The generalized connectivity of (n,k)-bubble-sort graphs

Shu-Li Zhao[1], Rong-Xia Hao[1]*, Lidong Wu[2]

[1] *Department of Mathematics, Beijing Jiaotong University, Beijing 100044, P.R. China*

[2] *Department of Computer Science, University of Texas at Tyler, Tyler, Texas 75799, USA*

Let $S \subseteq V(G)$ and $\kappa_G(S)$ denote the maximum number r of edge-disjoint trees $T_1, T_2, \cdots, T_r$ in G such that $V(T_i) \bigcap V(T_j) = S$ for any $i,j \in \{1,2,\cdots,r\}$ and $i \neq j$. For an integer k with $2 \leq k \leq n$, the *generalized k-connectivity* of a graph G is defined as $\kappa_k(G) = min\{\kappa_G(S) | S \subseteq V(G)$ and $|S| = k\}$. The generalized k-connectivity is a generalization of the traditional connectivity. In this paper, the generalized 3-connectivity of the (n,k)-bubble-sort graph $B_{n,k}$ is studied for $2 \leq k \leq n-1$. By proposing an algorithm to construct $n-1$ internally disjoint paths in $B_{n-1,k-1}$, we show that $\kappa_3(B_{n,k}) = n-2$ for $2 \leq k \leq n-1$, which generalizes the known result about the bubble-sort graph B_n [Applied Mathematics and Computation 274 (2016) 41-46] given by Li *et al.*, as the bubble-sort graph B_n is the special (n,k)-bubble-sort graph for $k = n-1$.

Keywords: Generalized connectivity; fault-tolerance; interconnection network; (n,k)-bubble-sort graph.

1 Introduction

For an interconnection network, one mainly concerns about the reliability and fault tolerance. An interconnection network is usually modelled as a connected graph $G = (V, E)$, where nodes represent processors and edges represent communication links between processors. *The connectivity* $\kappa(G)$ of a graph G is an important parameter to evaluate the reliability and fault tolerance of a network. It is defined as the minimum number of vertices whose deletion results in a disconnected graph. In addition, Whitney [16] defines the connectivity from local point of view. That is, for any subset $S = \{u, v\} \subseteq V(G)$, let $\kappa_G(S)$ denote the maximum number of internally disjoint paths between u and v in G. Then $\kappa(G) = min\{\kappa_G(S) | S \subseteq V(G)$ and $|S| = 2\}$. As a generalization of the traditional connectivity, Chartrand et al. [2] introduced the *generalized k-connectivity* in 1984. This parameter can measure the reliability of a network G to connect any k vertices in G. Let $S \subseteq V(G)$ and $\kappa_G(S)$ denote the maximum number r of edge-disjoint trees $T_1, T_2, \ldots, T_r$ in G such that $V(T_i) \bigcap V(T_j) = S$ for any $i,j \in \{1,2,\ldots,r\}$ and $i \neq j$. For an integer k with $2 \leq k \leq n$, the *generalized k-connectivity* of a graph G is defined as $\kappa_k(G) = min\{\kappa_G(S) | S \subseteq V(G)$ and $|S| = k\}$. The generalized 2-connectivity is exactly the traditional connectivity. Li [8]

*Corresponding author. Email:17118434@bjtu.edu.cn(S.-L. Zhao), rxhao@bjtu.edu.cn (R.-X. Hao), lw@uttyler.edu(L. Wu)

derived that it is NP-complete for a general graph G to decide whether there are k internally disjoint trees connecting S, where k is a fixed integer and $S \subseteq V(G)$. Some results [6, 10] about the upper and lower bounds of the generalized connectivity are obtained. In addition, there are some results of the generalized k-connectivity for some classes of graphs and most of them are about $k = 3$. For example, Chartrand *et al.* [3] studied the generalized connectivity of complete graphs; Li *et al.* [11] characterized the minimally 2-connected graphs with generalized connectivity $\kappa_3 = 2$; Li *et al.* [4] studied the generalized 3-connectivity of Cartesian product graphs; Li *et al.* [9] studied the generalized 3-connectivity of graph products; Li *et al.* [12] studied the generalized connectivity of the complete bipartite graphs; Li *et al.* [7] studied the generalized 3-connectivity of the star graphs and bubble-sort graphs; Li *et al.* [5] studied the generalized 3-connectivity of the Cayley graph generated by trees and cycles and Lin and Zhang [13] studied the generalized 4-connectivity of hypercubes etc.

In this paper, we focus on the (n,k)-bubble-sort graph, denoted by $B_{n,k}$. The complete graph K_n and the bubble-sort graph B_n are special (n,k)-bubble-sort graphs $B_{n,k}$ for $k = 1$ and $k = n-1$, respectively. In [3], it was shown that $\kappa_3(K_n) = n-2$ for $n \geq 3$ and in [7], it was shown that $\kappa_3(B_n) = n-2$ for $n \geq 3$. Following, we study the generalized 3-connectivity of $B_{n,k}$ for $2 \leq k \leq n-1$ and it is shown that $\kappa_3(B_{n,k}) = n-2$, which generalizes the known results about bubble-sort graphs [7].

The paper is organized as follows. In section 2, some notation and definitions are given. In section 3, the connectivity of (n,k)-bubble-sort graphs $B_{n,k}$ is determined for $2 \leq k \leq n-1$. In addition, the generalized 3-connectivity of $B_{n,k}$ is determined for $2 \leq k \leq n-1$ and an algorithm for constructing $n-1$ internally disjoint paths in $B_{n-1,k-1}$ was proposed. In section 4, the paper is concluded.

2 Preliminary

Let $G = (V,E)$ be a simple, undirected graph. Let $|V(G)|$ be the size of vertex set and $|E(G)|$ be the size of edge set. For a vertex v in G, we denote by $N_G(v)$ the *neighbourhood* of the vertex v in G and $N_G[v] = N_G(v) \bigcup \{v\}$. Let $U \subseteq V(G)$, denote $N_G(U) = \bigcup\limits_{v \in U} N_G(v) - U$. Let $d_G(v)$ denote the degree of the vertex v in G and $\delta(G)$ denote the *minimum degree* of the graph G. The subgraph induced by V' in G, denoted by $G[V']$, is a graph whose vertex set is V' and the edge set is the set of all the edges of G with both ends in V'. A graph is said to be *k-regular* if for any vertex v of G, $d_G(v) = k$. Two xy- paths P and Q in G are *internally disjoint* if they have no common internal vertices, that is $V(P) \bigcap V(Q) = \{x,y\}$. Let $Y \subseteq V(G)$ and $X \subset V(G) \setminus Y$, the (X,Y)-paths is a family of internally disjoint paths starting at a vertex $x \in X$, ending at a vertex $y \in Y$ and whose internal vertices belong neither to X nor Y. If $X = \{x\}$, the (X,Y)-paths is a family of internal disjoint paths whose starting vertex is x and the terminal vertices are distinct in Y, which is referred to as a *k-fan* from x to Y. For terminologies and notation not undefined here we follow the reference [1].

