On computational and combinatorial properties of the total co-independent domination number of graphs

Abel Cabrera Martínez⁽¹⁾, Frank A. Hernández Mira ⁽¹⁾ José M. Sigarreta Almira⁽¹⁾ and Ismael G. Yero⁽²⁾

 $^{(1)}$ Facultad de Matemáticas, Universidad Autónoma de Guerrero

Carlos E. Adame 5, Col. La Garita, Acapulco, Guerrero, Mexico

abecamar@qmail.com, fmira8906@qmail.com, josemariasiqarretaalmira@hotmail.com

⁽²⁾Departamento de Matemáticas, Escuela Politécnica Superior de Algeciras

Universidad de Cádiz, Av. Ramón Puyol s/n, 11202 Algeciras, Spain.

ismael.gonzalez@uca.es

Abstract

A subset D of vertices of a graph G is a total dominating set if every vertex of G is adjacent to at least one vertex of D. The total dominating set D is called a total coindependent dominating set if the subgraph induced by V - D is edgeless and has at least one vertex. The minimum cardinality of any total co-independent dominating set is the total co-independent domination number of G and is denoted by $\gamma_{t,coi}(G)$. In this work we study some complexity and combinatorial properties of $\gamma_{t,coi}(G)$. Specifically, we prove that deciding whether $\gamma_{t,coi}(G) \leq k$ for a given integer k is an NP-complete problem and give several bounds on $\gamma_{t,coi}(G)$. Also, since any total co-independent dominating set is also a total dominating set, we characterize all the trees having equal total co-independent domination number and total domination number.

Keywords: total co-independent domination; total domination; vertex independence; vertex cover.

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1 Introduction

Problems concerning domination in graphs are one of the most popular and highly investigated ones in the area of graph theory, and a rich literature in the topic is nowadays known in the research community. Such problems run from the theoretical point of view till several practical applications in real life situations. The most interesting cases of such applications are probably regarding problems in computer science. One of the most interesting features of domination in graphs involves the existence of a very high number of variants of domination parameters. A very common combination of domination is made with vertex independence of graphs, and the way of combining both concepts groups a considerable number of possibilities. In this work, we center our attention into precisely study one of these combinations between domination and independence in graphs, namely the total co-independence domination parameter. We focus the investigation on some computational complexity aspects of this parameter as well as on combinatorial properties of it.

Given a graph G with vertex set V(G) and edge set E(G), a set $D \,\subset V(G)$ is a *total* dominating set of G if every vertex in V(G) is adjacent to at least one vertex in D. The total domination number of G is the minimum cardinality of any total dominating set in G and is denoted by $\gamma_t(G)$. A $\gamma_t(G)$ -set is a total dominating set of cardinality $\gamma_t(G)$. For more information on total domination we suggest the recent and fairly complete survey [10] and the book [11]. A set S of vertices is independent if S induced an edgeless graph. An independent set of maximum cardinality is a maximum independent set of G. The independence number of G is the cardinality of a maximum independent set of G and is denoted by $\beta(G)$. An independent set of cardinality $\beta(G)$ is called a $\beta(G)$ -set. Relationships between (total) domination and independence in graphs have attracted the attention of several researchers in the last years. Several interesting connection among these parameters include independent dominating sets [2, 14], partitions into a dominating set and an independent set [13], (total) dominating sets which intersect every maximal independent set [1, 4, 9], and some other ones more, which we prefer to not mention here, since it is not the goal of this work.

A total dominating set D of a graph G is called a *total co-independent dominating set* (or TC-ID set for short) if the set of vertices of the subgraph induced by V - D is independent and not empty¹. The minimum cardinality of any TC-ID set is the *total co-independent domination number* of G and is denoted by $\gamma_{t,coi}(G)$. A TC-ID set of cardinality $\gamma_{t,coi}(G)$ is a $\gamma_{t,coi}(G)$ -set. These concepts were previously introduced and barely studied in [15]. Moreover, in [12], the same parameter was introduced under the name of total outer-independent domination number. Since this article ([12]) is not published in any journal and the other one ([15]) is already published, we precisely follow the terminology and notation of the latter. Since total domination is not defined for graphs having isolated vertices, all the graphs considered herein have not isolated vertices. Moreover, in order to satisfy the total domination property and that the complement of a TC-ID set is not empty, it is required that $2 \leq \gamma_{t,coi}(G) \leq n-1$, if n is the order of G. Such trivial bounds were already noted in the seminal work [15].

Throughout this work we consider G = (V, E) as a simple graph of order n and size m. That is, graphs that are finite, undirected, and without loops or multiple edges. Given a vertex vof G, $N_G(v)$ represents the open neighborhood of v, *i.e.*, the set of all neighbors of v in G and

¹Notice that the condition of V - D to be not empty is not exactly necessary. However, if such condition is not required, then we readily seen that the only graphs containing a TC-ID set of minimum cardinality with empty complement are the union of paths P_2 .

the degree of v is $\delta(v) = |N_G(v)|$. The minimum and maximum degrees of G are denoted by $\delta(G)$ and $\Delta(G)$, respectively (or δ and Δ , respectively, for short). If X and Y are two subsets of V(G), then we denote the set of all edges of G joining a vertex of X with a vertex of Y by E(X, Y). For a set $S \subset V(G)$, the complement of S is $\overline{S} = V(G) \setminus S$. In this work, we represent an edgeless graph G of order n as N_n . For any other graph theory terminology and notation we follow the book [11].

Let T be a tree (a connected graph without cycles). A *leaf* or a *pendant vertex* of T is a vertex of degree one (it is similarly defined for non tree graphs). A *support vertex* of T is a vertex adjacent to a leaf and a *semi-support vertex* is a vertex adjacent to a support vertex that is not a leaf. By an *isolated support vertex* of T we mean an isolated vertex of the subgraph induced by the support vertices of T. The set of leaves of T is denoted by L(T), the set of support vertices by S(T), and the set of semi-support vertices by SS(T). Moreover, $S^*(T)$ is the set of isolated support vertices of T.

We first notice that if H_1, H_2, \ldots, H_r with $r \ge 2$, are the connected components of a graph H, then any TC-ID set of minimum cardinality in H is formed by a minimum total dominating set in the subgraphs H_j where $|V(H_j)| = 2$ and a minimum TC-ID set in the remaining subgraphs H_i with $|V(H_i)| \ge 3$. That is stated in the following result.

Remark 1. Let H_1, H_2, \ldots, H_r with $r \ge 2$, be the connected components of a graph H different from the union of r copies of the path P_2 . Then

$$\gamma_{t,coi}(H) = \sum_{\substack{i \in \{1,\dots,r\} \\ |V(H_i)|=2}} \gamma_t(H_i) + \sum_{\substack{j \in \{1,\dots,r\} \\ |V(H_j)| \ge 3}} \gamma_{t,coi}(H_j).$$

Proof. Let D_j be a $\gamma_{t,coi}(H_j)$ -set for $j \in \{1, \ldots, r\}$ such that $|V(H_j)| \geq 3$. It is easy see that $(\bigcup_{\substack{i \in \{1,\ldots,r\}\\|V(H_i)|=2}} V(H_i)) \cup (\bigcup_{\substack{j \in \{1,\ldots,r\}\\|V(H_j)|\geq 3}} D_j)$ is a TC-ID set of H and we have

$$\gamma_{t,coi}(H) \le \sum_{\substack{i \in \{1,\dots,r\} \\ |V(H_i)|=2}} \gamma_t(H_i) + \sum_{\substack{j \in \{1,\dots,r\} \\ |V(H_j)| \ge 3}} \gamma_{t,coi}(H_j).$$

On the other hand, let A be a $\gamma_{t,coi}(H)$. Firstly, we observe that for every $i \in \{1, \ldots, r\}$ such that $|V(H_i)| = 2$, it is satisfied that $A \cap V(H_i) = V(H_i)$. Moreover, let $A_j = A \cap V(H_j)$ for every $j \in \{1, \ldots, r\}$ such that $|V(H_j)| \geq 3$. We notice that every A_j must be a TC-ID set of H_j . In this sense,

$$\gamma_{t,coi}(H) = |A| = \sum_{\substack{i \in \{1,\dots,r\} \\ |V(H_i)|=2}} |V(H_i)| + \sum_{\substack{j \in \{1,\dots,r\} \\ |V(H_j)|\geq 3}} |A_j|$$

$$\geq \sum_{\substack{i \in \{1,\dots,r\} \\ |V(H_i)|=2}} \gamma_t(H_i) + \sum_{\substack{j \in \{1,\dots,r\} \\ |V(H_j)|\geq 3}} \gamma_{t,coi}(H_j),$$

which completes the proof.

