Enumerating proofs of positive formulae

Gilles Dowek*and Ying Jiang[†]

Abstract

We provide a semi-grammatical description of the set of normal proofs of positive formulae in minimal predicate logic, *i.e.* a grammar that generates a set of *schemes*, from each of which we can produce a finite number of normal proofs. This method is complete in the sense that each normal proof-term of the formula is produced by some scheme generated by the grammar. As a corollary, we get a similar description of the set of normal proofs of positive formulae for a large class of theories including simple type theory and System F.

1 Introduction

A simple way to establish that provability in a logic is decidable is to develop a proof-search method, enumerating all the potential proofs of a given formula, and to prove that the search tree of this method is finite. In this case, when a formula is provable, we can even conclude that it has a finite number of proofs. This is typically the situation in some formulations of classical propositional sequent calculus [1].

In some other cases, typically in some formulations of intuitionistic or minimal propositional sequent calculus, the search tree is infinite but regular, i.e. it has only a finite number of distinct sub-trees [1]. In such a situation, provability is still decidable, but the sets of proofs may be infinite. Nevertheless, we can describe it with a context-free grammar.

In contrast to Kleene's result, Zaionc has proved that the set of normal proof-terms of a given formula in minimal propositional logic (*i.e.* the set of normal terms of a given type in simply typed lambda-calculus) is not a context-free language [2]. This result is a consequence of the undecidability of definability in simply typed lambda-calculus [3] (see also [4] for a minimal example) and it explains why previous grammatical descriptions of the set of normal terms of a given type had required an infinite number of symbols [5, 6, 7, 8].

The reason of this discrepancy between Kleene's and Zaionc's results is that the former applies to a notion of sequent whose left hand side is a set and the latter to one whose left hand side is a list. When using sets, there is no way to distinguish proof-terms such as $\lambda \alpha : P \lambda \beta : P \alpha$ and $\lambda \alpha : P \lambda \beta : P \beta$. These two proof-terms should be written in the same way using the schematic notation $\lambda \alpha : P \lambda \alpha : P \alpha$.

Using this idea, Takahashi, Akama, and Hirokawa [5] as well as Broda and Damas [9, 10] have shown that if we use such a schematic language for proof-terms, where identical hypotheses are referred to by the same name, the set of proof-terms of a given formula in minimal propositional logic becomes a context-free language. Moreover, each schematic proof-term of this context-free language corresponds to a finite number of genuine proof-terms. For instance, the schematic proof-term $\lambda \alpha : P \ \lambda \alpha : P \ \alpha$ corresponds to two proof-terms: $\lambda \alpha : P \ \lambda \beta : P \ \alpha$ and $\lambda \alpha : P \ \lambda \beta : P \ \beta$. More generally, each variable occurrence of a schematic proof-term may be replaced by a variable chosen in a finite set, yielding a finite number of proof-terms.

When such a grammar exists, we say that we have a *semi-grammatical description* of the set of proof-terms of a given formula. More precisely, a semi-grammatical description of a set is formed with a context-free grammar and an algorithm generating a finite number of elements of the set from each element of the language defined by the grammar.

In [11], we have given a new decidability proof for the fragment of minimal predicate logic where all quantifiers are positive and obtained, as a corollary, the decidability of type inhabitation for positive types in System F. The motivation for studying the positive fragment of minimal logic is twofold. First, in the classical case, it is well-known that the undecidability comes from the negative quantifiers and that

^{*}École polytechnique and INRIA, LIX, École polytechnique, 91128 Palaiseau Cedex, France. gilles.dowek@polytechnique.fr

[†]State Key Laboratory of Computer Science, Institute of Software, Chinese Academy of Sciences, Beijing 100190, P.R.China. jy@ios.ac.cn

$$\frac{\Delta, \alpha: A_1 \to \dots \to A_n \to P \vdash t_1: A_1 \quad \dots \quad \Delta, \alpha: A_1 \to \dots \to A_n \to P \vdash t_n: A_n}{\Delta, \alpha: A_1 \to \dots \to A_n \to P \vdash (\alpha \ t_1 \ \dots \ t_n): P} L \to$$

if P is atomic.

$$\frac{\Delta \vdash t : A}{\Delta \vdash \lambda x \ t : \forall x \ A} R \forall$$

if x is not free in Δ .

$$\frac{\Delta, \alpha: A \vdash t: B}{\Delta \vdash \lambda \alpha \ t: A \to B} R \to$$

Figure 1: The system LJ⁺: a sequent calculus for positive sequents

$$\frac{\Gamma^*{\downarrow}\vdash A_1\ \dots\ \Gamma^*{\downarrow}\vdash A_n}{\Gamma\vdash P}\,L\to$$

where

$$\begin{split} &\Gamma = \Gamma_1, [\Gamma_2, [...\Gamma_{i-1}, [\Gamma_i, A_1 \to ... \to A_n \to P]_{V_{i-1}}...]_{V_2}]_{V_1}, \\ &\Gamma^* = ([...[[\Gamma_1]_{V_1}, \Gamma_2]_{V_2}, ...\Gamma_{i-1}]_{V_{i-1}}, \Gamma_i, A_1 \to ... \to A_n \to P), \\ &\text{and } P \text{ is atomic and has no free variable in } V_1 \cup V_2 \cup ... \cup V_{i-1}. \end{split}$$

$$\frac{[\Gamma]_V \downarrow \vdash A}{\Gamma \vdash \forall x \ A} \ R \forall$$

where V is the set of all variables bound in $\forall x \ A$.

$$\frac{(\Gamma,A){\downarrow}\vdash B}{\Gamma\vdash A\to B}\,R\to$$

Figure 2: The system LJB: a sequent calculus with brackets

the positive fragment is decidable. The positive fragment, both for classical and minimal predicate logics, appears to be a large natural decidable fragment. Secondly, in System F, the datatypes are expressed as positive types. For instance, the type of unary natural numbers is encoded as $\forall X \ (X \to (X \to X) \to X)$ and that of binary numbers as $\forall X \ (X \to (X \to X) \to (X \to X) \to X)$. However, some positive types, such as $\forall X \ (X \to ((X \to X) \to X) \to X))$, are not datatypes. Nevertheless, we may want to describe the sets of normal terms of such types, because they are used in higher-order abstract syntax or as the input type of the algorithm, extracted from the constructive proof of the completeness theorem [12].

The algorithm defined in [11] consists in building a regular search tree, based on a careful handling of variable names with a system of brackets. In this paper, we extend the result and give a semigrammatical description for the set of β -normal η -long proof-terms of a given formula in the positive fragment of minimal predicate logic.