Let Γ be a finite group and S be a subset of Γ, where the identity of the group does not belong to S. The *Cayley graph* $Cay(\Gamma, S)$ is a digraph with vertex set Γ and arc set $\{(g, g.s) | g \in \Gamma, s \in S\}$. If $S = S^{-1}$, then $Cay(\Gamma, S)$ is an undirected graph, where $S^{-1} = \{s^{-1} | s \in S\}$.

Let $[n] = \{1, 2, \cdots, n\}$ and $Sym(n)$ denote the group of all permutations on $[n]$. Let $(p_1 p_2 \cdots p_n)$ denote a permutation on $[n]$ and (ij), which is called a transposition, denote the transposition that swaps the objects at positions i and j, that is, $(p_1 \cdots p_i \cdots p_j \cdots p_n)(ij) = (p_1 \cdots p_j \cdots p_i \cdots p_n)$. For the Cayley graph $Cay(Sym(n), T)$, where T is a set of transpositions of $Sym(n)$. Let $G(T)$ be the graph on n vertices $\{1, 2, \ldots, n\}$ such that there is an edge ij in $G(T)$ if and only if transposition $(ij) \in T$ [15]. The graph $G(T)$ is called *the transposition generating graph* of $Cay(Sym(n), T)$. It is well known that if $G(T) \cong P_n$, where P_n is a path with n vertices, then $Cay(Sym(n), T)$ is called an *n-dimensional bubble sort graph* and denoted by B_n.

As a generalization of B_n, the (n, k)-bubble-sort graph, denoted by $B_{n,k}$, was introduced by Shawash [14] in 2008. The (n, k)-bubble-sort graph $B_{n,k}$ is defined as follows.

Definition 1. *Given two positive integers n and k with $n > k$, let $[n]$ denote the set $\{1, 2, \cdots, n\}$ and $P_{n,k}$ be a set of arrangements of k elements in $[n]$. The (n, k)-bubble-sort graph $B_{n,k}$ has vertex set $P_{n,k}$, and two vertices $u = a_1a_2 \cdots a_k$ and $v = b_1b_2 \cdots b_k$ are adjacent if and only if one of the following conditions hold.*

(a) *There exists an integer $m \in [2, k]$ such that $a_{m-1} = b_m, a_m = b_{m-1}$ and $a_i = b_i$ for all $i \in [k] \setminus \{m - 1, m\}$.*

(b) *$a_i = b_i$ for all $i \in [k] \setminus \{1\}$ and $a_1 \neq b_1$.*

For two distinct i and j, where $i \in [n]$ and $j \in [k]$. Let $V_{n,k}^{j:i}$ be the set of vertices in $B_{n,k}$ with the jth position being i, that is, $V_{n,k}^{j:i} = \{p|p = p_1p_2 \cdots p_j \cdots p_k \in P_{n,k}$ and $p_j = i\}$. For a vertex $v = p_1p_2 \cdots p_i \cdots p_n$, we call p_i the element at position i of the vertex v. For a fixed position $j \in [k]$, $\{V_{n,k}^{j:i}|1 \le i \le n\}$ forms a partition of $V_{n,k}$. Let $B_{n,k}^{j:i}$ denote the subgraph of $B_{n,k}$ induced by $V_{n,k}^{j:i}$. Then for each $j \in [k]$, $B_{n,k}^{j:i}$ is isomorphic to $B_{n-1,k-1}$. Thus, $B_{n,k}$ can be recursively constructed from n copies of $B_{n-1,k-1}$. It is easy to check that each $B_{n,k}^{j:i}$ is a subgraph of $B_{n,k}$ and $B_{n,k}$ can be decomposed into n subgraphs $B_{n,k}^{j:i}$s according to the jth position. By the symmetry of $B_{n,k}$ and for simplicity, we shall take j as the last position k and use $B_{n,k}^i$ to denote $B_{n,k}^{k:i}$. For convenience, let $B_{n,k} = B_{n,k}^1 \bigoplus B_{n,k}^2 \bigoplus \cdots \bigoplus B_{n,k}^n$, where $\bigoplus$ just denotes the corresponding decomposition of $B_{n,k}$. Obviously, any vertex u of $B_{n,k}^i$ has $k-1$ neighbors in $B_{n,k}^i$ and one neighbor outside of $B_{n,k}^i$, which is called the outside neighbour of u. Let $E(i, j)$ be the set of edges between

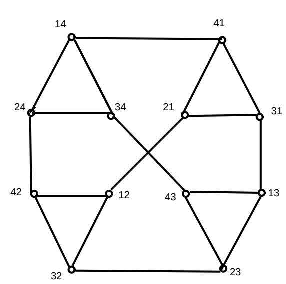

Figure 1: The $(4, 2)$-bubble-sort graph $B_{4,2}$

$B^i_{n,k}$ and $B^j_{n,k}$, that is, $E(i,j) = \{(p,q) \in E(B_{n,k}) | p \in V(B^i_{n,k})$ and $q \in V(B^j_{n,k})\}$. Clearly, $E(i,j)$ is a matching between $B^i_{n,k}$ and $B^j_{n,k}$ and $|E(i,j)| = \frac{(n-2)!}{(n-k)!}$. By the definition of $B_{n,k}$, $B_{n,1}$ is isomorphic to K_n and $B_{n,n-1}$ is isomorphic to B_n. It follows that $B_{n,k}$ is a generalization of the bubble-sort graph B_n. The $(4,2)$-bubble-sort graph $B_{4,2}$ is depicted in Figure 2.

3 The generalized 3-connectivity of the (n,k)-bubble-sort graph

In this section, the generalized 3-connectivity of the (n,k)-bubble-sort graph $B_{n,k}$ will be proved. To prove the result, the following lemmas are useful.

Lemma 1. *Let $B_{n,k} = B^1_{n,k} \bigoplus B^2_{n,k} \bigoplus \ldots \bigoplus B^n_{n,k}$ for $n \geq k+1$ and $1 \leq k \leq n-1$. Then the following results hold.*

(1) *For any vertex u of $B^i_{n,k}$, it has exactly one outside neighbour.*

(2) *For any copy $B^i_{n,k}$, no two vertices in $B^i_{n,k}$ have a common outside neighbour. In addition, $|N(B^i_{n,k})| = \frac{(n-1)!}{(n-k)!}$ and $|N(B^i_{n,k}) \bigcap V(B^j_{n,k})| = \frac{(n-2)!}{(n-k)!}$ for $i \neq j$.*

Proof. (1) By the definition of $B_{n,k}$, the result holds clearly.

(2) Let $u, v \in V(B^i_{n,k})$ and $u \neq v$. If they have a common outside neighbour w, then u and v are the two outside neighbours of w which lie in the same copy, which contradicts with (1). Thus, no two vertices in $B^i_{n,k}$ have a common outside neighbour.