In concordance with the result above, from now on, we only consider the study of the TC-ID sets of connected graphs and omit to refer to that fact throughout all our exposition.

2 Complexity of the decision problem

We begin our exposition by considering the problem of deciding whether the total co-independent domination number of a graph is less than a given integer. That is stated in the following decision problem.

TOTAL CO-INDEPENDENT DOMINATION PROBLEM INSTANCE: A non trivial graph G and a positive integer rPROBLEM: Deciding whether $\gamma_{t,coi}(G)$ is less than r

In order to deal with the complexity of the TOTAL CO-INDEPENDENT DOMINATION PROBLEM (TC-ID PROBLEM), we make a reduction from a very well known decision problem concerning the independence number of graphs.

MAXIMAL INDEPENDENT SET PROBLEM INSTANCE: A non trivial graph G and a positive integer rPROBLEM: Deciding whether the independence number of G is larger than r

The problem above is one of the classical NP-complete problems appearing in the book [8]. Moreover, it remains NP-complete even when restricted to planar graphs.

Theorem 2. [8] MAXIMAL INDEPENDENT SET PROBLEM is NP-complete even when restricted to planar graphs of maximum degree at most 3.

Now on, in order to present our complexity results we need to introduce a family of graphs which is next defined. Let T_6 be a tree with six vertices having two adjacent vertices u, v of degree three and the other four u_1, u_2, v_1, v_2 vertices are leaves. Clearly, each vertex of degree three has two adjacent leaves, say $u_1, u_2 \in N(u)$ and $v_1, v_2 \in N(v)$ (see Figure 1 (I)). Given a graph Gof order n and n trees $T_6^{(1)}, \ldots, T_6^{(n)}$ isomorphic to the tree T_6 , the graph G_T is constructed by adding edges between the i^{th} -vertex of G and the vertex u of the i^{th} -tree $T_6^{(i)}$. See Figure 1 (II) for an example.

We are now able to prove the NP-completeness of the TC-ID PROBLEM.

Theorem 3. TOTAL CO-INDEPENDENT DOMINATION PROBLEM is NP-complete even when restricted to planar graphs of maximum degree at most 3.

Proof. The problem is clearly in NP since verifying that a given set is indeed a TC-ID set can be done in polynomial time. Let us now make a reduction from the MAXIMAL INDEPENDENT SET PROBLEM. Let G be a not edgeless graph of order n and construct the graph G_T as



Figure 1: The graph T_6 (I) and a graph G_T (II) where G is a complete graph minus one edge.

described above. Let us denote by $u^{(i)}, v^{(i)}$ the vertices of degree three in the i^{th} copy $T_6^{(i)}$ of the tree T_6 used to generate G_T . We shall prove that $\gamma_{t,coi}(G_T) = 3n - \beta(G)$.

Let A be a $\beta(G)$ -set and let D be the set of vertices of G_T obtained from the complement of A in G together with the vertices u, v belonging to all the copies of the tree T_6 used to generate G_T , that is $D = (V(G) \setminus A) \cup \{\bigcup_{i=1}^n \{u^{(i)}, v^{(i)}\}\}$. This set is clearly a total dominating set and its complement is an independent set. Thus, D is a TC-ID set in G_T and, as a consequence,

$$\gamma_{t,coi}(G_T) \le n - |A| + \left| \bigcup_{i=1}^n \{u^{(i)}, v^{(i)}\} \right| = 3n - \beta(G).$$

On the other hand, let D' be a $\gamma_{t,coi}(G_T)$ -set. In order to totally dominate the leaves of every copy of T_6 in G_T , it must happen that $|D' \cap V(T_6^{(i)})| \ge 2$ for every $i \in \{1, \ldots, n\}$. Moreover, $V(G) \cap D' \neq \emptyset$, since otherwise the complement of D' would not be independent. Moreover the complement of $V(G) \cap D'$ in G is an independent set in G. Thus, $\beta(G) \ge n - |V(G) \cap D'|$ and we obtain the following.

$$\gamma_{t,coi}(G_T) = |D'| = |D' \cap V(G)| + \left| \bigcup_{i=1}^n \left(D' \cap V(T_6^{(i)}) \right) \right| \ge n - \beta(G) + 2n = 3n - \beta(G).$$

As a consequence, it follows that $\gamma_{t,coi}(G_T) = 3n - \beta(G)$.

Now, for j = 3n - k, it is readily seen that $\gamma_{t,coi}(G_T) \leq j$ if and only if $\beta(G) \geq k$, which complete the reduction. We also observe that, if G is a planar graph, then G_T is also planar. Therefore, since the MAXIMAL INDEPENDENT SET PROBLEM is NP-complete even when restricted to planar graphs of maximum degree at most 3, we also deduce that the TC-ID PROBLEM is NP-complete even when restricted to planar graphs of maximum degree at most 3 and the proof is completed.

As a consequence of the result above, we deduce the following consequence.

Corollary 4. The problem of computing the total co-independent domination number of graphs is NP-hard even when restricted to planar graphs of maximum degree at most 3.

3 Bounding the total co-independent domination number

In order to present the first bounds for $\gamma_{t,coi}(G)$ of any graph G, we need the next concepts. A set S of vertices of G is a vertex cover of G if every edge of G is incident with at least one vertex of S. The vertex cover number of G, denoted by $\alpha(G)$, is the smallest cardinality of a vertex cover of G. We refer to an $\alpha(G)$ -set in G as a vertex cover of cardinality $\alpha(G)$. The following well-known result, due to Gallai [7], states the relationship between the independence number and the vertex cover number of a graph.

Theorem 5. [7](Gallai, 1959) For any graph G of order n, $\alpha(G) + \beta(G) = n$.

On the other hand, it was shown in [15] the following relationship between $\gamma_{t,coi}(G)$ and $\beta(G)$.

Theorem 6. [15] For any graph G of order n, $\gamma_{t,coi}(G) \ge n - \beta(G)$.

By using the two theorems above, we can easily deduce the lower bound of our next result. However, an upper bound for $\gamma_{t,coi}(G)$ in terms of the vertex cover number can also be deduced. We first consider the case whether G is a star graph S_n for which is known that $\gamma_{t,coi}(S_n) = 2$ and $\alpha(S_n) = 1$.

Remark 7. For any star graph S_n , $\gamma_{t,coi}(S_n) = 2 = 2\alpha(S_n)$.

In concordance with the remark above, for our next result we exclude the case of star graphs and see that they behave in a different manner.

Theorem 8. For any graph G of order n different from a star graph,

$$\alpha(G) \le \gamma_{t,coi}(G) \le 2\alpha(G) - 1.$$

Proof. The lower bound follows from Theorems 5 and 6. If $\alpha(G) \ge n/2$, then $\gamma_{t,coi}(G) \le n-1 \le 2(n/2) - 1 \le 2\alpha(G) - 1$. Thus, from now on in this proof we consider $\alpha(G) < n/2$. Now, let C be an $\alpha(G)$ -set.

We choose two vertices $u, v \in C$ with the minimum possible distance between them and let P be a shortest u - v path. Clearly, $V(P) \cap C = \{u, v\}$ and the distance between u and v is one or two (notice this also means $2 \leq |V(P)| \leq 3$). For each vertex $x \in C - \{u, v\}$, choose a neighbor x' of x. Then $C \cup V(P) \cup \{x' : x \in C - \{u, v\}\}$ is a TC-ID set of cardinality $2|C| + |V(P)| - 4 \leq 2\alpha(G) - 1$, which completes the proof of the upper bound. \Box

The bounds above are tight. For instance, a characterization of that trees achieving the equality in the lower bound was given in [3] (note that in [3] the trees T of order n satisfying equality in the bound $\gamma_{t,coi}(T) = n - \beta(T)$ were characterized, which equals the lower bound of Theorem 8, in concordance with Theorem 5). The upper bound is attained for an infinite family of graphs, as we next show. To this end, we need the following operations for edges or induced paths P_3 of a graph G.