First, as the search-tree introduced in [11] is regular, we can define a grammar enumerating the schematic proof-terms. Then, we give an algorithm to generate a finite set of terms corresponding to a given scheme. This algorithm is more complex than that for the propositional case, because the types may be modified when a variable is replaced by another. The method obtained in this way is complete in the sense that each normal proof-term of the formula is produced from some scheme generated by the grammar. Finally, this semi-grammatical description of normal proof-terms of positive formulae also applies to several theories such as simple type theory and System F.

2 The systems LJ^+ and LJB

Leaving a more complete description to [11], we briefly recall, in this section, the notion of positive formula, the sequent calculi LJ^+ and LJB. We also introduce a notion of proof-term to represent derivations in each of these calculi. The proof-terms of LJ^+ are usual lambda-terms and are just called *proof-terms*, while the proof-terms of LJB are called *schemes*.

2.1 Positive formulae

Minimal predicate logic is the fragment of predicate logic with a single connector \rightarrow and a single quantifier \forall . Terms and formulas are defined as usual. A *context* is a finite multiset of formulae and a *sequent* $\Gamma \vdash A$ is a pair formed with a context Γ and a formula A.

A formula in minimal predicate logic is said to be positive if all its universal quantifier occurrences are positive. More precisely, the set of positive and negative formulae and positive sequents in minimal predicate logic are defined by induction as follows.

Definition 2.1 (Positive and negative formulae)

- An atomic formula is positive and negative.
- A formula of the form $A \to B$ is positive (resp. negative) if A is negative (resp. positive) and B is positive (resp. negative).
- A formula of the form $\forall x \ A$ is positive if A is positive.

As pointed out in [11], a negative formula has the form $A_1 \to ... \to A_n \to P$, where P is an atomic formula and $A_1, ..., A_n$ are positive formulae.

Definition 2.2 (Positive sequents) A sequent $A_1, ..., A_n \vdash B$ is positive if $A_1, ..., A_n$ are negative and B is positive.

2.2 LJ⁺: a sequent calculus for positive sequents

We use a cut-free sequent calculus for positive sequents in minimal predicate logic. This sequent calculus contains the usual right rule for the universal quantifier, but no left rule for this quantifier is needed because all sequents are positive. It contains also the usual right rule for the implication. But the left rule for implication

$$\frac{\Delta, A \to B \vdash A \quad \Delta, A \to B, B \vdash C}{\Delta, A \to B \vdash C}$$

and the axiom rule

$$\overline{\Delta, A \vdash A}$$

are replaced by a more restricted, but equivalent, rule

$$\frac{\Delta, A_1 \rightarrow \ldots \rightarrow A_n \rightarrow P \vdash A_1 \ \ldots \ \Delta, A_1 \rightarrow \ldots \rightarrow A_n \rightarrow P \vdash A_n}{\Delta, A_1 \rightarrow \ldots \rightarrow A_n \rightarrow P \vdash P}$$

where P is an atomic formula.

In order to associate lambda-terms to proofs, we must associate proof variables to formulae in contexts. A context with named formulae is a finite multiset of pairs, each of them formed with a proof variable and a formula, in such a way that each proof variables occurs at most once. A sequent with named formulae $\Delta \vdash A$ is a pair formed with a context Δ with named formulae and a formula A. These proof variables are distinguished from the usual term variables of predicate logic.

The rules of the system LJ^+ , equipped with proof-terms, are depicted in Figure 1. Notice that all these proof-terms are β -normal η -long. Ignoring these proof-terms, it yields the original presentation of LJ^+ given in [11]. When $\Delta \vdash t : A$ is derivable, we also say that t is a proof-term of the sequent $\Delta \vdash A$.

2.3 LJB: a sequent calculus with brackets

Search trees in LJ⁺ are not always finite or even regular. For instance, the search tree of the formula $((P \rightarrow Q) \rightarrow Q) \rightarrow Q$ is infinite and that of the formula $((\forall x \ (P(x) \rightarrow Q)) \rightarrow Q) \rightarrow Q$ is not even regular. To prove the decidability of the positive fragment of minimal predicate logic, we have introduced in [11] another sequent calculus called LJB.

In LJ⁺, to apply the R \forall rule to the sequent $\Gamma \vdash \forall x A$, we have to rename the variable x either in $\forall x A$ or in Γ so that the variable released by the rule does not appear in the context. In LJB, instead of renaming the variable x, we bind it in the context Γ with brackets and obtain the sequent $[\Gamma]_x \vdash A$. In fact, for technical reasons, we bind in Γ , not only the variable x, but also all the bound variables of A.

Definition 2.3 (LJB-contexts and items) LJB-contexts and items are mutually inductively defined as follows.

Figure 3: An example of search tree in LJB.

$$\frac{\Gamma^* \! \downarrow \vdash \pi_1 : A_1 \quad \dots \quad \Gamma^* \! \downarrow \vdash \pi_n : A_n}{\Gamma \vdash (\alpha \ \pi_1 \ \dots \ \pi_n) : P} L \rightarrow$$

where

$$\begin{split} &\Gamma = \Gamma_1, [\Gamma_2, [\dots \Gamma_{i-1}, [\Gamma_i, A_1 \to \dots \to A_n \to P]_{V_{i-1}} \dots]_{V_2}]_{V_1}, \\ &\Gamma^* = ([\dots [[\Gamma_1]_{V_1}, \Gamma_2]_{V_2}, \dots \Gamma_{i-1}]_{V_{i-1}}, \Gamma_i, A_1 \to \dots \to A_n \to P), \\ &P \text{ is atomic and has no free variable in } V_1 \cup V_2 \cup \dots \cup V_{i-1}, \\ &\text{and } \alpha \text{ is the canonical variable of type } A_1 \to \dots \to A_n \to P. \end{split}$$

$$\frac{[\Gamma]_V \downarrow \vdash \pi : A}{\Gamma \vdash \lambda x \ \pi : \forall x \ A} R \forall$$

where V is the set of all variables bound in $\forall x \ A$.

$$\frac{(\Gamma, A) \downarrow \vdash \pi : B}{\Gamma \vdash \lambda \alpha : A \; \pi : A \to B} \; R \to$$

where α is the canonical variable of type A.

Figure 4: The system LJB with schemes.

- A LJB-context Γ is a finite multiset of items $\{I_1, ..., I_n\}$.
- An item I is either a formula or an expression of the form $[\Gamma]_V$ where V is a set of variables and Γ a LJB-context.

In the item $[\Gamma]_V$, the variables of V are bound by the symbol [].

A LJB-sequent $\Gamma \vdash A$ is a pair formed by a LJB-context Γ and a formula A.