Since $|V(B^i_{n,k})| = \frac{(n-1)!}{(n-k)!}$ and no two vertices in $B^i_{n,k}$ have a common outside neighbor, $|N(B^i_{n,k})| = \frac{(n-1)!}{(n-k)!}$ and $|N(B^i_{n,k}) \bigcap V(B^j_{n,k})| = \frac{(n-2)!}{(n-k)!}$ for $i \neq j$. $\square$

Lemma 2. ([10]) *Let G be a connected graph and δ be its minimum degree. Then $\kappa_3(G) \leq \delta$. Further, if there are two adjacent vertices of degree δ, then $\kappa_3(G) \leq \delta - 1$.*

Lemma 3. ([10]) *Let G be a connected graph with n vertices. If $\kappa(G) = 4k + r$, where k and r are two integers with $k \geq 0$ and $r \in \{0,1,2,3\}$, then $\kappa_3(G) \geq 3k + \lceil \frac{r}{2} \rceil$. Moreover, the lower bound is sharp.*

Lemma 4. ([1]) *Let $G = (V,E)$ be a k-connected graph, and let X and Y be subsets of $V(G)$ of cardinality at least k. Then there exists a family of k pairwise disjoint (X,Y)-paths in G.*

Lemma 5. ([1]) *Let $G = (V,E)$ be a k-connected graph, let x be a vertex of G, and let $Y \subseteq V \setminus \{x\}$ be a set of at least k vertices of G. Then there exists a k-fan in G from x to Y, that is, there exists a family of k internally disjoint (x,Y)-paths whose terminal vertices are distinct in Y.*

Next, we determine the connectivity of $B_{n,k}$ for $k = 2$.

Lemma 6. $\kappa(B_{n,2}) = n - 1$ *for $n \geq 3$.*

Proof. Let $B_{n,2} = B^1_{n,2} \bigoplus B^2_{n,2} \bigoplus \ldots \bigoplus B^n_{n,2}$. Let F be a minimum vertex cut of $B_{n,2}$ and $u \in V(B_{n,2})$. Since $N_{B_{n,2}}(u)$ is a vertex cut of $B_{n,2}$ and $|N_{B_{n,2}}(u)| = n-1$, $|F| \leq n-1$.

Next, we show that $|F| \geq n-1$. Suppose to the contrary, that is, $|F| \leq n-2$. Let $F_i = F \bigcap V(B^i_{n,2})$ for each $i \in \{1, 2, \cdots, n\}$. Without loss of generality, let $|F_1| \geq |F_2| \geq \cdots \geq |F_n|$. Then $|F_{n-1}| = |F_n| = 0$. By Lemma 1(2), $B_{n,2}[V(B^{n-1}_{n,2}) \bigcup V(B^n_{n,2})]$ is connected. Let C be a component of $B_{n,2} - F$ that does not contain $B_{n,2}[V(B^{n-1}_{n,2}) \bigcup V(B^n_{n,2})]$ as a subgraph and $c_i = |V(C) \bigcap V(B^i_{n,2})|$ for each $i \in \{1, 2, \cdots, n-2\}$. Then there exists an integer $l \in \{1, 2, \cdots, n-2\}$ such that $c_l > 0$. Let $u \in V(B^l_{n,2}) \bigcap V(C)$ and $u' \in V(B^j_{n,2})$, where u' is the outside neighbour of u in $B^j_{n,2}$, $j \in [n]$ and $l \neq j$.

If $u' \in V(B^j_{n,2}) \setminus V(C)$, then $u' \in F_j$. It implies that $|F_j| \geq 1$.

If $u' \in V(C)$, then $N_{B^j_{n,2}}(V(B^{n-1}_{n,2}) \bigcup V(B^n_{n,2})) \subseteq F_j$. Otherwise, the component that contains $B_{n,2}[V(B^{n-1}_{n,2}) \bigcup V(B^n_{n,2})]$ will be C as $B^j_{n,2} \cong K_{n-1}$, which is a contradiction. By Lemma 2, $|N_{B^j_{n,2}}(V(B^{n-1}_{n,2}) \bigcup V(B^n_{n,2})| = 2$. It implies that $|F_j| \geq 2$.

Recall that $B^l_{n,2}$ is a complete graph, then $|F| = |F_1 \bigcup \cdots \bigcup F_n| \geq |V(B^l_{n,2})| - c_l + c_l = n-1$, a contradiction. Thus, $|F| \geq n-1$.

$\square$

Next, we determine the connectivity of $B_{n,k}$ for $2 \leq k \leq n-1$.

Lemma 7. *$\kappa(B_{n,k}) = n-1$ for $2 \leq k \leq n-1$.*

Proof. Let F be a minimum vertex cut of $B_{n,k}$ and $u \in V(B_{n,2})$. Since $N_{B_{n,k}}(u)$ is a vertex cut of $B_{n,k}$ and $|N_{B_{n,k}}(u)| = n-1$, $|F| \leq n-1$.

Next, we show that $\kappa(B_{n,k}) \geq n-1$. We prove the result by induction on k. When $n \geq 3$ and $k = 2$, by Lemma 6, the result holds. Suppose that the result holds for $B_{n',k-1}$, where $2 \leq k-1 \leq n'-2$. Now we consider $B_{n,k}$ for $3 \leq k \leq n-2$. Let $F_i = F \bigcap V(B^i_{n,k})$ for each $i \in \{1, 2, \cdots, n\}$. Without loss of generality, let $|F_1| \geq |F_2| \geq \cdots \geq |F_n|$. Suppose to the contrary, that is, $|F| \leq n-2$. Thus, $|F_{n-1}| = |F_n| = 0$.

If $|F_1| = n-2$, then $|F_i| = 0$ for each $i \in \{2, 3, \cdots, n\}$. By Lemma 1(2), $B_{n,k}[\bigcup_{i=2}^n V(B^i_{n,k})]$ is connected. As any vertex in $B^1_{n,k} \setminus F_1$ has an outside neighbour, $B_{n,k} - F$ is connected, a contradiction.

If $|F_1| \leq n-3$, then $|F_i| \leq n-3$ for each $i \in \{2, 3, \cdots, n\}$. By induction, $B^i_{n,k} - F_i$ is connected for each $i \in \{1, 2, \cdots, n\}$. As $|F_n| = 0$ and there are $\frac{(n-2)!}{(n-k)!}$ independent edges between $B^i_{n,k}$ and $B^n_{n,k}$. Note that $\frac{(n-2)!}{(n-k)!} - |F_i| \geq \frac{(n-2)!}{(n-3)!} - |F_i| \geq 1$ for each $i \in \{1, 2, \cdots, n-1\}$. Then there exists at least one edge between $B^i_{n,k} - F_i$ and $B^n_{n,k}$. It implies that $B_{n,k} - F$ is connected, a contradiction. Thus, $|F| \geq n-1$.

$\square$

To prove the main result, the following lemmas are useful.

Lemma 8. *Let $B_{n,k} = B^1_{n,k} \bigoplus B^2_{n,k} \bigoplus \ldots \bigoplus B^n_{n,k}$ and $H = B_{n,k}[V(B_{n,k}) \setminus V(B^i_{n,k})]$ for some $i \in [n]$. If $2 \leq k \leq n-1$, then $\kappa(H) = n-2$.*

Proof. Without loss of generality, let $H = B_{n,k}[V(B_{n,k}) \setminus V(B^n_{n,k})]$, that is, $H = B^1_{n,k} \bigoplus B^2_{n,k} \bigoplus \ldots \bigoplus B^{n-1}_{n,k}$. As there is some vertex $v \in V(H)$ whose outside neighbour belongs to $B^n_{n,k}$, $\delta(H) = n-2$. Hence, $\kappa(H) \leq \delta(H) = n-2$.

