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Inequalities on the Number of Connected Spanning Subgraphs in a Multigraph

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SUMMARY Consider an undirected multigraph $G = (V, E)$ with n vertices and m edges, and let N_i denote the number of connected spanning subgraphs with $i(m \geqq i \geqq n)$ edges in G. Recently, we showed in [3] the validity of $(m-i+1)N_{i-1} > (i-n+\lfloor \frac{3+\sqrt{9+8(i-n)}}{2} \rfloor)N_i$ for a simple graph and each $i(m \geqq i \geqq n)$. Note that, from this inequality, $\frac{(m-n)N_n}{2N_{n+1}} + \frac{N_n}{(m-n+1)N_{n-1}} \geqq 2$ is easily derived. In this paper, for a multigraph G and all $i(m \geqq i \geqq n)$, we prove $(m-i+1)N_{i-1} \geqq (i-n+2)N_i$, and give a necessary and sufficient condition by which $(m-i+1)N_{i-1} = (i-n+2)N_i$. In particular, this means that $(m-i+1)N_{i-1} > (i-n+\lfloor \frac{3+\sqrt{9+8(i-n)}}{2} \rfloor)N_i$ is not valid for all multigraphs, in general. Furthermore, we prove $\frac{(m-n)N_n}{2N_{n+1}} + \frac{N_n}{(m-n+1)N_{n-1}} \geqq 2$, which is not straightforwardly derived from $(m-i+1)N_{i-1} \geqq (i-n+2)N_i$, and also introduce a necessary and sufficent condition by which $\frac{(m-n)N_n}{2N_{n+1}} + \frac{N_n}{(m-n+1)N_{n-1}} = 2$. Moreover, we show a sufficient condition for a multigraph to have $N_n^2 > N_{n-1}N_{n+1}$. As special cases of the sufficient condition, we show that if G contains at least $\lceil \frac{2}{3}(m-n) \rceil + 1$ multiple edges between some pair of vertices, or if its underlying simple graph has no cycle with length more than 4, then $N_n^2 > N_{n-1}N_{n+1}$.

key words: *multigraph, the number of connected spanning subgraphs, network reliability polynomial, inequality*

1. Introduction

In network reliability analysis, a network is usually modeled by an undirected graph with n vertices and m edges, where all vertices are reliable, and each edge is either operational or failed with the same independently operational probability $p(0 < p < 1)$. The reader may refer to [5] for background on network reliability.

Let N_i for an integer $i(m \geqq i \geqq n-1)$ denote the number of connected spanning subgraphs with i edges in G. Then, $N_{n-1}, N_n, \cdots, N_m$, called the *coefficient sequence of all-terminal reliability polynomial* (see e.g., [1], [2], [5], [6], [8]), are used to estimate the all-terminal reliability $Rel_A(G, p)$ defined by

$$Rel_A(G,p) = \sum_{i=n-1}^{m} N_i p^i (1-p)^{m-i}. \tag{1}$$

It is well known that computing $Rel_A(G, p)$ is NP-hard, even if the graphs are restricted to be planar, since the problem of computing N_i's is #P-complete [9], [10]. Thus, it is important to find inequalities useful for approximately computing N_i's. Many extensive investigations on the computation problem have been done, and properties of N_i's have been summarized in [2], [5], [8].

Little, however, is known about the inequalities with respect to N_{i-1}, N_i or N_{i-1}, N_i, N_{i+1} other than Sperner's inequality $iN_i \geqq (m-i+1)N_{i-1}$ (see e.g., [5]). This may be a reason that it has been not shown whether G has unimodality or log-concavity on the sequence $N_{n-1}, N_n, \cdots, N_m$ in [5], [6]. Here, *unimodality* is a property that there is some index i such that $N_{n-1} \leqq N_n \leqq \cdots \leqq N_i \geqq N_{i+1} \geqq \cdots \geqq N_m$, and *log-concavity* is a property that $N_i^2 \geqq N_{i-1}N_{i+1}$ for $m > i \geqq n$. Furthermore, we easily see by (1) that for such a probabilistic graph (G, p) when p is very small, $Rel_A(G, p)$ is mainly determined by the terms on the three coefficients N_{n-1}, N_n, N_{n+1}. Then, it is also interesting to investigate a formula on N_{n-1}, N_n, N_{n+1}.