The system LJB is formed by two sets of rules: the usual deduction rules and additional transformation rules dealing with bracket manipulation. The transformation rules form a terminating rewrite system: the first rule allows to replace an item of the form $[I, \Gamma]_V$ by the two items I and $[\Gamma]_V$ provided no free variable of I is in V; the second one allows to remove trivial items; the third rule to replace two identical items by one.

Definition 2.4 (Cleaning LJB-contexts) The cleaning rules are

$$\begin{split} & [I,\Gamma]_V \longrightarrow I, [\Gamma]_V, \quad \textit{if } FV(I) \cap V = \varnothing \\ & [\]_V \longrightarrow \varnothing \\ & II \longrightarrow I \end{split}$$

where I is an item and Γ a LJB-context.

Instead of proving the confluence of the rewrite system of Definition 2.4, we fix an arbitrary strategy and define the normal form $\Gamma \downarrow$ of a context Γ as the normal form relative to this strategy. We may, for instance, proceed as follows. If $\Gamma = \emptyset$ then we let $\Gamma \downarrow = \emptyset$. Otherwise, we choose an item I in Γ and let $\Gamma' = \Gamma \setminus \{I\}$. Then, we normalize the item I and the LJB-context Γ' recursively. We let $\Gamma \downarrow = \Gamma' \downarrow$ if $I \downarrow$ is an element of $\Gamma' \downarrow$ and $\Gamma \downarrow = I \downarrow, \Gamma' \downarrow$ otherwise. To normalize an item I, we need to consider the two following cases. If I is a formula, then we let $I \downarrow = I$. If it has the form $[\Delta]_V$, we first normalize recursively Δ , then we let Δ_1 be the part of $\Delta \downarrow$ formed with the elements that have a free variable in $s_{\Gamma \vdash P} \longrightarrow \left(\alpha \ s_{\Gamma^* \downarrow \vdash A_1} \ \dots \ s_{\Gamma^* \downarrow \vdash A_n} \right)$

where

$$\begin{split} &\Gamma = \Gamma_1, [\Gamma_2, [...\Gamma_{i-1}, [\Gamma_i, A_1 \to ... \to A_n \to P]_{V_{i-1}}...]_{V_2}]_{V_1}, \\ &\Gamma^* = ([...[[\Gamma_1]_{V_1}, \Gamma_2]_{V_2}, ...\Gamma_{i-1}]_{V_{i-1}}, \Gamma_i, A_1 \to ... \to A_n \to P), \\ &P \text{ is atomic and has no free variable in } V_1 \cup V_2 \cup ... \cup V_{i-1}, \\ &\text{and } \alpha \text{ is the canonical variable of type } A_1 \to ... \to A_n \to P. \end{split}$$

$$s_{\Gamma \vdash \forall x \ A} \longrightarrow \lambda x \ s_{[\Gamma]_V \downarrow \vdash A}$$

where V is the set of all variables bound in $\forall x \ A$.

 $s_{\Gamma \vdash A \to B} \longrightarrow \lambda \alpha : A \; s_{(\Gamma, A) \downarrow \vdash B}$

where α is the canonical variable of type A.

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Figure 5: The scheme grammar.
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V and let $\Delta_2 = \Delta \downarrow \setminus \Delta_1$. Finally, we let $I \downarrow = \Delta_2$ if $[\Delta_1]_V$ is an element of Δ_2 and $I \downarrow = [\Delta_1]_V, \Delta_2$ otherwise.

The deduction rules apply to LJB-sequents with normalized contexts with respect to the cleanning rules and where the bound variables are named differently and differently from the free variables. It is easy to check that these properties are preserved by the rules. Moreover, in LJB we deal with formulae, not formulae modulo α -equivalence.

The rules of the system LJB are depicted in Figure 2. In the $L \to \text{rule}$, brackets are moved from some items of the LJB-context to others, bringing the formula $A_1 \to \ldots \to A_n \to P$ inside brackets to the surface, so that it can be used. For instance the LJB-sequent $Q(x), [Q(x) \to P]_x \vdash P$ is transformed (bottom-up) into $[Q(x)]_x, Q(x) \to P \vdash Q(x)$. The crucial point is that the two occurrences of x in Q(x)and $Q(x) \to P$ that are separated in the first LJB-sequent remain separated.

The main interest of the system LJB is that, as illustrated in the Example 2.3, the search tree in LJB of any positive formula is regular. This property is a consequence of the following proposition proved in [11] (Proposition 4.5).

Proposition 2.1 Let A be a positive formula. There exists a finite set S of sequents such that all the sequents occurring in a LJB-proof of the sequent $\vdash A$ are in S.

Example: Let $A = (B \to Q) \to Q$ where $B = \forall y ((P(y) \to Q) \to (P(y) \to Q))$. The search tree of the sequent $\vdash A$ is given in Figure 3.

Notice that when trying to prove the sequent $B \to Q, P(y) \to Q, P(y) \vdash Q$ we may apply the $L \to$ rule either with the proposition $B \to Q$ or with the proposition $P(y) \to Q$, yielding two branches in the search tree. The same holds with the sequent $B \to Q, [P(y) \to Q, P(y)]_y, P(y) \to Q, P(y) \vdash Q$. Notice also that the search tree is infinite and regular. We have cut the infinite branch when the sequent $B \to Q, [P(y) \to Q, P(y)]_y \vdash (P(y) \to Q) \to P(y) \to Q$ appeared for the second time.

2.4 Schemes

Now we introduce *schemes*, that are the proof-terms for the system LJB. Unlike what we did for LJ^+ , we do not assign names to hypotheses in LJB. Instead, we choose a canonical proof variable for each such formula. The rules of LJB with schemes are depicted in Figure 4.

3 A grammar to enumerate schemes

In this section, we prove that, although it may be infinite, the set of schemes of a given normalized LJB-sequent may be described by a context-free grammar.

Definition 3.1 (Scheme grammar) Let $\Gamma \vdash A$ be a normalized LJB-sequent and S be the finite set of sequents that may occur in a derivation of $\Gamma \vdash A$. To each sequent S of S, we associate a non-terminal symbol s_S and set up the rules displayed in Figure 5.

The grammar generating the schemes of the type A given in Example 2.3 and a scheme generated by this grammar are detailed in the example below.

Example: The grammar generating the schemes of the type $A = (B \to Q) \to Q$ where $B = \forall y ((P(y) \to Q) \to (P(y) \to Q))$ is

$$S \rightarrow \lambda \alpha (\alpha \lambda y \lambda \beta \lambda \gamma (\beta \gamma))$$

$$S \rightarrow \lambda \alpha (\alpha \lambda y \lambda \beta \lambda \gamma (\alpha \lambda y S_1))$$

$$S_1 \rightarrow \lambda \beta \lambda \gamma (\beta \gamma)$$

$$S_1 \rightarrow \lambda \beta \lambda \gamma (\alpha \lambda y S_1)$$

where S is the non-terminal associated to the sequent $\vdash A$, S_1 that associated to $B \to Q$, $[P(y) \to Q, P(y)]_y \vdash (P(y) \to Q) \to P(y) \to Q$, α is the canonical variable of type $B \to Q$, β that of type $P(y) \to Q$ and γ that of type P(y).