Subdivision: Given an edge uv, remove the edge, add a vertex w and the edges uw, wv.

Inflation of size k: Given an induced path $P_3 = uvw$ of G, in which v has degree two, remove the vertex v and the two incident edges, and replace them with k vertices v_1, v_2, \ldots, v_k and edges uv_i, v_iw for every $i \in \{1, \ldots, k\}$.

Addition of t pendant vertices: Given a vertex x add t new vertices y_1, \ldots, y_t and the edges xy_i for every $i \in \{1, \ldots, t\}$.

Now, a graph $H_{n,a,b} \in \mathcal{F}_1$ is a graph obtained from a star graph S_n by making the following sequence of operations, which we will call as **Sequence I**.

- (a) Apply the operation "Subdivision" to a $(1 \le a \le n)$ edges of S_n .
- (b) Apply the operation "Inflation of size k_i " with $k_i \ge 2$ to $b \ (0 \le b \le a)$ paths $P_3^{(i)}$ obtained from (a).
- (c) Apply the operation "Addition of q_i pendant vertices", $q_i \ge 0$, to the *b* vertices corresponding to leaves of S_n obtained in the step (b).
- (d) Apply the operation "Addition of t_i pendant vertices", $t_i \ge 1$, to the leaves v_i belonging to the remaining a b paths obtained from (a), which were not "inflated" in (b).
- (e) If a = n and b = 0 (notice that in this case $H_{n,a,b}$ is a tree such that the central vertex of the original star graph S_n has no adjacent leaves), then apply the operation "Addition of t pendant vertices", $t \ge 1$, to the vertex corresponding to the central vertex of S_n .
- (f) If a = n and b > 0, then apply the operation "Addition of t pendant vertices", $t \ge 0$, to the vertex corresponding to the central vertex of S_n .

As an example, to obtain the cycle C_4 (which belongs to \mathcal{F}_1) we begin with the star S_1 (a path P_2), next we apply the operation "Subdivision" to the unique edge of S_1 and then we apply the operation "Inflation of size 2" to the path P_3 obtained in the previous step. Note that different sequences of operations would lead to the same graph. For instance, the graph P_5 can be obtained from the star S_1 by subdividing its unique edge and then adding a pendant vertex to the leaf corresponding to the subdivision, as well as another pendant vertex to the center of S_1 (coincidentally such center is also a leaf). Moreover, the graph P_5 is obtained from the star P_3 by subdividing a pendant vertex to the leaf corresponding to such subdivision. On the other hand, we remark that three integers n, a, b would produce different graphs $H_{n,a,b}$ depending on the addition of pendant vertices that would be done. However, since it is not significant for our work to denote them, we skip to use the notations for the addition of pendant vertices. A fairly representative graph of the family \mathcal{F}_1 is given in Figure 2.

Remark 9. For any graph $H_{n,a,b} \in \mathcal{F}_1$, $\alpha(H_{n,a,b}) = a + 1$ and $\gamma_{t,coi}(H_{n,a,b}) = 2a + 1$.

Proof. For any edge of S_n which was subdivided in step (a), it appears either a path P_4 or a cycle C_4 and all these subgraphs have in common only one vertex (the corresponding one to the center of S_n). Thus, in order to cover all the edges of $H_{n,a,b}$, at least a + 1 vertices are required.



Figure 2: A graph $H_{5,5,3} \in \mathcal{F}_1$ where the six bolded vertices form an $\alpha(H)$ -set and gray vertices form a possible set to be added to the bolded vertices to get a $\gamma_{t,coi}(H)$ -set, which has cardinality eleven.

Moreover, a set given by those a leaves corresponding to the a edges of the star S_n which were subdivided together with the central vertex form a vertex cover of cardinality a + 1. Thus, the equality $\alpha(H_{n,a,b}) = a + 1$ follows.

Now, let D be a $\gamma_{t,coi}(H_{n,a,b})$ -set. We analyze the following situations for every edge wu (assume w is the center of S_n) of the star which is initially subdivided.

Case 1: There is only one path between w and u in $H_{n,a,b}$. Hence, the edge wu was subdivided with a vertex, say v, and not inflated, which made a required addition of at least one pendant vertex, say u', to the leaf u. Thus, in order to totally dominate u', $|D \cap \{v, u, u'\}| \ge 2$.

Case 2: There are at least two paths between w and u in $H_{n,a,b}$. Clearly, this means wu was subdivided and then inflated with at least two vertices, say $v_1, \ldots, v_r, r \ge 2$. Moreover, probably some pendant vertices were added to u. So, in order to totally dominate v_1, \ldots, v_r, u (and probably other extra leaves adjacent to u), at least two vertices of v_1, \ldots, v_r, u are required.

We next consider the vertex w separately. If a < n, then the vertex w has a least one adjacent leaf which needs to be totally dominated. Thus, w must belong to D. On the contrary, if a = n, then we must consider the value b. If b = 0, then no path P_3 was inflated $(H_{n,a,b}$ is a tree) and so, by step (e), w has at least one adjacent leaf which needs to be totally dominated, which means w must belong to D again. Finally, we assume b > 0. Thus, at least one path P_3 was inflated and there is a cycle $C_4^{(j)}$ to which w belongs. Also, it may happen w has no adjacent leaves. Now, note that if $w \notin D$, then the two vertices of $C_4^{(j)}$ adjacent to w must belong to D, since \overline{D} is an independent set. Moreover, the fourth vertex of $C_4^{(j)}$ must belong to D too, in order to get the vertices of D totally dominated. As a consequence, at least three vertices of the cycle are in D, which is equivalent to have in D the vertex w, one of its neighbors in $C_4^{(j)}$ and the vertex of $C_4^{(j)}$ which is not adjacent to w. Consequently, we can deduce that for any set of vertices of a subgraph of $H_{n,a,b}$, induced by the vertices obtained in a subdivision of one of the *a* leaves of S_n and probably the corresponding addition of some pendant vertices, at least two of these vertices are in *D*. Moreover, one extra vertex is required, which could mainly be the central vertex *w* of S_n . Thus, $\gamma_{t,coi}(H_{n,a,b}) = |D| \ge 2a + 1$.

On the other hand, by using Theorem 8, we obtain that $\gamma_{t,coi}(H_{n,a,b}) \leq 2\alpha(H_{n,a,b}) - 1 = 2a + 1$ and the equality follows for $\gamma_{t,coi}(H_{n,a,b})$.

Now, a graph $H \in \mathcal{F}_2$ is a graph obtained from the cycle C_6 by making the following sequence of operations, which we will call as **Sequence II**.

- (a) Apply the operation "Addition of t_i pendant vertices", $t_i \ge 0$ and $i \in \{1, 2, 3\}$, to the three vertices, say v_1, v_2, v_3 , of a $\beta(C_6)$ -set, respectively.
- (b) If there is a $t_i = 0$ from the above operation and the degree of v_i is two, then apply the operation "Inflation of size k" with $k \ge 2$ to one of the two possible paths of order three between v_i and the other two vertices in $\{v_1, v_2, v_3\} \{v_i\}$.
- (c) Apply the operation "Inflation of size k_i " with $k_i \ge 1$ and $i \in \{1, 2, 3\}$ to the three possible paths of order three between v_1, v_2, v_3 .

An example of a graph of the family \mathcal{F}_2 appears in Figure 3.



Figure 3: A graph $H \in \mathcal{F}_2$ where the three bolded vertices form an $\alpha(H)$ -set and the two gray vertices form a possible set to be added to the bolded vertices to get a $\gamma_{t,coi}(H)$ -set, which has cardinality five.

The following result concerning the values of $\alpha(H)$ and $\gamma_{t,coi}(H)$ for graphs $H \in \mathcal{F}_2$ is straightforward to observe.

