Clique-width of Graph Classes Defined by Two Forbidden Induced Subgraphs[⋆]

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Abstract. The class of H-free graphs has bounded clique-width if and only if H is an induced subgraph of the 4-vertex path P_4. We study the (un)boundedness of the clique-width of graph classes defined by two forbidden induced subgraphs H_1 and H_2. Prior to our study it was not known whether the number of open cases was finite. We provide a positive answer to this question. To reduce the number of open cases we determine new graph classes of bounded clique-width and new graph classes of unbounded clique-width. For obtaining the latter results we first present a new, generic construction for graph classes of unbounded clique-width. Our results settle the boundedness or unboundedness of the clique-width of the class of (H_1, H_2)-free graphs

(i) for all pairs (H_1, H_2), both of which are connected, except two non-equivalent cases, and
(ii) for all pairs (H_1, H_2), at least one of which is not connected, except 11 non-equivalent cases.

We also consider classes characterized by forbidding a finite family of graphs $\{H_1, \ldots, H_p\}$ as subgraphs, minors and topological minors, respectively, and completely determine which of these classes have bounded clique-width. Finally, we show algorithmic consequences of our results for the graph colouring problem restricted to (H_1, H_2)-free graphs.

Keywords: clique-width, forbidden induced subgraph, graph class

1 Introduction

Clique-width is a well-known graph parameter studied both in a structural and in an algorithmic context; we refer to the surveys of Gurski [2] and Kamiński, Lozin and Milanič [3] for an in-depth study of the properties of clique-width. However, our understanding of clique-width is still very limited. For example, no polynomial-time algorithms are known for computing the clique-width of very restricted graph classes, such as unit interval graphs, or for deciding whether a graph has clique-width at most c for any fixed $c \geq 4$ (as an aside, we note that such an algorithm does exist for $c = 3$ [4]).

In order to get more structural insight into clique-width, we are interested in determining whether the clique-width of some given class of graphs is *bounded*, that is, whether there exists a constant c such that every graph from the class has clique-width at most c (our secondary motivation is algorithmic, as we will explain in detail later). The graph classes that we consider consist of graphs in which one or more specified graphs are forbidden as a “pattern”. In particular, we consider classes of graphs that contain no graph from some specified family $\{H_1, \ldots, H_p\}$ as an *induced subgraph*; such graphs are said to be $(H_1, \ldots, H_p)$-free. Our research is well embedded in the literature, as there are many papers that determine the boundedness or unboundedness of the clique-width of graph classes characterized by one or more forbidden induced subgraphs; see e.g. [5,6,7,8,9,10,11,12,13,14,15,16,17,18,19,20,21,22,23,24].

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As we show later, it is not difficult to verify that the class of H-free graphs has bounded clique-width if and only if H is an induced subgraph of the 4-vertex path P_4 (note that P_4-free graphs are also known as cographs, which are the graphs obtainable from the single-vertex graph K_1 by taking disjoint unions and complements). Hence, it is natural to consider the following problem:

For which pairs (H_1, H_2) does the class of (H_1, H_2)-free graphs have bounded clique-width?

In this paper we address this question by narrowing the gap between the known and open cases significantly; in particular we show that the number of open cases is *finite* (i.e. there are only finitely many cases for which we do not know whether or not the clique-width is bounded). Collecting all known results for (H_1, H_2)-free graphs from various papers and showing that the number of open cases is finite was one of the main goals of this paper. We emphasize, however, that the *underlying* research question is: what kind of properties of a graph class ensure that its clique-width is bounded? Our paper is to be interpreted as a further step towards this direction, and in our research project (see also [7,8,17,19]) we aim to develop *general* techniques for attacking a number of the open cases simultaneously.

Algorithmic Motivation. For problems that are NP-complete in general, one naturally seeks to find subclasses of graphs on which they are tractable, and graph classes of bounded clique-width have been studied extensively for this purpose, as we discuss below.

Courcelle, Makowsky and Rotics [25] showed that all MSO_1 graph problems, which are problems definable in Monadic Second Order Logic using quantifiers on vertices but not on edges, can be solved in linear time on graphs with clique-width at most c, provided that a c-expression of the input graph is given. Later, Espelage, Gurski and Wanke [26], Kobler and Rotics [27] and Rao [28] proved the same result for many non-MSO_1 graph problems. Although computing the clique-width of a given graph is NP-hard, as shown by Fellows, Rosamond, Rotics and Szeider [29], it is possible to find an $(8^c - 1)$-expression for any n-vertex graph with clique-width at most c in cubic time. This is a result of Oum [30] after a similar result (with a worse bound and running time) had already been shown by Oum and Seymour [31]. Hence, the NP-complete problems considered in the aforementioned papers [25,26,27,28] are all polynomial-time solvable on any graph class of bounded clique-width even if no c-expression of the input graph is given.

As a consequence of the above, when solving an NP-complete problem on some graph class $\mathcal{G}$, it is natural to try to determine *first* whether the clique-width of $\mathcal{G}$ is bounded. In particular this is the case if we aim to determine the computational complexity of some NP-complete problem when restricted to graph classes characterized by some common type of property. This property may be the absence of a family of forbidden induced subgraphs $H_1, \ldots, H_p$ and we may want to classify for which families of graphs $H_1, \ldots, H_p$ the problem is still NP-hard and for which ones it becomes polynomial-time solvable (in order to increase our understanding of the hardness of the problem in general). We give examples later.

Our Results. In Section 2 we state a number of basic results on clique-width and two results on H-free bipartite graphs that we showed in a very recent paper [19]; we need these results for proving our new results. We then identify a number of new classes of (H_1, H_2)-free graphs of bounded clique-width (Section 3) and unbounded clique-width (Section 4). In particular, the new unbounded cases are obtained from a new, general construction for graph classes of unbounded clique-width. In Section 5, we first observe for which graphs H_1 the class of H_1-free graphs has bounded clique-width. We then present our main theorem, which gives a summary of our current knowledge of those pairs (H_1, H_2) for which the class of (H_1, H_2)-free graphs has bounded clique-width and unbounded clique-width, respectively.[1] In this way we are able to narrow the gap to 13 open cases (up to some natural equivalence relation, which we explain later); when we only consider pairs (H_1, H_2) of connected graphs the number of non-equivalent open cases is only two. In order to present our summary, we will need several results from the papers listed above. We will also need these results in Section 6, where we consider graph classes characterized by forbidding a finite family of graphs $\{H_1, \ldots, H_p\}$ as subgraphs, minors and topological minors, respectively. For these containment relations we are able to completely determine which of these classes have bounded clique-width.

Algorithmic Consequences. Our results are of interest for any NP-complete problem that is solvable in polynomial time on graph classes of bounded clique-width. In Section 7 we give a concrete application of our results by considering the well-known COLOURING problem, which is that of testing whether a graph can be coloured with at most k colours for some given integer k and which is solvable in polynomial time on any graph class of bounded clique-width [27]. The complexity of COLOURING has been studied extensively for (H_1, H_2)-free graphs [16,18,34,35,36,37], but a full classification is still far from being settled. Many of the polynomial-time results follow directly from bounding the clique-width in such classes. As such this forms a direct motivation for our research. Another example for which our study might be of interest is the LIST k-COLOURING problem (another problem mentioned in the paper of Kobler and Rotics [27]). The complexity of this problem was recently investigated for (H_1, H_2)-free graphs when H_1 is a path and H_2 is a cycle [38].

Related Work. We finish this section by briefly discussing some related results.

First, a graph class $\mathcal{G}$ has power-bounded clique-width if there is a constant r so that the class consisting of all r-th powers of all graphs from $\mathcal{G}$ has bounded clique-width. Recently, Bonomo, Grippo, Milanič and Safe [6] determined all pairs of connected graphs H_1, H_2 for which the class of (H_1, H_2)-free graphs has power-bounded clique-width. If a graph class has bounded clique-width, it has power-bounded clique-width. However, the reverse implication does not hold in general. The latter can be seen as follows. Bonomo et al. [6] showed that the class of H-free graphs has power-bounded clique-width if and only if H is a linear forest (recall that such a class has bounded clique-width if and only if H is an induced subgraph of P_4). Their classification for connected graphs H_1, H_2 is the following. Let $S_{1,i,j}$ be the graph obtained from a 4-vertex star by subdividing one leg $i - 1$ times and another leg $j - 1$ times. Let $T_{1,i,j}$ be the line graph of $S_{1,i,j}$. Then the class of (H_1, H_2)-free graphs has power-bounded clique-width if and only if one of the following two cases applies: (i) one of H_1, H_2 is a path or (ii) one of H_1, H_2 is isomorphic to $S_{1,i,j}$ for some $i, j \geq 1$ and the other one is isomorphic to $T_{1,i',j'}$ for some $i', j' \geq 1$. In particular, the classes of power-unbounded clique-width were already known to have unbounded clique-width.

Second, Kratsch and Schweitzer [39] initiated a study into the computational complexity of the GRAPH ISOMORPHISM problem (GI) for graph classes defined by two forbidden induced subgraphs. The exact number of open cases is still not known, but Schweitzer [40] very recently proved that this number is finite. There are similarities between classifying the boundedness of clique-width and solving GI for classes of graphs characterized by one or more forbidden induced subgraphs. This was noted by Schweitzer [40], who proved that any graph class that allows a so-called simple path encoding has unbounded clique-width. Indeed, a common technique (see e.g. [3]) for showing that a class of graphs has unbounded clique-width relies on showing that it contains simple path encodings of walls or of graphs in some other specific graph class known to have unbounded clique-width. Furthermore, Grohe and Schweitzer [41] recently proved that GRAPH ISOMORPHISM is polynomial-time solvable on graphs of bounded clique-width. For H-free graphs, GI is polynomial-time solvable if H is an

[1] Before finding the combinatorial proof of our main theorem we first obtained a computer-assisted proof using Sage [32] and the Information System on Graph Classes and their Inclusions [33] (which keeps a record of classes for which boundedness or unboundedness of clique-width is known). In particular, we would like to thank Nathann Cohen and Ernst de Ridder for their help.

induced subgraph of P_4 [42] and GI-complete otherwise [39]. Hence, if only one induced subgraph is forbidden, the dichotomy classifications for clique-width and GI are identical.

Finally, similar research to ours has also been done for other variants of clique-width, such as linear clique-width [43]. Moreover, clique-width is also closely related to other graph width parameters. For example every graph class of bounded treewidth has bounded clique-width, but the reverse is not true [44]. Also, for any graph class, having bounded clique-width is equivalent to having bounded rank-width [31] and to having bounded NLC-width [45].

2 Preliminaries

Below we define the graph terminology used throughout our paper. For any undefined terminology we refer to Diestel [46].

Let G be a graph. The set $N(u) = \{v \in V(G) \mid uv \in E(G)\}$ is the *(open) neighbourhood* of $u \in V(G)$ and $N[u] = N(u) \cup \{u\}$ is the *closed neighbourhood* of $u \in V(G)$. The *degree* of a vertex in a graph is the size of its neighbourhood. The *maximum degree* of a graph is the maximum vertex degree. For a subset $S \subseteq V(G)$, we let $G[S]$ denote the subgraph of G *induced* by S, which has vertex set S and edge set $\{uv \mid u, v \in S, uv \in E(G)\}$. If $S = \{s_1, \ldots, s_r\}$ then, to simplify notation, we may also write $G[s_1, \ldots, s_r]$ instead of $G[\{s_1, \ldots, s_r\}]$. Let H be another graph. We write $H \subseteq_i G$ to indicate that H is an induced subgraph of G.