Next, we show that $\kappa(H) \geq n-2$. To prove the result, we just need to show that for any two distinct vertices v_1 and v_2 of H, there exist at least $n-2$ internally disjoint paths between them. The result is proved by considering the following two cases.

Case 1. v_1 and v_2 belong to the same copy of $B_{n-1,k-1}$.

Without loss of generality, let $v_1, v_2 \in V(B^1_{n,k})$. By Lemma 7, $\kappa(B^1_{n,k}) = n-2$. Hence, there are $n-2$ internally disjoint paths between v_1 and v_2 in $B^1_{n,k}$.

Case 2. v_1 and v_2 belong to different copies of $B_{n-1,k-1}$.

Without loss of generality, let $v_1 \in V(B^1_{n,k})$ and $v_2 \in V(B^2_{n,k})$.

Subcase 2.1. $3 \leq k \leq n-1$

By Lemma 1(2), there are $\frac{(n-2)!}{(n-k)!}$ independent edges between $B^1_{n,k}$ and $B^2_{n,k}$. Choose $n-2$ vertices $u_1, u_2, u_3, \cdots, u_{n-2}$ from $B^1_{n,k}$ such that the outside neighbour u'_i of u_i belongs to $B^2_{n,k}$ for each $i \in \{1, 2, \cdots, n-2\}$. This can be done as $\frac{(n-2)!}{(n-k)!} \geq n-2$ for $k \geq 3$ and $n \geq k+1$. Let $S = \{u_1, u_2, u_3, \cdots, u_{n-2}\}$ and $S' = \{u'_1, u'_2, u'_3, \cdots, u'_{n-2}\}$. By Lemma 7, $\kappa(B^1_{n,k}) = \kappa(B^2_{n,k}) = n-2$. If $v_1 \notin S$, by Lemma 5, there exists a family of $n-2$ internally disjoint (v_1, S)-paths $P_1, P_2, \cdots, P_{n-2}$ whose terminal vertices are distinct in S. Note that if $v_1 \in S$, then there is a (v_1, S) path that contains the only vertex v_1. Similarly, if $v_2 \notin S'$, by Lemma 5, there exists a family of $n-2$ internally disjoint (v_2, S') paths $P'_1, P'_2, \cdots, P'_{n-2}$ whose terminal vertices are distinct in S'. Note that if $v_2 \in S'$, there is a (v_2, S') path that contains the only vertex v_2. Let $\widehat{P}_i = P_i \bigcup u_i u'_i \bigcup P'_i$ for each $i \in \{1, 2, \cdots, n-2\}$, then $n-2$ disjoint paths between v_1 and v_2 are obtained in H.

Subcase 2.2. $k = 2$ and $n \geq 3$

By Lemma 1(2), there is exactly one edge between $B^i_{n,k}$ and $B^j_{n,k}$ for $i \neq j$ and $i, j \in \{1, 2, \cdots, n-1\}$. Choose $n-2$ vertices $u_1, u_2, u_3, \cdots, u_{n-2}$ from $B^1_{n,k}$ such that the outside neighbour u'_i of u_i belongs to $B^{i+1}_{n,k}$ for each $i \in \{1, 2, \cdots, n-2\}$, and choose $n-3$ vertices $w_2, w_3, \cdots, w_{n-2}$ from $B^2_{n,k}$ such that the outside neighbour w'_i of w_i belongs to $B^{i+1}_{n,k}$ for each $i \in \{2, 3, \cdots, n-2\}$. Let $S = \{u_1, u_2, u_3, \cdots, u_{n-2}\}$ and $S' = \{u'_1, w_2, w_3, \cdots, w_{n-2}\}$. Note that $B^i_{n,k} \cong K_{n-1}$ for each $i \in \{1, 2, \cdots, n\}$. If $v_1 \notin S$, then $S = N_{B^1_{n,k}}(v_1)$. If $v_1 \in S$, let $v_1 = u_1$. Then $S \backslash \{u_1\} \subseteq N_{B^1_{n,k}}(v_1)$. Similarly, if $v_2 \notin S'$, then $S' = N_{B^2_{n,k}}(v_2)$. If $v_2 \in S'$, let $v_2 = u'_1$. Then $S' \setminus \{u'_1\} \subseteq N_{B^2_{n,k}}(v_2)$. Recall that $B^i_{n,k} \cong K_{n-1}$ for $i \in [n-1]$, then $u'_i w'_i$ is an edge in $B^{i+1}_{n,k}$ for each $i \in \{2, 3, \cdots, n-2\}$. Let $P_1 = v_1 u_1 u'_1 v_2$ and $P_i = v_1 u_i u'_i w'_i w_i v_2$ for each $2 \leq i \leq n-2$, then $n-2$ disjoint paths between v_1 and v_2 are obtained in H.

Hence, $\kappa(H) = n-2$. □

Lemma 9. *Let $B_{n,2} = B^1_{n,2} \bigoplus B^2_{n,2} \bigoplus \ldots \bigoplus B^n_{n,2}$. For any vertex $v \in V(B^i_{n,2})$ for $1 \leq i \leq n$, let $N_{B^i_{n,2}}[v] = N_{B^i_{n,2}}(v) \bigcup \{v\}$. Then $|N_{B^i_{n,2}}[v]| = n-1$ and the $n-1$ outside neighbours of vertices in $N_{B^i_{n,2}}[v]$ belong to different copies of $B_{n-1,1}$.*

Proof. Let $v \in V(B^i_{n,2})$, then $d_{B^i_{n,2}}(v) = n-2$. Thus, $|N_{B^i_{n,2}}[v]| = n-1$ holds clearly. Without loss of generality, assume $i = 2$ and $v = 12$. Then $N_{B^i_{n,2}}[v] = \{32, 42, \cdots, n2\}$. Let S be the set of outside neighbours of the vertices in $N_{B^i_{n,2}}[v]$, then $S = \{21, 23, 24, \cdots, 2n\}$. Hence, the outside neighbours are contained in $B^1_{n,2}, B^3_{n,2}, \cdots, B^n_{n,2}$, respectively. The result is desired. □

Following, we prove the generalized 3-connectivity of $B_{n,k}$ for $k = 2$.

Theorem 1. *$\kappa_3(B_{n,2}) = n-2$ for $n \geq 3$.*

Proof. As $B_{n,2}$ is $(n-1)$-regular. By Lemma 2, $\kappa_3(B_{n,2}) \leq \delta - 1 = n-2$. To complete the result, it suffices to show that $\kappa_3(B_{n,2}) \geq n-2$. We prove the result by induction on n.

For $n=3$, $B_{3,2}$ is connected. Then $\kappa_3(B_{3,2})\geq 1=n-2$.

For $n=4$, by Lemma 3 and Lemma 7, $\kappa_3(B_{n,2})\geq\lceil\frac{3}{2}\rceil=2=n-2$.

Next, suppose that $n\geq 5$. Let $B_{n,2}=B^1_{n,2}\bigoplus B^2_{n,2}\bigoplus\ldots\bigoplus B^n_{n,2}$ and v_1,v_2,v_3 be any three distinct vertices of $B_{n,2}$. For convenience, let $S=\{v_1,v_2,v_3\}$. We prove the result by considering the following three cases.