Remark 10. For any graph $H \in \mathcal{F}_2$, $\alpha(H) = 3$ and $\gamma_{t,coi}(H) = 5$.

According to the Remarks above, we can easily check that the upper bound of Theorem 8 is achieved for any graph $G \in \mathcal{F}_1 \cup \mathcal{F}_2$. Moreover, we next prove that precisely the graphs of these families are the only ones achieving the upper bound of Theorem 8. To this end, we need the following two lemmas whose proofs can be made by using some similar techniques as in the proof of Theorem 8.

Lemma 11. If a graph G contains an $\alpha(G)$ -set which is not independent, then $\gamma_{t,coi}(G) \leq 2\alpha(G) - 2$.

Proof. Let C be an $\alpha(G)$ -set which is not independent. We choose two adjacent vertices $u, v \in C$. For each vertex $x \in C - \{u, v\}$, choose a neighbor x' of x. Then $C \cup \{x' : x \in C - \{u, v\}\}$ is a TC-ID set of cardinality $2(|C|-2) + 2 \leq 2\alpha(G) - 2$.

Lemma 12. If an $\alpha(G)$ -set of a graph G contains four different vertices u_1, u_2, v_1, v_2 such that a shortest $u_1 - u_2$ path and a shortest $v_1 - v_2$ path have length two and are vertex disjoint, then $\gamma_{t,coi}(G) \leq 2\alpha(G) - 2$.

Proof. Let C be an $\alpha(G)$ -set such that $u_1, u_2, v_1, v_2 \in C$. For each vertex $x \in C - \{u_1, u_2, v_1, v_2\}$, choose a neighbor x' of x and let $w_1, w_2 \notin C$ be two vertices such that $w_1 \in N(u_1) \cap N(u_2)$ and $w_2 \in N(v_1) \cap N(v_2)$, which exist by assumption. Then $C \cup \{w_1, w_2\} \cup \{x' : x \in C - \{u_1, u_2, v_1, v_2\}\}$ is a TC-ID set of cardinality $2(|C| - 4) + 6 \leq 2\alpha(G) - 2$.

Theorem 13. Let G be a graph of order n such that $2\alpha(G) \leq n$. Then $\gamma_{t,coi}(G) = 2\alpha(G) - 1$ if and only if $G \in \mathcal{F}_1 \cup \mathcal{F}_2$.

Proof. In one hand, if $G \in \mathcal{F}_1 \cup \mathcal{F}_2$, then it clearly happens that $\gamma_{t,coi}(G) = 2\alpha(G) - 1$ according to Remarks 9 and 10.

On the second hand, assume $\gamma_{t,coi}(G) = 2\alpha(G) - 1$ and let D be any $\alpha(G)$ -set. We first notice that D must induce an independent set according to Lemma 11. We shall now proceed by proving some partial claims that will further give our required conclusion.

Claim 1: G has no triangles (cycles of order three).

Proof of Claim 1: If there is a triangle, then, in order to cover all its edges, at least two of its vertices must belong to D. So, this cover is not an independent set. Thus we get a contradiction by using Lemma 11. (\Box)

Claim 2: G has no induced cycles of order five or larger than six.

Proof of Claim 2: Suppose G contains a cycle C_r with r = 5 or $r \ge 7$. In order to cover all the edges of C_r , and since $r \ne 6$, there must be two vertices in $D \cap V(C_r)$ at distance one (which means D is not independent), or there are four different vertices $u_1, u_2, v_1, v_2 \in D \cap V(C_r)$ such that a shortest $u_1 - u_2$ path and a shortest $v_1 - v_2$ path have length two and are vertex disjoint. Thus, we obtain contradictions by using Lemmas 11 and 12. (\Box)

As a consequence of the Claims above, we have that G can only contain cycles of order four or six. We first analyze the case in which G contains a cycle of order six. Let $V(C_6) = \{v_1, \ldots, v_6\}$ where $v_1 \sim v_2 \sim \cdots \sim v_6 \sim v_1$ ($u \sim v$ means u, v are adjacent). According to Lemma 11, it must happen $D \cap V(C_6)$ is independent. Thus, without loss of generality we assume $D \cap V(C_6) = \{v_1, v_3, v_5\}$. We consider now several situations.

Suppose v_2 has degree larger than two. If $v_2 \sim v_j$ with $j \in \{4, 6\}$, then G has a triangle, which is not possible. If $v_2 \sim v_5$, then for each vertex $x \in C - \{v_1, v_3, v_5\}$, choose a neighbor x'of x and we observe that the set $D \cup \{v_2\} \cup \{x' : x \in D - \{v_1, v_3, v_5\}$ is a TC-ID set of cardinality $2(|D|-3)+4 \leq 2\alpha(G)-2$, a contradiction. Thus, v_2 has a neighbor $z \notin V(C_6)$. Since $v_2 \notin D$, it must happen $z \in D$. Since $z \not\sim v_1$ and $z \not\sim v_3$ (otherwise there would be a triangle), we obtain a contradiction with Lemma 12 by using the vertices z, v_1, v_3, v_5 . As a consequence, v_2 must have degree two, and by symmetry also v_4, v_6 are of degree two too.

Suppose v_1 has degree two. Hence, for each vertex $x \in C - \{v_1, v_3, v_5\}$, we choose a neighbor x' of x and observe that the set $(D - \{v_1\}) \cup \{v_2, v_6\} \cup \{x' : x \in D - \{v_1, v_3, v_5\}$ is a TC-ID set of cardinality $2(|D| - 3) + 4 \leq 2\alpha(G) - 2$, a contradiction. So, v_1 must have degree at least three and, by symmetry also v_3, v_5 are of degree at least three too.

We consider now a vertex $x \in N(v_1) - \{v_2, v_6\}$. Notice that $x \neq v_3, v_5$ (otherwise there would be a triangle). Also, $x \neq v_4$, by using the same idea as before whether v_2 has degree larger than two and $v_2 \sim v_5$. Suppose x has degree larger than one and let $x' \in N(x) - \{v_1\}$. Since D is independent and the edge xx' must be covered by D, $x' \in D$. If $x' \neq v_3$ and $x' \neq v_5$, then we obtain a contradiction with Lemma 12 by using the vertices x', v_1, v_3, v_5 (notice that $x' \not\sim v_1$ since D is independent). As a consequence, we obtain that any neighbor x of v_1 is either of degree one or has a neighbor in $V(C_6) - \{v_1\}$.

We next consider the latter situation whether $x' \in V(C_6) - \{v_1\}$. Clearly $x' \neq v_2, v_4, v_6$. Suppose x is neighbor of v_3 and of v_5 . We choose a neighbor y' of y for every $y \in D - \{v_1, v_3, v_5\}$ and observe that the set $D \cup \{x\} \cup \{y' : y \in D - \{v_1, v_3, v_5\}$ is a TC-ID set of cardinality $2(|D| - 3) + 4 \leq 2\alpha(G) - 2$, a contradiction. Thus, x is a neighbor of either v_3 or v_5 , in which case, it happens x has degree two. By symmetry, we obtain similar conclusions for v_3 and v_5 . That is, for any v_i with $i \in \{1, 3, 5\}$, $N(v_i)$ is given by leaves or vertices of degree two. In the latter case, if $x \in N(v_i) - V(C_6)$, then $N(x) = \{v_i, y\}$ where $y \in D \cap V(C_6) - \{v_i\}$.

As a consequence, we observe that G can be obtained from a cycle C_6 by the Sequence II of operations described above, or equivalently $G \in \mathcal{F}_2$. We may now consider the case in which G contains a cycle C_4 , but G does not contain the cycle C_6 .

Claim 3: G does not contain vertex disjoint cycles.

Proof of Claim 3: We directly obtain a contradiction by Lemma 12, since in this case there are four different vertices u_1, u_2, v_1, v_2 (two of them in one cycle, the other two in the other cycle) such that a shortest $u_1 - u_2$ path and a shortest $v_1 - v_2$ path have length two and are vertex disjoint. (\Box)

Thus, if G contains more than one cycle C_4 , then they are not vertex disjoint. Moreover, we

can next see that not two adjacent vertices of a cycle can be in any other cycle.

