belongs to $\Phi_{t}$. So the pair $\left(\xi\left(\left(M_{j}\right)_{j \in|u|}\right), \xi\left(\left(N_{j}\right)_{j \in|u|}\right)\right)$ satisfies the first condition for being related under $\Psi_{t}$.

Regarding the second condition, let us assume that, for $(w, r) \in S^{\star} \times S, \sigma \in \Sigma_{w, r}$, $R \in \operatorname{Subt}\left(c_{w, r}(\sigma)\right)_{t}, i \in|w|$, and $l_{i} \in k_{w_{i}}$, we have that

$$
\xi\left(\left(M_{j}\right)_{j \in|u|}\right) \in\left(\binom{v_{i}^{\varphi\left(w_{i}\right)}}{f_{w_{i}}^{\sharp}\left[\left[W_{w_{i}}, l_{i}\right] \Theta_{w_{i}}\right]}_{i \in|w|}\right)_{t}^{\sharp}(R) .
$$

However, given that the tree homomorphism $f^{\sharp}$ is linear, we have that, for every $(w, r) \in S^{\star} \times S$, every $\sigma \in \Sigma_{w, r}$, and every $i \in|w|$, no variable $v_{i}^{\varphi\left(w_{i}\right)}$ appears more than once in $c_{w, r}(\sigma)$. Thus, for every $i \in|w|,|R|_{v_{i}^{\varphi\left(w_{i}\right)}} \leq 1$. Hence there exists a family $\left(W_{i}\right)_{i \in|w|}$ in $\mathrm{T}_{\Sigma}(X)_{w}$ such that, for every $i \in|w|, W_{i} \in\left[W_{w_{i}, l_{i}}\right]_{\Theta_{w_{i}}}$ and

$$
\begin{equation*}
\xi\left(\left(M_{j}\right)_{j \in|u|}\right)=\binom{v_{i}^{\varphi\left(w_{i}\right)}}{f_{w_{i}}^{\sharp}\left(W_{i}\right)}_{i \in|w|}(R) . \tag{3.5}
\end{equation*}
$$

Now, for $R$, either (a) $\downarrow \varphi^{\star}(w) \cap \operatorname{Var}(R)=\varnothing^{T}$ or (b) $\downarrow \varphi^{\star}(w) \cap \operatorname{Var}(R) \neq \varnothing^{T}$.
In case (a), Equation [3.5 turns into $\xi\left(\left(M_{j}\right)_{j \in|u|}\right)=R$. It follows that, for every $i \in|w|$ and every $j \in|u|,\left|M_{j}\right|_{v^{\varphi}\left(w_{i}\right)}=0$. Moreover, since $R \in \operatorname{Subt}\left(c_{w, r}(\sigma)\right)_{t}$, we have that, for every $j \in|u|, M_{j} \in \operatorname{Subt}\left(c_{w, r}(\sigma)\right)_{u_{j}}$. In addition, for every $j \in|u|$,

$$
M_{j} \in\left(\binom{v_{i}^{\varphi\left(w_{i}\right)}}{f_{w_{i}}^{\sharp}\left[\left[W_{w_{i}}, l_{i}\right]_{\Theta_{w_{i}}}\right]}_{i \in|w|}\right)_{u_{j}}^{\sharp}\left(M_{j}\right)=\left\{M_{j}\right\} .
$$

Since, for every $j \in|u|,\left(M_{j}, N_{j}\right) \in \Psi_{u_{j}}$, it happens that $N_{j}=M_{j}$. Thus $\xi\left(\left(N_{j}\right)_{j \in|u|}\right)=\xi\left(\left(M_{j}\right)_{j \in|u|}\right)=R$. Therefore this case is settled.

In case (b), in its turn, either (b.1) there exists an $i \in|w|$ such that $R=v_{i}^{\varphi\left(w_{i}\right)}$ or (b.2) there exists a unique $\left(R_{j}\right)_{j \in|u|} \in \mathrm{T}_{\Xi}(Y)_{u}$ such that $R=\xi\left(\left(R_{j}\right)_{j \in|u|}\right)$. Note that if $R$ starts with an operation symbol from $\Xi$, then, from Equation 3.5, it follows that $\xi$ is the only possibility.

In case (b.1), Equation 3.5 turns into

$$
\begin{equation*}
\xi\left(\left(M_{j}\right)_{j \in|u|}\right)=f_{w_{i}}^{\sharp}\left(W_{i}\right) . \tag{3.6}
\end{equation*}
$$

Note that in this case $t$ must be equal to $\varphi\left(w_{i}\right)$.
Case (b.1) still needs to be further refined attending to the different possibilities for $W_{i}$ according to the definition of the tree homomorphism $f^{\sharp}$. Either (b.1.i) there exists a unique $x \in X_{w_{i}}$ such that $W_{i}=x$ or (b.1.ii) there exists a unique $w^{\prime} \in S^{\star}$ a unique $\nu \in \Sigma_{w^{\prime}, w_{i}}$, and a unique $\left(\left(Q_{i^{\prime}}\right)_{i^{\prime} \in\left|w^{\prime}\right|}\right) \in \mathrm{T}_{\Sigma}(X)_{w^{\prime}}$ such that $W_{i}=\nu\left(\left(Q_{i^{\prime}}\right)_{i^{\prime} \in\left|w^{\prime}\right|}\right)$.

In case (b.1.i), the value of the tree homomorphism $f_{w_{i}}^{\sharp}$ at $W_{i}$ is $f_{w_{i}}(x)$. Let us recall that $\Phi$ was defined so that the set $\left\{f_{w_{i}}(x)\right\}$ is recognized by $\Phi_{\varphi\left(w_{i}\right)}$. Since, for every $j \in|u|,\left(M_{j}, N_{j}\right) \in \Phi_{u}$, it follows that $\xi\left(\left(N_{j}\right)_{j \in|u|}\right) \in\left\{f_{w_{i}}(x)\right\}$. Therefore this case is settled.

In case (b.1.ii), the value of the tree homomorphism $f_{w_{i}}^{\sharp}$ at $W_{i}$ is given by

$$
f_{w_{i}}^{\sharp}\left(W_{i}\right)=\binom{c_{i^{\prime}}^{\varphi\left(w_{i^{\prime}}^{\prime}\right)}}{f_{w_{i^{\prime}}^{\prime}}^{\sharp}\left(Q_{i^{\prime}}\right)}_{i^{\prime} \in\left|w^{\prime}\right|}\left(d_{w^{\prime}, w_{i}}(\nu)\right) .
$$

In virtue of Equation [3.6] this last term is equal to $\xi\left(\left(M_{j}\right)_{j \in|u|}\right)$. On the other hand, since no substitution changes the operation symbol, we have that $d_{w^{\prime}, w_{i}}(\nu)$ has the form $\xi\left(\left(R_{j}\right)_{j \in|u|}\right)$ for a unique $\left(R_{j}\right)_{j \in|u|} \in \mathrm{T}_{\Xi}\left(Y \cup \downarrow \varphi^{\star}(w)\right)_{u}$. Moreover, for every $j \in|u|, R_{j}$ is a subterm of sort $u_{j}$ of $d_{w^{\prime}, w_{i}}(\nu)$. Hence, by the linearity of the tree homomorphism $f^{\sharp}$, we have that

$$
\left.\left(\begin{array}{c}
\left.v_{v_{i^{\prime}}\left(w_{i^{\prime}}^{\prime}\right)}^{f_{w_{i^{\prime}}^{\prime}}^{\prime}}\right)_{i^{\prime}}
\end{array}\right)_{i^{\prime} \in\left|w^{\prime}\right|}\left(\xi\left(\left(R_{j}\right)_{j \in|u|}\right)\right)=\xi\left(\binom{v_{i^{\prime}}^{\varphi\left(w_{i^{\prime}}^{\prime}\right)}}{f_{w_{i^{\prime}}^{\prime}}^{\prime}\left(Q_{i^{\prime}}\right)}_{i^{\prime} \in\left|w^{\prime}\right|}\left(R_{j}\right)\right)_{j \in|u|}\right) .
$$

Consequently, for every $j \in|u|, M_{j}=\binom{v_{i^{\prime}}^{\varphi\left(w_{i^{\prime}}^{\prime}\right)}}{f_{w_{i^{\prime}}^{\prime \prime}}^{f^{\prime}}\left(Q_{i^{\prime}}\right)}_{i^{\prime} \in\left|w^{\prime}\right|}\left(R_{j}\right)$. Thus, for every $j \in|u|, M_{j} \in\left(\binom{v_{i^{\prime}}^{\varphi\left(w_{i^{\prime}}^{\prime}\right)}}{\left.f_{w_{i^{\prime}}^{\prime}}^{\sharp}\left[Q_{i^{\prime}}\right] \Theta_{w_{i^{\prime}}^{\prime}}\right]}_{i^{\prime} \in\left|w^{\prime}\right|}\right)_{u_{j}}^{\#}\left(R_{j}\right)$. However, since, for every $j \in|u|,\left(M_{j}, N_{j}\right) \in \Psi_{u_{j}}$, we can assert that $N_{j} \in\left(\binom{v_{i^{\prime}}^{\varphi\left(w_{i^{\prime}}^{\prime}\right)}}{\left.f_{w_{i^{\prime}}^{\prime}}^{\sharp}\left[Q_{i^{\prime}}\right] \Theta_{w_{i^{\prime}}^{\prime}}\right]}_{i^{\prime} \in\left|w^{\prime}\right|}\right)_{u_{j}}^{\#}\left(R_{j}\right)$. Therefore, for every $i^{\prime} \in\left|w^{\prime}\right|$, there exists a $\bar{Q}_{i^{\prime}} \in\left[Q_{i^{\prime}}\right]_{\Theta_{w^{\prime}}^{\prime}}$ such that $N_{j}=$ $\binom{v_{i^{\prime}}^{\varphi\left(w_{i^{\prime}}^{\prime}\right)}}{f_{w_{i^{\prime}}^{\prime}}^{\sharp}\left(\bar{Q}_{i^{\prime}}\right)}_{i^{\prime} \in\left|w^{\prime}\right|}\left(R_{j}\right)$. Let $\bar{W}_{i}$ stand for $\nu\left(\left(\bar{Q}_{i^{\prime}}\right)_{i^{\prime} \in\left|w^{\prime}\right|}\right)$.

Before proceeding any further, let us note that

$$
f_{w_{i}}^{\sharp}\left(\bar{W}_{i}\right)=\binom{v_{i^{\prime}}^{\varphi\left(w_{i^{\prime}}^{\prime}\right)}}{f_{w_{i^{\prime}}^{\prime}}\left(\bar{Q}_{i^{\prime}}\right)}_{i^{\prime} \in\left|w^{\prime}\right|}\left(d_{w^{\prime}, w_{i}}(\nu)\right)=\xi\left(\left(N_{j}\right)_{j \in|u|}\right) .
$$

Now, since, for every $i^{\prime} \in w^{\prime}, \bar{Q}_{i^{\prime}} \in\left[Q_{i^{\prime}}\right]_{\Theta_{i^{\prime}}}$, it follows that $\bar{W}_{i} \in\left[W_{i}\right]_{\Theta_{w_{i}}}$ and $\xi\left(\left(N_{j}\right)_{j \in|u|}\right) \in\left(\binom{v_{i}^{\varphi\left(w_{i}\right)}}{f_{w_{i}}^{\sharp}\left[\left[W_{i}\right]_{\left.\Theta_{w_{i}}\right]}\right]}_{i \in|w|}\right)_{t}^{\sharp}\left(v_{i}^{\varphi\left(w_{i}\right)}\right)$. Therefore this case is settled.