A scheme generated by the grammar is

$$\lambda \alpha \ (\alpha \ \lambda y \lambda \beta \ \lambda \gamma \ (\alpha \ \lambda y \lambda \beta \ \lambda \gamma \ (\beta \ \gamma)))$$

Proposition 3.1 (Soundness) Let $\Gamma \vdash A$ be a normalized LJB-sequent. Then for any scheme π generated in $s_{\Gamma \vdash A}$, we have $\Gamma \vdash \pi : A$.

Proof. By induction on the derivation of π in the grammar.

Proposition 3.2 (Completeness) Let $\Gamma \vdash A$ be a normalized LJB-sequent. Then each scheme π such that $\Gamma \vdash \pi : A$ is generated in $s_{\Gamma \vdash A}$.

Proof. By induction on the derivation of $\Gamma \vdash \pi : A$ in the system LJB with schemes.

4 Generating proof-terms

Now we are ready to provide a term enumeration algorithm through the grammatical scheme enumeration algorithm described in the previous section. In this endeavor, we will define a function \mathcal{H} , which, roughly speaking, associates a finite set of terms to a scheme, in such a way that t is a proof-term if and only if there exists a scheme π such that $t \in \mathcal{H}(\pi)$. To define this function \mathcal{H} , we need a function \mathcal{G} handling context cleaning. When defining the function \mathcal{G} , the only non trivial case is that of the rule $II \longrightarrow I$, which is handled in turn by another function \mathcal{F} .

Definitions 4.1 and 4.2 below extend the usual notion of α -equivalence for formulae to sequents of LJ⁺ and LJB, and will be useful in the rest of the section.

Definition 4.1 (α -equivalence of sequents) Two sequents $\Gamma \vdash A$ and $\Gamma' \vdash A'$ are said to be α -equivalent if there exists a variable renaming σ of term variables (i.e. an injective substitution mapping variables to variables) such that Γ' is α -equivalent to $\sigma\Gamma$ and A' is α -equivalent to σA .

For instance, the sequents $P(x) \vdash P(x)$ and $P(y) \vdash P(y)$ are α -equivalent. The intuition is that the variables free in Γ and A are considered as implicitly bound by the symbol \vdash in the sequent $\Gamma \vdash A$.

We also extend the notion of α -equivalence to sequents of LJ⁺ with named formulae as follows.

Definition 4.2 (α -equivalence of sequents with named formulae) Two sequents $\Gamma \vdash A$ and $\Gamma' \vdash A'$ are said to be α -equivalent if there exists a variable renaming σ of term and proof variables such that Γ' is α -equivalent to $\sigma\Gamma$ and A' is α -equivalent to σA .

For instance, the sequents $\alpha : P(x) \vdash P(x)$ and $\beta : P(y) \vdash P(y)$ are α -equivalent.

Definition 4.3 (Fresh α -variant and flattening) Let $\Gamma \vdash A$ be a normalized LJB-sequent, a fresh α -variant $\Gamma' \vdash A'$ of $\Gamma \vdash A$ is an LJB-sequent, which is α -equivalent to $\Gamma \vdash A$ and where all bound variables are named differently.

A LJ^+ -sequent $\Delta \vdash B$ is said to be a flattening of a normalized LJB-sequent $\Gamma \vdash A$, if it is obtained by erasing all the brackets in a fresh α -variant of $\Gamma \vdash A$ and naming all the formulae in Γ with distinct proof variables. *Example:* A flattening of the LJB-sequent $[P(x), P(x) \to Q]_x, [P(x), P(x) \to Q]_x \vdash Q$ is the LJ⁺-sequent $\alpha_1 : P(x_1), \beta_1 : (P(x_1) \to Q), \alpha_2 : P(x_2), \beta_2 : (P(x_2) \to Q) \vdash Q$.

Remark that two flattenings of the same LJB-sequent are α -equivalent LJ⁺-sequents.

Definition 4.4 (Partial duplication) Let $\Sigma \vdash A$ be a sequent of LJ^+ . A sequent $\Delta \vdash B$ of LJ^+ is said to be a partial duplication of $\Sigma \vdash A$ if there exist two substitutions σ_1 and σ_2 of term-variables with the same domain, renaming the variables of their domain with fresh and distinct variables such that for each variable $\gamma : C$ of Σ , Δ contains either the variable $\gamma_1 : \sigma_1 C$ or the variable $\gamma_2 : \sigma_2 C$ or both, and B is either $\sigma_1 A$ or $\sigma_2 A$.

Example: If the sequent $\Sigma \vdash A$ is

$$\alpha: (Px \to Q), \beta: Px \vdash Q$$

and $\sigma_1 = \sigma_2 = id$, then one partial duplication is the sequent

$$\alpha_1 : (Px \to Q), \beta_1 : Px, \alpha_2 : (Px \to Q), \beta_2 : Px \vdash Q$$

If the sequent $\Sigma \vdash A$ is

$$\alpha: (Px \to Q), \beta: Px \vdash Q$$

but $\sigma_1 = x_1/x$ and $\sigma_2 = x_2/x$, then one partial duplication is the sequent

$$\alpha_1 : (Px_1 \to Q), \beta_1 : Px_1, \alpha_2 : (Px_2 \to Q), \beta_2 : Px_2 \vdash Q$$

If the sequent $\Sigma \vdash A$ is

$$\alpha: (Px \to Q), \beta: Px \vdash Px$$

and $\sigma_1 = x_1/x$ and $\sigma_2 = x_2/x$, then one partial duplication is the sequent

$$\alpha_1 : (Px_1 \to Q), \beta_1 : Px_1, \alpha_2 : (Px_2 \to Q), \beta_2 : Px_2 \vdash Px_1$$

Definition 4.5 (The function \mathcal{F}) Let $\Sigma \vdash A$ be a sequent of LJ^+ and $\Delta \vdash B$ a partial duplication of this sequent obtained with the substitutions σ_1 and σ_2 .

$$LJ^+$$

$$\Delta \vdash B \quad \mathcal{F}(u)$$

$$\uparrow$$

$$\Sigma \vdash A \quad u$$

Let u be a proof-term of $\Sigma \vdash A$. We define, by induction on the structure of u, a finite set $\mathcal{F}_{\Sigma \vdash A}^{\Delta \vdash B}(u)$ of proof-terms of $\Delta \vdash B$.