Claim 4: If two cycles C_4 of G has exactly two vertices in common, then these vertices are not adjacent.

Proof of Claim 4: Suppose there are two cycles C_4 having two adjacent vertices in common. Assume the cycles are $C_4^{(1)} = v_1 v_2 v_3 v_4 v_1$ and $C_4^{(2)} = v_1 v_2 v_5 v_6 v_1$. Hence, we note that exactly three vertices of $\{v_1, \ldots, v_6\}$ must belong to D, otherwise there are two adjacent vertices in D. Indeed, such vertices are either v_1, v_3, v_5 or v_2, v_4, v_6 , say for instance v_1, v_3, v_5 . We choose a neighbor x' of x for every $x \in D - \{v_1, v_3, v_5\}$ and observe that the set $D \cup \{v_2\} \cup \{x' : x \in D - \{v_1, v_3, v_5\}$ is a TC-ID set of cardinality $2(|D| - 3) + 4 \leq 2\alpha(G) - 2$, which is a contradiction. (\Box)

Now, according to the Claims above, if G contains more than one cycle C_4 , then only the following situations can occur.

- Any two cycles have exactly one vertex in common.
- Any two cycles have exactly two vertices in common which are not adjacent.
- Any two cycles have exactly three vertices in common.

We note that the situation in which two cycles of G have exactly three vertices in common can be understood as G has three cycles with two vertices in common. We now turn our attention on the following.

Claim 5: There is a vertex $w \in D$ such that d(w, x) = 2 for every $x \in D - \{w\}$.

Proof of Claim 5: We first note that there are at least two vertices $w, x \in D$ such that d(x, w) = 2, otherwise there would be an edge not covered by D. Let h be a vertex adjacent to w and x. Suppose there is a vertex $y \in D$ such that $d(w, y) \neq 2$ and $d(x, y) \neq 2$ (note that $d(w, y) \neq 1$ and $d(x, y) \neq 1$). Thus, since there are no cycles of order larger than four in G, there must happen one of the following situations.

- (a) There is a shortest path joining y and h not containing w nor x. Also, y is different from the neighbor of h, say h', in such path. In such case, in order to cover the edge hh', it must happen $h' \in D$. Thus, we obtain a contradiction by using Lemma 12 and the vertices h', w, x, z where z is a vertex at distance two from x in the x y path.
- (b) Without loss of generality, there is a shortest path joining y and x containing w. Thus, there must be a vertex $y' \in D$ belonging to this path such that d(y, y') = 2 (it cannot be d(y, y') = 1 since D is independent), otherwise there should be a not covered edge. Clearly $w \neq y'$. Thus, we obtain a contradiction by using Lemma 12 and the vertices y, y', w, x.

As a consequence, the vertex y has distance two to x or to w. Moreover, if d(w, y) = 2 and d(x, y) = 2, then we there is a cycle of order six, which is not possible. Thus, y has distance two to exactly one vertex of x and w. From now on, we assume d(y, w) = 2.

We next prove that for any vertex $z \in D - \{x, y, w\}$, it follows d(z, w) = 2 too. If $D = \{x, y, w\}$, then we are done. So, me may suppose there is a vertex $z' \in D - \{x, y, w\}$ such that $d(z', w) \neq 2$ (clearly d(z', w) > 2). Consider now the shortest path between z' and w, say $z'z'_1z'_2...z'_qw$. Notice that $q \geq 2$. In order to cover the edge $z'_1z'_2$, it must be $z'_2 \in D$. So, we obtain a contradiction by using Lemma 12 and the vertices x, w, z', z'_2 . Therefore, for any vertex $z \in D - \{w\}$, we obtain that d(z, w) = 2 and the claim is proved. (\Box)

Next step gives some result on the distances between any two vertices $x, y \in D - \{w\}$.

Claim 6: For any two vertices $x, y \in D - \{w\}$, any shortest path between x and y passes through w.

Proof of Claim 6: From Claim 5, we know that d(x, w) = d(y, w) = 2. Thus $d(x, y) \le 4$. Clearly d(x, y) > 1, since x, y cannot be adjacent. Let $x', y' \in N(w)$ such that $x' \in N(x)$ and $y' \in N(y)$. If $x \sim y'$ or $y \sim x'$ (say $x \sim y'$), then we choose a neighbor z' of z for every $z \in D - \{w, x, y\}$ and observe that the set $D \cup \{y'\} \cup \{z' : z \in D - \{w, x, y\}$ is a TC-ID set of cardinality $2(|D| - 3) + 4 \le 2\alpha(G) - 2$, which is a contradiction. Thus, neither $x \sim y'$ nor $y \sim x'$. If there is a vertex $z \in N(x) \cap N(y)$, then wx'xzyy'w is a cycle C_6 in G, which is not possible. Thus $d(x, y) \neq 2$. By using a similar reasoning, it can be deduced that $d(x, y) \neq 3$ and so, d(x, y) = 4. If there is another path of length four between x and y not containing w, then we have one of the following situations.

- There is a vertex $w' \in D$ such that $x', y' \in N(w')$ (note that w' must be in D in order to cover the edges w'y', w'x'). In such case, we obtain a contradiction by using Lemma 12 and the vertices x, w, w', y.
- There are three vertices $x_1, y_1, w'' \neq x', y', w$ such that $x_1 \in N(x), y_1 \in N(y)$ and $x_1, y_1 \in N(w'')$. In such situation, $wx'xx_1w''y_1yy'w$ is an induced cycle of order eight in G, which is not possible.
- Similarly to the case above, if either $x_1 = x'$ or $y_1 = y'$, then we obtain an induced cycle of order six in G, which is also not possible.

Therefore, any shortest path between x and y passes throughout w. (\Box)

We now give several facts which are consequences of the Claims above, in order to deduce the structure of the graph G.

- The set V(G) D is independent (otherwise there is an edge not covered by D).
- If $x, y \in D \{w\}$, then $N(x) \cap N(y) = \emptyset$.
- If $z \in N(x)$ for some $x \in D \{w\}$, then either $z \in N(w)$ and z has degree two, or z is a vertex of degree one.

• If $z' \in N(w)$ is not a vertex of degree one, then there is exactly one vertex $x \in D$ such that $N(z') = \{w, x\}$ (equivalently z' has degree two).

As a consequence of the items above, as well as from the Claims, and all the reasoning till this point, we observe that D is formed by w and a set of vertices $v_1, v_2 \ldots, v_r$ (satisfying the properties above). Clearly, for any vertex v_i , the set of its neighbors are either leaves or vertices of degree two adjacent to w. Moreover, if v_i has only one neighbor of degree two, then it must have at least one adjacent leaf (otherwise one can find a cover set of smaller cardinality). In this sense, such set of vertices can clearly be obtained from a leaf of a star by making a subdivision of the corresponding edge, an inflation of the path P_3 obtained from the subdivision and a subsequent addition of some pendant vertices. On the other hand, if w has some adjacent leaves, then they could be obtained directly from a star, if subdivisions were not done to all the leaves of the star or, by a subsequent addition of leaves to the center of the original star, if all its leaves would have been subdivided. Therefore, it is then concluded that the graph G was obtained from a star by making the Sequence I of operations previously described, which means $G \in \mathcal{F}_1$ and the proof is completed.

We close this section with two bounds for $\gamma_{t,coi}(G)$ in terms of order, size and minimum and maximum degrees.

Proposition 14. Let G be a graph of order n, minimum and maximum degrees δ and Δ , respectively. Then $\gamma_{t,coi}(G) \geq \frac{n\delta}{\Delta + \delta - 1}$.

Proof. Let D be a $\gamma_{t,coi}(G)$ -set. Hence, the subgraph induced by V(G) - D is edgeless. So, $(n - |D|)\delta = (|V(G) - D|)\delta \leq E(V(G) - D, D) \leq |D|(\Delta - 1)$. Furthermore, it follows that $\gamma_{t,coi}(G) \geq \frac{n\delta}{\Delta + \delta - 1}$.

Proposition 15. Let G be a graph of order n, size m, minimum and maximum degrees δ and Δ , respectively. Then $\gamma_{t,coi}(G) \geq \frac{2m+n\delta}{3\Delta+\delta-2}$.