Let $\{H_1, \ldots, H_p\}$ be a set of graphs. We say that a graph G is $(H_1, \ldots, H_p)$*-free* if G has no induced subgraph isomorphic to a graph in $\{H_1, \ldots, H_p\}$. If $p = 1$, we may write H_1-free instead of (H_1)-free. The *disjoint union* $G + H$ of two vertex-disjoint graphs G and H is the graph with vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H)$. We denote the disjoint union of r vertex-disjoint copies of G by rG.

For positive integers s and t, the *Ramsey number* $R(s, t)$ is the smallest number n such that all graphs on n vertices contain an independent set of size s or a clique of size t. Ramsey's Theorem [47] states that such a number exists for all positive integers s and t.

The *clique-width* of a graph G, denoted $\mathrm{cw}(G)$, is the minimum number of labels needed to construct G by using the following four operations:

1. creating a new graph consisting of a single vertex v with label i (denoted by $i(v)$);
2. taking the disjoint union of two labelled graphs G_1 and G_2 (denoted by $G_1 \oplus G_2$);
3. joining each vertex with label i to each vertex with label j ($i \neq j$, denoted by $\eta_{i,j}$);
4. renaming label i to j (denoted by $\rho_{i \to j}$).

An algebraic term that represents such a construction of G and uses at most k labels is said to be a k*-expression* of G (i.e. the clique-width of G is the minimum k for which G has a k-expression). For instance, an induced path on four consecutive vertices a, b, c, d has clique-width equal to 3, and the following 3-expression can be used to construct it:

$$\eta_{3,2}(3(d) \oplus \rho_{3\to 2}(\rho_{2\to 1}(\eta_{3,2}(3(c) \oplus \eta_{2,1}(2(b) \oplus 1(a)))))).$$

Alternatively, any k-expression for a graph G can be represented by a rooted tree, where the leaves correspond to the operations of vertex creation and the internal nodes correspond to the other three operations. The rooted tree representing the above k-expression is depicted in Fig. 1. A class of graphs $\mathcal{G}$ has *bounded* clique-width if there is a constant c such that the clique-width of every graph in $\mathcal{G}$ is at most c; otherwise the clique-width of $\mathcal{G}$ is *unbounded*.

Let G be a graph. The *complement* of G, denoted by $\overline{G}$, has vertex set $V(\overline{G}) = V(G)$ and an edge between two distinct vertices if and only if these vertices are not adjacent in G.

Let G be a graph. We define the following five operations. The *contraction* of an edge uv removes u and v from G, and replaces them by a new vertex made adjacent to precisely those vertices that were

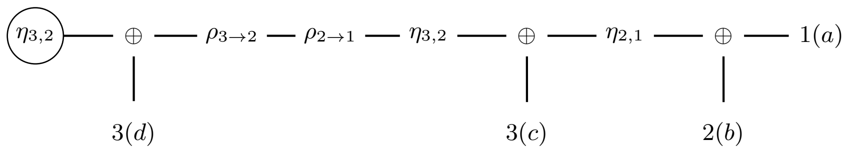


Fig. 1: The rooted tree representing a 3-expression for P_4.

adjacent to u or v in G. By definition, edge contractions create neither self-loops nor multiple edges. The *subdivision* of an edge uv replaces uv by a new vertex w with edges uw and vw. Let $u \in V(G)$ be a vertex that has exactly two neighbours v, w, and moreover let v and w be non-adjacent. The *vertex dissolution* of u removes u and adds the edge vw. For an induced subgraph $G' \subseteq_i G$, the *subgraph complementation* operation (acting on G with respect to G') replaces every edge present in G' by a non-edge, and vice versa. Similarly, for two disjoint vertex subsets X and Y in G, the *bipartite complementation* operation with respect to X and Y acts on G by replacing every edge with one end-vertex in X and the other one in Y by a non-edge and vice versa.

We now state some useful facts for dealing with clique-width. We will use these facts throughout the paper. Let $k \geq 0$ be a constant and let γ be some graph operation. We say that a graph class $\mathcal{G}'$ is (k, γ)*-obtained* from a graph class $\mathcal{G}$ if the following two conditions hold:

(i) every graph in $\mathcal{G}'$ is obtained from a graph in $\mathcal{G}$ by performing γ at most k times, and
(ii) for every $G \in \mathcal{G}$ there exists at least one graph in $\mathcal{G}'$ obtained from G by performing γ at most k times.

If we do not impose a finite upper bound k on the number of applications of γ then we write that $\mathcal{G}'$ is (∞, γ)-obtained from $\mathcal{G}$.

We say that γ *preserves* boundedness of clique-width if for any finite constant k and any graph class $\mathcal{G}$, any graph class $\mathcal{G}'$ that is (k, γ)-obtained from $\mathcal{G}$ has bounded clique-width if and only if $\mathcal{G}$ has bounded clique-width.

Fact 1. Vertex deletion preserves boundedness of clique-width [21,48].

Fact 2. Subgraph complementation preserves boundedness of clique-width [3].

Fact 3. Bipartite complementation preserves boundedness of clique-width [3].

Fact 4. For a class of graphs $\mathcal{G}$ of *bounded* maximum degree, let $\mathcal{G}'$ be a class of graphs that is $(\infty, \mathtt{es})$-obtained from $\mathcal{G}$, where $\mathtt{es}$ is the edge subdivision operation. Then $\mathcal{G}$ has bounded clique-width if and only if $\mathcal{G}'$ has bounded clique-width [3].

It is easy to show that the condition on the maximum degree in Fact 4 is necessary for the reverse (i.e. the "only if") direction: for a graph G of arbitrarily large clique-width, take a clique K (which has clique-width at most 2) with vertex set $V(K) = V(G)$, apply an edge subdivision on an edge uv in K if and only if uv is not an edge in G and, in order to obtain G from this graph, remove any vertex introduced by an edge subdivision (this does not increase the clique-width). As another aside, note that the reverse direction of Fact 4 also holds if we replace "edge subdivisions" by "edge contractions".[2] It was an open problem [2] whether the condition on maximum degree was also necessary in this case. This was recently solved by Courcelle [50], who showed that if $\mathcal{G}$ is the class of graphs of

[2] Combine the fact that a class of graphs of bounded maximum degree has bounded clique-width if and only if it has bounded treewidth [49] with the well-known fact that edge contractions do not increase the treewidth of a graph.

clique-width 3 and $\mathcal{G}'$ is the class of graphs obtainable from graphs in $\mathcal{G}$ by applying one or more edge contraction operations then $\mathcal{G}'$ has unbounded clique-width.

We also use a number of other elementary results on the clique-width of graphs. The first one is well known (see e.g. [48]) and straightforward to check.

Lemma 1. *The clique-width of a graph with maximum degree at most* 2 *is at most* 4*.*

We also need the well-known notion of a *wall*. We do not formally define this notion, but instead refer to Fig. 2, in which three examples of walls of different height are depicted (see e.g. [51] for a formal definition). The class of walls is well known to have unbounded clique-width; see for example [3]. (Note that walls have maximum degree at most 3, hence the degree bound in Lemma 1 is tight.)

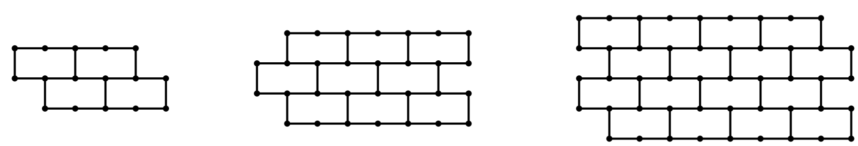


Fig. 2: Walls of height 2, 3, and 4, respectively.

A k*-subdivided wall* is a graph obtained from a wall after subdividing each edge exactly k times for some constant $k \geq 0$.

The following lemma is well known and follows from combining Fact 4 with the aforementioned fact that walls have maximum degree at most 3 and unbounded clique-width.

Lemma 2 ([22]). *For any constant* $k \geq 0$*, the class of* k*-subdivided walls has unbounded clique-width.*

For $r \geq 1$, the graphs C_r, K_r, P_r denote the cycle, complete graph and path on r vertices, respectively, and the graph $K_{1,r}$ denotes the star on $r+1$ vertices. The graph $K_{1,3}$ is also called the *claw*. For $1 \leq h \leq i \leq j$, let $S_{i,j,k}$ denote the tree that has only one vertex x of degree 3 and that has exactly three leaves, which are of distance i, j and k from x, respectively. Observe that $S_{1,1,1} = K_{1,3}$. A graph $S_{i,j,k}$ is said to be a *subdivided claw*. We let $\mathcal{S}$ be the class of graphs each connected component of which is either a subdivided claw or a path.

Like Lemma 1, the following lemma is also well known and follows from Lemma 2, by choosing appropriate values for k.

Lemma 3 ([22]). *Let* $\{H_1, \ldots, H_p\}$ *be a finite set of graphs. If* $H_i \notin \mathcal{S}$ *for* $i = 1, \ldots, p$ *then the class of* $(H_1, \ldots, H_p)$*-free graphs has unbounded clique-width.*

We say that G is *bipartite* if its vertex set can be partitioned into two (possibly empty) independent sets B and W. We say that (B, W) is a *bipartition* of G. Let H be a bipartite graph with a fixed partition (B_H, W_H). A bipartite graph G is *strongly* H-free if no bipartition of G contains an induced copy of H in a way that respects the bipartition of H (as we do not need this notion in this paper, we refer to [19] for a more formal discussion of it) Lozin and Volz [23] characterized all bipartite graphs H for which the class of strongly H-free bipartite graphs has bounded clique-width. Recently, we proved a similar characterization for H-free bipartite graphs; we will use this result in Section 5.

Lemma 4 **([19]).** *Let H be a graph. The class of H-free bipartite graphs has bounded clique-width if and only if one of the following cases holds:*

- $H = sP_1$ *for some* $s \geq 1$
- $H \subseteq_i K_{1,3} + 3P_1$
- $H \subseteq_i K_{1,3} + P_2$
- $H \subseteq_i P_1 + S_{1,1,3}$
- $H \subseteq_i S_{1,2,3}$.

From the same paper we will also need the following lemma.

Lemma 5 **([19]).** *Let $H \in \mathcal{S}$. Then H is $(2P_1 + 2P_2, 2P_1 + P_4, 4P_1 + P_2, 3P_2, 2P_3)$-free if and only if $H = sP_1$ for some integer $s \geq 1$ or H is an induced subgraph of one of the graphs in $\{K_{1,3} + 3P_1$, $K_{1,3} + P_2, P_1 + S_{1,1,3}, S_{1,2,3}\}$.*

We say that a graph G is *complete multipartite* if $V(G)$ can be partitioned into k independent sets $V_1, \ldots, V_k$ for some integer k, such that two vertices are adjacent if and only if they belong to two different sets V_i and V_j. The next result is due to Olariu [52] (the graph $\overline{P_1 + P_3}$ is also called the *paw*).

Lemma 6 **([52]).** *Every connected $(\overline{P_1 + P_3})$-free graph is either complete multipartite or K_3-free.*

Every complete multipartite graph has clique-width at most 2. Also, the definition of clique-width directly implies that the clique-width of any graph is equal to the maximum clique-width of its connected components. Hence, Lemma 6 immediately implies the following (well-known) result.

Lemma 7. *For any graph H, the class of $(\overline{P_1 + P_3}, H)$-free graphs has bounded clique-width if and only if the class of (K_3, H)-free graphs has bounded clique-width.*

Kratsch and Schweitzer [39] proved that the GRAPH ISOMORPHISM problem is GRAPH ISOMORPHISM-complete for the class of $(K_4, P_1 + P_4)$-free graphs. It is a straightforward exercise to simplify their construction and use analogous arguments to prove that the class of $(K_4$, $P_1 + P_4)$-free graphs has unbounded clique-width. Recall that Schweitzer [40] proved that any graph class that allows a so-called simple path encoding has unbounded clique-width, implying this result as a direct consequence.