It only remains to consider the case (b.2), in which $R=\xi\left(\left(R_{j}\right)_{j \in|u|}\right)$. Again, by the linearity of the tree homomorphism $f^{\sharp}$, it follows, from Equation [3.5] that

$$
\xi\left(\left(M_{j}\right)_{j \in|u|}\right)=\xi\left(\left(\binom{v_{i}^{\varphi}\left(w_{i}\right)}{f_{w_{i}}^{\#}\left(Q_{i}\right)}_{i \in|w|}\left(R_{j}\right)\right)_{j \in|u|}\right) .
$$

Hence, for every $j \in|u|, M_{j}=\binom{\varphi_{i}^{\varphi\left(w_{i}\right)}}{f_{w_{i}}^{\ddagger}\left(Q_{i}\right)}_{i \in|w|}\left(R_{j}\right)$. Then the statement may be handled in much the same way as in case (b.1.ii).

Therefore $\Psi$ is a congruence on $\mathbf{T}_{\Xi}(Y)$ of $T$-finite index.
Finally, we prove that $\left[f_{s}^{\sharp}[L]\right]_{\varphi(s)}^{\Psi}=f_{s}^{\sharp}[L]$, i.e., that $f_{s}^{\sharp}[L]$ is $\Psi_{\varphi(s)}$-saturated. Since, obviously, $f_{s}^{\sharp}[L] \subseteq\left[f_{s}^{\sharp}[L]\right]_{\varphi(s)}^{\Psi}$, we restrict ourselves to prove the other inclusion.

Let $M$ be a term in $\left[f_{s}^{\sharp}[L]\right]_{\varphi(s)}^{\Psi}$, then there exists a term $P \in L$ such that $M$ and $f_{s}^{\sharp}(P)$ are $\Psi_{\varphi(s)}$-related. Now, for $P$, either (a) $P=x$, for a unique variable $x \in X_{s}$, or (b) $P=\sigma\left(\left(P_{i}\right)_{i \in|w|}\right)$, for a unique $w \in S^{\star}$, a unique $\sigma \in \Sigma_{w, s}$, and a unique $\left(P_{i}\right)_{i \in|w|} \in \mathrm{T}_{\Sigma}(X)_{w}$. Before proceeding any further, let us explain the reason why, in the second case, it is unnecessary to take into account sorts $r \in S-\{s\}$. The reason is simple, terms of the form $\sigma\left(\left(P_{i}\right)_{i \in|w|}\right)$, for $\sigma \in \Sigma_{w, r}$ with $r \neq s$ and $\left(P_{i}\right)_{i \in|w|} \in \mathrm{T}_{\Sigma}(X)_{w}$ are terms of sort $r$, and, consequently, terms that are not in $L$.

In case (a), $f_{s}^{\sharp}(x)=f_{s}(x)$. By construction of $\Phi$, the set $\left\{f_{s}(x)\right\}$ is $\Phi_{\varphi(s)^{-}}$ saturated. Moreover, since $\Psi_{\varphi(s)}$ is a refinement of $\Phi_{\varphi(s)}$, we conclude that $\left\{f_{s}(x)\right\}$ is $\Psi_{\varphi(s)}$-saturated. In this case, sinece $M$ is related to $f_{s}(x)$ for $\Psi_{\varphi(s)}$, we conclude that $M$ must be equal to $f_{s}(x)$ and, consequently, $M$ is an element in $f_{s}^{\sharp}[L]$.

In case $(\mathrm{b}), f_{s}^{\sharp}\left(\sigma\left(\left(P_{i}\right)_{i \in|w|}\right)\right)=\binom{v_{i}^{\varphi\left(w_{i}\right)}}{f_{w_{i}}^{\sharp}\left(P_{i}\right)}_{i \in|w|}\left(c_{w, s}(\sigma)\right)$. Note that

$$
\binom{v_{i}^{\varphi\left(w_{i}\right)}}{f_{w_{i}}^{\sharp}\left(P_{i}\right)}_{i \in|w|}\left(c_{w, s}(\sigma)\right) \in\left(\binom{v_{i}^{\varphi\left(w_{i}\right)}}{\left.f_{w_{i}}^{\sharp}\left[P P_{i}\right]_{\Theta_{i}}\right]}_{i \in|w|}\right)_{s}^{\#}\left(c_{w, s}(\sigma)\right) .
$$

By construction of $\Psi$, it follows that $M \in\left(\binom{v_{i}^{\varphi\left(w_{i}\right)}}{f_{w_{i}}^{\sharp}\left[\left[P_{i}\right]_{\Theta_{i}}\right]}_{i \in|w|}\right)_{s}^{\sharp}\left(c_{w, s}(\sigma)\right)$. Therefore, there exists, for every $i \in|w|$, terms $Q_{i}$ in $\left[P_{i}\right]_{\Theta_{w_{i}}}$ such that

$$
M=\binom{v_{i}^{\varphi\left(w_{i}\right)}}{f_{w_{i}}^{t_{i}}\left(Q_{i}\right)}_{i \in|w|}\left(c_{w, s}(\sigma)\right) .
$$

It follows that $M=f_{s}^{\sharp}\left(\sigma\left(\left(Q_{i}\right)_{i \in|w|}\right)\right)$. Since $Q_{i} \in\left[P_{i}\right]_{\Theta_{w_{i}}}$, and $\Theta$ is a congruence on $\mathrm{T}_{\Sigma}(X)$, we conclude that $\sigma\left(\left(P_{i}\right)_{i \in|w|}\right)$ and $\sigma\left(\left(Q_{i}\right)_{i \in|w|}\right)$ are $\Theta_{s}$-related. Moreover, since $L$ is $\Theta_{s}$-saturated and $\sigma\left(\left(P_{i}\right)_{i \in|w|}\right)$ is a term in $L$, we can assert that $\sigma\left(\left(Q_{i}\right)_{i \in|w|}\right)$ is also a term in $L$. Consequently, $M$ is a term in $f_{s}^{\sharp}[L]$.

Hence, $f_{s}^{\sharp}[L]$ is $\Psi_{\varphi(s)}$-saturated.
In [15], Proposition 7.7, on p. 18, Gécseg and Steinby state that, for a homomorphism $f: \mathbf{T}_{\Sigma}(X) \longrightarrow \mathbf{T}_{\Sigma}(Y)$ between single-sorted algebras and a subset $L$ of $\mathrm{T}_{\Sigma}(X)$, if $L \in \operatorname{Rec}\left(\mathbf{T}_{\Sigma}(X)\right)$, then $f[L] \in \operatorname{Rec}\left(\mathbf{T}_{\Sigma}(Y)\right)$. Moreover, they, seemingly, provide a proof of it. One can obtain a plain proof of such a result, as an immediate corollary of Proposition 3.28 and particularizing it to the single-sorted case, taking
into account the following fact: Every homomorphism $f: \mathbf{T}_{\Sigma}(X) \longrightarrow \mathbf{T}_{\Sigma}(Y)$ between many-sorted algebras is an instance of a linear tree homomorphism. Indeed, for $\left(\left(\operatorname{id}_{S}, c\right), f \circ \eta_{X}\right)$, where $c=\left(c_{w, s}\right)_{(w, s) \in S^{\star} \times S}$ is the family of mappings defined, for every $(w, s) \in S^{\star} \times S$, as follows:

$$
c_{w, s}\left\{\begin{aligned}
\Sigma_{w, s} & \longrightarrow \mathrm{~T}_{\Sigma}(Y \cup \downarrow w)_{s} \\
\sigma & \longmapsto \sigma\left(v_{0}^{w_{0}}, \ldots, v_{|w|-1}^{w_{|w|-1}}\right)
\end{aligned}\right.
$$

we have that $\left(f \circ \eta_{X}\right)^{\sharp}: \mathbf{T}_{\Sigma}(X) \longrightarrow \mathbf{T}_{\Sigma}(Y)$, the tree homomorphism determined by the pair $\left(\left(\operatorname{id}_{S}, c\right), f \circ \eta_{X}\right)$, is $f$.

Corollary 3.29. Let $f$ be a homomorphism from $\mathbf{T}_{\Sigma}(X)$ to $\mathbf{T}_{\Sigma}(Y), s \in S$ and $L \in \mathrm{~T}_{\Sigma}(X)_{s}^{\wp}$. If $L \in \operatorname{Rec}_{s}\left(\mathbf{T}_{\Sigma}(X)\right)$, then $f_{s}[L] \in \operatorname{Rec}_{s}\left(\mathbf{T}_{\Sigma}(Y)\right)$.

We next provide, among other things, a categorial rendering of the just indicated result, but for a suitable class of homomorphisms.

Proposition 3.30. Let $\operatorname{Rec}_{\Sigma}$ be the mapping that sends an $S$-sorted set $X$ to the $\Sigma$-algebra Rec. $\left(\mathbf{T}_{\Sigma}(X)\right)$ and an $S$-sorted mapping $f$ from $X$ to $Y$ to the $S$ sorted mapping $\left(f^{@}\right)^{\wp}=\left(f_{s}^{@}[\cdot]\right)_{s \in S}$ from $\left(\operatorname{Rec}_{s}\left(\mathbf{T}_{\Sigma}(X)\right)\right)_{s \in S}$ to $\left(\operatorname{Rec}_{s}\left(\mathbf{T}_{\Sigma}(Y)\right)\right)_{s \in S}$, where, we recall, $f^{@}$ is the unique homomorphism from $\mathbf{T}_{\Sigma}(X)$ to $\mathbf{T}_{\Sigma}(Y)$ such that $f^{@} \circ \eta_{X}=\eta_{Y} \circ f$-let us note that $\left(f^{@}\right)^{\wp}$ is an $S$-sorted mapping from $\mathrm{T}_{\Sigma}(X)^{\wp}$ to $\mathrm{T}_{\Sigma}(Y)^{6}$, and that to simplify notation we have used the same symbol for its restriction to $\left(\operatorname{Rec}_{s}\left(\mathbf{T}_{\Sigma}(X)\right)\right)_{s \in S}$ and $\left(\operatorname{Rec}_{s}\left(\mathbf{T}_{\Sigma}(Y)\right)\right)_{s \in S}$. Then, for every $(w, s) \in$ $S^{\star} \times S$ and every $\sigma \in \Sigma_{w, s}, f_{s}^{@}[\cdot] \circ \sigma^{\wp}=\sigma^{\wp} \circ f_{w}^{@}[\cdot]$, i.e., for every $\left(L_{i}\right)_{i \in|w|} \in$ $\prod_{i \in|w|} \operatorname{Rec}_{w_{i}}\left(\mathbf{T}_{\Sigma}(X)\right)$, the sets

$$
\begin{gathered}
\left(f_{s}^{@}[\cdot] \circ \sigma^{\wp}\right)\left(\left(L_{i}\right)_{i \in|w|}\right)=\left\{f_{s}^{@}\left(\sigma\left(\left(P_{i}\right)_{i \in n}\right)\right) \mid\left(P_{i}\right)_{i \in|w|} \in \prod_{i \in|w|} L_{i}\right\} \text { and } \\
\left(\sigma^{\wp} \circ f_{w}^{@}[\cdot]\right)\left(\left(L_{i}\right)_{i \in|w|}\right)=\left\{\sigma\left(\left(Q_{i}\right)_{i \in n}\right) \mid\left(Q_{i}\right)_{i \in|w|} \in \prod_{i \in|w|} f_{w_{i}}\left[L_{i}\right]\right\}
\end{gathered}
$$

are equal. Thus $\left(f^{@}\right)^{\wp}$ is a homomorphism from the $\Sigma$-algebra $\operatorname{Rec} .\left(\mathbf{T}_{\Sigma}(X)\right)$ to the $\Sigma$-algebra Rec. $\left(\mathbf{T}_{\Sigma}(Y)\right)$. Therefore $\operatorname{Rec}_{\Sigma}$ is a functor from $\mathbf{S e t}^{S}$ to $\operatorname{Alg}(\Sigma)$ and a subfunctor of $(\cdot)^{\wp} \circ \mathbf{T}_{\Sigma}$.