Proof. Let D be a $\gamma_{t,coi}(G)$ -set. Hence, the subgraph induced by V(G) - D is edgeless. So, E(V(G) - D, D) + E(D, D) = m. Now, notice that $E(V(G) - D, D) \leq |D|(\Delta - 1)$ and $E(D, D) \leq \frac{|D|\Delta - (n-|D|)\delta}{2}$. Adding this inequations, we have $m = E(V(G) - D, D) + E(D, D) \leq \frac{|D|\Delta - (n-|D|)\delta}{2} + |D|(\Delta - 1)$. Therefore, it follows that $\gamma_{t,coi}(G) \geq \frac{2m + n\delta}{3\Delta + \delta - 2}$.

The two bounds above are attained for instance for the double stars $S_{k,k}$ (each non leaf vertex is adjacent to k leaves), which has order 2(k + 1), size m = 2k + 1, minimum degree $\delta = 1$, maximum degree $\Delta = k + 1$ and $\gamma_{t,coi}(S_{k,k}) = 2$.

4 The case of trees

In order to easily proceed with our exposition, and based on the following known bound, from now on we say that a tree T belongs to the family \mathcal{T}_{γ_t} , if $\gamma_{t,coi}(T) = \gamma_t(T)$. Moreover, we assume in this section that $|S(T)| \ge 2$, since the case |S(T)| = 0 (T is a P_2 and $\gamma_{t,coi}(T)$ is not defined) and |S(T)| = 1 (T is a star graph S_n and $\gamma_{t,coi}(T) = 2$) are straightforward to study. **Theorem 16.** [15] For any graph G, $\gamma_{t,coi}(G) \geq \gamma_t(G)$.

It is now our goal to characterize the family of trees achieving the equality in the bound above. To this end, we observe the following basic results, which can easily be obtained by using some known properties of minimum total dominating sets.

Proposition 17. [6] If S is a minimal total dominating set of a connected graph G = (V, E), then each $v \in S$ has at least one of the following two properties.

- (i) There exists a vertex $w \in V S$ such that $N(w) \cap S = \{v\}$.
- (ii) The subgraph induced by $S \{v\}$ contains an isolated vertex.

The next remark is one useful consequence of the proposition above.

Remark 18. Let D be a $\gamma_{t,coi}(T)$ -set of cardinality $\gamma_t(T)$. Then, for every $v \in D$, at least one of the following conditions is satisfied.

- (i) There exists a vertex $u \in D$ such that $N(u) \cap D = \{v\}$.
- (ii) There exists a vertex $w \in V D$ such that $N(w) \cap D = \{v\}$.

We may recall to notice that condition (ii) implies that vertex v is a support, because the set \overline{D} is independent.

Lemma 19. Let $T \in \mathcal{T}_{\gamma_t}$ and let D be a $\gamma_{t,coi}(T)$ -set containing no leaves. Then for every $v \in V(T) - (D \cup L(T))$ there exist a leaf h such that $d(v, h) \leq 3$.

Proof. Let $v \in V(T) - (D \cup L(T))$. Since $|N(v)| \ge 2$, we consider $N(v) = \{v_1, v_2, \ldots, v_r\}$ with $r \ge 2$. Clearly, $N(v) \subset D$ since \overline{D} is independent. For every v_i , with $i \in \{1, \ldots, r\}$, by Remark 18, v_i is adjacent to a leaf or there exist a vertex $s_i \in D$ such that $N(s_i) \cap D = \{v_i\}$. Hence, as $s_i \in D$, $N(s_i) \subset V(T) - D$. We assume that for every $i \in \{1, \ldots, r\}$, v_i is not adjacent to a leaf h, otherwise d(v, h) = 2. Now, we suppose that $(N(s_i) - \{v_i\}) \cap L(T) = \emptyset$. Also note that, by condition above, the vertices belonging to $N(s_i)$ are totally dominated by other vertices of D. So, we observe that the set $(D - \{s_1, s_2, \ldots, s_r\}) \cup \{v\}$ is a total dominating set of T of cardinality smaller than |D|, a contradiction. Furthermore, there exist $i \in \{1, \ldots, r\}$ such that $(N(s_i) - \{v_i\}) \cap L(T) \neq \emptyset$. Thus, for any $h \in (N(s_i) - \{v_i\}) \cap L(T)$, it follows d(v, h) = 3, and this completes the proof.

From this point, the set of leaves having distance three with respect to at least one other leaf is denoted by $L_3(T)$, and given a $\gamma_{t,coi}(T)$ -set D, we denote by $V_{2,3}(T) \subset V(T) - D$ the set of vertices having distance two or three to some leaf and by $V_6(T) \subset V(T) - D$ the set of vertices having distance three to some vertex of $V_{2,3}(T)$.

In order to provide a constructive characterization of the trees belonging to the family \mathcal{T}_{γ_t} , we need the following five operations F_1 , F_2 , F_3 , F_4 and F_5 on a tree T (by attaching a path P to a vertex v of T we mean adding the path P and joining v to a vertex of P). Moreover, through all the next results we make use of the fact that any tree T always contains a $\gamma_{t,coi}(T)$ -set which does not contain leaves. **Operation** F_1 : Attach a path P_1 to a vertex of T, which is in some $\gamma_{t,coi}(T)$ -set.

Operation F_2 : Attach a path P_1 to a vertex of T, which is in $L_3(T)$.

Operation F_3 : Attach a path P_2 to a vertex of T, which is in $L_3(T)$.

Operation F_4 : Attach a path P_3 to a vertex of T, which is in $V_{2,3}(T)$.

Operation F_5 : Attach a path P_3 to a vertex of T, which is in $V_6(T)$.

Let \mathcal{F} be the family of trees defined as $\mathcal{F} = \{T \mid T \text{ is obtained from } P_4 \text{ by a finite sequence}$ of operations F_1, F_2, F_3, F_4 or $F_5\}$. The Figure 4 contains a fairly representative example of a tree $T \in \mathcal{F}$. We first show that every tree of the family \mathcal{F} belongs to the family \mathcal{T}_{γ_t} .



Figure 4: A tree T obtained from a path $P_4 = v_1 v_2 v_3 v_4$, applying the five operations F_1 , F_2 , F_3 , F_4 and F_5 . Firstly, operations F_4 and F_5 are applied by adding the path $P_3 = v_5 v_6 v_7$ to the vertex v_4 through the edge v_4v_5 and the path $P_3 = v_8 v_9 v_{10}$ to the vertex v_7 through the edge v_7v_8 . Next, we apply the operation F_1 twice by attaching the vertices w_1 and w_2 to the vertices v_3 and v_8 , respectively. Moreover, we apply the operation F_2 by adding the vertex u_3 to w_2 . Finally, we apply the operation F_3 by adding the path $P_2 = u_1 u_2$ to the vertex w_1 through the edge w_1u_1 .

Lemma 20. If $T \in \mathcal{F}$, then $T \in \mathcal{T}_{\gamma_t}$.

Proof. We proceed by induction on the number r(T) of operations required to construct the tree T. If r(T) = 0, then $T = P_4$ and $T \in \mathcal{T}_{\gamma_t}$. This establishes the base case. Hence, we now assume that $k \geq 1$ is an integer and that each tree $T' \in \mathcal{F}$ with r(T') < k satisfies that $T' \in \mathcal{T}_{\gamma_t}$. Let $T \in \mathcal{F}$ be a tree for wich r(T) = k. Since T can be obtained from a tree $T' \in \mathcal{F}$ with r(T') = k - 1 by one of the operations F_1, F_2, F_3, F_4 or F_5 , we shall prove that $T \in \mathcal{T}_{\gamma_t}$, by considering a $\gamma_{t,coi}(T')$ -set D' containing no leaves and through the following situations.