Lemma 8 **([40]).** *The class of $(K_4, P_1 + P_4)$-free graphs has unbounded clique-width.*

3 New Classes of Bounded Clique-width

In this section we identify two new graph classes that have bounded clique-width, namely the classes of $(\overline{P_1 + P_3}, P_1 + S_{1,1,2})$-free graphs and $(\overline{P_1 + P_3}, K_{1,3} + 3P_1)$-free graphs.

We first prove that the class of $(\overline{P_1 + P_3}, P_1 + S_{1,1,2})$-free graphs has bounded clique-width. To do so we use a similar approach to that used by Dabrowski, Lozin, Raman and Ries [18] to prove that the classes of $(K_3, S_{1,1,3})$-free and $(K_3, K_{1,3} + P_2)$-free graphs have bounded clique-width.

Theorem 1. *The class of $(\overline{P_1 + P_3}, P_1 + S_{1,1,2})$-free graphs has bounded clique-width.*

Proof. Let G be a $(\overline{P_1 + P_3}, P_1 + S_{1,1,2})$-free graph. By Lemma 7 we may assume G is $(K_3$, $P_1 + S_{1,1,2})$-free. Without loss of generality, we may also assume that G is connected (as otherwise we could consider each connected component of G separately). If G is bipartite, then G has bounded clique-width by Lemma 4. For the remainder of the proof we assume that G is not bipartite, that is, G contains an induced odd cycle $C = v_1 v_2 \cdots v_k v_1$. Because G is K_3-free, $k \geq 5$.

First, suppose that $k \geq 7$. We claim that $G = C$. Indeed, suppose not. Since G is connected, G must have a vertex $x \notin V(C)$ that is adjacent to a vertex of C. Since G is K_3-free, x cannot be adjacent to any two consecutive vertices of the cycle C. Since C is an odd cycle, x must therefore have two consecutive non-neighbours on the cycle. Without loss of generality we assume that x is adjacent to v_1 and non-adjacent to v_{k-1} and v_k. Then x must be adjacent to v_4, otherwise $G[v_1, x, v_2, v_k, v_{k-1}, v_4]$ would be isomorphic to $P_1 + S_{1,1,2}$. Now x cannot be adjacent to v_3 or v_5, since G is K_3-free. However, then $G[v_1, x, v_k, v_2, v_3, v_5]$ would be a $P_1 + S_{1,1,2}$, which is a contradiction. Hence, $G = C$ and as such has clique-width at most 4 by Lemma 1.

From now on we assume that $k = 5$. Every vertex not on C has at most two neighbours on the cycle, and if it has two, then these neighbours on C cannot be consecutive vertices of C (since G is K_3-free). We now partition the vertices of G not in C into sets, depending on their neighbourhood in C. We let X denote the vertices with no neighbours on the cycle. We let V_i denote the set of all vertices not on the cycle C that are adjacent to both v_{i-1} and v_{i+1}, where subscripts are interpreted modulo 5. We let W_i denote the set of all vertices that are adjacent to v_i but to no other vertices of C. We say that a set V_i or W_i is *large* if it contains at least two vertices, otherwise we say that it is *small*. We say that a set in $\{V_i, W_i\}$ and a set in $\{V_j, W_j\}$ are *consecutive* if v_i and v_j are consecutive vertices on C, otherwise, we say that they are *opposite*. Note that each V_i and each W_i is an independent set, since G is K_3-free. We now investigate the possible adjacencies between vertices of these sets through a series of eight claims.

1. *X is an independent set and every vertex in X is adjacent to every vertex in V_i and W_i.* Suppose there is a vertex $x \in X$. Since G is connected, there must be a vertex $y \notin V(C)$ with a neighbour on the cycle. We may assume without loss of generality that y is adjacent to v_1, but not to v_2, v_3 or v_5. Then x must be adjacent to y, otherwise $G[v_1, y, v_5, v_2, v_3, x]$ would be isomorphic to $P_1 + S_{1,1,2}$. Hence every vertex in X is adjacent to every vertex in V_i and W_i for all i. Because of the fact that if X is non-empty then some V_i or W_i must also be non-empty and the fact that G is K_3-free, X must be an independent set.
2. *If V_i and V_j are opposite then no vertex of V_i is adjacent to a vertex of V_j.* This follows from the fact that any two such vertices have a common neighbour on C and the fact that G is K_3-free.
3. *If V_i and V_j are consecutive and V_j is large then every vertex of V_i is adjacent to every vertex of V_j.* Without loss of generality, let $i = 1, j = 2$. Suppose $y \in V_1$ is not adjacent to $z_1, z_2 \in V_2$. Then $G[v_1, z_1, z_2, v_2, y, v_4]$ is a $P_1 + S_{1,1,2}$. Now suppose that y is adjacent to z_1, but not to z_2, then $G[y, v_2, z_1, v_5, v_4, z_2]$ is isomorphic to $P_1 + S_{1,1,2}$, which is a contradiction.
4. *If V_i and W_j are consecutive then one of them must be empty.* Suppose, for contradiction, that there exist vertices $x \in V_1$ and $y \in W_2$. Then x and y are non-adjacent, as G is K_3-free. However, then $G[v_5, v_1, x, v_4, v_3, y]$ is isomorphic to $P_1 + S_{1,1,2}$, which is a contradiction.
5. *If V_i and W_j are opposite and W_j is large then no vertex of V_i has a neighbour in W_j.* Let $y \in V_1$ and $z_1, z_2 \in W_3$. If y is adjacent to both z_1 and z_2, then $G[y, z_1, z_2, v_2, v_1, v_4]$ is isomorphic to $P_1 + S_{1,1,2}$. So y is adjacent to at most one vertex of W_3, say y is adjacent to z_1, but not to z_2. Then $G[v_5, v_1, v_4, y, z_1, z_2]$ is isomorphic to $P_1 + S_{1,1,2}$, which is a contradiction.
6. *Every vertex in V_i has at most one non-neighbour in W_i and vice versa.* If $y_1 \in V_1$ has two non-neighbours $z_1, z_2 \in W_1$ then the graph $G[v_1, z_1, z_2, v_2, y_1, v_4]$ is isomorphic to $P_1 + S_{1,1,2}$, which is a contradiction. If $z_1 \in W_1$ has two non-neighbours $y_1, y_2 \in V_1$ then $G[v_2, y_1, y_2, v_1, z_1, v_4]$ is isomorphic to $P_1 + S_{1,1,2}$, which is again a contradiction.
7. *If W_i and W_j are consecutive and W_j is large then W_i is empty.* Without loss of generality, let $i = 1$ and $j = 2$. Suppose, for contradiction, that $y \in W_1$ and $z_1, z_2 \in W_2$. If y is adjacent to both z_1 and z_2 then $G[y, z_1, z_2, v_1, v_5, v_3]$ is isomorphic to $P_1 + S_{1,1,2}$. Without loss of generality, we therefore assume that y is not adjacent to z_1. If y is not adjacent to z_2 then $G[v_2, z_1, z_2, v_1, y, v_4]$ is isomorphic to $P_1 + S_{1,1,2}$. If y is adjacent to z_2, then $G[v_2, v_3, z_1, z_2, y, v_5]$ is isomorphic to $P_1 + S_{1,1,2}$. Hence in all three cases we have a contradiction.

8. *If W_i and W_j are opposite then every vertex of W_i must be adjacent to every vertex of W_j.* Without loss of generality, let $i = 1$, $j = 3$, $x \in W_1$, and $y \in W_3$. If x and y are not adjacent, then $G[v_1, v_2, x, v_5, v_4, y]$ is isomorphic to $P_1 + S_{1,1,2}$, which is not possible.

We now do as follows. First, we remove the vertices of C and all small sets V_i or W_i if they exist. In this way we remove at most $5+5+5 = 15$ vertices. Hence, G has bounded clique-width if and only if the resulting graph G' has bounded clique-width, by Fact 1. We then consider the remaining sets X, V_i and W_i in G'. We complement the edges between the vertices in X and the vertices not in X. If V_i and V_j are consecutive, we complement the edges between them. If W_i and W_j are opposite, we complement the edges between them. Finally, for any pair V_i and W_i, we complement the edges between them. Then G' has bounded clique-width if and only if the resulting graph G^* has bounded clique-width, by Fact 3. If two vertices are adjacent in G^*, then they must be members of some V_i and W_i, respectively. By construction, $G^*[V_i \cup W_i]$ is a (not necessarily perfect) matching. Thus G^* has clique-width at most 2, completing the proof. □

Next, we prove that the class of $(\overline{P_1 + P_3}, K_{1,3} + 3P_1)$-free graphs has bounded clique-width. To do so we first prove Lemma 9, which says that the class of $(\overline{P_1 + P_3}, K_{1,3} + 2P_1)$-free graphs has bounded clique-width. We then use this result to prove Theorem 2, which says that the larger class of $(\overline{P_1 + P_3}, K_{1,3} + 3P_1)$-free graphs also has bounded clique-width.

Lemma 9. *The class of $(\overline{P_1 + P_3}, K_{1,3} + 2P_1)$-free graphs has bounded clique-width.*

Proof. Let G be a $(\overline{P_1 + P_3}, K_{1,3} + 2P_1)$-free graph. By Lemma 7, we may assume G is $(K_3, K_{1,3} + 2P_1)$-free. Let x be an arbitrary vertex in G. Let $N_1 = N(x)$ and $N_2 = V(G) \setminus N[x]$. Since G is K_3-free, N_1 must be an independent set. Since G is $(K_{1,3} + 2P_1)$-free, $G[N_2]$ must be $(K_{1,3} + P_1)$-free. Then $G[N_2]$ must have bounded clique-width by Theorem 1.

Suppose that $|N_1| \leq 2$. Then we delete x and the vertices of N_1 and obtain a graph of bounded clique-width, namely $G[N_2]$. By Fact 1, we find that G also has bounded clique-width. Hence we may assume that $|N_1| \geq 3$.

We prove the following claim.

Claim 1. *Let $S \subseteq N_2$ with $|S| \leq k$ for some k. If $G[N_2 \setminus S]$ is complete bipartite, then the clique-width of G is bounded by a function of k. In particular, this includes the case where $G[N_2 \setminus S]$ is an independent set.*

To prove Claim 1, suppose that $G[N_2 \setminus S]$ is complete bipartite. No vertex in N_1 has a neighbour in both partition classes of $G[N_2 \setminus S]$, due to the fact that G is K_3-free. Because N_1 is an independent set, this means that $G[N_1 \cup (N_2 \setminus S)]$ is bipartite, in addition to being $(K_3, K_{1,3} + 2P_1)$-free. Hence, $G[N_1 \cup (N_2 \setminus S)]$ has bounded clique-width by Lemma 4. Then by Fact 1, $G = G[N_1 \cup (N_2 \setminus S) \cup S \cup \{x\}]$ has clique-width bounded by some function of $|S|$. This proves Claim 1.

We will use Claim 1 later in the proof and now proceed as follows. We fix three arbitrary vertices $x_1, x_2, x_3 \in N_1$; such vertices exist because $|N_1| \geq 3$. Let y_1, y_2, y_3 be three arbitrary vertices of N_2. We will show that at least one of them is adjacent to at least one of x_1, x_2, x_3. Because G is K_3-free, two of y_1, y_2, y_3 are not pairwise adjacent, say $y_1y_2 \notin E(G)$. If both y_1 and y_2 have no neighbour in $\{x_1, x_2, x_3\}$, then $G[x, x_1, x_2, x_3, y_1, y_2]$ is isomorphic to $K_{1,3} + 2P_1$, a contradiction. Hence, all vertices of N_2 except at most two have at least one neighbour in $\{x_1, x_2, x_3\}$. Then, by Fact 1, we may assume without loss of generality that all vertices of N_2 have at least one neighbour in $\{x_1, x_2, x_3\}$.

