# Aperiodic Tilings: Breaking Translational Symmetry 

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#### Abstract

Classical results on aperiodic tilings are rather complicated and not widely understood. Below, an alternative approach is discussed in hope to provide additional intuition not apparent in classical works.


## 1 Palettes and Tilings

Physical computing media are asymmetric. Their symmetry is broken by irregularities, physical boundaries, external connections, and so on. Such peculiarities, however, are highly variable and extraneous to the fundamental nature of the media. Thus, they are pruned from theoretical models, such as cellular automata, and reliance on them is frowned upon in programming practice.

Yet, computation, like many other highly organized activities, is incompatible with perfect symmetry. Some standard mechanisms must assure breaking the symmetry inherent in idealized computing models. A famous example of such mechanisms is aperiodic tiling: hierarchical self-similar constructions, first used for computational purposes in a classical - although rather complicated - work Berger 66 and further developed in Robinson 71, Myers 74, Gurevich Koriakov 72. Allauzen Durand 96 give a helpful exposition.

Definition 1 Let $G$ be the grid of unit length edges between integer points on an infinite plane.
$A$ tiling $T$ is its mapping into a finite set of colors. Its crosses and tiles are ordered color combinations of four edges sharing a corner or forming a square, respectively. A palette $P$ of $T$ is a set including all its tiles (+palette for crosses). We say $P$ with a mapping $f$ of its colors into a smaller color alphabet enforces a set $S$ of tilings if replacing colors according to $f$ turns each $P$-tiling into a tiling in $S$.

Turning each edge orthogonally around its center turns $G$ into its dual graph and palettes into + palettes and vice versa. Thus, one can use either type as convenient.

## 2 2-Adic Coordinates

The set of all tilings with a given palette $P$ has translational symmetry, i.e. any shift produces another $P$-tiling. We want a palette that forces a complete spontaneous breaking of this symmetry, i.e. prevents individual tilings from being periodic. Accordingly, each location in a given tiling will be uniquely characterized by a sort of coordinates. Their infinitely many values cannot be reflected in the finite variety of the tile's colors. They will be represented distributively, i.e. in the colors of the surrounding tiles, and computable from them to any given number of digits.

Let us first so distribute the horizontal Cartesian integer coordinates $x=(2 i+1) 2^{k}$ of vertical edges by reflecting one bit $(i \bmod 2)$ in their color. We view this bit as the direction of a bracket. In this 1-dimensional tiling $C_{1}$, the brackets of the same rank $k$ are equidistant (Figure (1).

It is convenient to visualize the bits of even rank, picturing them $\} /\{$ or red, or dotted, separately from odd, depicted $] /[$. The bits of either shape at each side of the origin form a progression of balanced parenthetical expressions, called domains. Each domain has four grandchildren of the second lower rank: two within its outermost brackets and one to each side. The two children have the other shape and are centered at each border of the parent, thus connecting it with its grandchildren.

Handling the vertical coordinate similarly yields a neat 2 -dimensional tiling $C$ called central. Figure 2 marks the borders of intersections of its vertical and horizontal domains of equal ranks with a boldness bit.

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Figure 1: Brackets of $C_{1}$ split by rank.
Figure 2: (Right) Boldness bit: bold lines in $C$. Courtesy of A. Shen and B. Durand.

$C_{1}$ has a special, i.e. unmatched, bracket in the origin, directed arbitrarily and unranked. No palette can enforce a set of tilings with unique special points (designated by a Borel function commuting with shifts) since the set of all tilings is compact ${ }^{1}$ whereas the set of locations of their special point and the group $\mathbb{Z}$ of their shifts is not. We will extend $\mathbb{Z}$ to a compact group and also define ranks in other tilings, e.g., shifted $C_{1}$, using the following property.

Remark 1 A shift by $(2 i+1) 2^{k}$ in $C_{1}$ reverses all brackets of rank $k-1$, none of lower ranks, and every second bracket of any rank $>k$.