Case 1. T is obtained from T' by operation F_1 . Let u be the vertex added to T in order to obtain T'. Since u is a leaf of T and is adjacent to a vertex of D', the set D' remains to be a total dominating set in T. Moreover, D' is a $\gamma_t(T)$ -set, since otherwise we would find a total dominating set in T' of cardinality smaller than $\gamma_t(T')$. On the other hand, since $(V(T') - D') \cup \{u\}$ is independent, we deduce D' is a TC-ID set in T. Thus, $\gamma_{t,coi}(T) \leq |D'| = \gamma_{t,coi}(T') = \gamma_t(T') = \gamma_t(T)$ (by also using the inductive hypothesis). Thus, by

Theorem 16, we get the equality $\gamma_{t,coi}(T) = \gamma_t(T)$, which means $T \in \mathcal{T}_{\gamma_t}$.

Case 2. T is obtained from T' by operation F_2 . Assume T is obtained from T' by adding the vertex u and the edge uv where $v \in L_3(T')$. As $v \in L_3(T')$, there exist a path vu_1u_2h in T' where h is a leaf and u_1, u_2 are support vertices adjacent to v, h, respectively. Now, in T, the vertices u_2, v are supports and belong to any TC-ID set in T. Hence, the set $D = D' \cup \{v\}$ is a TC-ID set in T, and so

$$\gamma_t(T) \le \gamma_{t,coi}(T) \le \gamma_{t,coi}(T') + 1 = \gamma_t(T') + 1 \tag{1}$$

(by also using Theorem 16 and the inductive hypothesis). Now, let A be a $\gamma_t(T)$ -set containing no leaves. Notice that the vertex v is a support and so, it belong to A, also the vertex u_1 belongs to A too, because v has degree two. Moreover, note that the set $A - \{v\}$ is a total dominating set in T', which leads to $\gamma_t(T') \leq \gamma_t(T) - 1$. By using this, it follows that all the inequalities in (1) must be equalities. Thus $\gamma_{t,coi}(T) = \gamma_t(T)$, and $T \in \mathcal{T}_{\gamma_t}$.

Case 3. T is obtained from T' by operation F_3 . Assume T is obtained from T' by adding the path $P_2 = h_1 h_2$ to a vertex $v \in L_3(T')$ through the edge vh_1 . By using some similar reasons as in the case above (now we must use $D = D' \cup \{v, h_1\}$ instead of $D = D' \cup \{v\}$), it is observed that $T \in \mathcal{T}_{\gamma_t}$.

Case 4. *T* is obtained from *T'* by operation F_4 . Assume *T* is obtained from *T'* by adding the path $P_3 = h_1 u_1 h_2$ to a vertex $v \in V_{2,3}(T')$ through the edge vh_1 . We notice that u_1, h_1 belong to any TC-ID set containing no leaves of *T*. Hence, the set $D = D' \cup \{u_1, h_1\}$ is a TC-ID set in *T*. Thus $\gamma_t(T) \leq \gamma_{t,coi}(T) \leq \gamma_{t,coi}(T') + 2 = \gamma_t(T') + 2$ (by also using Theorem 16 and the inductive hypothesis). Now, let *A* be a $\gamma_t(T)$ -set. Since the vertex u_1 is a support, it belongs to *A* and so, $|A \cap \{h_1, u_1, h_2\}| \geq 2$. Moreover, note that $|A \cap V(T')| \geq \gamma_t(T')$. Hence, $\gamma_t(T) = |A| \geq \gamma_t(T') + 2$. Again, as in Case 2, we deduce $\gamma_{t,coi}(T) = \gamma_t(T)$, which means $T \in \mathcal{T}_{\gamma_t}$.

Case 5. T is obtained from T' by operation F_5 . Assume T is obtained from T' by adding the path $P_3 = h_1 u_1 h_2$ to a vertex $v \in V_6(T')$ through the edge vh_1 . By using some similar reasons as in the case above, it can be deduced that $\gamma_{t,coi}(T) = \gamma_t(T)$, which gives $T \in \mathcal{T}_{\gamma_t}$.

We now turn our attention to the opposite direction concerning the lemma above. In this sense, from now on we shall need the following terminology and notation in our results. Given a tree T and a set $S \subset V(T)$, by T - S we denote a tree obtained from T by removing from T all the vertices in S and all its incident edges (if $S = \{v\}$ for some vertex v, then we simply write T - v). For an integer $r \geq 2$, by Q_r we mean a graph which is obtained from a path $P_{r+2} = vss_1s_2\ldots s_r$ by attaching a path P_1 to every vertex of $P_{r+2} - v$. In Figure 5 we show the example of Q_5 .

We next show that every tree of the family \mathcal{T}_{γ_t} belongs to the family \mathcal{F} .



Figure 5: The structure of the tree Q_5 .

Lemma 21. If $T \in \mathcal{T}_{\gamma_t}$, then $T \in \mathcal{F}$.

Proof. We proceed by induction on the order $n \ge 4$ of the trees $T \in \mathcal{T}_{\gamma_t}$. If T is a double star, then T can be obtained from P_4 by repeatedly applying operation F_1 . This establishes the base case. We assume next that k > 4 is an integer and that each tree $T' \in \mathcal{T}_{\gamma_t}$ with |V(T')| < k satisfies $T' \in \mathcal{F}$.

Let T be a tree such that $T \in \mathcal{T}_{\gamma_t}$ and |V(T)| = k. Let D be a $\gamma_{t,coi}(T)$ -set containing no leaves and let B = V(T) - D. We analyze the following situations.

Case 1: |S(T)| < |L(T)|. We consider a support vertex v that is adjacent to at least two leaves. Let $h \in N(v) \cap L(T)$ and T' = T - h. Thus, the set D is a $\gamma_t(T')$ -set too, and by inductive hypothesis, $T' \in \mathcal{F}$. Therefore, since T can be obtained from T' by operation F_1 , it follows $T \in \mathcal{F}$.

Case 2: |S(T)| = |L(T)| and |SS(T)| = 0. In this case we note that $V(T) = S(T) \cup L(T)$ and clearly, S(T) is a $\gamma_{t,coi}(T)$ -set (moreover $|S(T)| \ge 3$ since otherwise T is a double star). Let $s \in S(T)$ such that $|N(s) \cap S(T)| = 1$ (note that such s always exists) and let $h \in L(T)$ be the leaf adjacent to s. We first notice that there exists a leaf having distance three to the support s. Thus, we deduce that $S'(T) = S(T) - \{s\}$ is a $\gamma_{t,coi}(T')$ -set, where T' = T - h. By induction hypothesis $T' \in \mathcal{F}$ and, since T can be obtained from T' by operation F_2 , we get $T \in \mathcal{F}$.

Case 3: |S(T)| = |L(T)| and |SS(T)| > 0. Herein we denote by P(x, y) the set of vertices of one shortest path between x and y, including x and y. Let h, h' be two leaves at the maximum possible distance in T such that there is $v \in SS(T) \cap P(h, h')$ with d(v, h) = 2 or d(v, h') = 2. Without loss of generality assume that d(v, h) = 2 and let s be the support adjacent to h. Since |S(T)| = |L(T)| and by the maximality of the path between h and h', we observe that $N(s) \subset S(T) \cup \{h, v\}$ and also, that every support vertex is adjacent to exactly one leaf. We have now some possible scenarios.

Case 3.1 $|N(s) \cap S(T)| = 1$. Hence, by the maximality of the path P(h, h'), it must happen that T has an induced subgraph isomorphic to a graph Q_r , as previously described, obtained from the vertices v, s, h and some supports, say $s_1, s_2, \ldots, s_r \in S(T)$, with the leaves h_1, h_2, \ldots, h_r , adjacent to the supports s_1, s_2, \ldots, s_r , respectively, and such that $\{s_1, \ldots, s_r, h_1, \ldots, h_r\} \cap P(h, h') = \emptyset$.

Assume r = 1. Note that $s, s_1 \in D$ and that $h, h_1 \notin D$. Let T' = T - h. Notice that D is

also a TC-ID set in T', and so

$$\gamma_t(T') \le \gamma_{t,coi}(T') \le \gamma_{t,coi}(T) = \gamma_t(T) \tag{2}$$

(by using Theorem 16 and hypothesis). On the other hand, let A be a $\gamma_t(T')$ -set containing no leaves. We observe that $s_1 \in A$ because s_1 is a support in T', and $s \in A$ because $\delta(s_1) = 2$. Thus, clearly A is also a total dominating set in T. Hence $\gamma_t(T) \leq |A| = \gamma_t(T')$. Thus, all the inequalities in the relation (2) must be equalities, from which follows $\gamma_{t,coi}(T') = \gamma_t(T')$ and by the inductive hypothesis $T' \in \mathcal{F}$. Since T can be obtained from T' by operation F_1 , we obtain $T \in \mathcal{F}$.