Let A consist of those vertices of N_2 that are adjacent to x_1. Let B consist of those vertices of N_2 that are adjacent to x_2 but not to x_1. Let $C = N_2 \setminus (A \cup B)$. Note that every vertex in C is adjacent to x_3 but not to x_1 or x_2. Moreover, A, B, C are three independent sets due to the fact that G is K_3-free. If C contains at least three vertices, say c_1, c_2, c_3, then $G[x_3, c_1, c_2, c_3, x_1, x_2]$ is isomorphic to $K_{1,3}+2P_1$. Thus $|C| \leq 2$. If $|A| \leq 7$, then $|A \cup C| \leq 9$. Moreover, $G[N_2 \setminus (A \cup C)] = G[B]$ is complete bipartite, because B is an independent set. Hence, we may apply Claim 1. From now on we assume that $|A| \geq 8$, and similarly, that $|B| \geq 8$.

At least one vertex of any pair from B must be adjacent to at least one vertex of any triple from A; otherwise these five vertices, together with x_1, induce a subgraph isomorphic to $K_{1,3}+2P_1$, since A and B are independent sets and x_1 is adjacent to all vertices of A and to none of B. Fix three vertices $a_1, a_2, a_3 \in A$. Then at most one vertex of B has no neighbours in $\{a_1, a_2, a_3\}$. Because $|B| \geq 8$, this means that at least one of a_1, a_2, a_3 must have at least three neighbours in B. By repeating this argument with different choices of a_1, a_2, a_3, we find that all but at most two vertices in A have at least three neighbours in B. So, at least six vertices in A have at least three neighbours in B, and vice versa.

Let $a \in A$ be adjacent to at least three vertices b_1, b_2, b_3 of B. If a is not adjacent to some $b_4 \in B$, then $G[a_1, b_1, b_2, b_3, b_4, x]$ is isomorphic to $K_{1,3}+2P_1$. Hence, every vertex of A with at least three neighbours in B is adjacent to all vertices of B. By reversing the roles of A and B, we find that every vertex in B with at least three neighbours in A must be adjacent to all vertices of A. Because there are at least six vertices in A with at least three neighbours in B, and vice versa, we conclude that all vertices of A are adjacent to all vertices of B, that is, $G[N_2 \setminus C] = G[A \cup B]$ is complete bipartite. Because $|C| \leq 2$, we may apply Claim 1 to complete the proof. $\square$

Theorem 2. *The class of ($\overline{P_1+P_3}, K_{1,3}+3P_1$)-free graphs has bounded clique-width.*

Proof. Let G be a ($\overline{P_1+P_3}, K_{1,3}+3P_1$)-free graph. By Lemma 7, we may assume G is (K_3, $K_{1,3}+3P_1$)-free. Suppose that G contains a vertex of degree at most 18. If we remove this vertex and its neighbours, we obtain a ($K_3, K_{1,3}+2P_1$)-free graph, which has bounded clique-width by Lemma 9. Hence, G also has bounded clique-width, by Fact 1. From now on we assume that G has minimum degree at least 19 (the reason for choosing this number becomes clear later).

Let $x \in V(G)$. Let $N_1 = N(x)$ and $N_2 = V(G) \setminus N[x]$. Note that $|N_1| \geq 19$ and fix three arbitrarily-chosen vertices $x_1, x_2, x_3 \in N_1$. Let Y be the set of vertices in N_2 that have no neighbour in $\{x_1, x_2, x_3\}$. We will need the following claim.

Claim 1. $|Y| \leq 5$.

We prove Claim 1 as follows. Suppose that there are three vertices $y_1, y_2, y_3 \in N_2$ that are pairwise non-adjacent. Then at least one of y_1, y_2, y_3 must be adjacent to at least one of x_1, x_2, x_3, as otherwise $G[x, x_1, x_2, x_3, y_1, y_2, y_3]$ would be isomorphic to $K_{1,3}+3P_1$. Hence $G[Y]$ is $3P_1$-free. Because $G[Y]$ is also K_3-free, we apply Ramsey's Theorem and find that $|Y| \leq R(3,3)-1 = 6-1 = 5$. This proves Claim 1.

We proceed as follows. Let $N_2' = N_2 \setminus Y$. Let A consist of those vertices of N_2' that are adjacent to x_1. Let B consist of those vertices of N_2' that are adjacent to x_2 but not to x_1. Let $C = N_2' \setminus (A \cup B)$. Note that every vertex in C is adjacent to x_3, but not to x_1 or x_2. Moreover, A, B, C are three independent sets due to the fact that G is K_3-free.

We need the following claim.

Claim 2. *Let $S, T \in \{A, B, C\}$ with $S \neq T$, $|S| \geq 9$ and $|T| \geq 9$. Then there exist vertices $s \in S$ and $t \in T$ such that $G[(S \setminus \{s\}) \cup (T \setminus \{t\})]$ is a complete bipartite graph minus a matching.*

We prove Claim 2 as follows. Suppose $S = A$ and $T = B$ with $|A| \geq 9$ and $|B| \geq 9$. Let $a, a', a'' \in A$ and $b, b', b'' \in B$ be pairwise distinct. Recall that A and B are independent sets. Then at least one of a, a', a'' must be adjacent to at least one of b, b', b'', as otherwise the graph $G[x_1, a, a', a'', b, b', b'']$ would be isomorphic to $K_{1,3}+3P_1$. This means that at most two vertices in B have no neighbour in $\{a, a', a''\}$. Hence, as $|B| \geq 9$, at least one of a, a', a'' has at least three neighbours in B. Repeating this argument with different choices of a, a', a'', we find that all but at most two vertices in A have at least three neighbours in B.

Every vertex $a' \in A$ that is adjacent to at least three vertices of B, say b_1, b_2, b_3, must be adjacent to all but at most one vertex of B, since if a' is not adjacent to $b_4, b_5 \in B$, then $G[a', b_1, b_2, b_3, x, b_4, b_5]$ would be a $K_{1,3} + 3P_1$. Because all but at most two vertices in A have at least three neighbours in B, this means that all but at most two vertices of A are adjacent to all but at most one vertex of B. Because $|A| \geq 9 > 7$, this means that every vertex of B except at most one has at least three neighbours in A. Let $b \in B$ be this exceptional vertex; if it does not exist then we pick $b \in B$ arbitrarily. If $b' \in B \setminus \{b\}$, let a_1, a_2, a_3 be three of its neighbours in A. Then b' cannot be non-adjacent to two vertices, say a_4, a_5 in A, otherwise $G[b', a_1, a_2, a_3, x, a_4, a_5]$ would be a $K_{1,3} + 3P_1$. Thus every vertex in $B \setminus \{b\}$ is adjacent to all but at most one vertex of A. Since $|B \setminus \{b\}| \geq 8 > 5$, every vertex in A, except at most one has at least three neighbours in $B \setminus \{b\}$ and as stated above must therefore be adjacent to all but at most one vertex of B. We let $a \in A$ denote this exceptional vertex; if it does not exist, then we pick $a \in A$ arbitrarily. Because A and B are independent sets, we conclude that $G[(A \setminus \{a\}) \cup (B \setminus \{b\})]$ is a complete bipartite graph minus a (not necessarily perfect) matching. If a different pair of sets in $\{A, B, C\}$ both have at least nine vertices, the claim follows by the same arguments.

We now consider three different cases.

Case 1. *At least two sets out of A, B, C have less than nine vertices.*
Suppose $|A| \leq 8$ and $|B| \leq 8$. Recall that C, N_1 are independent sets and that G is $(K_{1,3}+3P_1)$-free. Then $G[V(G) \setminus (\{x\} \cup A \cup B \cup Y)] = G[C \cup N_1]$ is bipartite and $(K_{1,3} + 3P_1)$-free. Consequently, it has bounded clique-width by Lemma 4. We have $|Y| \leq 5$ by Claim 1. Then $|\{x\} \cup A \cup B \cup Y| \leq 1 + 8 + 8 + 5 = 22$. Hence, G has bounded clique-width by Fact 1. If a different pair of sets in $\{A, B, C\}$ both have less than nine vertices, we apply the same arguments.

Case 2. *Exactly one set out of A, B, C has less than nine vertices.*
Suppose $|C| \leq 8$. Hence $|A| \geq 9$ and $|B| \geq 9$. By Claim 2 we find that there exist two vertices $a \in A$ and $b \in B$ such that $G[(A \setminus \{a\}) \cup (B \setminus \{b\})]$ is a complete bipartite graph minus a matching. Let $x' \in N_1$. Suppose, for contradiction, that x' is adjacent to a vertex $a' \in A \setminus \{a\}$ and to a vertex $b' \in B \setminus \{b\}$. Then x' is not adjacent to any other vertices of $(A \setminus \{a\}) \cup (B \setminus \{b\})$, otherwise G would not be K_3-free. Recall that N_1 is an independent set. Hence $N(x') \subseteq \{a, b, a', b', x\} \cup C \cup Y$. We have $|Y| \leq 5$ by Claim 1. Hence, $|N(x')| \leq 5+8+5 = 18$, which is a contradiction since G has minimum degree at least 19. We conclude that no vertex in N_1 has neighbours in both $A \setminus \{a\}$ and $B \setminus \{b\}$. Because N_1 is independent and G is $(K_{1,3}+3P_1)$-free, this means that $G[V(G) \setminus (\{a, b, x\} \cup C \cup Y)] = G[N_1 \cup (A \setminus \{a\}) \cup (B \setminus \{b\})]$ is bipartite and $(K_{1,3} + 3P_1)$-free. Consequently, it has bounded clique-width by Lemma 4. Because $|\{a, b, x\} \cup C \cup Y| \leq 3 + 8 + 5 = 16$, we conclude that G has bounded clique-width by Fact 1. If $|A| \leq 8$ or $|B| \leq 8$, we repeat the above arguments with A and B replaced by B and C, or A and C, respectively.

Case 3. *None of the sets A, B, C has less than nine vertices.*
By Claim 2, we find that there exist vertices a, a', b, b', c, c' such that $G[(A \setminus \{a\}) \cup (B \setminus \{b\})]$, $G[(A \setminus \{a'\}) \cup (C \setminus \{c\})]$, and $G[(B \setminus \{b'\}) \cup (C \setminus \{c'\})]$ are complete bipartite graphs minus a matching. Hence $G[(A \setminus \{a, a'\}) \cup (B \setminus \{b, b'\})]$, $G[(A \setminus \{a, a'\}) \cup (C \setminus \{c, c'\})]$, and $G[(B \setminus \{b, b'\}) \cup (C \setminus \{c, c'\})]$ are also complete bipartite graphs minus a matching. Because $|A| \geq 9 > 2$, $|B| \geq 9 > 3$ and $|C| \geq 9 > 4$, there exist vertices $a_1 \in A \setminus \{a, a'\}$, $b_1, b_2 \in B \setminus \{b, b'\}$ and $c_1, c_2, c_3 \in C \setminus \{c, c'\}$. Then a_1 is adjacent to at least one of b_1, b_2 and to at least two of c_1, c_2, c_3. Moreover, b_1 and b_2 are each adjacent to at least two of c_1, c_2, c_3. Hence G is not K_3-free. This contradiction completes the proof. $\square$

4 New Classes of Unbounded Clique-width

In order to prove our results, we first present a general construction for obtaining graph classes of unbounded clique-width. We then show how we can use our construction to obtain two new classes of unbounded clique-width. Our construction generalizes the constructions used by Golumbic and Rotics [20],[3] Brandstädt et al. [9] and Lozin and Volz [23] to prove that the classes of square grids, K_4-free co-chordal graphs and $2P_3$-free graphs, respectively, have unbounded clique-width. It can also be used to show directly that the classes of k-subdivided walls have unbounded clique-width (Lemma 2).