Therefore, the shifts by $(2 i+1) 2^{k}$ change our bits only at a $2 / 2^{k}$ fraction of locations. This fraction can be used as a metric on the group of shifts which can then be completed for it. The result is a remarkable compact group $g$ of 2-adic integers, or $\mathfrak{2}$-adics, acting on a similarly completed set of 2 -adic coordinates. A 2-adic $a$ is a formal infinite sum $\sum_{i \geq 0} 2^{i} a_{i}=\ldots+4 a_{2}+2 a_{1}+a_{0}$, where $a_{i} \in\{0,1\}$, viewed as an infinite to the left sequence of bits. The usual algorithms for addition and multiplication make $g$ a ring with $\mathbb{Z}$ as a subring $(e . g .,-1=\ldots+8+4+2+1) .{ }^{2}$

The natural action of $\mathbb{Z}$ (by shifts) can be extended to the action of the whole $g$ on $C_{1}$ and its images. Indeed, the brackets of rank $k$ are unaffected by terms $a_{i}$ with $i>k+1$. Thus, a 2-adic shift $a$ of $C_{1}$ can be defined as the pointwise limit of the sequence of shifts by integers approximating $a$.

With inverse shifts, this sequence diverges for the unranked bracket in the origin of $C_{1}$ and of its integer shifts. The direction of this bracket is determined not by its location, but by an arbitrary default included as an additional (external, unmoved by shifts) point in the tilings. The reflection reverses the default, all brackets, and the signs of their locations. With added reflection, the action of $g$ is free and transitive: each of these tilings can be obtained from any other, e.g., from $C_{1}$. In 2 dimensions we can also add the diagonal reflection exchanging the vertical and horizontal coordinates. ${ }^{3}$

## 3 Enforcing the Coordinate System with a Palette

Theorem 1 The set of 2-adic shifts and reflections of tiling $C$ can be enforced by a palette.
To prove it, we use multicomponent colors in $T$ which $f$ projects onto their first component - bracket bit. The second component includes two enforcement bits that extend $C$ to the enhanced tiling $C E$, with a +palette ce of 7 crosses modulo 8 reflections. ${ }^{4}$ One of its bits is the already described boldness bit (Figure 2). The other is a pointer in the direction of the nearest orthogonal line of the same rank. On the (unranked) axes these bits are set by a default central cross.

[^1]Definition $2 A$ box is an open ce-tiled rectangle, i.e. one with the border edges removed. Its $k$-block is a square with monochromatic sides that is ${ }^{5}$ a tile (for $k=0$ ) or a combination of four $(k-1)$-blocks sharing a corner. The four segments connecting the block's center to its sides are called ( $k-1$ )-medians. The rest of the open block is called a frame. We call a box $k$-tiled if removing an outer layer which is thinner than a $k$-block turns it into a box composed of (open at the box border) $k$-blocks. ${ }^{6}$

Lemma 1 (i) Borders between open blocks in a box are monochromatic. (ii) All k-frame patterns in a box are enclosed in its open $k$-blocks. (iii) All open $k$-blocks are congruent and have equal frames.

Proof. (i) comes from all ce-crosses having one or four inward pointers. (ii,iii) for $k>2$ follow from $k-1$ by viewing 1-blocks as tiles. Let 1-blocks $a$ and $b$ be adjacent in a 2-block $c ;(L, l)$ and $(R, r)$ be pairs of medians of $a$ and $b$ with $l, r$ directed to the side $s$ of $c, L-R$ crossing a median $m$ of $c$ at a cross $x$. $x$ forces $L, R$ to be both pale or both bold. This forces opposite brackets on $l, r$ which, too, must be both pale or both bold depending on the bracket of $L-R$. $l, r$ cannot be both bold which would require the pointer of $s$ to agree with their opposite brackets. Thus, all external medians of 2-frames are pale, internal medians bold, their brackets face the frame's center forcing inward pointers on the 1-medians, like $m$.