Let $f$ be an $S$-sorted mapping from $X$ to $Y,\{\cdot\}_{X}^{\sharp}$ the canonical extension of the $S$-sorted mapping $\{\cdot\}_{X}$ from $X$ to $\left(\operatorname{Rec}_{s}\left(\mathbf{T}_{\Sigma}(X)\right)\right)_{s \in S}$ that, for every $s \in S$, sends $x$ in $X_{s}$ to $\{x\}$ in $\operatorname{Rec}_{s}\left(\mathbf{T}_{\Sigma}(X)\right)$, and $\{\cdot\}_{Y}^{\#}$ the canonical extension of the $S$-sorted mapping $\{\cdot\}_{Y}$ from $Y$ to $\left(\operatorname{Rec}_{s}\left(\mathbf{T}_{\Sigma}(Y)\right)\right)_{s \in S}$. Then the following diagram commutes


Therefore $\left(\{\cdot\}_{X}^{\sharp}\right)_{X \in \mathcal{U}}$ is a natural transformation from $\mathbf{T}_{\Sigma}$ to $\operatorname{Rec}{ }_{\Sigma}$.
Let $X$ be an $S$-sorted set, $(w, s) \in S^{\star} \times S, \sigma \in \Sigma_{w, s}, i \in|w|$, and $\left(L_{i}\right)_{i \in|w|} \in$ $\prod_{i \in|w|} \operatorname{Rec}_{w_{i}}\left(\mathbf{T}_{\Sigma}(X)\right)$ such that $L_{i}=L^{\prime} \cup L^{\prime \prime}$. Then

$$
\sigma^{\wp}\left(\left(L_{i}\right)_{i \in|w|}\right)=\sigma^{\wp}\left(L_{0}, \ldots, L^{\prime}, \ldots, L_{|w|-1}\right) \cup \sigma^{\wp}\left(L_{0}, \ldots, L^{\prime \prime}, \ldots, L_{|w|-1}\right)
$$

Hence Rec. $\left(\mathbf{T}_{\Sigma}(X)\right)$ is a many-sorted Boolean Algebra with operators, i.e., a $\Sigma$ algebra such that, for every $s \in S, \mathbf{R e c}_{s}\left(\mathbf{T}_{\Sigma}(X)\right.$ ) is a Boolean algebra and the structures of $\Sigma$-algebra and of Boolean algebra are compatibles as stated above (for the concept of single-sorted Boolean algebra with operators see [23] and [24]).

Finally, let $f: X \longrightarrow Y$ an $S$-sorted mapping. Then the homomorphism $\left(f^{@}\right)^{\wp}=$ $\left(f_{s}^{@}[\cdot]\right)_{s \in S}$ is such that, for every $s \in S$, $f_{s}^{@}[\cdot]$ is a Boolean algebra homomorphism from $\boldsymbol{\operatorname { R e c }}_{s}\left(\mathbf{T}_{\Sigma}(X)\right)$ to $\boldsymbol{\operatorname { R e c }}_{s}\left(\mathbf{T}_{\Sigma}(Y)\right)$.

From the just stated proposition and Proposition 2.69 we obtain the following corollary.

Corollary 3.31. Let $X$ be an $S$-sorted set. Then the Boolean algebra $\boldsymbol{\operatorname { R e c }}\left(\mathbf{T}_{\Sigma}(X)\right)$ is a subdirect product of the family of Boolean algebras $\left(\boldsymbol{R e c}_{s}\left(\mathbf{T}_{\Sigma}(X)\right)\right)_{s \in S}$. Moreover, if $f: X \longrightarrow Y$ is an $S$-sorted mapping, then $f^{@}[\cdot]$ and $\prod\left(f^{@}\right)^{\wp}=\prod_{s \in S} f_{s}^{@}[\cdot]$ are compatible with the subdirect embeddings of $\boldsymbol{\operatorname { R e c }}\left(\mathbf{T}_{\Sigma}(X)\right)$ and $\boldsymbol{\operatorname { R e c }}\left(\mathbf{T}_{\Sigma}(Y)\right)$ into $\prod_{s \in S} \boldsymbol{R e c}_{s}\left(\mathbf{T}_{\Sigma}(X)\right)$ and $\prod_{s \in S} \mathbf{R e c}_{s}\left(\mathbf{T}_{\Sigma}(Y)\right)$, respectively.
Remark. From the results stated above it seems natural to consider categories of the type BoolOp ${ }^{S}$, where BoolOp is a category of Boolean algebras with operators, and to investigate a duality theory for them. Toward this end it is perhaps worth pointing out that, for a finite set of sorts $S$, since the discrete category determined by $S$ is loopless (it has no non-identity endomorphisms), $\operatorname{Pro}\left(\operatorname{Set}_{\mathrm{f}}^{S}\right)$ is equivalent to $\left(\mathbf{P r o}\left(\mathbf{S e t}_{\mathrm{f}}\right)\right)^{S}$. But $\left(\mathbf{P r o}\left(\mathbf{S e t}_{\mathrm{f}}\right)\right)^{S}$ and $\mathbf{S t o n e}^{S}$ are equivalent, because $\operatorname{Pro}\left(\mathbf{S e t}_{f}\right)$ and Stone are equivalent. Thus $\operatorname{Pro}\left(\operatorname{Set}_{\mathrm{f}}^{S}\right)$ and $\mathbf{S t o n e}^{S}$ are equivalent. Therefore $\left(\text { Bool }^{S}\right)^{\text {op }}$ and Stone ${ }^{S}$ are equivalent.
3.6. Derivors and recognizability. Thatcher in 40, on p. 132, wrote: "Generally, the term 'transformation' will mean any map from $\mathrm{T}_{\Sigma}\left[\right.$ the free algebra $\mathrm{T}_{\Sigma}(\varnothing)$, we add] into $\mathrm{T}_{\Omega}\left[\right.$ the free algebra $\mathrm{T}_{\Omega}(\varnothing)$, we add] where $\Sigma=(\Sigma, r)$ and $\Omega=(\Omega, s)$ are ranked alphabets. We will, in the sequel, define several types of transformations which are of particular interest. In formulating these definitions, there were three principal considerations or objectives: (1) It was intended that the transformations be, in some sense, natural in that they would fit the algebraic framework within which we are working; (2) We should be able to generalize the conventional concept of finite state mapping . . . to the case for trees; and (3) It was hoped that the end result would be a unified approach taking into account various formulations of 'transformation,' 'transduction,' and 'translation' which have appeared in the literature."

Following Thatcher's first dictum, in this subsection, after defining the variety of Hall algebras, we introduce the notion of derivor between many-sorted signatures. This will allow us, using the homomorphisms between Hall algebras, to obtain $\mathbf{S i g}_{\mathfrak{d}}$, the category of many-sorted signatures and derivors-we recall that we were unable to show that hyperderivors are the morphisms of a category with objects ordered pairs $(\boldsymbol{\Sigma}, X)$, where $\boldsymbol{\Sigma}=(S, \Sigma)$ is a many-sorted signature and $X$ an $S$-sorted set. Next, after defining a suitable contravariant functor from $\mathbf{S i g}_{\mathfrak{d}}$ to Cat, the category of $\mathcal{U}$-locally small categories and functors, we obtain, by means of the Grothendieck construction, $\mathbf{A l g}_{\mathfrak{d}}$, the category with objects the ordered pairs $((S, \Sigma),(A, F))$, with $(S, \Sigma)$ a many-sorted signature and $(A, F)$ a $\Sigma$-algebra, and morphisms from $((S, \Sigma),(A, F))$ to $((T, \Lambda),(B, G))$ the ordered pairs $((\varphi, d), f)$, with $(\varphi, d)$ a derivor from $(S, \Sigma)$ to $(T, \Lambda)$ and $f$ a homomorphism of $\Sigma$-algebras from $(A, F)$ to $\left(B_{\varphi}, G^{(\varphi, d)}\right)$, a canonically derived algebra of $(B, G)$. Finally, after showing that every derivor is a hyperderivor, we state the counterparts of Propositions 3.27 and 3.28 for suitable morphisms of $\mathbf{A l g}_{\mathfrak{d}}$.

Before defining the notion of Hall algebra, we recall that (1) a finitary specification is an ordered triple $(S, \Sigma, \mathcal{E})$, where $S$ is a set of sorts, $\Sigma$ an $S$-sorted signature, and $\mathcal{E} \subseteq \operatorname{Eq}_{V^{S}}(\Sigma)=\left(\mathrm{T}_{\Sigma}(X)_{s}^{2}\right)_{(X, s) \in \operatorname{Sub}_{\mathrm{f}}\left(V^{S}\right) \times S}$, i.e., a set of finitary $\Sigma$-equations, where $V^{S}$, the $S$-sorted set of the variables, is a fixed $S$-countably infinite $S$-sorted set; and that (2) a $\Sigma$-algebra $\mathbf{A}$ is a $(S, \Sigma, \mathcal{E})$-algebra if $\mathbf{A} \models^{\Sigma} \mathcal{E}$, i.e., if, for every
$(X, s) \in \operatorname{Sub}_{\mathrm{f}}\left(V^{S}\right) \times S$ and every $(P, Q) \in \mathcal{E}_{X, s}, \mathbf{A}$ is a model of $(P, Q)$, in symbols $\mathbf{A} \models_{X, s}^{\Sigma}(P, Q)$, which in turn means that, for every $a \in \operatorname{Hom}(X, A), a_{s}^{\sharp}(P)=a_{s}^{\sharp}(Q)$.