Theorem 3. *For $m \geq 0$ and $n > m+1$ the clique-width of a graph G is at least $\lfloor \frac{n-1}{m+1} \rfloor + 1$ if $V(G)$ has a partition into sets $V_{i,j} (i, j \in \{0, \ldots, n\})$ with the following properties:*

1. $|V_{i,0}| \leq 1$ *for all* $i \geq 1$.
2. $|V_{0,j}| \leq 1$ *for all* $j \geq 1$.
3. $|V_{i,j}| \geq 1$ *for all* $i, j \geq 1$.
4. $G[\cup_{j=0}^n V_{i,j}]$ *is connected for all* $i \geq 1$.
5. $G[\cup_{i=0}^n V_{i,j}]$ *is connected for all* $j \geq 1$.
6. *For $i, j, k \geq 1$, if a vertex of $V_{k,0}$ is adjacent to a vertex of $V_{i,j}$ then $i \leq k$.*
7. *For $i, j, k \geq 1$, if a vertex of $V_{0,k}$ is adjacent to a vertex of $V_{i,j}$ then $j \leq k$.*
8. *For $i, j, k, \ell \geq 1$, if a vertex of $V_{i,j}$ is adjacent to a vertex of $V_{k,\ell}$ then $|k-i| \leq m$ and $|\ell-j| \leq m$.*

Proof. Fix integers n, m with $m \geq 0$ and $n > m+1$, and let G be a graph with a partition as described above. For $i > 0$ we let $R_i = \cup_{j=0}^n V_{i,j}$ be a *row* of G and for $j > 0$ we let $C_j = \cup_{i=0}^n V_{i,j}$ be a *column* of G. Note that $G[R_i]$ and $G[C_j]$ are non-empty by Property 3. They are connected graphs by Properties 4 and 5, respectively.

Consider a k-expression for G. We will show that $k \geq \lfloor \frac{n-1}{m+1} \rfloor + 1$. As stated in Section 2, this k-expression can be represented by a rooted tree T, whose leaves correspond to the operations of vertex creation and whose internal nodes correspond to the other three operations (see Fig. 1 for an example). We denote the subgraph of G that corresponds to the subtree of T rooted at node x by $G(x)$. Note that it is possible that $G(x)$ is not an induced subgraph of G, as missing edges can be added by operations corresponding to $\eta_{i,j}$ nodes higher up in T.

Recall that $\oplus$ represents the disjoint union operation in the definition of clique-width. Let x be a deepest (i.e. furthest from the root) $\oplus$ node in T such that $G(x)$ contains an entire row or an entire column of G (the node x may not be unique). Let y and z be the children of x in T. Colour all vertices in $G(y)$ blue and all vertices in $G(z)$ red. Colour all remaining vertices of G yellow. Note that a vertex of G appears in $G(x)$ if and only if it is coloured either red or blue and that there is no edge in $G(x)$ between a red and a blue vertex. Due to our choice of x, the graph G contains a row or a column none of whose vertices are yellow, but no row or column of G is entirely blue or entirely red. Without loss of generality, assume that G contains a non-yellow column.

Because G contains a non-yellow column, each row of G contains a non-yellow vertex, by Property 3. Since no row is entirely red or entirely blue, every row of G is therefore coloured with at least two colours. Let R_i be an arbitrary row. Since $G[R_i]$ is connected, there must be two adjacent vertices $v_i, w_i \in R_i$ in G, such that v_i is either red or blue and w_i has a different colour than v_i. Note that v_i and w_i are therefore not adjacent in $G(x)$ (recall that if w_i is yellow then it is not even present as a vertex of $G(x)$).

[3] The class of (square) grids was first shown to have unbounded clique-width by Makowsky and Rotics [24]. The construction of [20] determines the exact clique-width of square grids and narrows the clique-width of non-square grids to two values.

Now consider indices $i, k \geq 1$ with $k > i + m$. By Properties 6 and 8, no vertex of R_i is adjacent to a vertex of $R_k \setminus V_{k,0}$ in G. Therefore, since $|V_{k,0}| \leq 1$ by Property 1, we conclude that either v_i and w_i are not adjacent to v_k in G, or v_i and w_i are not adjacent to w_k in G. In particular, this implies that w_i is not adjacent to v_k in G or that w_k is not adjacent to v_i in G. Recall that v_i and w_i are adjacent in G but not in $G(x)$, and the same holds for v_k and w_k. Hence, a $\eta_{i,j}$ node higher up in the tree, makes w_i adjacent to v_i but not to v_k, or makes w_k adjacent to v_k but not to v_i. This means that v_i and v_k must have different labels in $G(x)$. We conclude that $v_1, v_{(m+1)+1}, v_{2(m+1)+1}, v_{3(m+1)+1}, \ldots, v_{(\lfloor \frac{n-1}{m+1} \rfloor)(m+1)+1}$ must all have different labels in $G(x)$. Hence, the k-expression of G uses at least $\lfloor \frac{n-1}{m+1} \rfloor + 1$ labels. □

We now use Theorem 3 to determine two new graph classes that have unbounded clique-width.

Theorem 4. *The class of* $(P_6, \overline{2P_1 + P_2})$*-free graphs has unbounded clique-width.*

Proof. Let $n \geq 1$ be an integer. Using the notation of Theorem 3, we construct a graph G_n as follows. We define vertex subsets

$$
\begin{aligned}
V_{0,0} &= \emptyset \\
V_{i,0} &= \{b_i\} \text{ for } i \geq 1 \\
V_{0,j} &= \{w_j\} \text{ for } j \geq 1 \\
V_{i,j} &= \{b_{i,j}, r_{i,j}, w_{i,j}\} \text{ for } i, j \geq 1.
\end{aligned}
$$

We define edge subsets

$$
\begin{aligned}
E_1 &= \{b_{i,j}r_{i,j}, r_{i,j}w_{i,j} \mid i, j \in \{1, \ldots, n\}\} \\
E_2 &= \{b_k w_{i,j} \mid i, j, k \in \{1, \ldots, n\}, i \leq k\} \\
E_3 &= \{w_k b_{i,j} \mid i, j, k \in \{1, \ldots, n\}, j \leq k\}.
\end{aligned}
$$

Let $V(G_n)$ be the union of the sets $V_{i,j}$ for $i, j \in \{0, \ldots, n\}$, and let $E(G_n) = E_1 \cup E_2 \cup E_3$. Note that in G_n the constructed sets $V_{i,j}$ fulfil the conditions of Theorem 3 when $m = 0$. Therefore G_n has clique-width at least n.

We now define the sets

$$
\begin{aligned}
B_1 &= \{b_i \mid i \in \{1, \ldots, n\}\} \\
W_1 &= \{w_j \mid j \in \{1, \ldots, n\}\} \\
B_2 &= \{b_{i,j} \mid i, j \in \{1, \ldots, n\}\} \\
R_2 &= \{r_{i,j} \mid i, j \in \{1, \ldots, n\}\} \\
W_2 &= \{w_{i,j} \mid i, j \in \{1, \ldots, n\}\}.
\end{aligned}
$$

Let H_n be the graph obtained from G_n by complementing the edges between B_2 and W_2. By Fact 3, the class of graphs $\{H_n\}_{n \geq 1}$ has unbounded clique-width. Note that $H_n[B_1 \cup W_2]$ and $H_n[B_2 \cup W_1]$ are $2P_2$-free bipartite graphs. We claim that every H_n is $(P_6, \overline{2P_1 + P_2})$-free.

First we show that H_n is $(\overline{2P_1 + P_2})$-free. For contradiction, suppose that $\overline{2P_1 + P_2}$ is present as an induced subgraph. Consider one of the vertices of degree 3 in the $\overline{2P_1 + P_2}$. It cannot be in B_1 or W_1 since those vertices have neighbourhoods that are independent sets. It cannot be a vertex in R_2, since those vertices have degree 2. Therefore one of these vertices must be in B_2 and the other in W_2. Therefore the other two vertices in the diamond must both be in R_2, which is a contradiction, since every vertex in B_2 has a unique neighbour in R_2. Therefore H_n is indeed $(\overline{2P_1 + P_2})$-free.

We now show that H_n is P_6-free. For contradiction, suppose that P_6 is present as an induced subgraph. We will first show that no vertex of the P_6 may contain a vertex of R_2. Indeed, if one of the vertices in the P_6 is in R_2, it must be an end-vertex of the P_6 (since the neighbourhood of any vertex in R_2 induces a P_2, but P_6 does not contain a K_3). Let $x_1,\ldots,x_6$ be the vertices of the P_6, in order. Note that $x_2,x_3,x_4,x_5 \notin R_2$. Suppose that $x_1 \in R_2$. Without loss of generality, we may assume $x_2 \in W_2$. If $x_3 \in B_1$, then we must have $x_4 \in W_2$. But then there is no possible choice for x_5: we cannot have $x_5 \in R_2$ (as noted above), we cannot have $x_5 \in B_2$ (since then x_2 would be adjacent to x_5) and we cannot have $x_5 \in B_1$, since then x_6 would be in W_2 and $H_n[x_2,x_3,x_5,x_6]$ would be a $2P_2$, contradicting the fact that $H_n[B_1 \cup W_2]$ is a $2P_2$-free bipartite graph. Thus if $x_1 \in R_2$, $x_2 \in W_2$ then $x_3 \in B_2$ (since every vertex in W_2 has a unique neighbour in R_2). Now $x_4 \notin W_1$ (otherwise x_5 would be in B_2, which would mean that x_2 would be adjacent to x_5) and $x_4 \notin R_2$ (as explained above), so $x_4 \in W_2$. But this cannot happen, since $x_5 \notin R_2$ (as explained above), $x_5 \notin B_2$ (since x_5 is not adjacent to x_2), so $x_5 \in B_1$, so $x_6 \in W_2$, contradicting the fact that x_3 and x_6 are not adjacent. We conclude that no P_6 in H_n can include a vertex of R_2.

By symmetry, we may therefore assume that every induced P_6 contains at least three vertices in $W_1 \cup B_2$. In this case, it must have at least two vertices in B_2 since W_1 is an independent set. If the P_6 also has a vertex in W_2 then it must have exactly one vertex in W_2, two in B_2, none in B_1 and three in W_1, which is impossible, by a parity argument. Thus the whole of the P_6 must be contained in $H_n[W_1 \cup B_2]$, which leads to $H_n[W_1 \cup B_2]$ containing a $2P_2$, which contradicts the fact that $H_n[W_1 \cup B_2]$ is a $2P_2$-free bipartite graph. This completes the proof. □

Theorem 5. *The class of* $(3P_2, P_2+P_4, P_6, \overline{P_1+P_4})$*-free graphs has unbounded clique-width.*

Proof. Let $n \geq 1$ be an integer. Using the notation of Theorem 3, we construct a graph G_n as follows. We define vertex subsets

$$\begin{aligned} V_{0,0} &= \emptyset \\ V_{i,0} &= \{b_i\} \text{ for } i \geq 1 \\ V_{0,j} &= \{w_j\} \text{ for } j \geq 1 \\ V_{i,j} &= \{x_{i,j}\} \text{ for } i,j \geq 1. \end{aligned}$$

We define edge subsets

$$\begin{aligned} E_1 &= \{b_ib_j \mid i,j \in \{1,\ldots,n\}, i \neq j\} \\ E_2 &= \{w_iw_j \mid i,j \in \{1,\ldots,n\}, i \neq j\} \\ E_3 &= \{b_kx_{i,j} \mid i,j,k \in \{1,\ldots,n\}, i \leq k\} \\ E_4 &= \{w_kx_{i,j} \mid i,j,k \in \{1,\ldots,n\}, j \leq k\}. \end{aligned}$$

Let $V(G_n)$ be the union of the sets $V_{i,j}$ for $i,j \in \{0,\ldots,n\}$, and let $E(G_n) = E_1 \cup E_2 \cup E_3 \cup E_4$. Note that in G_n the constructed sets $V_{i,j}$ fulfil the conditions of Theorem 3 when $m = 0$. Therefore G_n has clique-width at least n.