Case 1. v_1,v_2 and v_3 belong to the same copy of $B_{n-1,1}$.

Without loss of generality, let $v_1,v_2,v_3\in V(B^1_{n,2})$. By the inductive hypothesis, $\kappa_3(B^1_{n,2})\geq n-3$. That is, there are $n-3$ internally disjoint trees $T_1,T_2\cdots,T_{n-3}$ connecting S in $B^1_{n,2}$. Let v'_1,v'_2 and v'_3 be the outside neighbours of v_1,v_2 and v_3, respectively. Then $\{v'_1,v'_2,v'_3\}\subseteq V(B_{n,2})\setminus V(B^1_{n,2})$. As $B_{n,2}[V(B_{n,2})\setminus V(B^1_{n,2})]$ is connected, there exists a tree T connecting v'_1,v'_2 and v'_3 in $B_{n,2}[V(B_{n,2})\setminus V(B^1_{n,2})]$. Let $T_{n-2}=T\bigcup v_1v'_1\bigcup v_2v'_2\bigcup v_3v'_3$, then it is a tree connecting S and $V(T_{n-2})\bigcap V(B^1_{n,2})=S$. Hence, there exist $n-2$ internally disjoint trees connecting S in $B_{n,2}$ and the result is desired.

Case 2. v_1,v_2 and v_3 belong to two different copies of $B_{n-1,1}$.

Without loss of generality, let $v_1,v_2\in V(B^1_{n,2})$ and $v_3\in V(B^2_{n,2})$. By Lemma 7, $\kappa(B^1_{n,2})=n-2$. Hence, there exist $n-2$ internally disjoint paths $P_1,P_2,\ldots,P_{n-2}$ between v_1 and v_2 in $B^1_{n,2}$. Choose $n-2$ distinct vertices $x_1,x_2,\ldots,x_{n-2}$ from $P_1,P_2,\ldots,P_{n-2}$ such that $x_i\in V(P_i)$ for each $i\in\{1,2,\cdots,n-2\}$. Note that at most one of these paths has length 1. If there is one path with length 1, say P_1 and let $x_1=v_1$. Let x'_i be the outside neighbour of x_i for each $i\in\{1,2,\cdots,n-2\}$. Let $X'=\{x'_1,x'_2,\cdots,x'_{n-2}\}$, then $X'\subset V(B_{n,2})\setminus V(B^1_{n,2})$. By Lemma 1, $|X'|=n-2$. By Lemma 8, $B_{n,2}[V(B_{n,2})\setminus V(B^1_{n,2})]$ is $n-2$ connected. By Lemma 5, there exist $n-2$ internally disjoint (v_3,X')-paths $P'_1,P'_2,\ldots,P'_{n-2}$ in $B_{n,2}[V(B_{n,2})\setminus V(B^1_{n,2})]$ whose terminal vertices are distinct in X'. Note that if $v_3\in X'$, then there is a (v_3,X')-path that contains exactly one vertex v_3. Let $T_i=P_i\bigcup x_ix'_i\bigcup P'_i$ for each $i\in\{1,2,\cdots,n-2\}$. Then $n-2$ internally disjoint trees connecting S in $B_{n,2}$ are obtained.

Case 3. v_1,v_2 and v_3 belong to three different copies of $B_{n-1,1}$, respectively.

Without loss of generality, let $v_1\in V(B^1_{n,2}),v_2\in V(B^2_{n,2})$ and $v_3\in V(B^3_{n,2})$. Let $N_{B^i_{n,2}}[v_i]=N_{B^i_{n,2}}(v_i)\bigcup\{v_i\}$ for $i=1,2,3$. By Lemma 9, for each $i\in\{1,2,3\}$ and $j\in\{4,5,\cdots,n\}$, there exists one vertex in $N_{B^i_{n,2}}[v_i]$, say u^j_i, such that the outside neighbour $(u^j_i)'$ of u^j_i belongs to $B^j_{n,2}$. As $B^j_{n,2}$ is connected, we can find a tree $\widehat{T}_j$ connecting $(u^j_1)',(u^j_2)'$ and $(u^j_3)'$ for each $j\in\{4,5,\cdots,n\}$. Let $T_j=\widehat{T}_j\bigcup u^j_1(u^j_1)'\bigcup u^j_2(u^j_2)'\bigcup u^j_3(u^j_3)'\bigcup v_1u^j_1\bigcup v_2u^j_2\bigcup v_3u^j_3$ as $B_{n-1,1}\cong K_{n-1}$, then $n-3$ internally disjoint trees connecting S are obtained. Let $\widehat{B}^i_{n,2}=B^i_{n,2}-(\{u^4_i,u^5_i,\cdots,u^n_i\}\setminus\{v_i\})$. Then there are at most $n-3$ vertices deleted from $B^i_{n,2}$ for each $i\in\{1,2,3\}$. As $B^i_{n,2}$ is $n-2$ connected, $\widehat{B}^i_{n,2}$ is still connected. For $i,j\in\{1,2,3\}$ and $i\neq j$, there is exactly an edge between $B^i_{n,2}$ and $B^j_{n,2}$. Thus, $B_{n,2}[\bigcup_{i=1}^3 V(\widehat{B}^i_{n,2})]$ is connected and there is a tree T_{n-2} connecting S. Hence, there exist $n-2$ internally disjoint trees connecting S in $B_{n,2}$ and the result is desired. □

Next, we prove the generalized 3-connectivity of $B_{n,k}$ for $3\leq k\leq n-1$.

Theorem 2. *$\kappa_3(B_{n,k})=n-2$ for $3\leq k\leq n-1$.*

Proof. As $B_{n,k}$ is $(n-1)$-regular. By Lemma 2, $\kappa_3(B_{n,k})\leq\delta-1=n-2$. To complete the result, it suffices to show that $\kappa_3(B_{n,k})\geq n-2$. We prove the result by induction on n.

For $n=3$, $B_{3,k}$ is connected. Then $\kappa_3(B_{3,k})\geq 1=n-2$.

For $n=4$, by Lemma 3 and Lemma 7, $\kappa_3(B_{n,k})\geq\lceil\frac{3}{2}\rceil=2=n-2$.

Next, suppose that $n \geq 5$. Let $B_{n,k} = B^1_{n,k} \bigoplus B^2_{n,k} \bigoplus \ldots \bigoplus B^n_{n,k}$ and v_1, v_2, v_3 be any three distinct vertices of $B_{n,k}$. For convenience, let $S = \{v_1, v_2, v_3\}$. We prove the result by considering the following three cases.

Case 1. v_1, v_2 and v_3 belong to the same copy of $B_{n-1,k-1}$.

Case 2. v_1, v_2 and v_3 belong to two different copies of $B_{n-1,k-1}$.

Case 3. v_1, v_2 and v_3 belong to three different copies of $B_{n-1,k-1}$, respectively.

The proofs of Case 1 and Case 2 are the same as the proof of Case 1 and Case 2 in Theorem 1. Thus, only the Case 3 is considered.