Definition 3.32. Let $S$ be a set of sorts and $V^{\mathrm{H}_{S}}$ the $S^{\star} \times S$-sorted set of variables $\left(V_{w, s}\right)_{(w, s) \in S^{\star} \times S}$ where $V_{w, s}=\left\{v_{n}^{w, s} \mid n \in \mathbb{N}\right\}$, for every $(w, s) \in S^{\star} \times S$. A Hall algebra for $S$ is a $\mathrm{H}_{S}=\left(S^{\star} \times S, \Sigma^{\mathrm{H}_{S}}, \mathcal{E}^{\mathrm{H}_{S}}\right)$-algebra, where $\Sigma^{\mathrm{H}_{S}}$ is the $S^{\star} \times S$-sorted signature, i.e., the $\left(S^{\star} \times S\right)^{\star} \times\left(S^{\star} \times S\right)$-sorted set, defined as follows:
$\mathrm{HS}_{1}$. For every $w \in S^{\star}$ and $i \in|w|$,

$$
\pi_{i}^{w}: \lambda \longrightarrow\left(w, w_{i}\right)
$$

where $|w|$ is the length of the word $w$ and $\lambda$ the empty word in the underlying set of the free monoid on $S^{\star} \times S$.
$\mathrm{HS}_{2}$. For every $u, w \in S^{\star}$ and $s \in S$,

$$
\xi_{u, w, s}:\left((w, s),\left(u, w_{0}\right), \ldots,\left(u, w_{|w|-1}\right)\right) \longrightarrow(u, s)
$$

while $\mathcal{E}^{\mathrm{H}_{S}}$ is the sub- $\left(S^{\star} \times S\right)^{\star} \times\left(S^{\star} \times S\right)$-sorted set of $\operatorname{Eq}\left(\Sigma^{\mathrm{H}_{S}}\right)$, where

$$
\operatorname{Eq}\left(\Sigma^{\mathrm{H}_{S}}\right)=\left(\mathrm{T}_{\Sigma^{\mathrm{H}_{S}}}(\downarrow \bar{w})_{(u, s)}^{2}\right)_{(\bar{w},(u, s)) \in\left(S^{\star} \times S\right)^{\star} \times\left(S^{\star} \times S\right)},
$$

defined as follows:
$\mathrm{H}_{1}$. Projection. For every $u, w \in S^{\star}$ and $i \in|w|$, the equation

$$
\xi_{u, w, w_{i}}\left(\pi_{i}^{w}, v_{0}^{u, w_{0}}, \ldots, v_{|w|-1}^{u, w_{|w|-1}}\right)=v_{i}^{u, w_{i}}
$$

of type $\left(\left(\left(u, w_{0}\right), \ldots,\left(u, w_{|w|-1}\right)\right),\left(u, w_{i}\right)\right)$.
$\mathrm{H}_{2}$. Identity. For every $u \in S^{\star}$ and $j \in|u|$, the equation

$$
\xi_{u, u, u_{j}}\left(v_{j}^{u, u_{j}}, \pi_{0}^{u}, \ldots, \pi_{|u|-1}^{u}\right)=v_{j}^{u, u_{j}}
$$

of type $\left(\left(\left(u, u_{j}\right)\right),\left(u, u_{j}\right)\right)$.
$\mathrm{H}_{3}$. Associativity. For every $u, v, w \in S^{\star}$ and $s \in S$, the equation

$$
\begin{array}{r}
\xi_{u, v, s}\left(\xi_{v, w, s}\left(v_{0}^{w, s}, v_{1}^{v, w_{0}}, \ldots, v_{|w|}^{v, w_{|w|-1}}\right), v_{|w|+1}^{u, v_{0}}, \ldots, v_{|w|+|v|}^{u, v_{|v|-1}}\right)= \\
\xi_{u, w, s}\left(v_{0}^{w, s}, \xi_{u, v, w_{0}}\left(v_{1}^{v, w_{0}}, v_{|w|+1}^{u, v_{0}}, \ldots, v_{|w|+|v|}^{u, v_{|v|-1}}\right), \ldots,\right. \\
\left.\xi_{u, v, w_{|w|-1}}\left(v_{|w|}^{v, w_{|w|-1}}, v_{|w|+1}^{u, v_{0}}, \ldots, v_{|w|+|v|}^{u, v_{|v|-1}}\right)\right)
\end{array}
$$

of type $\left(\left((w, s),\left(v, w_{0}\right), \ldots,\left(v, w_{|w|-1}\right),\left(u, v_{0}\right), \ldots,\left(u, v_{|v|-1}\right)\right),(u, s)\right)$.
We call the formal constants $\pi_{i}^{w}$ projections, and the formal operations $\xi_{u, w, s}$ substitution operators. Furthermore, we will denote by $\operatorname{Alg}\left(\mathrm{H}_{S}\right)$ the category of Hall algebras for $S$ and homomorphisms between Hall algebras. Since $\operatorname{Alg}\left(\mathrm{H}_{S}\right)$ is a variety, the forgetful functor $\mathrm{G}_{\mathrm{H}_{S}}$ from $\operatorname{Alg}\left(\mathrm{H}_{S}\right)$ to $\operatorname{Set}^{S^{\star} \times S}$ has a left adjoint $\mathbf{T}_{\mathrm{H}_{S}}$, situation denoted by $\mathbf{T}_{\mathrm{H}_{S}} \dashv \mathrm{G}_{\mathrm{H}_{S}}$, or diagrammatically by

which assigns to an $S^{\star} \times S$-sorted set $\Sigma$ the corresponding free Hall algebra $\mathbf{T}_{\mathrm{H}_{S}}(\Sigma)$.
Remark. From $\mathrm{H}_{3}$, for $w=\lambda$, the empty word on $S$, we get the invariance of constant functions axiom in [17]: For every $u, v \in S^{\star}$ and $s \in S$, we have the equation

$$
\xi_{u, v, s}\left(\xi_{v, \lambda, s}\left(v_{0}^{\lambda, s}\right), v_{1}^{u, v_{0}}, \ldots, v_{|v|}^{u, v_{|v|-1}}\right)=\xi_{u, \lambda, s}\left(v_{0}^{\lambda, s}\right)
$$

of type $\left(\left((\lambda, s),\left(u, v_{0}\right), \ldots,\left(u, v_{|v|-1}\right)\right),(u, s)\right)$.

For every $S$-sorted set $A, \mathrm{Op}_{\mathrm{H}_{S}}(A)=\left(\operatorname{Hom}\left(A_{w}, A_{s}\right)\right)_{(w, s) \in S^{\star} \times S}$, the $S^{\star} \times S$ sorted set of operation for $A$, is naturally equipped with a structure of Hall algebra, as stated in the following proposition, if we realize the projections as the true projections and the substitution operators as the generalized composition of mappings.

Proposition 3.33. Let $A$ be an $S$-sorted set and $\mathbf{O} \mathbf{p}_{\mathrm{H}_{S}}(A)$ the $\Sigma^{\mathrm{H}_{S}}$-algebra with underlying many-sorted set $\mathrm{Op}_{\mathrm{H}_{S}}(A)$ and algebraic structure defined as follows:
(1) For every $w \in S^{\star}$ and $i \in|w|,\left(\pi_{i}^{w}\right)^{\mathbf{O}_{\mathbf{H}_{S}}(A)}=\operatorname{pr}_{w, i}^{A}: A_{w} \longrightarrow A_{w_{i}}$.
(2) For every $u, w \in S^{\star}$ and $s \in S, \xi_{u, w, S}^{\mathbf{O p}_{\mathrm{H}_{S}}(A)}$ is defined, for every $f \in A_{s}^{A_{w}}$ and $g \in A_{w}^{A_{u}}$, as $\xi_{u, w, s}^{\mathbf{O p}_{\mathrm{H}_{S}}(A)}\left(f, g_{0}, \ldots, g_{|w|-1}\right)=f \circ\left\langle g_{i}\right\rangle_{i \in|w|}$, where $\left\langle g_{i}\right\rangle_{i \in|w|}$ is the unique mapping from $A_{u}$ to $A_{w}$ such that, for every $i \in|w|$, we have that

$$
\operatorname{pr}_{w, i}^{A} \circ\left\langle g_{i}\right\rangle_{i \in|w|}=g_{i} .
$$

Then $\mathbf{O} \mathbf{p}_{\mathrm{H}_{S}}(A)$ is a Hall algebra, the Hall algebra for $(S, A)$.
Remark. The closed sets of the Hall algebra $\mathbf{O} \mathbf{p}_{\mathrm{H}_{S}}(A)$ for $(S, A)$ are precisely the clones of (many-sorted) operations on the $S$-sorted set $A$. On the other hand, every $\Sigma$-algebra $\mathbf{A}$ has associated a Hall algebra. In fact, it suffices to consider $\mathbf{O p}_{\mathrm{H}_{S}}(A)$, denoted by $\mathbf{O p}_{\mathrm{H}_{S}}(\mathbf{A})$. Moreover, the finitary term operations on $\mathbf{A}$ and the finitary algebraic operations on $\mathbf{A}$ are subalgebras of the Hall algebra $\mathbf{O p}_{\mathrm{H}_{S}}(\mathbf{A})$.

For every $S$-sorted signature $\Sigma, \operatorname{Ter}_{H_{S}}(\Sigma)=\left(\mathrm{T}_{\Sigma}(\downarrow w)_{s}\right)_{(w, s) \in S^{\star} \times S}$ is also equipped with a structure of Hall algebra that formalizes the concept of substitution as stated in the following proposition.