We define the sets

$$\begin{aligned} B &= \{b_i \mid i \in \{1,\ldots,n\}\} \\ W &= \{w_i \mid i \in \{1,\ldots,n\}\} \\ X &= \{x_{i,j} \mid i,j \in \{1,\ldots,n\}\}. \end{aligned}$$

Note that two vertices in B (respectively X) cannot each have private neighbours in X (respectively B). (When considering a pair of vertices v_1, v_2, a *private neighbour* of v_1 is a vertex adjacent to v_1, but not to v_2.) We will show that every G_n is $(3P_2, P_2+P_4, P_6, \overline{P_1+P_4})$-free.

First we show that G_n is $(3P_2)$-free. For contradiction, suppose that G_n contains an induced $3P_2$. Then, since X is an independent set and both B and W are cliques, at most one of the P_2 components could occur in $G_n[B \cup X]$ and at most one of the P_2 components could occur in $G_n[W \cup X]$. Since no vertex of B is adjacent to a vertex of W, we find that G_n therefore cannot contain an induced $3P_2$.

We now show that G_n is $(P_2 + P_4)$-free. For contradiction, suppose that G_n contains an induced $P_2 + P_4$. Since X is an independent set, we may assume that the P_4 contains at least one vertex of B. The P_4 can have at most two vertices in B and if it has two such vertices, one of them must be the end-vertex of the P_4; otherwise the two vertices in B would each have a private neighbour in X which cannot happen. Thus if the P_4 has a vertex in B then it must have a vertex in X and another in W (since X is an independent set). Thus the P_4 must have both a vertex in B and a vertex in W. Then an independent P_2 cannot be found since B and W are cliques and X is an independent set.

We now show that G_n is P_6-free. For contradiction, suppose that G_n contains an induced P_6. Any P_6 can contain at most two vertices of B (respectively W), at most one of which can be adjacent to any vertex of X in the P_6. Let $v_1, \ldots, v_6$ be the vertices of the P_6 in order. If the P_6 contains two vertices of B (respectively W), then these two vertices must be adjacent and one of them must be an end-vertex of the P_6. In this case, assume without loss of generality that $v_1, v_2 \in B$. Then $v_3 \in X$, so $v_4 \in W$. Since v_4 is a middle-vertex of the P_6, neither $v_5, v_6 \notin W$. This means $v_5, v_6 \in X$, which cannot happen since X is an independent set. This contradiction means that at most one vertex of the P_6 can be in each of B and W, so at least four vertices of the P_6 are members of X. This is impossible since X is an independent set. Thus G_n is indeed P_6-free.

Finally, we show that G_n is $(\overline{P_1 + P_4})$-free. For contradiction, suppose that G_n contains an induced $\overline{P_1 + P_4}$. If the dominating vertex of the $\overline{P_1 + P_4}$ is in X then, since no vertex in B is adjacent to a vertex in W, the other vertices must either be all in B or all be in W, which is a contradiction. Thus the dominating vertex must be (without loss of generality) in B and the other vertices in the $\overline{P_1 + P_4}$ must therefore all be in $B \cup X$. At most two of the other vertices can be in X (since X is an independent set and $\overline{P_4}$ has independence number 2) and at most two of them can be in B (since B is a clique). So exactly three vertices of the $\overline{P_1 + P_4}$ must be in B and two must be in X. Since X is an independent set and B is a clique, the two vertices in X must be the two vertices of degree 2 in the $\overline{P_1 + P_4}$. However, this means that each of these two vertices in X has a private neighbour in B, which is a contradiction. This shows that G_n is indeed $(\overline{P_1 + P_4})$-free, which completes the proof. □

5 Classifying Classes of (H_1, H_2)-Free Graphs

In this section we study the boundedness of clique-width of classes of graphs defined by two forbidden induced subgraphs. Recall that this study is partially motivated by the fact that it is easy to obtain a full classification for the boundedness of clique-width of graph classes defined by one forbidden induced subgraph, as shown in the next theorem. This classification does not seem to have previously been explicitly stated in the literature.

Theorem 6. *Let H be a graph. The class of H-free graphs has bounded clique-width if and only if H is an induced subgraph of P_4.*

Proof. First suppose that H is an induced subgraph of P_4. Then the class of H-free graphs is a subclass of the class of P_4-free graphs. The class of P_4-free graphs is precisely the class of graphs of clique-width at most 2 [48].

Now suppose that H is a graph such that the class of H-free graphs has bounded clique-width. By Fact 2, the class of $\overline{H}$-free graphs has bounded clique-width. By Lemma 3, $H, \overline{H} \in \mathcal{S}$. Since $\overline{H} \in \mathcal{S}$, the graph $\overline{H}$ must be (K_3, C_4)-free. Thus H must be a $2P_2$-free forest whose maximum independent set has size at most 2. Therefore H must be one of the following graphs: $P_1, 2P_1, P_1 + P_2, P_2, P_3, P_4$. All these graphs are induced subgraphs of P_4. □

We are now ready to systematically consider classes of graphs defined by two forbidden induced subgraphs. Given four graphs H_1, H_2, H_3, H_4, we say that the class of (H_1, H_2)-free graphs and the class of (H_3, H_4)-free graphs are *equivalent* if the pair (H_3, H_4) can be obtained from the pair (H_1, H_2) by some combination of the following operations:

1. reversing the order of the two graphs in the pair;
2. complementing both graphs in the pair;
3. if one of the graphs in the pair is K_3, replacing it with $\overline{P_1 + P_3}$ or vice versa.

By Fact 2 and Lemma 7, if two classes are equivalent then one has bounded clique-width if and only if the other one does. Given this definition, we now give a partial classification of the (un)boundedness of the clique-width for classes defined by two forbidden induced subgraphs. This includes both already-known results and our new results. We will later show that (up to equivalence) our classification leaves only 13 open cases.

Theorem 7. *Let $\mathcal{G}$ be a class of graphs defined by two forbidden induced subgraphs. Then:*

(i) $\mathcal{G}$ has bounded clique-width if it is equivalent to a class of (H_1, H_2)-free graphs such that one of the following holds:
   1. *H_1 or $H_2 \subseteq_i P_4$;*
   2. *$H_1 = sP_1$ and $H_2 = K_t$ for some s, t;*
   3. *$H_1 \subseteq_i P_1 + P_3$ and $\overline{H_2} \subseteq_i K_{1,3} + 3P_1,\ K_{1,3} + P_2,\ P_1 + S_{1,1,2},\ P_6$ or $S_{1,1,3}$;*
   4. *$H_1 \subseteq_i 2P_1 + P_2$ and $\overline{H_2} \subseteq_i 2P_1 + P_3,\ 3P_1 + P_2$ or $P_2 + P_3$;*
   5. *$H_1 \subseteq_i P_1 + P_4$ and $\overline{H_2} \subseteq_i P_1 + P_4$ or P_5;*
   6. *$H_1 \subseteq_i 4P_1$ and $\overline{H_2} \subseteq_i 2P_1 + P_3$;*
   7. *$H_1, \overline{H_2} \subseteq_i K_{1,3}$.*

(ii) $\mathcal{G}$ has unbounded clique-width if it is equivalent to a class of (H_1, H_2)-free graphs such that one of the following holds:
   1. *$H_1 \notin \mathcal{S}$ and $H_2 \notin \mathcal{S}$;*
   2. *$\overline{H_1} \notin \mathcal{S}$ and $\overline{H_2} \notin \mathcal{S}$;*
   3. *$H_1 \supseteq_i K_{1,3}$ or $2P_2$ and $\overline{H_2} \supseteq_i 4P_1$ or $2P_2$;*
   4. *$H_1 \supseteq_i 2P_1 + P_2$ and $\overline{H_2} \supseteq_i K_{1,3},\ 5P_1,\ P_2 + P_4$ or P_6;*
   5. *$H_1 \supseteq_i 3P_1$ and $\overline{H_2} \supseteq_i 2P_1 + 2P_2,\ 2P_1 + P_4,\ 4P_1 + P_2,\ 3P_2$ or $2P_3$;*
   6. *$H_1 \supseteq_i 4P_1$ and $\overline{H_2} \supseteq_i P_1 + P_4$ or $3P_1 + P_2$.*

Proof. We first consider the bounded cases. Statement (i).1 follows from Theorem 6. To prove Statement (i).2 note that if $H_1 = sP_1$ and $H_2 = K_t$ for some s, t then by Ramsey's Theorem, all graphs in the class of (H_1, H_2)-free graphs have a bounded number of vertices and therefore the clique-width of graphs in this class is bounded. By the definition of equivalence, when proving Statement (i).3, we may assume that $H_1 = K_3$. Then Statement (i).3 follows from Fact 2 combined with the fact that (K_3, H)-free graphs have bounded clique-width if H is $K_{1,3} + 3P_1$ (Theorem 2), $K_{1,3} + P_2$ [18], $P_1 + S_{1,1,2}$ (Theorem 1), P_6 [10] or $S_{1,1,3}$ [18]. Statement (i).4 follows from Fact 2 and the fact that $(\overline{2P_1 + P_2}, 2P_1 + P_3)$-free, $(\overline{2P_1 + P_2}, 3P_1 + P_2)$-free and $(\overline{2P_1 + P_2}, P_2 + P_3)$-free graphs have bounded clique-width [17]. Statement (i).5 follows from Fact 2 and the fact that both $(P_1 + P_4, \overline{P_1 + P_4})$-free graphs [12] and $(P_5, \overline{P_1 + P_4})$-free graphs [13] have bounded clique-width. Statement (i).6 follows from Fact 2 and the fact that $(2P_1 + P_3, K_4)$-free graphs have bounded clique-width [7]. Statement (i).7 follows from the fact that $(K_{1,3}, \overline{K_{1,3}})$-free graphs have bounded clique-width [5,14].