Without loss of generality, let $v_1 \in V(B^1_{n,k}), v_2 \in V(B^2_{n,k})$ and $v_3 \in V(B^3_{n,k})$. Let $v_1 = p_1p_2 \cdots p_{k-1}1$ and $v_i = p_ip_2 \cdots p_{k-1}1$ for $k+1 \leq i \leq n$, where $p_{k+1}, p_{k+2}, \cdots, p_n$ are distinct elements in $[n] \setminus \{p_1, p_2, \cdots, p_{k-1}, 1\}$. We now present the algorithm, called (n-1)IDP, that constructs $n-1$ internally disjoint paths $P^1_2, P^1_3, \cdots, P^1_n$ in B^1_n such that the outside neighbour of each terminal vertex of the $n-1$ paths belong to different copies of $B_{n-1,k-1}$.

Algorithm 1 (n-1)IDP(k)

Input: n, k, where $3 \leq k \leq n-1$, $v_1 = p_1p_2 \cdots p_{k-1}1$;
Output: $n-1$ pairwise disjoint path $P^1_2, P^1_3, \cdots, P^1_k, P^1_{k+1}, \cdots, P^1_n$;

```
for i = 2 to k − 1 do
    P_i^1 = v_1, t = v_1;
    for j = i to k − 1 do
        t = t(j − 1, j) // where (j − 1, j) is a transposition
        P_i^1 = P_i^1 ∪ t;
    end for
end for
P_k^1 = v_1;
for i = k + 1 to n do
    P_i^1 = v_1 v_i, t = v_i = p_i p_2 ··· p_{k−1} 1;
    for j = 1 to k − 2 do
        t = t(j, j + 1) // where (j, j + 1) is a transposition
        P_i^1 = P_i^1 ∪ t;
    end for
end for
```

By the above algorithm, there are the following $n-1$ paths $P^1_2, P^1_3, \cdots, P^1_n$ starting at the vertex v_1 in $B^1_{n,k}$, where $p_{k+1}, p_{k+2}, \cdots, p_n$ are distinct elements in $[n] \setminus \{p_1, p_2, \cdots, p_{k-1}, 1\}$.

$P^1_2 = (\underline{p_1}p_2p_3 \cdots p_{k-1}1)(i_2\underline{p_1}p_3 \cdots p_{k-1}1)(p_2p_3\underline{p_1} \cdots p_{k-1}1) \cdots (p_2p_3 \cdots p_{k-1}\underline{p_1}1)$;

$P^1_3 = (p_1\underline{p_2}p_3 \cdots p_{k-1}1)(p_1p_3\underline{p_2} \cdots p_{k-1}1) \cdots (p_1p_3 \cdots p_{k-1}\underline{p_2}1)$;

$\cdots$

$P^1_{k-1} = (p_1p_2p_3 \cdots \underline{p_{k-2}}p_{k-1}1)(p_1p_2p_3 \cdots p_{k-1}\underline{p_{k-2}}1)$;

$P^1_k = (p_1p_2p_3 \cdots \underline{p_{k-1}}1)$;

$P^1_{k+1} = (p_1p_2p_3 \cdots p_{k-1}1)(\underline{p_{k+1}}p_2p_3 \cdots p_{k-1}1)(p_2\underline{p_{k+1}}p_3 \cdots p_{k-1}1)(p_2p_3\underline{p_{k+1}} \cdots p_{k-1}1) \cdots (p_2 p_3p_4 \cdots \underline{p_{k+1}}1)$;

$P^1_{k+2} = (p_1p_2p_3 \cdots p_{k-1}1)(\underline{p_{k+2}}p_2p_3 \cdots p_{k-1}1)(p_2\underline{p_{k+2}}p_3 \cdots p_{k-1}1)(p_2p_3\underline{p_{k+2}} \cdots p_{k-1}1) \cdots (p_2 p_3p_4 \cdots \underline{p_{k+2}}1)$;

$\cdots$

$P_n^1 = (p_1p_2p_3\cdots p_{k-1}1)(\underline{p_n}p_2p_3\cdots p_{k-1}1)(p_2\underline{p_n}p_3\cdots p_{k-1}1)(p_2p_3\underline{p_n}\cdots p_{k-1}1)\cdots(p_2p_3p_4\cdots \underline{p_n}$ $1)$.

Claim 1. *For every $a, b \in \{2, 3, \cdots, n\}$ and $a \neq b, V(P_a^1)\bigcap V(P_b^1) = \{v_1\}$.*

The proof of the Claim 1. Without loss of generality, suppose that $a < b$.

If $a, b \in \{2, 3, \cdots, k\}$, then for any vertex $y \in V(P_a^1) \setminus \{v_1\}$, the $a-1$ elements at positions $1, 2, \cdots, a-1$ of y are $p_1p_2\cdots p_{a-2}p_a$. However, for any vertex $z \in V(P_b^1) \setminus \{v_1\}$, the $a-1$ elements at positions $1, 2, \cdots, a-1$ of z are $p_1p_2\cdots p_{a-2}p_{a-1}$. As $p_a \neq p_{a-1}$, then $y \neq z$. Hence, the claim holds.

If $a, b \in \{k+1, \cdots, n\}$, then for any vertex $y \in V(P_a^1) \setminus \{v_1\}$, it is the permutation of $\{p_a, p_2\cdots, p_{k-1}, 1\}$. For any vertex $z \in V(P_b^1) \setminus \{v_1\}$, it is the permutation of $\{p_b, p_2\cdots, p_{k-1}, 1\}$. As $p_a, p_b \in [n] \setminus \{p_1, p_2\cdots, p_{k-1}, 1\}$ and $p_a \neq p_b$, then $y \neq z$. Thus, the claim holds.

If $a \in \{2, 3, \cdots, k\}$ and $b \in \{k+1, \cdots, n\}$, then for any vertex $y \in V(P_a^1) \setminus \{v_1\}$, it is the permutation of $\{p_1, p_2\cdots, p_{k-1}, 1\}$ and for any vertex $z \in V(P_b^1) \setminus \{v_1\}$, it is the permutation of $\{p_b, p_2\cdots, p_{k-1}, 1\}$. As $p_b \in [n] \setminus \{p_1, p_2\cdots, p_{k-1}, 1\}$, then $p_1 \neq p_b$ and $y \neq z$. Thus, the claim holds.

The proof of the Claim 1 is complete.

Claim 2. *Let $X^1 = \{u_i^1 | u_i^1$ is the terminal vertex of the path P_i^1 for each $i \in \{2, 3, \cdots, n\}\}$. Then the outside neighbours of vertices in X^1 belong to different copies of $B_{n-1,k-1}$, respectively.*

The proof of the Claim 2. By Lemma 1(2), the outside neighbours of vertices in X^1 are in $B_{n,k}^2, B_{n,k}^3, \cdots, B_{n,k}^n$, respectively. The proof of the Claim 2 is complete.

Without loss of generality, suppose that the outside neighbour $(u_i^1)'$ of u_i^1 is in $B_{n,k}^i$ for each $i \in \{2, 3, 4, \cdots, n\}$. Otherwise, we can reorder the paths accordingly.

