

Aperiodic Tilings: Breaking Translational Symmetry

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Abstract

Classical results on aperiodic tilings are rather complicated and not widely understood. Below, an alternative approach is discussed in hope to provide additional intuition not apparent in classical works.

1 Palettes and Tilings

Physical computing media are asymmetric. Their symmetry is broken by irregularities, physical boundaries, external connections, and so on. Such peculiarities, however, are highly variable and extraneous to the fundamental nature of the media. Thus, they are pruned from theoretical models, such as cellular automata, and reliance on them is frowned upon in programming practice.

Yet, computation, like many other highly organized activities, is incompatible with perfect symmetry. Some standard mechanisms must assure breaking the symmetry inherent in idealized computing models. A famous example of such mechanisms is aperiodic tiling: hierarchical self-similar constructions, first used for computational purposes in a classical – although rather complicated – work [Berger 66] and further developed in [Robinson 71, Myers 74, Gurevich Koriakov 72]. [Allauzen Durand 96] give a helpful exposition.

Definition 1 Let G be the grid of unit length edges between integer points on an infinite plane.

A **tiling** T is its mapping into a finite set of **colors**. Its **crosses** and **tiles** are ordered color combinations of four edges sharing a corner or forming a square, respectively. A **palette** P of T is a set including all its tiles (+**palette** for crosses). We say P with a mapping f of its colors into a smaller color alphabet **enforces** a set S of tilings if replacing colors according to f turns each P -tiling into a tiling in S .

Turning each edge orthogonally around its center turns G into its dual graph and palettes into +palettes and vice versa. Thus, one can use either type as convenient.

2 2-Adic Coordinates

The set of all tilings with a given palette P has translational symmetry, *i.e.* any shift produces another P -tiling. We want a palette that forces a complete spontaneous breaking of this symmetry, *i.e.* prevents individual tilings from being *periodic*. Accordingly, each location in a given tiling will be uniquely characterized by a sort of *coordinates*. Their infinitely many values cannot be reflected in the finite variety of the tile's colors. They will be represented *distributively*, *i.e.* in the colors of the surrounding tiles, and computable from them to any given number of digits.

Let us first so distribute the horizontal Cartesian integer coordinates $x = (2i+1)2^k$ of vertical edges by reflecting one bit ($i \bmod 2$) in their color. We view this bit as the direction of a **bracket**. In this 1-dimensional tiling C_1 , the brackets of the same **rank** k are equidistant (Figure 1).

It is convenient to visualize the bits of even rank, picturing them $\}/\{$ or red, or dotted, separately from odd, depicted $]/[$. The bits of either shape at each side of the origin form a progression of balanced parenthetical expressions, called **domains**. Each domain has four **grandchildren** of the second lower rank: two within its outermost brackets and one to each side. The two **children** have the other shape and are centered at each border of the **parent**, thus connecting it with its grandchildren.

Handling the vertical coordinate similarly yields a neat 2-dimensional tiling C called **central**. Figure 2 marks the borders of intersections of its vertical and horizontal domains of equal ranks with a **boldness** bit.

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Definition 2 A **box** is an **open** *ce*-tiled rectangle, i.e. one with the border edges removed. Its **k -block** is a square with monochromatic sides that is⁵ a tile (for $k=0$) or a combination of four $(k-1)$ -blocks sharing a corner. The four segments connecting the block's center to its sides are called **$(k-1)$ -medians**. The rest of the open block is called a **frame**. We call a box **k -tiled** if removing an outer layer which is thinner than a k -block turns it into a box composed of (open at the box border) k -blocks.⁶

Lemma 1 (i) Borders between open blocks in a box are monochromatic. (ii) All k -frame patterns in a box are enclosed in its open k -blocks. (iii) All open k -blocks are congruent and have equal frames.