We now consider the unbounded cases. Statements (ii).1 and (ii).2 follow from Lemma 3 and Fact 2. Statement (ii).3 follows from the fact that the classes of $(C_4, K_{1,3}, K_4, \overline{2P_1 + P_2})$-free [9], $(K_4, 2P_2)$-free [9] and $(C_4, C_5, 2P_2)$-free graphs (or equivalently, split graphs) [24] have unbounded clique-width. Statement (ii).4 follows from Fact 2 and the fact that $(C_4, K_{1,3}, K_4, \overline{2P_1 + P_2})$-free [9],

$(5P_1, \overline{2P_1 + P_2})$-free [16], $(\overline{2P_1 + P_2}, P_2 + P_4)$-free (see arXiv version of [17]) and $(P_6, \overline{2P_1 + P_2})$-free (Theorem 4) graphs have unbounded clique-width. To prove Statement (ii).5, suppose $H_1 \supseteq_i 3P_1$ and $\overline{H_2} \supseteq_i 2P_1 + 2P_2, 2P_1 + P_4, 4P_1 + P_2, 3P_2$ or $2P_3$. Then $\overline{H_1} \notin \mathcal{S}$, so $\overline{H_2} \in \mathcal{S}$, otherwise we are done by Statement (ii).2. By Lemma 5, $\overline{H_2}$ is not an induced subgraph of any graph in $\{K_{1,3} + 3P_1, K_{1,3} + P_2, P_1 + S_{1,1,3}, S_{1,2,3}\}$. The class of (H_1, H_2)-free graphs contains the class of complements of $\overline{H_2}$-free bipartite graphs. By Fact 2 and Lemma 4, this latter class has unbounded clique-width. Statement (ii).6 follows from the Fact 2 and the fact that the classes of $(K_4, P_1 + P_4)$-free graphs (Lemma 8) and $(4P_1, \overline{3P_1 + P_2})$-free graphs [16] have unbounded clique-width. $\square$

As we will prove in Theorem 8, the above classification leaves exactly 13 open cases (up to equivalence).

Open Problem 1 *Does the class of* (H_1, H_2)*-free graphs have bounded clique-width when:*

1. $H_1 = 3P_1, \overline{H_2} \in \{P_1 + P_2 + P_3, P_1 + 2P_2, P_1 + P_5, P_1 + S_{1,1,3}, P_2 + P_4, S_{1,2,2}, S_{1,2,3}\}$*;*
2. $H_1 = 2P_1 + P_2, \overline{H_2} \in \{P_1 + P_2 + P_3, P_1 + 2P_2, P_1 + P_5\}$*;*
3. $H_1 = P_1 + P_4, \overline{H_2} \in \{P_1 + 2P_2, P_2 + P_3\}$ *or*
4. $H_1 = \overline{H_2} = 2P_1 + P_3$.

Note that the two pairs $(3P_1, \overline{S_{1,1,2}})$ and $(3P_1, \overline{S_{1,2,3}})$, or equivalently, the two pairs $(K_3, S_{1,2,2})$ and $(K_3, S_{1,2,3})$ are the only pairs that correspond to open cases in which both H_1 and H_2 are connected. We also observe the following. Let $H_2 \in \{P_1 + P_2 + P_3, P_1 + 2P_2, P_1 + P_5, P_1 + S_{1,1,3}, P_2 + P_4, S_{1,2,2}, S_{1,2,3}\}$. Lemma 4 shows that all bipartite H_2-free graphs have bounded clique-width. Moreover, the graph $P_1 + 2P_2$ is an induced subgraph of H_2. Hence, for investigating whether the boundedness of the clique-width of bipartite H_2-free graphs can be extended to (K_3, H_2)-free graphs, the $H_2 = P_1 + 2P_2$ case is the starting case.

Theorem 8. *Let $\mathcal{G}$ be a class of graphs defined by two forbidden induced subgraphs. Then $\mathcal{G}$ is not equivalent to any of the classes listed in Theorem 7 if and only if it is equivalent to one of the 13 cases listed in Open Problem 1.*

Proof. It is easy to verify that none of the classes listed in Open Problem 1 are equivalent to classes listed in Theorem 7.

Let H_1, H_2 be graphs and let $\mathcal{G}$ be the class of (H_1, H_2)-free graphs. Suppose $\mathcal{G}$ is not equivalent to any class listed in Theorem 7. Then $H_1 \in \mathcal{S}$ or $H_2 \in \mathcal{S}$, otherwise Theorem 7.(ii).1 applies. Similarly, $\overline{H_1} \in \mathcal{S}$ or $\overline{H_2} \in \mathcal{S}$. If $H_i, \overline{H_i} \in \mathcal{S}$ for some $i \in \{1, 2\}$, then $H_i \subseteq_i P_4$ (as shown in the proof of Theorem 6), in which case Theorem 7.(i).1 applies.

Due to the definition of equivalence, for the remainder of the proof we may assume without loss of generality that $H_1, \overline{H_2} \in \mathcal{S}$, but neither is an induced subgraph of P_4. Furthermore, we may assume that neither H_1 nor $\overline{H_2}$ is isomorphic to $P_1 + P_3$, as in this case the definition of equivalence would allow us to replace $P_1 + P_3$ by $3P_1$. Also note that the situation for H_1 and $\overline{H_2}$ is symmetric, i.e. if we exchanged these graphs, the resulting class would be equivalent.

Suppose that $3P_1 \not\subseteq_i H_1$. Then we must have that $H_1 = 2P_2$ (as $H_1 \not\subseteq_i P_4$). If $\overline{H_2} \supseteq_i K_{1,3}, 4P_1$ or $2P_2$ then Theorem 7.(ii).3 applies. Since $\overline{H_2} \in \mathcal{S}$, we may therefore assume that $\overline{H_2}$ is a linear forest which is $(4P_1, 2P_2)$-free. This means that $\overline{H_2}$ is an induced subgraph of $P_1 + P_4$, in which case Theorem 7.(i).5 applies (since $2P_2 \subseteq_i P_5$).

We therefore assume that $3P_1 \subseteq_i H_1, \overline{H_2}$. Now $H_1, \overline{H_2}$ must be $(2P_1 + 2P_2, 2P_1 + P_4, 4P_1 + P_2, 3P_2, 2P_3)$-free, otherwise Theorem 7.(ii).5 would apply. Since $H_1, \overline{H_2} \in \mathcal{S}$, by Lemma 5, each of $H_1, \overline{H_2}$ must either contain no edges or be an induced subgraph of (possibly different) graphs in $\{K_{1,3} + 3P_1, K_{1,3} + P_2, P_1 + S_{1,1,3}, S_{1,2,3}\}$. The induced subgraphs of graphs in $\{K_{1,3} + 3P_1, K_{1,3} + P_2, P_1 + S_{1,1,3}, S_{1,2,3}\}$ are listed in Table 1.

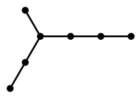
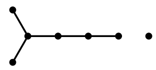
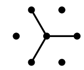
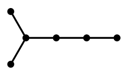
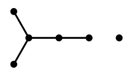
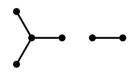
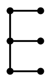
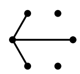
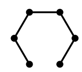
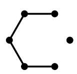
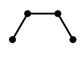
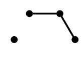
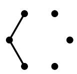
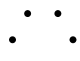
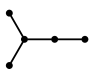
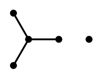
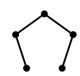
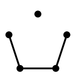
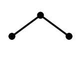
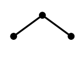
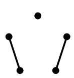
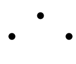
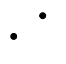
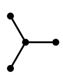
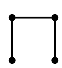
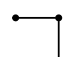
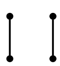
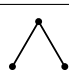
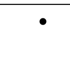
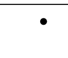
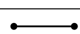
Graph	Name	Graph	Name	Graph	Name
	$S_{1,2,3}$		$P_1 + S_{1,1,3}$		$K_{1,3} + 3P_1$
	$S_{1,1,3}$		$P_1 + S_{1,1,2}$		$K_{1,3} + P_2$
	$S_{1,2,2}$		$K_{1,3} + 2P_1$		P_6
	$P_1 + P_5$		$P_2 + P_4$		$P_1 + P_2 + P_3$
	$3P_1 + P_3$		$6P_1$		
	$S_{1,1,2}$		$K_{1,3} + P_1$		P_5
	$P_1 + P_4$		$P_2 + P_3$		$2P_1 + P_3$
	$P_1 + 2P_2$		$3P_1 + P_2$		$5P_1$
	$K_{1,3}$		P_4		$P_1 + P_3$
	$2P_2$		$2P_1 + P_2$		$4P_1$
	P_3		$P_1 + P_2$		$3P_1$
	P_2		$2P_1$		P_1

Table 1: The induced subgraphs of $S_{1,2,3}$, $S_{1,1,3} + P_1$, $K_{1,3} + 3P_1$ and $K_{1,3} + P_2$, arranged by number of vertices.

First suppose that H_1 contains no edges. Then $\overline{H_2}$ must contain an edge, otherwise Theorem 7.(i).2 would apply. We first assume that $H_1 = 3P_1$. If $\overline{H_2} \subseteq_i K_{1,3} + 3P_1, K_{1,3} + P_2, P_1 + S_{1,1,2}, P_6$ or $S_{1,1,3}$, then Theorem 7.(i).3 applies. This leaves the cases where $\overline{H_2} \in \{P_1 + P_2 + P_3, P_1 + 2P_2, P_1 + P_5, P_1 + S_{1,1,3}, P_2 + P_4, S_{1,2,2}, S_{1,2,3}\}$, all of which are stated in Open Problem 1.1. Now assume $H_1 = kP_1$ for $k \geq 4$. If $\overline{H_2} \supseteq_i K_{1,3}, P_1 + P_4, 3P_1 + P_2$ or $2P_2$, Theorem 7.(ii).3 or 7.(ii).6 applies. Otherwise, $\overline{H_2}$ must be a $(P_1 + P_4, 3P_1 + P_2, 2P_2)$-free linear forest, which (by assumption) is not an edgeless graph. As $\overline{H_2} \not\subseteq_i P_4$ and $\overline{H_2} \neq P_1 + P_3$, this means that $\overline{H_2} \in \{2P_1 + P_2, 2P_1 + P_3\}$. In both these cases, if $k = 4$ then $\overline{H_2} \subseteq_i 2P_1 + P_3$, so Theorem 7.(i).6 applies; if $k \geq 5$ then $2P_1 + P_2 \subseteq_i \overline{H_2}$, so Theorem 7.(ii).4 applies.

By symmetry, we may therefore assume that neither H_1 nor $\overline{H_2}$ are edgeless. As stated above, in this case we may assume that both H_1 and $\overline{H_2}$ are induced subgraphs of (possibly different) graphs in $\{K_{1,3} + 3P_1, K_{1,3} + P_2, P_1 + S_{1,1,3}, S_{1,2,3}\}$. Combining this with our previous assumptions that neither H_1 nor $\overline{H_2}$ is equal to $P_1 + P_3$ or an induced subgraph of P_4 means that $H_1, \overline{H_2} \in \{K_{1,3}, K_{1,3}+P_1, K_{1,3}+2P_1, K_{1,3}+3P_1, K_{1,3}+P_2, P_1+P_2+P_3, P_1+2P_2, P_1+P_4, P_1+P_5, P_1+S_{1,1,2}, P_1+S_{1,1,3}, 2P_1+P_2, 2P_1+P_3, 3P_1+P_2, 3P_1+P_3, P_2+P_3, P_2+P_4, P_5, P_6, S_{1,1,2}, S_{1,1,3}, S_{1,2,2}, S_{1,2,3}\}$ (see also Table 1 and recall that $3P_1 \subseteq_i H_1, \overline{H_2}$). *In particular, this shows that the number of open cases is finite*.

Suppose H_1 is not a linear forest. Then $K_{1,3} \subseteq_i H_1$. If $\overline{H_2} \supseteq_i 2P_1 + P_2, 4P_1$ or $2P_2$ then Theorem 7.(ii).3 or 7.(ii).4 applies. The only remaining choice for $\overline{H_2}$ is $K_{1,3}$. Then, by symmetry, we may assume that H_1 is isomorphic to $K_{1,3}$, in which case Theorem 7.(i).7 applies.