Similarly, let $v_2 = p_1p_2p_3\cdots p_{k-1}2$, then there are $n-1$ paths $P_1^2, P_3^2, \cdots, P_n^2$ starting at the vertex v_2 in $B_{n,k}^2$. Let $X^2 = \{u_1^2, u_3^2, \cdots, u_n^2\}$ such that u_i^2 is the terminal vertex of the path P_i^2 and the outside neighbour $(u_i^2)'$ of u_i^2 is in $B_{n,k}^i$ for each $i \in \{1, 3, 4, \cdots, n\}$. In addition, there are $n-1$ paths $P_1^3, P_2^3, \cdots, P_n^3$ starting at the vertex v_3 in $B_{n,k}^3$. Let $X^3 = \{u_1^3, u_2^3, \cdots, u_n^3\}$ such that u_i^3 is the terminal vertex of the path P_i^3 and the outside neighbour $(u_i^3)'$ of u_i^3 is in $B_{n,k}^i$ for each $i \in \{1, 2, 4, \cdots, n\}$.

Obviously, the outside neighbour $(u_1^3)'$ of u_1^3 is in $B_{n,k}^1$ and the outside neighbour $(u_2^3)'$ of u_2^3 is in $B_{n,k}^2$. As $B_{n,k}^1$ is connected, there is a $((u_1^3)', v_1)$-path $\widehat{P}_1$ in $B_{n,k}^1$. Let t_1 be the first vertex of the path $\widehat{P}_1$ which is in $\bigcup_{l\in\{2,3,\cdots,n\}} V(P_l^1)$. Similarly, there is a $((u_2^3)', v_2)$-path $\widehat{P}_2$ in $B_{n,k}^2$ as $B_{n,k}^2$ is connected. Let t_2 be the first vertex of the path $\widehat{P}_2$ which is in $\bigcup_{l\in\{1,3,\cdots,n\}} V(P_l^2)$.

To prove the result for $3 \leq k \leq n-1$, the following two subcases are considered.

Subcase 3.1. $t_1 \in \bigcup_{l\in\{2,3\}} V(P_l^1)$ and $t_2 \in \bigcup_{l\in\{1,3\}} V(P_l^2)$.

In this case, the induced subgraph $B_{n,k}[V(P_1^3)\bigcup V(P_2^1)\bigcup V(P_3^1)\bigcup V(\widehat{P}_1[(u_1^3)', t_1])]$ of $B_{n,k}$ contains a (v_3, v_1)-path, where $\widehat{P}_1[(u_1^3)', t_1]$ is the subpath of $\widehat{P}_1$ starting at $(u_1^3)'$ and ending at t_1. Similarly, the induced subgraph $B_{n,k}[V(P_2^3)\bigcup V(P_1^2)\bigcup V(P_3^2)\bigcup V(\widehat{P}_2[(u_2^3)', t_2])]$ of $B_{n,k}$ contains a (v_3, v_2)-path, where $\widehat{P}_2[(u_2^3)', t_2]$ is the subpath of $\widehat{P}_2$ starting at $(u_2^3)'$ and

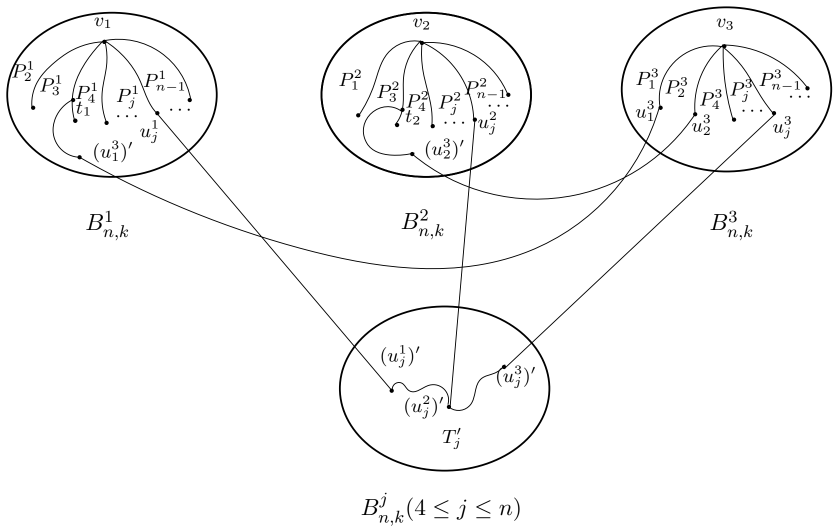

Figure 2: The illustration of Subcase 3.1 for $t_1 \in V(P_3^1)$ and $t_2 \in V(P_3^2)$

ending at t_2. The union of the (v_3, v_1)-path and the (v_3, v_2)-path forms a tree T_1 connecting S in $B_{n,k}$. See Figure 2.

In addition, as $(u_j^1)', (u_j^2)', (u_j^3)' \in V(B^j_{n,k})$ for each $j \in \{4, 5, \cdots, n\}$ and $B^j_{n,k}$ is connected, there is a tree T'_j connecting $(u_j^1)'$, $(u_j^2)'$ and $(u_j^3)'$ in $B^j_{n,k}$. Let $T_j = T'_j \bigcup P_j^1 \bigcup P_j^2 \bigcup P_j^3 \bigcup u_j^1(u_j^1)' \bigcup u_j^2(u_j^2)' \bigcup u_j^3(u_j^3)'$ for each $j \in \{4, 5, \cdots, n\}$. Combining the trees $T_j s$ for $4 \le j \le n$ and the tree T_1, and $n-2$ internally disjoint trees connecting S in $B_{n,k}$ are obtained.

Subcase 3.2. $t_1 \in \bigcup_{l \in \{4,5,\cdots,n\}} V(P_l^1)$ or $t_2 \in \bigcup_{l \in \{4,5,\cdots,n\}} V(P_l^2)$.

Without loss of generality, let $t_1 \in V(P_4^1)$. Note that $v_1 = p_1 p_2 \cdots p_{k-1} 1$. By the assumption that the outside neighbor of the terminal vertex in P_i^1 is in $B^i_{n,k}$ for $i \in \{2, 3, \ldots, k\}$, one has that $v_1 = 23 \cdots k1$. It implies that $p_i = i+1$ for $1 \le i \le k-1$.

If $k \ge 4$, we obtain that $p_{k-1} \ne 2$ and $p_3 = 4$. For any vertex $v \in V(P_4^1)$, v is a permutation of $\{p_1, p_2, \cdots, p_{k-1}, 1\}$. Next, we consider the path P_2^1. Note that u_2^1 is the terminal vertex of P_2^1 and $u_2^1 = p_2 p_3 \cdots p_{k-1} p_1 1 = 34 \cdots k21$. We can extend the path P_2^1 starting from u_2^1 as follows: $(3\underline{4}56 \cdots k21)(35\underline{4}6 \cdots k21) \cdots (35 \cdots 26\underline{k}41)$. Let $\widehat{u}_2^1 = 35 \cdots 241$ and the extended path starting at v_1 and ending at $\widehat{u}_2^1$ be $\widehat{P}_2^1$. Then the outside neighbour of $\widehat{u}_2^1$ is in $B^4_{n,k}$.