- If $u = (\alpha \ u_1 \ ... \ u_n)$, then A is atomic. Let $C_1 \to ... \to C_n \to A$ be the type of α . For $i \in \{1, 2\}$, if Δ contains a variable $\alpha_i : \sigma_i C_1 \to ... \to \sigma_i C_n \to \sigma_i A$ and $\sigma_i A = B$, then we take all terms of the form $(\alpha_i \ u'_1 \ ... \ u'_n)$ where u'_1 is an element of $\mathcal{F}_{\Sigma \vdash C_1}^{\Delta \vdash \sigma_i C_1}(u_1), \ ..., \ u'_n$ is an element of $\mathcal{F}_{\Sigma \vdash C_n}^{\Delta \vdash \sigma_i C_n}(u_n)$, otherwise we take no term with head variable α_i .
- If $u = \lambda x \ u_1$, then A has the form $\forall x \ A_1$ and B has the form $\forall x \ B_1$, where B_1 is either $\sigma_1 A_1$ or $\sigma_2 A_1$, we take all terms of the form $\lambda x \ u'_1$ where u'_1 is an element of $\mathcal{F}_{\Sigma \vdash A_1}^{\Delta \vdash B_1}(u_1)$.
- If $u = \lambda \alpha \ u_1$, then A has the form $A_1 \to A_2$ and B has the form $B_1 \to B_2$, where B_1 is either $\sigma_1 A_1$ or $\sigma_2 A_1$ and B_2 is either $\sigma_1 A_2$ or $\sigma_2 A_2$, we take all terms of the form $\lambda \alpha' \ u'_1$ with u'_1 an element of $\mathcal{F}_{\Sigma,\alpha:A_1\vdash A_2}^{\Delta,\alpha':B_1\vdash B_2}(u_1)$.

Example: If the sequent $\Sigma \vdash A$ is

$$\alpha: (Px \to Q), \beta: Px \vdash Q$$

 $\sigma_1 = \sigma_2 = id$ and one partial duplication is the sequent

$$\alpha_1 : (Px \to Q), \beta_1 : Px, \alpha_2 : (Px \to Q), \beta_2 : Px \vdash Q$$

then

$$\mathcal{F}_{\Sigma \vdash Q}^{\Delta \vdash Q}((\alpha \ \beta)) = \{ (\alpha_1 \ \beta_1), (\alpha_1 \ \beta_2), (\alpha_2 \ \beta_1), (\alpha_2 \ \beta_2) \}$$

If the sequent $\Sigma \vdash A$ is

$$\alpha: (Px \to Q), \beta: Px \vdash Q$$

 $\sigma_1 = x_1/x$ and $\sigma_2 = x_2/x$ and one partial duplication is the sequent

$$\alpha_1 : (Px_1 \to Q), \beta_1 : Px_1, \alpha_2 : (Px_2 \to Q), \beta_2 : Px_2 \vdash Q$$

then

$$\mathcal{F}_{\Sigma \vdash Q}^{\Delta \vdash Q}((\alpha \ \beta)) = \{(\alpha_1 \ \beta_1), (\alpha_2 \ \beta_2)\}$$

Notice that, after having chosen α_1 , in the first case, we obtain

$$\mathcal{F}_{\Sigma \vdash Px}^{\Delta \vdash Px}(\beta) = \{\beta_1, \beta_2\}$$

while in the second, we obtain

$$\mathcal{F}_{\Sigma \vdash Px}^{\Delta \vdash Px_1}(\beta) = \{\beta_1\}$$

Our relatively liberal notion of partial duplication allows the "pathological" example where the set $\mathcal{F}_{\Sigma \vdash A}^{\Delta \vdash B}(u)$ is empty: if the sequent $\Sigma \vdash A$ is

$$\alpha: (Px \to Q), \beta: Px \vdash Q$$

and $\sigma_1 = x_1/x$ and $\sigma_2 = x_2/x$, then one partial duplication is the sequent

$$\alpha_1: (Px_1 \to Q), \beta_2: Px_2 \vdash Q$$

and $\mathcal{F}_{\Sigma \vdash A}^{\Delta \vdash B}((\alpha \ \beta)) = \emptyset$.

Proposition 4.1 (Soundness) Let $\Delta \vdash B$ be a partial duplication of $\Sigma \vdash A$. If u is a proof of $\Sigma \vdash A$, and $t \in \mathcal{F}_{\Sigma \vdash A}^{\Delta \vdash B}(u)$, then t is a proof of $\Delta \vdash B$.

Proof. By induction on the structure of u.

Proposition 4.2 (Completeness) Let $\Delta \vdash B$ be a partial duplication of $\Sigma \vdash A$. If t is a proof of $\Delta \vdash B$ then there exists a proof u, of the same height as t, of $\Sigma \vdash A$ such that $t \in \mathcal{F}_{\Sigma \vdash A}^{\Delta \vdash B}(u)$.

Proof. By induction on the structure of t. The term u is obtained by replacing each variable of the form $\sigma_1 x$ or $\sigma_2 x$ by x.

Definition 4.6 (The function \mathcal{G}) Let $\Gamma \vdash A$ be a normalized LJB-sequent and $\Gamma \downarrow \vdash A$ its normal form. Let $\Delta \vdash B$ be a flattening of $\Gamma \vdash A$ and $\Delta' \vdash B'$ a flattening of $\Gamma \downarrow \vdash A$.

For any proof-term u of $\Delta' \vdash B'$, we construct a set $\mathcal{G}_{\Delta' \vdash B'}^{\Delta \vdash B}(u)$ of proof-terms of $\Delta \vdash B$ by induction on the length of the reduction from Γ to $\Gamma \downarrow$.

- If $\Gamma \downarrow = \Gamma$, then $\Delta' \vdash B'$ and $\Delta \vdash B$ are α -equivalent, thus there exists a renaming σ of the free variables of Δ and B such that Δ is α -equivalent to $\sigma\Delta'$ and B is α -equivalent to $\sigma B'$. We take $\mathcal{G}_{\Delta'\vdash B'}^{\Delta\vdash B}(u) = \{\sigma u\}.$
- If Γ rewrites to Γ_1 in one cleaning step and then Γ_1 rewrites to Γ_{\downarrow} , then let $\Delta_1 \vdash B_1$ be a flattening of $\Gamma_1 \vdash A$ and let $S = \mathcal{G}_{\Delta' \vdash B'}^{\Delta_1 \vdash B_1}(u)$. Now consider the rule used to reduce Γ to Γ_1 . If this rule is $[]_V \to \emptyset$ or $[\Gamma, I]_V \to [\Gamma]_V$, I then $\Delta \vdash B$ and $\Delta_1 \vdash B_1$ are α -equivalent, thus there exists a renaming σ of the free variables of Δ and B such that Δ is α -equivalent to $\sigma\Delta_1$ and B is α equivalent to σB_1 . We take $\mathcal{G}_{\Delta' \vdash B'}^{\Delta \vdash B}(u) = \{\sigma t \mid t \in S\}$. If this rule is $II \to I$ then $\Delta \vdash B$ is a partial duplication of $\Delta_1 \vdash B_1$. We take $\mathcal{G}_{\Delta' \vdash B'}^{\Delta \vdash B}(u) = \bigcup_{t \in S} \mathcal{F}_{\Delta_1 \vdash B_1}^{\Delta \vdash B}(t)$.