We may now assume that H_1 and $\overline{H_2}$ are both linear forests, each containing at least one edge. In other words, $H_1, \overline{H_2} \in \{P_1 + P_2 + P_3, P_1 + 2P_2, P_1 + P_4, P_1 + P_5, 2P_1 + P_2, 2P_1 + P_3, 3P_1 + P_2, 3P_1+P_3, P_2+P_3, P_2+P_4, P_5, P_6\}$. Note that both of these graphs must therefore either be isomorphic to P_5 or contain $2P_1 + P_2$ as an induced subgraph. If $H_1 = P_5$ then $\overline{H_2}$ must be $(4P_1, 2P_2)$-free otherwise Theorem 7.(ii).3 applies. Thus $\overline{H_2} \in \{P_1 + P_4, 2P_1 + P_2\}$, in which case Theorem 7.(i).5 applies. We may therefore assume that neither H_1 nor $\overline{H_2}$ is isomorphic to P_5, and both must therefore contain $2P_1 + P_2$ as an induced subgraph. Therefore, neither H_1 nor $\overline{H_2}$ may contain $5P_1, P_2 + P_4$ or P_6 as an induced subgraph, otherwise Theorem 7.(ii).4 would apply. We therefore conclude that $H_1, \overline{H_2} \in \{P_1 + P_2 + P_3, P_1 + 2P_2, P_1 + P_4, P_1 + P_5, 2P_1 + P_2, 2P_1 + P_3, 3P_1 + P_2, P_2 + P_3\}$.

Suppose $H_1 = 2P_1 + P_2$. If $\overline{H_2} \in \{P_1 + P_4, 2P_1 + P_2\}$, then Theorem 7.(i).5 would apply. If $\overline{H_2} \in \{2P_1 + P_3, 3P_1 + P_2, P_2 + P_3\}$, then Theorem 7.(i).4 would apply. This leaves the cases where $\overline{H_2} \in \{P_1 + P_2 + P_3, P_1 + 2P_2, P_1 + P_5\}$, which appear as Open Problem 1.2. We now assume neither H_1 nor $\overline{H_2}$ is isomorphic to $2P_1 + P_2$.

Suppose $H_1 = P_1+P_4$. If $\overline{H_2} \in \{P_1+P_2+P_3, P_1+P_5, 2P_1+P_3, 3P_1+P_2\}$ then Theorem 7.(ii).6 would apply. If $\overline{H_2} = P_1 + P_4$, then Theorem 7.(i).5 applies. This leaves the case where $\overline{H_2} \in \{P_1 + 2P_2, P_2 + P_3\}$, both of which appear in Open Problem 1.3. We may therefore assume that H_1 and $\overline{H_2}$ are not isomorphic to $P_1 + P_4$.

We have now that H_1 and $\overline{H_2} \in \{P_1 + P_2 + P_3, P_1 + 2P_2, P_1 + P_5, 2P_1 + P_3, 3P_1 + P_2, P_2 + P_3\}$. Note that each of these graphs contains either $4P_1$ or $2P_2$ as an induced subgraph. If either H_1 or $\overline{H_2}$ contains an induced $2P_2$, then in all these cases Theorem 7.(ii).3 would apply. We may therefore assume that $H_1, \overline{H_2} \in \{2P_1 + P_3, 3P_1 + P_2\}$. However, both these graphs contain $4P_1$, so if $H_1 = 3P_1 + P_2$, then Theorem 7.(ii).6 applies. Therefore $H_1 = \overline{H_2} = 2P_1 + P_3$, which is Open Problem 1.4. This completes the proof. $\square$

6 Forbidding Other Patterns

Instead of forbidding one or more graphs as an induced subgraph of some other graph G, we could also forbid graphs under other containment relations. For example, a graph G is $(H_1, \ldots, H_p)$*-subgraph-free* if G has no subgraph isomorphic to a graph in $\{H_1, \ldots, H_p\}$. In this section we consider this containment relation and two other well-known containment relations, which we define below.

Let G and H be graphs. Then G contains H as a *minor* or *topological minor* if G can be modified into H by a sequence that consists of edge contractions, edge deletions and vertex deletions, or by a sequence that consists of vertex dissolutions, edge deletions and vertex deletions, respectively. If G does not contain any of the graphs $H_1, \ldots, H_p$ as a (topological) minor, we say that G is $(H_1, \ldots, H_p)$*-(topological-)minor-free*.

When we forbid a finite collection of either minors, subgraphs or topological minors, we can completely characterize those graph classes that have bounded clique-width. Before we prove these results we first state four known results, the last of which can be found in the textbook of Diestel [46]. For a graph G, let $\mathrm{tw}(G)$ denote the treewidth of G (see, for example, Diestel [46] for a definition).

Lemma 10 ([5]). *Let $H \in \mathcal{S}$. Then the class of H-subgraph-free graphs has bounded clique-width.*

Lemma 11 ([44]). *Let G be a graph. Then* $\mathrm{cw}(G) \leq 3 \times 2^{\mathrm{tw}(G)-1}$.

Lemma 12 ([53]). *Let H be a planar graph. Then the class of H-minor-free graphs has bounded treewidth.*

Lemma 13. *Let H be a graph of maximum degree at most* 3*. Then any graph that contains H as a minor contains H as a topological minor.*

We are now ready to state the three dichotomy results. These classifications do not seem to have previously been explicitly stated in the literature.

Theorem 9. *Let $\{H_1, \ldots, H_p\}$ be a finite set of graphs. Then the following statements hold:*

(i) The class of $(H_1, \ldots, H_p)$-subgraph-free graphs has bounded clique-width if and only if $H_i \in \mathcal{S}$ for some $1 \leq i \leq p$.

(ii) The class of $(H_1, \ldots, H_p)$-minor-free graphs has bounded clique-width if and only if H_i is planar for some $1 \leq i \leq p$.

(iii) The class of $(H_1, \ldots, H_p)$-topological-minor-free graphs has bounded clique-width if and only if H_i is planar and has maximum degree at most 3 *for some $1 \leq i \leq p$.*

Proof. We first prove (i). First suppose that $H_i \in \mathcal{S}$ for some i. Then the class of $(H_1, \ldots, H_p)$-subgraph-free graphs has bounded clique-width, by Lemma 10. Now suppose that $H_i \notin \mathcal{S}$ for all i. For $j \geq 0$, let I_j be the graph formed from $2P_3$ by joining the central vertices of the two P_3's by a path of length j (so $I_0 = K_{1,4}$). Since $H_i \notin \mathcal{S}$, every H_i contains an induced subgraph isomorphic to some C_j or to some I_j. Let g be the maximum number of vertices of such an induced subgraph in $H_1 + \cdots + H_p$. Then the class of $(H_1, \ldots, H_p)$-subgraph-free graphs contains the class of g-subdivided walls. Hence, it has unbounded clique-width by Lemma 2.

We now prove (ii). First suppose that H_i is planar for some i. Then the class of H_i-minor-free graphs, and thus the class of $(H_1, \ldots, H_p)$-minor-free graphs, has bounded treewidth by Lemma 12. Consequently, it has bounded clique-width, by Lemma 11. Now suppose that H_i is non-planar for all i. Because planar graphs are closed under taking minors, every planar graph is $(H_1, \ldots, H_p)$-minor-free. Hence, the class of $(H_1, \ldots, H_p)$-minor-free graphs contains the class of walls, and thus has unbounded clique-width by Lemma 2.

Finally, we prove (iii). First suppose that H_i is a planar graph of maximum degree at most 3 for some i. By Lemma 13, any H_i-topological-minor-free is H_i-minor-free. Hence, we can repeat the arguments from above to find that the class of $(H_1, \ldots, H_p)$-free graphs has bounded clique-width. Now suppose that H_i is either non-planar or contains a vertex of degree at least 4 for all i. Consider some H_i. First assume that H_i is not planar. Because planar graphs are closed under taking topological minors, every planar graph, and thus every wall, is H_i-topological-minor-free. Now suppose that H_i is planar. Then H_i must have maximum degree at least 4. Because every wall has minimum degree at most 3, it is H_i-topological-minor-free. We conclude that the class of $(H_1, \ldots, H_p)$-topological-minor-free graphs contains the class of walls, and thus has unbounded clique-width by Lemma 2. □

7 Consequences for Colouring

One of the motivations of our research was to further the study of the computational complexity of the COLOURING problem for (H_1, H_2)-free graphs. Recall that COLOURING is polynomial-time solvable on any graph class of bounded clique-width by combining results of Kobler and Rotics [27] and Oum [30]. By combining a number of known results [18,34,35,36,37,54,55,56,57] with new results, Dabrowski, Golovach and Paulusma [16] presented a summary of known results for COLOURING restricted to (H_1, H_2)-free graphs. Combining Theorem 7 with the results of Kobler and Rotics [27] and Oum [30] and incorporating a number of recent results leads to an updated summary. This updated summary (and a proof of it) can be found in the recent survey paper of Golovach, Johnson, Paulusma and Song [58].

From this summary we note that not only the case when $H_1 = P_4$ or $H_2 = P_4$ but thirteen other maximal classes of (H_1, H_2)-free graphs for which COLOURING is known to be polynomial-time solvable can be obtained by combining Theorem 7 with the results of Kobler and Rotics [27] and Oum [30] (see also [58]). One of these thirteen classes is one that we obtained in this paper (Theorem 2), namely the class of $(K_{1,3} + 3P_1, \overline{P_1 + P_3})$-free graphs, for which COLOURING was not previously known to be polynomial-time solvable. Note that Dabrowski, Lozin, Raman and Ries [18] already showed that COLOURING is polynomial-time solvable for $(\overline{P_1 + P_3}, P_1 + S_{1,1,2})$-free graphs, but in Theorem 1 we strengthened their result by showing that the clique-width of this class is also bounded.

Theorem 8 shows that there are 13 classes of (H_1, H_2)-free graphs (up to equivalence) for which we do not know whether their clique-width is bounded. These classes correspond to 28+6+4+1=39 distinct classes of (H_1, H_2)-free graphs. As can be readily verified from [58], the complexity of COLOURING is unknown for only 15 of these classes. We list these cases below:

1. $\overline{H_1} \in \{3P_1, P_1 + P_3\}$ and $H_2 \in \{P_1 + S_{1,1,3}, S_{1,2,3}\}$;
2. $H_1 = 2P_1 + P_2$ and $\overline{H_2} \in \{P_1 + P_2 + P_3, P_1 + 2P_2, P_1 + P_5\}$;
3. $H_1 = \overline{2P_1 + P_2}$ and $H_2 \in \{P_1 + P_2 + P_3, P_1 + 2P_2, P_1 + P_5\}$;
4. $H_1 = P_1 + P_4$ and $\overline{H_2} \in \{P_1 + 2P_2, P_2 + P_3\}$;
5. $\overline{H_1} = P_1 + P_4$ and $H_2 \in \{P_1 + 2P_2, P_2 + P_3\}$;
6. $H_1 = \overline{H_2} = 2P_1 + P_3$.

Note that Case 1 above reduces to two subcases by Lemma 6. All classes of (H_1, H_2)-free graphs, for which the complexity of COLOURING is still open and which are not listed above have unbounded clique-width. Hence, new techniques will need to be developed to deal with these classes.

8 Conclusions

We have determined for which pairs (H_1, H_2) the class of (H_1, H_2)-free graphs has bounded clique-width, and for which pairs (H_1, H_2) it has unbounded clique-width except for 13 non-equivalent cases, which we posed as open problems. We completely classified the (un)boundedness of the clique-width of those classes of graphs in which we forbid a finite family of graphs $\{H_1, \ldots, H_p\}$ as subgraphs, minors and topological minors, respectively. Finally, we showed the implications of our results for the complexity of the COLOURING problem restricted to (H_1, H_2)-free graphs. In particular we identified all 15 additional classes of (H_1, H_2)-free graphs for which COLOURING could potentially be solved in polynomial time if their clique-width turns out to be bounded.

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