If $k = 3$ and $t_1 \ne v_1$, then $v_1 = 231$ and $4 \in [n] \setminus \{p_1, p_2, 1\} = \{4, 5, \ldots, n\}$ and the vertex t_1 is a permutation of $\{4, p_2, 1\} = \{4, 3, 1\}$. Note that $u_2^1 = p_2 21 = 321$. Now, we extend the path P_2^1 starting from u_2^1 to $\widehat{P}_2^1$, where $\widehat{P}_2^1 = P_2^1(421)(241)$. Let $\widehat{u}_2^1 = 241$. Now replacing P_2^1 with $\widehat{P}_2^1$, The outside neighbor of terminal vertex $\widehat{u}_2^1$ of $\widehat{P}_2^1$ is in $B^4_{n,k}$.

Next, we prove the following claim.

Claim 3. *$V(\widehat{P}_2^1) \bigcap V(P_j^1) = \{v_1\}$ for each $j \in \{3, 4, \cdots, n\}$ for $k \ge 3$.*

The proof of Claim 3. For $k \ge 4$, we prove the result by contradiction. Suppose that there exists $l \in \{3, 4, \cdots, n\}$ such that $|V(\widehat{P}_2^1) \bigcap V(P_l^1)| \geq 2$. Assume that $u \in V(\widehat{P}_2^1) \bigcap V(P_l^1)$ and $u \neq v_1$. Since $V(P_2^1) \bigcap V(P_l^1) = \{v_1\}, u \notin V(P_2^1)$. Thus, $u \in V(\widehat{P}_2^1) \setminus V(P_2^1)$.

If $u \neq \widehat{u}_2^1$, then the element at position $k-1$ of u is 2. However, the element at position $k-1$ of each vertex in $V(P_l^1)$ is p_{k-1} or k. As $k \neq 2$ and $p_{k-1} \neq 2$, a contradiction.

Next, suppose $u = \widehat{u}_2^1$. The $k = 4$ and $u = u_4^1$. However, the element at position $k-2$ of u_4^1 is i_{k-1}, a contradiction.

For $k = 3$, let $x \in V(P_m^1)$ for $4 \leq m \leq n$, then it is a permutation of $\{m, 3, 1\}$. However, for any vertex $y \in V(\widehat{P}_2^1 \setminus P_2^1)$, it is a permutation of $\{4, 2, 1\}$. Thus, $x \neq y$. The proof of the claim is complete.

Similarly, if $t_2 \in V(P_\ell^2)$ and $\ell \in \{4, 5, \cdots, n\}$, we can extend the path P_2^2 to obtain the extended path, say $\widehat{P}_2^2$, such that the outside neighbour of the terminal vertex of the extended path $\widehat{P}_2^2$ is in $B_{n,k}^\ell$ and there is only one common vertex v_2 between the extended path and other paths $P_j s$ in $B_{n,k}^2$.

Since the induced subgraph $B_{n,k}[V(P_1^3) \bigcup V(\widehat{P}_1[(u_1^3)', t_1]) \bigcup V(P_4^1)]$ contains a (v_3, v_1)-path, say D_1. Similarly, the induced subgraph $B_{n,k}[V(P_2^3) \bigcup V(\widehat{P}_2[(u_2^3)', t_2]) \bigcup V(P_4^1)]$ contains a (v_3, v_2)-path, say D_2. A tree, say T_1, by combining D_1 and D_2 is obtained and the tree T_1 connects S in $B_{n,k}$.

Similar as subcase 3.1 just by replacing P_4^1 with $\widehat{P}_2^1$ as $t_1 \in V(P_4^1)$ or replacing P_ℓ^2 with $\widehat{P}_2^2$ if $t_2 \in V(P_\ell^2)$ for $\ell \in \{4, 5, \cdots, n\}$, there is a tree T_j connecting $S \bigcup V(B_{n,k}^j)$ for each $j \in \{4, 5, \cdots, n\}$ and $T_j s$ are internally disjoint S-trees. Combining the trees $T_j s$ for $4 \leq j \leq n$ and the tree T_1, $n-2$ internally disjoint trees connecting S in $B_{n,k}$ are obtained. Thus, the result is desired.

□

4 Concluding remarks

The generalized k-connectivity is a generalization of traditional connectivity. In this paper, we focus on the (n, k)-bubble-sort graph, denoted by $B_{n,k}$. We study the generalized 3-connectivity of $B_{n,k}$ and show that $\kappa_3(B_{n,k}) = n - 2$ for $2 \leq k \leq n-1$. So far, there are few results about the generalized k-connectivity for larger k. We are interested in this topic and we would like to study in this direction to show the corresponding results of $B_{n,k}$ for $k \geq 4$.

Acknowledgments

This work was supported by the National Natural Science Foundation of China (No. 11731002), the Fundamental Research Funds for the Central Universities (No. 2016JBM071, 2016JBZ012) and the 111 Project of China (B16002).

References

[1] J. Bondy and U. Murty, Graph Theory, Springer, 2008.

[2] G. Chartrand, S.F. Kapoor, L. Lesniak, D.R. Lick, Generalized connectivity in graphs, Bull. Bombay Math. Colloq. 2 (1984) 1-6.

[3] G. Chartrand, F. Okamoto and P. Zhang, Rainbow trees in graphs and generalized connectivity, Networks 55 (4) (2010) 360-367.

[4] H. Li, X. Li and Y. Sun, The generalized 3-connectivity of Cartesian product graphs, Discret. Math. Theor. Comput. Sci. 14 (1) (2012) 43-54.

[5] S. Li, Y. Shi and J. Tu, The generalized 3-connectivity of Cayley graphs on Symmetric Groups generated by trees and cycles, Graph. Combinator., 33 (2017) 1195-1209.

[6] H. Li, X. Li, Y. Mao and Y. Sun, Note on the generalized connectivity, Ars Comb. 114 (2014) 193-202.

[7] S. Li, J. Tu and C. Yu, The generalized 3-connectivity of star graphs and bubble-sort graphs, Appl. Math. Comput. 274 (2016) 41-46.

[8] S. Li and X. Li, Note on the hardness of generalized connectivity, J. Comb. Optim. 24 (2012) 389-396.

[9] H. Li, Y. Ma, W. Yang and Y. Wang, The generalized 3-connectivity of graph products, Appl. Math. Comput. 295 (2017) 77-83.

[10] S. Li, X. Li and W. Zhou, Sharp bounds for the generalized connectivity $\kappa_3(G)$, Discret. Math. 310 (2010) 2147-2163.

[11] S. Li, W. Li, Y. Shi and H. Sun, On the mininmally 2-connected graphs with generalized connectivity $\kappa_3 = 2$, J. Comb. Optim. 34 (2017) 141-164.

[12] S. Li, W. Li and X. Li, The generalized connectivity of complete bipartite graphs, Ars Comb. 104 (2012) 65-79.

[13] S. Lin and Q. Zhang, The generalized 4-connectivity of hypercubes, Discret. Appl. Math. 220 (2017) 60-67.

[14] N. Shavash, Relationships among popular interconnection networks and their common generalization, Ph.D. thesis, Oakland University, 2008.

[15] H.-Z. Shi and J.-B. Lu, On conjectures of interconnection networks, Comput. Eng. Appl. 44 (2008) 199-200.

[16] H. Whitney, Congruent graphs and connectivity of graphs, Am. J. Math., 54 (1932) 150-168.