Example: The sequent

$$[P(x), P(x) \to Q]_x, [P(x), P(x) \to Q]_x \vdash Q$$

normalizes to

$$[P(x), P(x) \to Q]_x \vdash Q$$

A flattening of the first sequent is $\Delta \vdash Q$ where Δ is the context

$$\alpha_1 : P(x_1), \beta_1 : (P(x_1) \to Q), \alpha_2 : P(x_2), \beta_2 : (P(x_2) \to Q)$$

and a flattening of the second one is the sequent $\Delta' \vdash Q$ where Δ' is the context

$$\alpha: P(x), \beta: (P(x) \to Q)$$

Then

$$\mathcal{G}_{\Delta'\vdash Q}^{\Delta\vdash Q}((\alpha \ \beta)) = \{(\beta_1 \ \alpha_1), (\beta_2 \ \alpha_2)\}$$

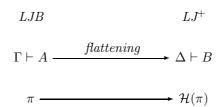
Proposition 4.3 (Soundness) Let $\Gamma \vdash A$ be a normalized LJB-sequent and $\Gamma \downarrow \vdash A$ its normal form. Let $\Delta \vdash B$ be a flattening of $\Gamma \vdash A$ and $\Delta' \vdash B'$ a flattening of $\Gamma \downarrow \vdash A$. Let u be a proof-term of $\Delta' \vdash B'$ and $t \in \mathcal{G}_{\Delta' \vdash B'}^{\Delta \vdash B}(u)$. Then t is a proof-term of $\Delta \vdash B$.

Proof. By induction on the length of the reduction from Γ to $\Gamma\downarrow$, using Proposition 4.1 for the case of the rule $II \longrightarrow I$.

Proposition 4.4 (Completeness) Let $\Gamma \vdash A$ be a normalized LJB-sequent and $\Gamma \downarrow \vdash A$ its normal form. Let $\Delta \vdash B$ be a flattening of $\Gamma \vdash A$ and $\Delta' \vdash B'$ a flattening of $\Gamma \downarrow \vdash A$. If t is a proof of $\Delta \vdash B$, then there exists a proof u, of the same height as t, of $\Delta' \vdash B'$ such that $t \in \mathcal{G}_{\Delta' \vdash B'}^{\Delta \vdash B'}(u)$.

Proof. By induction on the length of the reduction from Γ to $\Gamma\downarrow$, using Proposition 4.2 for the case of the rule $II \longrightarrow I$.

Definition 4.7 (The function \mathcal{H}) Let $\Gamma \vdash A$ be a normalized LJB-sequent and $\Delta \vdash B$ a flattening of $\Gamma \vdash A$.



Let π be a scheme of the sequent $\Gamma \vdash A$, we associate to π a set $\mathcal{H}_{\Gamma \vdash A}^{\Delta \vdash B}(\pi)$ of proof-terms of type $\Delta \vdash B$ in LJ^+ by induction on the structure of π .

• If $\pi = (\alpha \ \pi_1 \ \dots \ \pi_n)$, then let $A_1 \to \dots \to A_n \to A$ be the type of α . Select the occurrences of the formula $A_1 \to \dots \to A_n \to A$ in Γ , such that the rule $L \to can$ be applied to this occurrence, and for all i, the scheme π_i has type $\Gamma^* \downarrow \vdash A_i$ where $\Gamma^* \downarrow$ is the context obtained by applying $L \to to$ this occurrence. For each selected occurrence, let $\alpha' : B_1 \to \dots \to B_n \to B$ be the corresponding declaration in Δ . The sequent $\Delta \vdash B$ is also a flattening of $\Gamma^* \vdash A$ and the sequent $\Delta \vdash B_i$ is one of $\Gamma^* \vdash A_i$. Consider a flattening $\Delta' \vdash B'_i$ of $\Gamma^* \downarrow \vdash A_i$, set up $S_i = \mathcal{H}_{\Gamma^* \downarrow \vdash A_i}^{\Delta' \vdash B'_i}(\pi_i)$ and $S'_i = \bigcup_{t \in S_i} \mathcal{G}_{\Delta' \vdash B'_i}^{\Delta \vdash B_i}(t)$. The set $\mathcal{H}_{\Gamma \vdash A}^{\Delta \vdash B}(\pi)$ contains the terms of the form $(\alpha' \ t_1 \ \dots \ t_n)$ for some $\alpha' : B_1 \to \dots \to B_n \to B$ in Δ corresponding to a selected occurrence and $t_i \in S'_i$.

- If $\pi = \lambda x \ \pi_1$, then $A = \forall x \ A_1$, $B = \forall y \ B_1$ and π_1 is a scheme of $[\Gamma]_V \downarrow \vdash A_1$. The sequent $\Delta \vdash B_1$ is a flattening of $[\Gamma]_V \vdash A_1$. Let $\Delta' \vdash B'_1$ be a flattening of $[\Gamma]_V \downarrow \vdash A_1$, set up $S = \mathcal{H}_{[\Gamma]_V \downarrow \vdash A_1}^{\Delta' \vdash B'_1}(\pi_1)$ and $S' = \bigcup_{t \in S} \mathcal{G}_{\Delta' \vdash B'_1}^{\Delta \vdash B_1}(t)$. The set $\mathcal{H}_{\Gamma \vdash A}^{\Delta \vdash B}(\pi)$ is the set of the terms of the form $\lambda y \ t$ for t in S'.
- If $\pi = \lambda \alpha : A_1 \pi_1$, then $A = A_1 \to A_2$ and $B = B_1 \to B_2$ and π_1 is a scheme of $(\Gamma, A_1) \downarrow \vdash A_2$. The sequent $\Delta, \alpha' : B_1 \vdash B_2$ is a flattening of $\Gamma, A_1 \vdash A_2$. Let $\Delta' \vdash B'_2$ be a flattening of $(\Gamma, A_1) \downarrow \vdash A_2$, set $up \ S = \mathcal{H}_{(\Gamma, A_1) \downarrow \vdash A_2}^{\Delta' \vdash B'_2}(\pi_1)$ and $S' = \bigcup_{t \in S} \mathcal{G}_{\Delta' \vdash B'_2}^{\Delta, \alpha' : B_1 \vdash B_2}(t)$. The set $\mathcal{H}_{\Gamma \vdash A}^{\Delta \vdash B}(\pi)$ is the set of the terms of the form $\lambda \alpha' : B_1$ t for t in S'.