Assume now $r \geq 2$. Note that $s, s_1, \ldots, s_r \in D$ and that there is a leaf at distance three from s_r . Let $T' = T - h_r$. Hence, $D - \{s_r\}$ is a TC-ID set in T', and so $\gamma_t(T') \leq \gamma_{t,coi}(T') \leq \gamma_{t,coi}(T) - 1 = \gamma_t(T) - 1$ (by using Theorem 16 and hypothesis). Moreover, the set $D - \{s_r\}$ is a $\gamma_t(T')$ -set, otherwise we would find a total dominating set of T of cardinality smaller than $\gamma_t(T)$, which is not possible. So, $\gamma_t(T') = \gamma_t(T) - 1$ which leads to $\gamma_{t,coi}(T') = \gamma_t(T')$, as in the previous case. Now, by the inductive hypothesis $T' \in \mathcal{F}$, and since T can be obtained from T'by operation F_2 , we deduce $T \in \mathcal{F}$.

Case 3.2 $|N(s) \cap S(T)| > 1$. An analogous procedure to the one above (**Case 3.1**) leads to our desired conclusion, based on the fact that s must have at least two neighbors $s'_1, s''_1 \in S(T)$ and there are at least two induced subgraphs isomorphic to the graphs $Q_{r'}$ and $Q_{r''}$, which can be used instead of Q_r of Case 3.1.

Case 3.3: $|N(s) \cap S(T)| = 0$. Clearly, s has degree two since it has one leaf neighbor, no support neighbors and cannot have more than one (it has exactly one) semi-support neighbor due to the maximality of P(h, h'). Also, it must happen $v \in D$, $h \in B$ and $s \in D$. Assume the subgraph induced by P(h, h') is $h s v u_1 u_2 u_3 u_4 \dots s' h'$, where $h, h' \in L(T)$ and $s, s' \in S(T)$. Note that $N(v) \subset S(T) \cup \{u_1\}$. We consider again some possible scenarios.

Case 3.3.1: $|N(v) \cap S(T)| > 1$. In this case, the vertex v is also totally dominated by another support s_v different from s. Let h_v be the leaf adjacent to the support s_v . Notice that $D' = D - \{s\}$ is a TC-ID set of T' = T - h. Moreover, we note that the vertex s is a leaf in T'having distance three to the leaf h_v . So, by using a similar procedure as above (**Case 3.1** and $r \geq 2$) we obtain $T' \in \mathcal{F}$. Therefore, due to that T can be obtained from T' by operation F_2 , it follows $T \in \mathcal{F}$.

Case 3.3.2: $|N(v) \cap S(T)| = 1$ and $|N(u_1)| \ge 3$. Clearly s, v have degree two and belong to D. We firstly consider the case whether $u_1 \in D$. By Remark 18 we note that $N(u_1) \cap L(T) \neq \emptyset$ or that there is a vertex $r \in D$ with $N(r) \cap D = \{u_1\}$. If $N(u_1) \cap L(T) \neq \emptyset$, then $D' = D - \{s\}$ is a TC-ID in T' = T - h set, and by using a similar procedure as above (**Case 3.1** and $r \ge 2$) we obtain $T' \in \mathcal{F}$. Since T can be obtained from T' by operation F_2 , we are able to claim $T \in \mathcal{F}$.

On the other hand, assume that $N(u_1) \cap L(T) = \emptyset$ and there is a vertex $r \in D$ such that $N(r) \cap D = \{u_1\}$. We note that, by Remark 18, $(N(r) - \{u_1\}) \cap L(T) \neq \emptyset$. Hence, $D' = D - \{s\}$

is a TC-ID set in $T' = T - \{h, s\}$. Thus, $\gamma_t(T') \leq \gamma_{t,coi}(T') \leq \gamma_{t,coi}(T) - 1 = \gamma_t(T) - 1$ (by using Theorem 16 and hypothesis). Again, by using a similar procedure as above (**Case 3.1** and $r \geq 2$) we get that $\gamma_t(T') = \gamma_t(T) - 1$. So, $\gamma_{t,coi}(T') = \gamma_t(T')$, and by inductive hypothesis, $T' \in \mathcal{F}$. Also, it can relatively clearly be seen that v is having distance three to a leaf. This means T can be obtained from T' by operation F_3 , and so $T \in \mathcal{F}$.

Now, consider the case in which $u_1 \in B$. By the maximality of P(h, h') and by the fact that $|N(u_1)| \geq 3$, there is a leaf distinct to h at distance two or three from u_1 . Hence, the set $D' = D - \{s, v\}$ is a TC-ID set in $T' = T - \{h, s, v\}$. Again, by using a similar procedure as above (**Case 3.1** and $r \geq 2$) we obtain $T' \in \mathcal{F}$ and, due to that T can be obtained from T' by operation F_4 , we get $T \in \mathcal{F}$.

Case 3.3.3: $|N(v) \cap S(T)| = 1$ and $|N(u_1)| = 2$. Clearly s, v, u_1 have degree two and s, v belong to D. We only consider the case whether $u_1 \in B$, otherwise $u_1 \in D$ implies that u_2 is a leaf and T is P_5 , which can be obtained by operation F_2 from P_4 . As $u_1 \in B$, we get $u_2 \in D$. Notice that, as u_2 has to be totally dominated, there exist a vertex $r \in D$ such that $N(r) \cap D = \{u_2\}$. So, by Remark 18 and Lemma 19, it follows $(N(r) - \{u_2\}) \subset L(T) \cup V_{2,3}(T)$.

If $(N(r) - \{u_2\}) \cap L(T)) \neq \emptyset$, then this case is analogous to the **Case 3.3.2** and $u_1 \in B$. If $(N(r) - \{u_2\}) \subset V_{2,3}(T)$, then we see that $u_1 \in V_6(T)$. So, the set $D' = D - \{s, v\}$ is a TC-ID set in $T' = T - \{h, s, v\}$ and again, by using a similar procedure as above (**Case 3.1** and $r \geq 2$) we obtain $T' \in \mathcal{F}$. Finally, due to that T can be obtained from T' by operation F_5 , we have $T \in \mathcal{F}$, which completes the proof.

As an immediate consequence of Lemma 20 and Lemma 21 we have the following characterization.

Theorem 22. Let T be a tree. Then $T \in \mathcal{T}_{\gamma_t}$ if and only if $T \in \mathcal{F}$.

We next see that all the operations F_1 to F_4 are required in the characterization above. First, we see that operation F_1 is required to obtain a double star from the path P_4 . The operations F_2, F_3, F_4 are required to obtain the paths P_5, P_6, P_7 , respectively, from the path P_4 , and the path P_{10} can only be obtained from P_4 by a sequence of operations F_4, F_5 .

Concluding remarks

We have study several combinatorial and complexity properties of the total co-independent domination number of graphs. As a consequence of the study a couple of questions could be remarked as a possible future research lines.

• We have proved that computing the total co-independent domination number of graphs is NP-hard even when restricted to planar graphs of maximum degree at most 3. However, it would be interesting to find some non trivial families of graphs in which the problem above can be solved in polynomial time. On the other hand, the bounds of Theorem 8

together with the fact that the problem of computing the vertex cover number can be approximated within a factor of 2, allow to claim that the problem of computing the total co-independent domination number can be approximated within a constant factor. In this sense, it would be interesting to give some other approximation (or inapproximation) results on this parameter.

• We have characterized the family of graphs achieving the upper bound of Theorem 8. According to the construction of such family, it seems one could also characterize the graphs G for which $\gamma_{t,coi}(G) = 2\alpha(G) - k$ for some values of k like for instance k = 2 or k = 3. Moreover, it would be of interest to characterize the family of graphs attaining the lower bound of Theorem 8 (note that for instance the trees satisfying such bound were characterized in [3]).

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