Proposition 3.34. Let $\Sigma$ be an $S$-sorted signature and $\operatorname{Ter}_{\mathrm{H}_{S}}(\Sigma)$ the $\Sigma^{\mathrm{H}_{S}}$-algebra with underlying many-sorted set $\operatorname{Ter}_{H_{S}}(\Sigma)$ and algebraic structure defined as follows:
(1) For every $w \in S^{\star}$ and $i \in|w|$, $\left(\pi_{i}^{w}\right)^{\mathbf{T e r}_{\mathrm{H}_{S}}(\Sigma)}$ is the image under $\eta_{\downarrow w, w_{i}}$ of the variable $v_{i}^{w_{i}}$, where $\eta_{\downarrow w}=\left(\eta_{\downarrow w, s}\right)_{s \in S}$ is the canonical embedding of $\downarrow w$ into $\mathrm{T}_{\Sigma}(\downarrow w)$. Sometimes, to abbreviate, we will write $\pi_{i}^{w}$ instead of $\left(\pi_{i}^{w}\right)^{\mathbf{T e r}_{H_{S}}(\Sigma)}$.
(2) For every $u, w \in S^{\star}$ and $s \in S, \xi_{u, w, s}^{\mathbf{T e r}_{H_{S}}(\Sigma)}$ is the mapping

$$
\underset{\xi_{u, w, s}^{\operatorname{Ter}_{H_{S}}(\Sigma)}}{ }\left\{\begin{array}{cl}
\mathrm{T}_{\Sigma}(\downarrow w)_{s} \times \mathrm{T}_{\Sigma}(\downarrow u)_{w_{0}} \times \cdots \times \mathrm{T}_{\Sigma}(\downarrow u)_{w_{|w|-1}} & \longrightarrow \mathrm{~T}_{\Sigma}(\downarrow u)_{s} \\
\left(P,\left(Q_{i}\right)_{i \in|w|}\right) & \longmapsto\left(\binom{v_{i}^{w_{i}}}{Q_{i}}_{i \in|w|}\right)_{s}(P)
\end{array}\right.
$$

where, for $\binom{v_{i}^{w_{i}}}{Q_{i}}_{i \in|w|}$, the $S$-sorted mapping from $\downarrow w$ to $\mathrm{T}_{\Sigma}(\downarrow u)$ canonically associated to the family $\left(Q_{i}\right)_{i \in|w|},\left(\binom{v_{i}^{w_{i}}}{Q_{i}}_{i \in|w|}\right)^{\sharp}$, also denoted by $\mathcal{Q}^{\sharp}$, is the unique homomorphism from $\mathbf{T}_{\Sigma}(\downarrow w)$ into $\mathbf{T}_{\Sigma}(\downarrow u)$ such that

$$
\left(\binom{v_{i}^{w_{i}}}{Q_{i}}_{i \in|w|}\right)^{\sharp} \circ \eta_{\downarrow w}=\binom{v_{i}^{w_{i}}}{Q_{i}}_{i \in|w|} .
$$

Sometimes, to abbreviate, we will write $\xi_{u, w, s}$ instead of $\xi_{u, w, s}^{\mathbf{T e r}_{H_{S}}(\Sigma)}$.
Then $\operatorname{Ter}_{\mathrm{H}_{S}}(\Sigma)$ is a Hall algebra, the Hall algebra for $(S, \Sigma)$.
Remark. For every $\Sigma$-algebra $\mathbf{A}$ there exists a homomorphism from the Hall algebra $\operatorname{Ter}_{H_{S}}(\Sigma)$ to the Hall algebra $\mathbf{O} \mathbf{p}_{H_{S}}(\mathbf{A})$ and its image is the Hall subalgebra of the finitary term operations on $\mathbf{A}$.

Proposition 3.35. Let $\Sigma$ be an $S$-sorted signature. Then $\operatorname{Ter}_{\mathrm{H}_{S}}(\Sigma)^{\wp}$ is a Hall algebra.

Our next goal is to prove that, for every $S^{\star} \times S$-sorted set $\Sigma, \mathbf{T}_{\mathrm{H}_{S}}(\Sigma)$, the free Hall algebra on $\Sigma$, is isomorphic to $\mathbf{T e r}_{\mathrm{H}_{S}}(\Sigma)$. We remark that the existence of this isomorphism is interesting because it enables us to get a more tractable description of the terms in $\mathbf{T}_{\mathrm{H}_{S}}(\Sigma)$.

To attain the goal just stated we begin by defining, for a Hall algebra A, an $S$-sorted signature $\Sigma$, an $S^{\star} \times S$-mapping $f: \Sigma \longrightarrow A$, and a word $u \in S^{\star}$, the concept of derived $\Sigma$-algebra of $\mathbf{A}$ for $(f, u)$, since it will be used afterwards in the proof of the isomorphism between $\mathbf{T}_{\mathrm{H}_{S}}(\Sigma)$ and $\operatorname{Ter}_{\mathrm{H}_{S}}(\Sigma)$.

Definition 3.36. Let A be a Hall algebra and $\Sigma$ an $S$-sorted signature. Then, for every $f: \Sigma \longrightarrow A$ and $u \in S^{\star}, \mathbf{A}^{f, u}$, the derived $\Sigma$-algebra of $\mathbf{A}$ for $(f, u)$, is the $\Sigma$-algebra with underlying $S$-sorted set $A^{f, u}=\left(A_{u, s}\right)_{s \in S}$ and algebraic structure $F^{f, u}$, defined, for every $(w, s) \in S^{\star} \times S$, as

$$
F_{w, s}^{f, u}\left\{\begin{aligned}
\Sigma_{w, s} & \longrightarrow \operatorname{Op}_{w}\left(A^{f, u}\right)_{s} \\
\sigma & \longmapsto\left\{\begin{aligned}
\prod_{i \in|w|} A_{u, w_{i}} & \longrightarrow A_{u, s} \\
\left(a_{0}, \ldots, a_{|w|-1}\right) & \longmapsto \xi_{u, w, s}^{\mathbf{A}}\left(f_{(w, s)}(\sigma), a_{0}, \ldots, a_{|w|-1}\right)
\end{aligned}\right.
\end{aligned}\right.
$$

where $\operatorname{Op}_{w}\left(A^{f, u}\right)_{s}=A_{u, s}^{\prod_{i \in|w|} A_{u, w_{i}}}$.
Furthermore, we will denote by $p^{u}$ the $S$-sorted mapping from $\downarrow u$ to $A^{f, u}$ defined, for every $s \in S$ and $i \in|u|$, as $p_{s}^{u}\left(v_{i}^{s}\right)=\left(\pi_{i}^{u}\right)^{\mathbf{A}}$, and by $\left(p^{u}\right)^{\sharp}$ the unique homomorphism from $\mathbf{T}_{\Sigma}(\downarrow u)$ to $\mathbf{A}^{f, u}$ such that $\left(p^{u}\right)^{\sharp} \circ \eta_{\downarrow u}=p^{u}$.

Remark. For a $\Sigma$-algebra $\mathbf{B}=(B, G)$, we have that $G: \Sigma \longrightarrow \mathrm{Op}_{\mathrm{H}_{S}}(B)$ and $\mathbf{B} \cong \mathbf{O p}_{\mathrm{H}_{S}}(B)^{G, \lambda}$, where $\lambda$ is the empty word on $S$. Besides, for every $u \in S^{\star}$, we have that $\mathbf{B}^{B_{u}}$, the direct $B_{u}$-power of $\mathbf{B}$, is isomorphic to $\mathbf{O} \mathbf{p}_{\mathrm{H}_{S}}(B)^{G, u}$.

Lemma 3.37. Let $\Sigma$ be an $S$-sorted signature, A a Hall algebra, $f: \Sigma \longrightarrow A$ and $u \in S^{\star}$. Then, for every $(w, s) \in S^{\star} \times S, P \in \mathrm{~T}_{\Sigma}(\downarrow w)_{s}$ and $a \in \prod_{i \in|w|} A_{u, w_{i}}$, we have that

$$
P^{\mathbf{A}^{f, u}}\left(a_{0}, \ldots, a_{|w|-1}\right)=\xi_{u, w, s}^{\mathbf{A}}\left(\left(p^{w}\right)_{s}^{\sharp}(P), a_{0}, \ldots, a_{|w|-1}\right) .
$$

Proof. By algebraic induction on the complexity of $P$. If $P$ is a variable $v_{i}^{s}$, with $i \in|w|$, then

$$
\begin{aligned}
v_{i}^{s, \mathbf{A}^{f, u}}\left(a_{0}, \ldots, a_{|w|-1}\right) & =a_{w_{i}}^{\sharp}\left(v_{i}^{s}\right) \\
& =a_{i} \\
& =\xi_{u, w, s}^{\mathbf{A}}\left(\left(\pi_{i}^{w}\right)^{\mathbf{A}}, a_{0}, \ldots, a_{|w|-1}\right) \quad\left(\text { by } \mathrm{H}_{1}\right) \\
& =\xi_{u, w, s}^{\mathbf{A}}\left(\left(p^{w}\right)_{s}^{\sharp}\left(v_{i}^{s}\right), a_{0}, \ldots, a_{|w|-1}\right) .
\end{aligned}
$$

Let us assume that $P=\sigma\left(Q_{0}, \ldots, Q_{|x|-1}\right)$, with $\sigma: x \longrightarrow s$ and that, for every $j \in|x|, Q_{j} \in \mathrm{~T}_{\Sigma}(\downarrow w)_{x_{j}}$ fulfills the induction hypothesis. Then we have that

$$
\begin{aligned}
& \left(\sigma\left(Q_{0}, \ldots, Q_{|x|-1}\right)\right)^{\mathbf{A}^{f, u}}\left(a_{0}, \ldots, a_{|w|-1}\right) \\
& =\sigma^{\mathbf{A}^{f, u}}\left(Q_{0}^{\mathbf{A}^{f, u}}\left(a_{0}, \ldots, a_{|w|-1}\right), \ldots, Q_{|x|-1}^{\mathbf{A}^{f, u}}\left(a_{0}, \ldots, a_{|w|-1}\right)\right) \\
& =\xi_{u, x, s}^{\mathbf{A}}\left(f(\sigma), Q_{0}^{\mathbf{A}^{f, u}}\left(a_{0}, \ldots, a_{|w|-1}\right), \ldots, Q_{|x|-1}^{\mathbf{A}^{f, u}}\left(a_{0}, \ldots, a_{|w|-1}\right)\right) \\
& =\xi_{u, x, s}^{\mathbf{A}}\left(f(\sigma), \xi_{u, w, x_{0}}^{\mathbf{A}}\left(\left(p^{w}\right)_{x_{0}}^{\sharp}\left(Q_{0}\right), a_{0}, \ldots, a_{|w|-1}\right), \ldots,\right. \\
& \left.\quad \quad \xi_{u, w, x|x|-1}^{\mathbf{A}}\left(\left(p^{w}\right)_{x_{|x|-1}}^{\sharp}\left(Q_{|x|-1}\right), a_{0}, \ldots, a_{|w|-1}\right)\right) \quad \text { (by Ind. Hypothesis) } \\
& =\xi_{u, w, s}^{\mathbf{A}}\left(\xi_{w, x, s}^{\mathbf{A}}\left(f(\sigma),\left(p^{w}\right)_{x_{0}}^{\sharp}\left(Q_{0}\right), \ldots,\left(p^{w}\right)_{x_{|x|-1}}^{\sharp}\left(Q_{|x|-1}\right)\right), a_{0}, \ldots, a_{|w|-1}\right)\left(\text { by } H_{3}\right) \\
& =\xi_{u, w, s}^{\mathbf{A}}\left(\sigma^{\mathbf{A}_{w}}\left(\left(p^{w}\right)_{x_{0}}^{\sharp}\left(Q_{0}\right), \ldots,\left(p^{w}\right)_{x_{|x|-1}}^{\sharp}\left(Q_{|x|-1}\right)\right), a_{0}, \ldots, a_{|w|-1}\right) \\
& =\xi_{u, w, s}^{\mathbf{A}}\left(\left(p^{w}\right)_{s}^{\sharp}\left(\sigma, Q_{0}, \ldots, Q_{|x|-1}\right), a_{0}, \ldots, a_{|w|-1}\right) \\
& =\xi_{u, w, s}^{\mathbf{A}}\left(\left(p^{w}\right)_{s}^{\sharp}(P), a_{0}, \ldots, a_{|w|-1}\right) .
\end{aligned}
$$

Next we prove that, for every $S^{\star} \times S$-sorted set $\Sigma$, the Hall algebra for $(S, \Sigma)$ is isomorphic to the free Hall algebra on $\Sigma$.