Example: Continuing the Example 2.3, let

$$\pi = \lambda \alpha \, \left(\alpha \, \lambda y \lambda \beta \, \lambda \gamma \, \left(\alpha \, \lambda y \lambda \beta \, \lambda \gamma \, \left(\beta \, \gamma \right) \right) \right)$$

The set $\mathcal{H}_{\vdash A}^{\vdash A}(\pi)$ contains the two terms

$$\lambda \alpha \ (\alpha \ \lambda y_1 \lambda \beta_1 \lambda \gamma_1 \ (\alpha \ \lambda y_2 \lambda \beta_2 \lambda \gamma_2 \ (\beta_1 \ \gamma_1))) \\\lambda \alpha \ (\alpha \ \lambda y_1 \lambda \beta_1 \lambda \gamma_1 \ (\alpha \ \lambda y_2 \lambda \beta_2 \lambda \gamma_2 \ (\beta_2 \ \gamma_2)))$$

where $\alpha: B \to Q, \ \beta_1: P(y_1) \to Q, \ \gamma_1: P(y_1), \ \beta_2: P(y_2) \to Q, \ \gamma_2: P(y_2).$

Proposition 4.5 (Soundness) Let $\Gamma \vdash A$ be a normalized LJB-sequent and $\Delta \vdash B$ be a sequent of LJ^+ that is a flattening of $\Gamma \vdash A$. Then for each scheme π of $\Gamma \vdash A$, every proof-term in $\mathcal{H}_{\Gamma \vdash A}^{\Delta \vdash B}(\pi)$ is a proof-term of $\Delta \vdash B$.

Proof. By induction on the height of π , using Proposition 4.3 for context cleaning.

Proposition 4.6 (Completeness) Let $\Gamma \vdash A$ be a normalized LJB-sequent and $\Delta \vdash B$ a sequent of LJ^+ such that $\Delta \vdash B$ is a flattening of $\Gamma \vdash A$. Then for each proof-term t of $\Delta \vdash B$, there exists a scheme π of $\Gamma \vdash A$ such that $t \in \mathcal{H}_{\Gamma \vdash A}^{\Delta \vdash B}(\pi)$.

Proof. By induction on the structure of t.

• If $t = (\alpha' t_1 \dots t_n)$, then the variable $\alpha' : B_1 \to \dots \to B_n \to B$ is declared in Δ and t_i is a proof-term of $\Delta \vdash B_i$. The variable α' corresponds to an occurrence of a formula $A_1 \to \dots \to A_n \to A$ in Γ and Γ has the form $\Gamma_1, [\Gamma_2, [\dots, \Gamma_{i-1}, [\Gamma_i, A_1 \to \dots \to A_n \to A]_{V_{i-1}} \dots]_{V_2}]_{V_1}$. As $\Delta \vdash B$ is a flattening of $\Gamma \vdash A$ and this occurrence of $A_1 \to \dots \to A_n \to A$ corresponds to $B_1 \to \dots \to B_n \to B$, A has no free variable in $V_1 \cup V_2 \cup \dots \cup V_{i-1}$. Thus, the sequent $\Delta \vdash B$ is also a flattening of $\Gamma^* \vdash A$ and $\Delta \vdash B_i$ is a flattening of $\Gamma^* \vdash A_i$.

Let $\Delta' \vdash B'_i$ be a flattening of $\Gamma^* \downarrow \vdash A_i$. By Proposition 4.4, there exists a proof-term u_i of $\Delta' \vdash B'_i$ of the same height as t_i such that $t_i \in \mathcal{G}_{\Delta' \vdash B'_i}^{\Delta \vdash B_i}(u_i)$. By induction hypothesis, for each $i \in \{1, \ldots, n\}$, there exists scheme π_i of $\Gamma^* \downarrow \vdash A_i$ such that $u_i \in \mathcal{H}_{\Gamma^* \downarrow \vdash A_i}^{\Delta' \vdash B'_i}(\pi_i)$. So, if α is the canonical variable of type $A_1 \to \ldots \to A_n \to A$, then $(\alpha \ \pi_1 \ \ldots \ \pi_n)$ is a scheme of $\Gamma \vdash A$ and $(\alpha' \ t_1 \ \ldots \ t_n) \in \mathcal{H}_{\Gamma \vdash A}^{\Delta \vdash B}(\alpha \ \pi_1 \ \ldots \ \pi_n)$.

- If $t = \lambda y t_1$, then $B = \forall y B_1$, $A = \forall x A_1$ and t_1 is a proof-term of $\Delta \vdash B_1$ that is a flattening of $[\Gamma]_V \vdash A_1$. Let $\Delta' \vdash B'_1$ be a flattening of $[\Gamma]_V \downarrow \vdash A_1$. By Proposition 4.4, there exists a proof-term u_1 of $\Delta' \vdash B'_1$ of the same height as t_1 such that $t_1 \in \mathcal{G}_{\Delta' \vdash B'_1}^{\Delta \vdash B_1}(u_1)$. By induction hypothesis, there exists a scheme π_1 of $[\Gamma]_V \downarrow \vdash A_1$ such that $u_1 \in \mathcal{H}_{[\Gamma]_V \downarrow \vdash A_1}^{\Delta' \vdash B'_1}(\pi_1)$. This implies $\lambda y t_1 \in \mathcal{H}_{\Gamma \vdash A}^{\Delta \vdash B}(\lambda x \pi_1)$.
- If $t = \lambda \alpha' : B_1 \ t_1$, then $B = B_1 \to B_2$, $A = A_1 \to A_2$ and t_1 is a proof-term of $\Delta, B_1 \vdash B_2$ that is a flattening of $\Gamma, A_1 \vdash A_2$. Let $\Delta' \vdash B'_2$ be a flattening of $(\Gamma, A_1) \downarrow \vdash A_2$. By Proposition 4.4, there exists a proof-term u_1 of $\Delta' \vdash B'_2$ of the same height as t_1 such that $t_1 \in \mathcal{G}_{\Delta' \vdash B'_2}^{\Delta, B_1 \vdash B_2}(u_1)$. By induction hypothesis, there exists a scheme π_1 of $(\Gamma, A_1) \downarrow \vdash A_2$ such that $u_1 \in \mathcal{H}_{(\Gamma, A_1) \downarrow \vdash A_2}^{\Delta' \vdash B'_2}(\pi_1)$. Let α be the canonical variable of type A_1 , we have $\lambda \alpha' \ t_1 \in \mathcal{H}_{\Gamma \vdash A}^{\Delta \vdash B}(\lambda \alpha \ \pi_1)$.