Proof. (i) comes from all *ce*-crosses having one or four inward pointers. (ii,iii) for $k > 2$ follow from $k-1$ by viewing 1-blocks as tiles. Let 1-blocks a and b be adjacent in a 2-block c ; (L, l) and (R, r) be pairs of medians of a and b with l, r directed to the side s of c , $L-R$ crossing a median m of c at a cross x . x forces L, R to be both pale or both bold. This forces opposite brackets on l, r which, too, must be both pale or both bold depending on the bracket of $L-R$. l, r cannot be both bold which would require the pointer of s to agree with their opposite brackets. Thus, all external medians of 2-frames are pale, internal medians bold, their brackets face the frame's center forcing inward pointers on the 1-medians, like m . ■

Lemma 2 Any 1-tiled box is k -tiled. (Follows from $k=2$ case by seeing $(k-2)$ -blocks as tiles.)

Proof. The eight colors of edges fix their location in 2-frames, forcing open 1-blocks to alternate in the pattern of 2-frames which, by Lemma 1, extend to 2-blocks. ■

Corollary 1 Any $2^k \times 2^k$ box, extendible to a 3 times wider cocentric 1-tiled box, extends to a $(k+4)$ -block.

Induction Basis. For the simplest enforcement of tiling decomposition into 1-blocks we can use a 2-periodic **parity** bit to mark **odd** lines carrying 0-medians. All pointers on odd lines point to odd crossing lines, thus forcing a period 2 on them. One needs only to assure an odd line exists. This can be easily done with a **parity pointer** on even lines, pointing to a crossing odd line.

Proof of Corollary. The box is k -tiled covering the inner box with four blocks sharing a cross. The extension comes by viewing them as tiles with parity bit reflecting the blocks' orientation and noting that each cross of C with parity appears in its open 4-blocks. ■

Proof of Theorem 1. Let T be a *ce*-tiling decomposed, for each k , into k -blocks with equal frames. Then a shift of CE matches T on all lines of rank $< k$. The shifted CE converge pointwise to T , except possibly on their (unranked) axes. By Remark 1, the shift increments grow in rank, and so sum to one 2-adic shift. Finally, reflections match the brackets on axes. ■

3.1 Parsimonious Enforcement of the Grid of 1-Blocks

First, we reduce the needed parity colors. A parity pointer on a single edge suffices, so it needs to accompany only one color if we show that *ce*-tilings cannot skip colors. Indeed, all *ce*-crosses are either **bends**, i.e. have 4 inward pointers, or **passes**, i.e. have 1. Thus, a third of crosses are bends, up to $O(n)$ accuracy for $n \times n$ boxes. Moreover, all orientations of bends are equally frequent, alternating on each line. ■

Tedious case investigation of [Levitsky 04] shows *ce* bits themselves forcing 1-blocks, rendering parity bits redundant. A **k -bar** is a maximal bold or pale segment, k being its length. $k > 1$ and no bold 3-bar exists since it is easy to see that its middle link would be connected by a tile to a 1-bar. Levitsky first proves that each *ce*-tiling has bold 2-bars. Here is a simpler argument for this.

A third of crosses are bends, so the average bar length is 3. Absent bold 2-bars, this average would allow positive density only of bold 4-bars and pale 2-bars. Tilings with such bars have period 6 and map onto a 6×6 torus with 3 bold 4×4 squares. But Z_6 cannot have three disjoint pairs of points of equal parity! ■

The rest of [Levitsky 04] analysis assures bold 2-bars two tiles away at each side of any bold 2-bar. This involves a case-by-case demonstration that no violation can be centered in a 10×10 box. The analysis is laborious but may be verifiable by a computer check.

⁵The rest of the requirement is redundant but useful in the proof.

⁶*ce* prevents crossing of monochromatic segments, making decompositions of boxes into blocks unique.

4 Acknowledgments

These remarks were developed in my attempts to understand the classical constructions of aperiodic tilings while working on [Durand, Levin, Shen 01]. My main source of information was [Allauzen Durand 96] and its explanations by B. Durand and A. Shen to whom I owe all my knowledge on this topic.

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