Proposition 3.38. Let $\Sigma$ be an $S$-sorted signature, i.e., an $S^{\star} \times S$-sorted set. Then the Hall algebra $\mathbf{T e r}_{\mathrm{H}_{S}}(\Sigma)$ is isomorphic to $\mathbf{T}_{\mathrm{H}_{S}}(\Sigma)$.

Proof. It is enough to prove that $\operatorname{Ter}_{H_{S}}(\Sigma)$ has the universal property of the free Hall algebra on $\Sigma$. Therefore we have to specify an $S^{\star} \times S$-sorted mapping $h$ from $\Sigma$ to $\operatorname{Ter}_{H_{S}}(\Sigma)$ such that, for every Hall algebra $\mathbf{A}$ and $S^{\star} \times S$-sorted mapping $f$ from $\Sigma$ to $A$, there is a unique homomorphism $\widehat{f}$ from $\operatorname{Ter}_{H_{S}}(\Sigma)$ to $\mathbf{A}$ such that $\widehat{f} \circ h=f$. Let $h$ be the $S^{\star} \times S$-sorted mapping defined, for every $(w, s) \in S^{\star} \times S$, as

$$
h_{w, s}\left\{\begin{aligned}
\Sigma_{w, s} & \longrightarrow \mathrm{~T}_{\Sigma}(\downarrow w)_{s} \\
\sigma & \longmapsto \sigma\left(v_{0}^{w_{0}}, \ldots, v_{|w|-1}^{w_{|w|-1}}\right)
\end{aligned}\right.
$$

Let A be a Hall algebra, $f: \Sigma \longrightarrow A$ an $S^{\star} \times S$-sorted mapping and $\widehat{f}$ the $S^{\star} \times S$ sorted mapping from $\operatorname{Ter}_{\mathrm{H}_{S}}(\Sigma)$ to $A$ defined, for every $(w, s) \in S^{\star} \times S$, as $\widehat{f}_{w, s}=$ $\left(p^{w}\right)_{s}^{\sharp}$, where, we recall, $\left(p^{w}\right)^{\sharp}$ is the unique homomorphism from $\mathbf{T}_{\Sigma}(\downarrow w)$ to $\mathbf{A}^{f, w}$ such that $\left(p^{w}\right)^{\sharp} \circ \eta_{\downarrow w}=p^{w}$. Then $\widehat{f}$ is a homomorphism of Hall algebras, because, on the one hand, for $w \in S^{\star}$ and $i \in|w|$ we have that

$$
\begin{aligned}
\widehat{f}_{w, w_{i}}\left(\left(\pi_{i}^{w}\right)^{\mathbf{T e r}_{\mathrm{H}_{S}}(\Sigma)}\right) & =\widehat{f}_{w, w_{i}}\left(v_{i}^{w_{i}}\right) \\
& =p_{w_{i}}^{w}\left(v_{i}^{w_{i}}\right) \\
& =\left(\pi_{i}^{w}\right)^{\mathbf{A}},
\end{aligned}
$$

and, on the other hand, for $P \in \mathrm{~T}_{\Sigma}(\downarrow w)_{s}$ and $\left(Q_{i}\right)_{i \in|w|} \in \mathrm{T}_{\Sigma}(\downarrow u)_{w}$ we have that

$$
\begin{aligned}
& \widehat{f}_{u, s}\left(\xi_{u, w, s}^{\mathbf{T e r}_{H_{S}}(\Sigma)}\left(P, Q_{0}, \ldots, Q_{|w|-1}\right)\right) \\
& =\left(p^{u}\right)_{s}^{\sharp}\left(\mathcal{Q}_{s}^{\sharp}(P)\right) \\
& =\left(\left(p^{u}\right)^{\sharp} \circ \mathcal{Q}\right)_{s}^{\sharp}(P) \quad\left(\text { because }\left(p^{u}\right)^{\sharp} \circ \mathcal{Q}^{\sharp}=\left(\left(p^{u}\right)^{\sharp} \circ \mathcal{Q}\right)^{\sharp}\right) \\
& =P^{\mathbf{A}^{f, u}}\left(\left(p^{u}\right)_{w_{0}}^{\sharp}\left(Q_{0}\right), \ldots,\left(p^{u}\right)_{w_{|w|-1}}^{\sharp}\left(Q_{|w|-1}\right)\right) \\
& =\xi_{u, w, s}^{\mathbf{A}}\left(\left(p^{w}\right)_{s}^{\sharp}(P),\left(p^{u}\right)_{w_{0}}^{\sharp}\left(Q_{0}\right), \ldots,\left(p^{u}\right)_{w_{|w|-1}}^{\sharp}\left(Q_{|w|-1}\right)\right) \quad \quad \text { (by Lemma 3.37) } \\
& =\xi_{u, w, s}^{\mathbf{A}}\left(\widehat{f}_{w, s}(P), \widehat{f}_{u, w_{0}}\left(Q_{0}\right), \ldots, \widehat{f}_{u, w_{|w|-1}}\left(Q_{|w|-1}\right)\right) .
\end{aligned}
$$

Therefore the $S^{\star} \times S$-sorted mapping $\widehat{f}$ is a homomorphism. Furthermore, $\widehat{f} \circ h=f$, because, for every $w \in S^{\star}, s \in S$, and $\sigma \in \Sigma_{w, s}$, we have that

$$
\begin{aligned}
\widehat{f}_{w, s}\left(h_{w, s}(\sigma)\right) & =\left(p^{w}\right)_{s}^{\sharp}\left(\sigma\left(v_{0}^{w_{0}}, \ldots, v_{|w|-1}^{w_{|w|-1}}\right)\right) \\
& =\sigma^{\mathbf{A}_{w}}\left(p_{w_{0}}^{w}\left(v_{0}^{w_{0}}\right), \ldots, p_{w_{|w|-1}}^{w}\left(v_{|w|-1}^{w_{|w|-1}}\right)\right) \\
& =\xi_{w, w, s}^{\mathbf{A}}\left(f_{(w, s)}(\sigma),\left(\pi_{0}^{w}\right)^{\mathbf{A}}, \ldots,\left(\pi_{|w|-1}^{w}\right)^{\mathbf{A}}\right) \\
& =f_{w, s}(\sigma) \quad\left(\text { by } \mathrm{H}_{2}\right) .
\end{aligned}
$$

It is obvious that $\widehat{f}$ is the unique homomorphism such that $\widehat{f} \circ h=f$. Henceforth $\operatorname{Ter}_{\mathrm{H}_{S}}(\Sigma)$ is isomorphic to $\mathbf{T}_{\mathrm{H}_{S}}(\Sigma)$.

This isomorphism together with the adjunction $\mathbf{T}_{H_{S}} \dashv \mathrm{G}_{\mathrm{H}_{S}}$ has as an immediate consequence that, for every $S$-sorted set $A$ and every $S$-sorted signature $\Sigma$, the sets $\operatorname{Hom}\left(\Sigma, \mathrm{Op}_{\mathrm{H}_{S}}(A)\right)$, in the category $\mathbf{S e t}^{S^{\star} \times S}$, and $\operatorname{Hom}\left(\mathbf{T e r}_{\mathrm{H}_{S}}(\Sigma), \mathbf{O p}_{\mathrm{H}_{S}}(A)\right)$, in the category $\operatorname{Alg}\left(\mathrm{H}_{S}\right)$, are naturally isomorphic. Actually, (1) the mapping that assigns, for an $S$-sorted set $A$, to a structure of $\Sigma$-algebra $F$ on $A$ (i.e., an $S^{\star} \times S$-sorted mapping $F$ from $\Sigma$ to $\mathrm{Op}_{\mathrm{H}_{S}}(A)$ ) the homomorphism of Hall algebras $\operatorname{Tr}^{(A, F)}=\left(\operatorname{Tr}_{s}^{\downarrow w,(A, F)}\right)_{(w, s) \in S^{\star} \times S}$ from $\operatorname{Ter}_{\mathrm{H}_{S}}(\Sigma)$ to $\mathbf{O p}_{\mathrm{H}_{S}}(A)$, where, for every $w \in S^{\star}$, the subfamily $\operatorname{Tr}^{\downarrow w,(A, F)}=\left(\operatorname{Tr}_{s}^{\downarrow w,(A, F)}\right)_{s \in S}$ of $\operatorname{Tr}^{(A, F)}$ is the unique homomorphism from $\mathbf{T}_{\Sigma}(\downarrow w)$ to $(A, F)^{A_{w}}$, the direct $A_{w}$-power of $(A, F)$, such that $\operatorname{Tr}^{\downarrow w,(A, F)} \circ \eta_{\downarrow w}=\mathrm{p}_{\downarrow w}^{A}$, where $\mathrm{p}_{\downarrow w}^{A}$ is the $S$-sorted mapping from $\downarrow w$ to $A^{A_{w}}$ defined, for every $s \in S$ and $v_{i}^{s} \in(\downarrow w)_{s}$, as $\mathrm{p}_{\downarrow w, s}^{A}\left(v_{i}^{s}\right)=\operatorname{pr}_{w, i}^{A}$; together with (2) the mapping that sends an homomorphism $h$ from $\operatorname{Ter}_{H_{S}}(\Sigma)$ to $\mathbf{O p}_{\mathrm{H}_{S}}(A)$ to, essentially, the algebraic structure $\mathrm{G}_{\mathrm{H}_{S}}(h) \circ \eta_{\Sigma}$ on $A$, where $\eta_{\Sigma}$ is the canonical embedding of $\Sigma$ into $\mathrm{T}_{\mathrm{H}_{S}}(\Sigma)$, which are mutually inverse bijections, provide the mentioned natural isomorphism.

The derived operation symbols of an signature can be considered as the operation symbols of a new signature. The mappings that assign to operation symbols of a signature terms relative to another signature, together with mappings between the corresponding sets of sorts, form a new class of morphisms denominated derivors.

Definition 3.39. Let $\boldsymbol{\Sigma}=(S, \Sigma)$ and $\boldsymbol{\Lambda}=(T, \Lambda)$ be many-sorted signatures. A derivor from $\boldsymbol{\Sigma}$ to $\boldsymbol{\Lambda}$ is an ordered pair $\mathbf{d}=(\varphi, d)$, where $\varphi: S \longrightarrow T$ and $d: \Sigma \longrightarrow \operatorname{Ter}_{\mathrm{H}_{T}}(\Lambda)_{\varphi^{\star} \times \varphi}$. Thus, for every $(w, s) \in S^{\star} \times S$,

$$
d_{w, s}: \Sigma_{w, s} \longrightarrow \operatorname{Ter}_{\mathrm{H}_{T}}(\Lambda)_{\varphi^{\star}(w), \varphi(s)}=\mathrm{T}_{\Lambda}\left(\downarrow \varphi^{\star}(w)\right)_{\varphi(s)} .
$$

Remark. While derivors are not the most general type of morphism that might be considered between many-sorted signatures-for instance, one could consider, in addition to hyperderivors, polyderivors, see 7-, they are an important class of such morphisms. One reason for its relevance is its formal properties (see below), another that there are many mathematical examples of them which are of interest (see [7]).