Theorem 4.1 Let A be a formula. Then t is a proof-term of $\vdash A$ in LJ^+ if and only if there exists a scheme π generated by the grammar given in Definition 3.1 such that $t \in \mathcal{H}_{\vdash A}^{\vdash A}(\pi)$.

Proof. From Propositions 3.1, 3.2, 4.5, and 4.6.

5 Enumerating normal terms of a positive type in System F

As remarked in [11], to each positive type T of System F, we can associate a formula $\Phi(T)$ in predicate logic with a single unary predicate ε .

$$\Phi(X) = \varepsilon(X)$$

$$\Phi(T \to U) = \Phi(T) \to \Phi(U)$$

$$\Phi(\forall X \ T) = \forall X \ \Phi(T)$$

and the normal terms of type T in System F are exactly the proof-terms of $\Phi(T)$ in predicate logic. Thus, the enumeration algorithm described in the previous sections applies immediately to System F. The examples below (where we write X for $\varepsilon(X)$) illustrate the algorithm.

Example: Let $A = \forall X((\forall Y((Y \to X) \to (Y \to X)) \to X) \to X)$. Let $\alpha : \forall Y((Y \to X) \to (Y \to X)) \to X, \beta : Y \to X$ and $\gamma : Y$. Let $S = S_{\vdash A}$ and $S_1 = S_{B \to X, [Y \to X, Y]_Y \vdash (Y \to X) \to Y \to X}$. The scheme grammar is given by

$$\begin{array}{rcl} S & \to & \lambda X \ \lambda \alpha \ (\alpha \ \lambda Y \lambda \beta \ \lambda \gamma \ (\beta \ \gamma)) \\ S & \to & \lambda X \ \lambda \alpha \ (\alpha \ \lambda Y \lambda \beta \ \lambda \gamma \ (\alpha \ \lambda Y \ S_1)) \\ S_1 & \to & \lambda \beta \ \lambda \gamma \ (\beta \ \gamma) \\ S_1 & \to & \lambda \beta \ \lambda \gamma \ (\alpha \ \lambda Y S_1) \end{array}$$

It is easy to check that the scheme below is generated by the grammar

 $\lambda X \ \lambda \alpha \ (\alpha \ \lambda Y \ \lambda \beta \ \lambda \gamma \ (\alpha \ \lambda Y \ \lambda \beta \ \lambda \gamma \ (\beta \ \gamma)))$

And this scheme generates in turn two proof-terms:

$$\lambda X \ \lambda \alpha \ (\alpha \ \lambda Y_1 \ \lambda \beta_1 \ \lambda \gamma_1 \ (\alpha \ \lambda Y_2 \ \lambda \beta_2 \ \lambda \gamma_2 \ (\beta_1 \ \gamma_1))) \\ \lambda X \ \lambda \alpha \ (\alpha \ \lambda Y_1 \ \lambda \beta_1 \ \lambda \gamma_1 \ (\alpha \ \lambda Y_2 \ \lambda \beta_2 \ \lambda \gamma_2 \ (\beta_2 \ \gamma_2)))$$

where $\alpha: B \to X, \, \beta_1: Y_1 \to X, \, \gamma_1: Y_1, \, \beta_2: Y_2 \to X, \, \gamma_2: Y_2.$

More generally, one scheme of depth n generated by this grammar, yields n-1 proof-terms of type A.

Example: Consider now the prenex form of the formula of the previous example. Let $A = \forall X \forall Y ((B \rightarrow X) \rightarrow X)$ where $B = (Y \rightarrow X) \rightarrow (Y \rightarrow X)$. The search tree of A is given in Figure 6.

Let $\alpha : ((Y \to X) \to (Y \to X)) \to X, \beta : Y \to X$ and $\gamma : Y$. Let $S = S_{\vdash A}$ and $S_1 = S_{B \to X, Y \to X, Y \vdash X}$. The corresponding scheme grammar is given by

$$\begin{array}{rcl} S & \to & \lambda X \ \lambda Y \ \lambda \alpha \ (\alpha \ \lambda \beta \ \lambda \gamma \ S_1) \\ S_1 & \to & (\beta \ \gamma) \\ S_1 & \to & (\alpha \ \lambda \beta \ \lambda \gamma \ S_1) \end{array}$$

It is easy to check that the scheme below is generated by the grammar

$$\lambda X \lambda Y \lambda \alpha (\alpha \lambda \beta \lambda \gamma (\alpha \lambda \beta \lambda \gamma (\beta \gamma)))$$

And this scheme generates in turn four proof-terms

 $\begin{array}{l} \lambda X \ \lambda Y \ \lambda \alpha \ (\alpha \ \lambda \beta_1 \ \lambda \gamma_1 \ (\alpha \ \lambda \beta_2 \ \lambda \gamma_2 \ (\beta_1 \ \gamma_1))) \\ \lambda X \ \lambda Y \ \lambda \alpha \ (\alpha \ \lambda \beta_1 \ \lambda \gamma_1 \ (\alpha \ \lambda \beta_2 \ \lambda \gamma_2 \ (\beta_1 \ \gamma_2))) \\ \lambda X \ \lambda Y \ \lambda \alpha \ (\alpha \ \lambda \beta_1 \ \lambda \gamma_1 \ (\alpha \ \lambda \beta_2 \ \lambda \gamma_2 \ (\beta_2 \ \gamma_1))) \\ \lambda X \ \lambda Y \ \lambda \alpha \ (\alpha \ \lambda \beta_1 \ \lambda \gamma_1 \ (\alpha \ \lambda \beta_2 \ \lambda \gamma_2 \ (\beta_2 \ \gamma_2))) \end{array}$

where $\alpha: B \to X, \, \beta_1: Y \to X, \, \gamma_1: Y, \, \beta_2: Y \to X, \, \gamma_2: Y.$

More generally, one scheme of depth n generated by this grammar, yields $(n-1)^2$ proof-terms.

Conclusion

Once more, the complexity of predicate logic comes from the negative quantifiers: when they are removed, not only the logic becomes decidable, but also the proofs have a simple structure.

The usual interpretations of proofs as terms are based on formulations of deduction where contexts are multisets or lists. The schemes are the counterpart to these terms when contexts are sets. Their structure is even simpler than that of terms and their interest may go beyond the proof enumeration problem.

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