Lemma 2 Any 1-tiled box is $k$-tiled. (Follows from $k=2$ case by seeing ( $k-2$ )-blocks as tiles.)

Proof. The eight colors of edges fix their location in 2-frames, forcing open 1-blocks to alternate in the pattern of 2-frames which, by Lemma 1 extend to 2-blocks.

Corollary 1 Any $2^{k} \times 2^{k}$ box, extendible to a 3 times wider cocentric 1-tiled box, extends to a $(k+4)$-block.
Induction Basis. For the simplest enforcement of tiling decomposition into 1-blocks we can use a 2periodic parity bit to mark odd lines carrying 0 -medians. All pointers on odd lines point to odd crossing lines, thus forcing a period 2 on them. One needs only to assure an odd line exists. This can be easily done with a parity pointer on even lines, pointing to a crossing odd line.

Proof of Corollary. The box is $k$-tiled covering the inner box with four blocks sharing a cross. The extension comes by viewing them as tiles with parity bit reflecting the blocks' orientation and noting that each cross of $C$ with parity appears in its open 4-blocks.

Proof of Theorem 1. Let $T$ be a ce-tiling decomposed, for each $k$, into $k$-blocks with equal frames. Then a shift of $C E$ matches $T$ on all lines of rank $<k$. The shifted $C E$ converge pointwise to $T$, except possibly on their (unranked) axes. By Remark the shift increments grow in rank, and so sum to one 2-adic shift. Finally, reflections match the brackets on axes.

### 3.1 Parsimonious Enforcement of the Grid of 1-Blocks

First, we reduce the needed parity colors. A parity pointer on a single edge suffices, so it needs to accompany only one color if we show that ce-tilings cannot skip colors. Indeed, all ce-crosses are either bends, i.e. have 4 inward pointers, or passes, i.e. have 1. Thus, a third of crosses are bends, up to $O(n)$ accuracy for $n \times n$ boxes. Moreover, all orientations of bends are equally frequent, alternating on each line.

Tedious case investigation of Levitsky 04 shows ce bits themselves forcing 1-blocks, rendering parity bits redundant. A $k$-bar is a maximal bold or pale segment, $k$ being its length. $k>1$ and no bold 3 -bar exists since it is easy to see that its middle link would be connected by a tile to a 1-bar. Levitsky first proves that each ce-tiling has bold 2-bars. Here is a simpler argument for this.

A third of crosses are bends, so the average bar length is 3 . Absent bold 2-bars, this average would allow positive density only of bold 4 -bars and pale 2-bars. Tilings with such bars have period 6 and map onto a $6 \times 6$ torus with 3 bold $4 \times 4$ squares. But $Z_{6}$ cannot have three disjoint pairs of points of equal parity! ■

The rest of Levitsky 04 analysis assures bold 2-bars two tiles away at each side of any bold 2-bar. This involves a case-by-case demonstration that no violation can be centered in a $10 \times 10$ box. The analysis is laborious but may be verifiable by a computer check.

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## 4 Acknowledgments

These remarks were developed in my attempts to understand the classical constructions of aperiodic tilings while working on Durand, Levin, Shen 01. My main source of information was Allauzen Durand 96 and its explanations by B. Durand and A. Shen to whom I owe all my knowledge on this topic.

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[^1]:    ${ }^{1}$ and so has finite shift-invariant measures, e.g., defined by condensation points of frequencies of finite configurations in some quasiperiodic tiling
    ${ }^{2}$ Odd 2-adics have inverses. This allows extending $g$ to a famous locally compact field with fractions $a / 2^{i}$.
    ${ }^{3}$ We allow fewer tilings than Allauzen Durand 96 which permits different shifts at each side of the origin.
    ${ }^{4}$ Robinson 71] uses only six tiles (with reflections) but colors their corners, in addition to sides.

[^2]:    ${ }^{5}$ The rest of the requirement is redundant but useful in the proof.
    ${ }^{6}$ ce prevents crossing of monochromatic segments, making decompositions of boxes into blocks unique.