We next introduce the linear derivors, which, as we will see later on, are a remarkable subclass of the class of the derivors, especially in regards to recognizability, the same as linear hyperderivors are with respect to hyperderivors.

Definition 3.40. Let $\mathbf{d}: \boldsymbol{\Sigma} \longrightarrow \boldsymbol{\Lambda}$ be a derivor from $\boldsymbol{\Sigma}$ to $\boldsymbol{\Lambda}$. We will say that $\mathbf{d}$ is linear if, for every $(w, s) \in S^{\star} \times S$, every $\sigma \in \Sigma_{w, s}$, and every $i \in|w|$, no variable $v_{i}^{\varphi\left(w_{i}\right)}$ appears more than once in $d_{w, s}(\sigma)$, i.e., $\left|d_{w, s}(\sigma)\right|_{v_{i}^{\varphi\left(w_{i}\right)}} \leq 1$.

For every many-sorted signature $\boldsymbol{\Lambda}, \operatorname{Ter}_{\mathrm{H}_{T}}(\Lambda)$ is the underlying many-sorted set of $\operatorname{Ter}_{H_{T}}(\Lambda)$, the Hall algebra for $T$. Since by, Proposition 3.38, $\operatorname{Ter}_{H_{T}}(\Lambda)$ is
isomorphic to $\mathbf{T}_{\mathrm{H}_{T}}(\Lambda)$, the derivors can be defined, alternative, but equivalently, as ordered pairs $\mathbf{d}=(\varphi, d)$ with $d: \Sigma \longrightarrow \mathrm{T}_{\mathrm{H}_{T}}(\Lambda)$.

On the other hand, every mapping $\varphi: S \longrightarrow T$ determines a functor from the category $\operatorname{Alg}\left(\mathrm{H}_{T}\right)$ to the category $\operatorname{Alg}\left(\mathrm{H}_{S}\right)$, so $\operatorname{Ter}_{\mathrm{H}_{T}}(\Lambda)_{\varphi^{\star} \times \varphi}$ is in its turn equipped with a structure of Hall algebra for $S$, which will allow us, in particular, to define the composition of derivors. We next show the existence of such a functor by defining a morphism of algebraic presentations from $\left(\Sigma^{\mathrm{H}_{S}}, \mathcal{E}^{\mathrm{H}_{S}}\right)$ to $\left(\Sigma^{\mathrm{H}_{T}}, \mathcal{E}^{\mathrm{H}_{T}}\right)$.

Proposition 3.41. Let $\varphi: S \longrightarrow T$ be a mapping between sets of sorts and $h^{\varphi}$ the $S^{\star} \times S$-sorted mapping from $\Sigma^{\mathrm{H}_{S}}$ to $\Sigma_{\varphi^{\star} \times \varphi}^{\mathrm{H}_{T}}$ defined as follows:
(1) For every $w \in S^{\star}$ and every $i \in|w|, h^{\varphi}\left(\pi_{i}^{w}\right)=\pi_{i}^{\varphi^{\star}(w)}$.
(2) For every $u, w \in S^{\star}$ and every $s \in S, h^{\varphi}\left(\xi_{u, w, s}\right)=\xi_{\varphi^{\star}(u), \varphi^{\star}(w), \varphi(s)}$.

Then $\left(\varphi^{\star} \times \varphi, h^{\varphi}\right):\left(S^{\star} \times S, \Sigma^{\mathrm{H}_{S}}, \mathcal{E}^{\mathrm{H}_{S}}\right) \longrightarrow\left(T^{\star} \times T, \Sigma^{\mathrm{H}_{T}}, \mathcal{E}^{\mathrm{H}_{T}}\right)$ is a morphism of algebraic presentations.

The morphisms of algebraic presentations determine functors in the opposite direction between the associated categories of algebras. Therefore, each mapping $\varphi: S \longrightarrow T$ between sets of sorts, determines a functor $\left(\varphi^{\star} \times \varphi, h^{\varphi}\right)^{*}$ from $\mathbf{A l g}\left(\mathrm{H}_{T}\right)$ to $\operatorname{Alg}\left(\mathrm{H}_{S}\right)$, which transforms Hall algebras for $T$ into Hall algebras for $S$. The action of the functor on the free Hall algebra on a $T$-sorted signature $\Lambda$ is a Hall algebra for $S$, whose underlying $S^{\star} \times S$-sorted set is $\operatorname{Ter}_{H_{T}}(\Lambda)_{\varphi^{\star} \times \varphi}$.

If $\mathbf{d}: \boldsymbol{\Sigma} \longrightarrow \boldsymbol{\Lambda}$ is a derivor, then $d: \Sigma \longrightarrow \operatorname{Ter}_{\mathrm{H}_{T}}(\Lambda)_{\varphi^{*} \times \varphi}$ determines a homomorphism of Hall algebras $d^{\sharp}: \operatorname{Ter}_{H_{S}}(\Sigma) \longrightarrow \operatorname{Ter}_{H_{T}}(\Lambda)_{\varphi^{\star} \times \varphi}$. Thus, for every $(w, s) \in S^{\star} \times S, d_{w, s}^{\sharp}$ sends terms in $\mathrm{T}_{\Sigma}(\downarrow w)_{s}$ to terms in $\mathrm{T}_{\Lambda}\left(\downarrow \varphi^{\star}(w)\right)_{\varphi(s)}$.

Definition 3.42. Let $\mathbf{d}: \boldsymbol{\Sigma} \longrightarrow \boldsymbol{\Lambda}$ and $\mathbf{d}: \boldsymbol{\Lambda} \longrightarrow \boldsymbol{\Omega}$ be derivors. Then $\mathbf{e} \circ \mathbf{d}=$ $(\psi, e) \circ(\varphi, d)$, the composition of $\mathbf{d}$ and $\mathbf{e}$, is the derivor $\left(\psi \circ \varphi, e_{\varphi^{\star} \times \varphi}^{\sharp} \circ d\right)$, where $e_{\varphi^{\star} \times \varphi}^{\sharp} \circ d$ is obtained from

as

being $e^{\sharp}$ the canonical extension of $e$ to the free Hall algebra on $\Lambda$.
For every many-sorted signature $\boldsymbol{\Sigma}=(S, \Sigma)$, the identity at $(S, \Sigma)$ is $\left(\operatorname{id}_{S}, \eta_{\Sigma}\right)$.
The just stated definition allows us to form a category of many-sorted signatures whose morphisms are the derivors.

Proposition 3.43. The many-sorted signatures together with the derivors constitute a category, denoted by $\mathbf{S i g}_{\mathfrak{0}}$.

Remark. Let Sig be the category whose objects are the many-sorted signatures and whose morphisms from $\boldsymbol{\Sigma}$ to $\boldsymbol{\Lambda}$ are the ordered pairs $(\varphi, d)$, where $\varphi$ is a mapping from $S$ to $T$ and $d$ an $S^{\star} \times S$-sorted mapping from $\Sigma$ to $\Lambda_{\varphi^{\star} \times \varphi}$ (thus a mapping in $\operatorname{Sig}(S)$ ). Note that $\mathbf{S i g}$ is the category obtained by means of the Grothendieck construction for an appropriate contravariant functor from Set to Cat. We next show that the category $\mathbf{S i g}_{\mathfrak{d}}$ can be obtained as the Kleisli category for a suitable monad. In fact, for every set of sorts $S$, we have the adjoint situation $\mathbf{T}_{\mathrm{H}_{S}} \dashv \mathrm{G}_{\mathrm{H}_{S}}$ and, thus, a monad on $\operatorname{Sig}(S)$ which we will denote by $\mathbb{T}_{\mathrm{H}_{S}}=\left(\mathrm{T}_{\mathrm{H}_{S}}, \eta^{\mathrm{H}_{S}}, \mu^{\mathrm{H}_{S}}\right)$. Then the ordered triple $\mathfrak{d}=(\mathfrak{d}, \eta, \mu)$ in which (1) $\mathfrak{d}$ is the endofunctor at Sig that sends $(S, \Sigma)$ to $\left(S, \mathrm{~T}_{\mathrm{H}_{S}}(\Sigma)\right)$ and a morphism $(\varphi, d)$ from $(S, \Sigma)$ to $(T, \Lambda)$ to the
morphism $\left(\varphi, d^{\sharp}\right)$ from $\left(S, \mathrm{~T}_{\mathrm{H}_{S}}(\Sigma)\right)$ to $\left(T, \mathrm{~T}_{\mathrm{H}_{T}}(\Lambda)\right)$, (2) $\eta$ the natural transformation from $\mathrm{Id}_{\mathbf{S i g}}$ to $\mathfrak{d}$ that sends $(S, \Sigma)$ to $\eta_{(S, \Sigma)}=\left(\mathrm{id}, \eta_{\Sigma}^{\mathrm{H}_{S}}\right)$, and (3) $\mu$ the natural transformation from $\mathfrak{d} \circ \mathfrak{d}$ to $\mathfrak{d}$ that sends $(S, \Sigma)$ to $\mu_{(S, \Sigma)}=$ (id, $\left.\mu_{\Sigma}^{\mathrm{H}_{S}}\right)$, is a monad on Sig and $\mathbf{K l}(\mathfrak{d})$, the Kleisli category for $\mathfrak{d}$, is isomorphic to $\mathbf{S i g}_{\mathfrak{d}}$. This shows, in our opinion, the mathematical naturalness of the notion of derivor. Moreover, by defining a suitable notion of transformation between derivors one can equip the category $\mathbf{S i g}_{\mathfrak{d}}$ with a structure of 2-category (this was, in fact, already done in 7 for polyderivors).

Remark. Since $\mathbf{S i g}$ has coproducts, $\mathbf{K l}(\mathfrak{d}) \cong \mathbf{S i g}_{\mathfrak{d}}$ has coproducts.
We next associate to every derivor $\mathbf{d}: \boldsymbol{\Sigma} \longrightarrow \boldsymbol{\Lambda}$ a functor from $\mathbf{A l g}(\boldsymbol{\Lambda})(=\mathbf{A l g}(\Lambda))$ to $\boldsymbol{\operatorname { A l g }}(\boldsymbol{\Sigma})(=\boldsymbol{\operatorname { A l g }}(\Sigma))$.

Proposition 3.44. Let $\mathbf{d}: \boldsymbol{\Sigma} \longrightarrow \boldsymbol{\Lambda}$ be a morphism in $\mathbf{S i g}_{\mathfrak{\jmath}}$. Then $\operatorname{Alg}_{\mathfrak{d}}(\mathbf{d})$ is the functor from $\mathbf{A l g}(\boldsymbol{\Lambda})$ to $\mathbf{A l g}(\boldsymbol{\Sigma})$ that sends $(B, G)$ to $\left(B_{\varphi}, G^{\mathbf{d}}\right)$ and a homomorphism $f$ from $(B, G)$ to $\left(B^{\prime}, G^{\prime}\right)$ to the homomorphism $f_{\varphi}$ from $\left(B_{\varphi}, G^{\mathbf{d}}\right)$ to $\left(B_{\varphi}^{\prime}, G^{\prime \mathbf{d}}\right)$, where, for every $\boldsymbol{\Lambda}$-algebra $(B, G), G^{\mathbf{d}}=G_{\varphi^{\star} \times \varphi}^{\sharp} \circ d$ is obtained from

as

$\operatorname{Ter}_{\mathrm{H}_{T}}(\Lambda)_{\varphi^{\star} \times \varphi} \longleftarrow \quad d \quad \Sigma$

Proof. For every $\boldsymbol{\Lambda}$-algebra $(B, G), G: \Lambda \longrightarrow \mathrm{Op}_{\mathrm{H}_{T}}(B)$, and since $\mathbf{O p}_{\mathrm{H}_{T}}(B)$ is a Hall algebra, $G$ can be extended up to the free Hall algebra on $\Lambda$. Moreover, we have that $\mathrm{Op}_{\mathrm{H}_{T}}(B)_{\varphi^{\star} \times \varphi}=\mathrm{Op}_{\mathrm{H}_{S}}\left(B_{\varphi}\right)$ since, for every $(w, s) \in S^{\star} \times S$ it holds that

$$
\begin{aligned}
\left(\mathrm{Op}_{\mathrm{H}_{T}}(B)_{\varphi^{\star} \times \varphi}\right)_{w, s} & =\mathrm{Op}_{\mathrm{H}_{T}}(B)_{\varphi^{\star}(w), \varphi(s)} \\
& =B_{\varphi^{\star}(w)} \longrightarrow B_{\varphi(s)} \\
& =B_{\left(\varphi\left(w_{0}\right), \ldots, \varphi\left(w_{|w|-1}\right)\right)} \longrightarrow B_{\varphi(s)} \\
& =\prod\left(B_{\varphi\left(w_{i}\right)}|i \in| w \mid\right) \longrightarrow B_{\varphi(s)} \\
& =\prod\left(\left(B_{\varphi}\right)_{w_{i}}|i \in| w \mid\right) \longrightarrow\left(B_{\varphi}\right)_{s} \\
& =\left(B_{\varphi}\right)_{w} \longrightarrow\left(B_{\varphi}\right)_{s} \\
& =\mathrm{Op}_{\mathrm{H}_{S}}\left(B_{\varphi}\right)_{w, s},
\end{aligned}
$$

thus, so defined, $G^{(\varphi, d)}$ is an algebraic structure on $B_{\varphi}$.
Let $f:(B, G) \longrightarrow\left(B^{\prime}, G^{\prime}\right)$ be a homomorphism of $\boldsymbol{\Lambda}$-algebras, $(w, s) \in S^{\star} \times S$, and $\sigma \in \Sigma_{w, s}$. Then $f_{\varphi}:\left(B_{\varphi}, G^{\mathbf{d}}\right) \longrightarrow\left(B_{\varphi}^{\prime}, G^{\prime \mathbf{d}}\right)$ is a homomorphism of $\boldsymbol{\Sigma}$-algebras, because $G^{\mathbf{d}}(\sigma)$ is a term operation and, consequently, the following diagram

commutes. Moreover, $(g \circ f)_{\varphi}=g_{\varphi} \circ f_{\varphi}$, so that $\operatorname{Alg}_{\mathfrak{d}}(\mathbf{d})$ is a functor.

From the definition of the functor $\operatorname{Alg}_{\mathfrak{d}}$, for every derivor $\mathbf{d}: \boldsymbol{\Sigma} \longrightarrow \boldsymbol{\Lambda}$, it is obvious that the following diagram

commutes.
The previous construction can be extended up to a contravariant functor from the category $\mathbf{S i g}_{\mathfrak{d}}$ to the category Cat.
Proposition 3.45. From $\mathbf{S i g}_{\mathfrak{d}}$ to $\mathbf{C a t}$ there exists a contravariant functor, denoted by $\operatorname{Alg}_{\mathfrak{d}}$, that sends $(S, \Sigma)$ to $\operatorname{Alg}(S, \Sigma)$ and a morphism $(\varphi, d)$ from $(S, \Sigma)$ to $(T, \Lambda)$ to the functor $\operatorname{Alg}_{\mathfrak{d}}(\varphi, d)$ from $\operatorname{Alg}(T, \Lambda)$ to $\operatorname{Alg}(S, \Sigma)$.
Proof. Given $(\varphi, d):(S, \Sigma) \longrightarrow(T, \Lambda)$ and $(\psi, e):(T, \Lambda) \longrightarrow(U, \Omega)$, we show that $\operatorname{Alg}_{\mathfrak{d}}(\varphi, d) \circ \operatorname{Alg}_{\mathfrak{d}}(\psi, e)=\operatorname{Alg}_{\mathfrak{d}}((\psi, e) \circ(\varphi, d))$.

Let $(A, F)$ be a $(U, \Omega)$-algebra. Then $A_{\psi}=A_{\psi \circ \varphi}$. Moreover, we have that

$$
\left.\begin{array}{rl}
F^{(\psi, e)^{(\varphi, d)}} & =\left(F_{\psi^{\star} \times \psi}^{\sharp} \circ e\right)^{(\varphi, d)} \\
& =\left(F_{\psi^{\star} \times \psi}^{\sharp} \circ e\right)_{\varphi^{\star} \times \varphi}^{\sharp} \circ d \\
& =\left(F_{\psi^{\star} \times \psi}^{\sharp} \circ \varphi^{\star} \times \varphi\right.
\end{array} e_{\varphi^{\star} \times \varphi}^{\sharp}\right) \circ d ~\left(e^{\sharp}\right)
$$

 $(U, \Omega)$-algebras, then $f_{\psi \varphi}=f_{\psi \circ \varphi}$.

Definition 3.46. The category $\operatorname{Alg}_{\mathfrak{d}}$ is $\int^{\mathbf{S i g}_{\mathfrak{d}}} \mathrm{Alg}_{\mathfrak{d}}$, i.e., the category obtained by means of the Grothendieck construction applied to the contravariant functor $\mathrm{Alg}_{\mathfrak{j}}$.

The category $\mathbf{A l g}_{\mathfrak{d}}$ has as objects the ordered pairs $(\boldsymbol{\Sigma}, \mathbf{A})$, with $\boldsymbol{\Sigma}$ a manysorted signature and $\mathbf{A}$ a $\Sigma$-algebra, and as morphisms from $(\boldsymbol{\Sigma}, \mathbf{A})$ to $(\boldsymbol{\Lambda}, \mathbf{B})$ the ordered pairs $(\mathbf{d}, f)$, with $\mathbf{d}$ a derivor from $\boldsymbol{\Sigma}$ to $\boldsymbol{\Lambda}$ and $f$ a homomorphism of $\Sigma$-algebras from $(A, F)$ to $\left(B_{\varphi}, G^{(\varphi, d)}\right)$.
Remark. There exists a pseudo-functor Alg $_{\mathfrak{d}}$ from the 2-category $\mathbf{S i g}_{\mathfrak{d}}$ to the 2 category Cat, contravariant in the 1-cells, i.e., the derivors, and covariant in the 2-cells, i.e., the transformations between derivors (this was already done in 7 for polyderivors).
Proposition 3.47. Let $\mathbf{d}=(\varphi, d)$ be a derivor (resp., a linear derivor) from $\boldsymbol{\Sigma}$ to $\Lambda, X$ an $S$-sorted set, $Y$ a $T$-sorted set, and $f$ an $S$-sorted mapping from $X$ to $\mathrm{T}_{\Lambda}(Y)_{\varphi}$. Then $\left(\mathbf{d}^{Y}, f\right)=\left(\left(\varphi, d^{Y}\right), f\right)$, where $d^{Y}$ is, for every $(w, s) \in S^{\star} \times$ $S$, the composition of $d_{w, s}$, which is a mapping from $\Sigma_{w, s}$ to $\mathrm{T}_{\Lambda}\left(\downarrow \varphi^{\star}(w)\right)_{\varphi(s)}$, and $\left(\operatorname{in}_{\downarrow \varphi^{\star}(w), Y \cup \downarrow \varphi^{\star}(w)}^{@}\right)_{\varphi(s)}$, the underlying mapping of the component at $\varphi(s)$ of the canonical $\Lambda$-homomorphism $\operatorname{in}_{\downarrow \varphi^{\star}(w), Y \cup \downarrow \varphi^{\star}(w)}^{@}$ from $\mathbf{T}_{\Lambda}\left(\downarrow \varphi^{\star}(w)\right)$ to $\mathbf{T}_{\Lambda}(Y \cup \downarrow$ $\varphi^{\star}(w)$ ), is a hyperderivor (resp., a linear hyperderivor) from $(\boldsymbol{\Sigma}, X)$ to $(\boldsymbol{\Lambda}, Y)$.

In other words, some hyperderivors, but not all of them, are factorizable through a derivor.

Proposition 3.48. Let $\mathbf{d}$ be a derivor from $\boldsymbol{\Sigma}$ to $\boldsymbol{\Lambda}, X$ an $S$-sorted set, $Y$ a $T$-sorted set, $f$ an $S$-sorted mapping from $X$ to $\mathrm{T}_{\Lambda}(Y)_{\varphi}, f^{\sharp}$ the canonical homomorphism of $\Sigma$-algebras from $\mathbf{T}_{\Sigma}(X)$ to $\operatorname{Alg}_{\mathfrak{d}}(\mathbf{d})\left(\mathbf{T}_{\Lambda}(Y)\right)$, $s \in S$, and $L \in$ $\operatorname{Rec}_{\varphi(s)}\left(\mathbf{T}_{\Lambda}(Y)\right)$. Then $\left(f_{\varphi(s)}^{\sharp}\right)^{-1}[L] \in \operatorname{Rec}_{s}\left(\mathbf{T}_{\Sigma}(X)\right)$.

Proof. It follows from Proposition 3.27,
Assumption. For the following proposition, as was the case with Proposition 3.28, we will assume that $S, T, \Sigma$ and $X$ are finite.

Proposition 3.49. Let $\mathbf{d}$ be a linear derivor from $\boldsymbol{\Sigma}$ to $\boldsymbol{\Lambda}, X$ an $S$-sorted set, $Y$ a $T$-sorted set, $f$ an $S$-sorted mapping from $X$ to $\mathrm{T}_{\Lambda}(Y)_{\varphi}, f^{\sharp}$ the canonical homomorphism of $\Sigma$-algebras from $\mathbf{T}_{\Sigma}(X)$ to $\operatorname{Alg}_{\mathfrak{d}}(\mathbf{d})\left(\mathbf{T}_{\Lambda}(Y)\right), s \in S$, and $L \in$ $\mathrm{T}_{\Sigma}(X)_{s}^{\wp}$. If $L \in \operatorname{Rec}_{s}\left(\mathbf{T}_{\Sigma}(X)\right)$, then $f_{s}^{\sharp}[L] \in \operatorname{Rec}_{\varphi(s)}\left(\mathbf{T}_{\Lambda}(Y)\right)$.
Proof. It follows from Proposition 3.28
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