In this paper, by introducing the average value $h(\Phi_G^i; d)$ of $N(G_r^i; i-d)$'s, where $N(G_r^i; i-d)$ for each $r(N_i \geqq r \geqq 1)$ denotes the number of connected spanning $(i-d)$-edge subgraphs in a connected spanning i-edge subgraph G_r^i of G, we establish two formulas $h(\Phi_G^i; 1)N_i = (m-i+1)N_{i-1}$ and $i \geqq h(\Phi_G^i; 1) \geqq i-n+2$ for all $m \geqq i \geqq n$. In particular, we show the characterizations of multigraphs where $h(\Phi_G^i; 1) = (i-n+2)$ for all $m \geqq i \geqq n$, and $h(\Phi_G^i; 1)$ nearly equals to $(i-n+2)$ for a fixed i, respectively.

As a result, for a multigraph G and all $m \geqq i \geqq n$, we obtain $(m-i+1)N_{i-1} \geqq (i-n+2)N_i$ in Sect. 2, and characterize the multigraphs with $(m-i+1)N_{i-1} = (i-n+2)N_i$ in Sect. 3. It implies that for all multigraphs, $(m-i+1)N_{i-1} > (i-n+\lfloor \frac{3+\sqrt{9+8(i-n)}}{2} \rfloor)N_i$ does not hold in general, even though it has been applied to show several simple graphs with unimodality on $N_{n-1}, N_n, \cdots, N_m$ in [4]. This, in fact, means that there is a difference in inequalities of N_{i-1}, N_i between simple graphs and multigraphs.

On the other hand, when G is a simple graph, $\frac{(m-n)N_n}{2N_{n+1}} \geqq 2$ has been shown in [3] by the fact that the length of every cycle in a simple graph is at least 3. It is clear that $\frac{(m-n)N_n}{2N_{n+1}} + \frac{N_n}{(m-n+1)N_{n-1}} > 2$ by $\frac{N_n}{(m-n+1)N_{n-1}} > 0$. However, when G is a multigraph, since G contains cycles of length 2, $(m-i+1)N_{i-1} \geqq (i-n+2)N_i$ holds with equality. In addition, we show that there exist some multigraphs so that not only $\frac{N_n}{(m-n+1)N_{n-1}} < \frac{1}{2}$, but also $\frac{(m-n)N_n}{2N_{n+1}}$ nearly equals to $\frac{3}{2}$. Then, it is not necessarily obvious whether $\frac{(m-n)N_n}{2N_{n+1}} + \frac{N_n}{(m-n+1)N_{n-1}} \geqq 2$ for some multigraphs. In Sect. 4,

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we prove that it is true for all multigraphs as well, and show that $\frac{(m-n)N_n}{2N_{n+1}} + \frac{N_n}{(m-n+1)N_{n-1}} = 2$ iff G is a connected multigraph containing a pair of vertices with $m-n+2$ multiple edges. It is well known that such a multigraph is the least reliable one for all-terminal network reliability (see e.g., [1], [2]).

Before closing Sect. 4, we propose a sufficient condition for a multigraph G with $N_n^2 > N_{n-1}N_{n+1}$. That is, G contains at least $\left\lceil \frac{2(h(\Phi_G^n;1)-3)(m-n)}{(h(\Phi_G^n;1)-2)h(\Phi_G^{n+1};1)} \right\rceil + 1$ multiple edges between some pair of vertices. Moreover, we show that if G has at least $\left\lceil \frac{2}{3}(m-n) \right\rceil + 1$ multiple edges between some pair of vertices, or, no simple cycle with length more than 4, then it satisfies the sufficient condition.

2. Preliminaries

Consider an undirected multigraph $G = (V, E)$ with no loop. Unless defined otherwise, graph theoretic terminology used in this paper follows Harary [7]. We always assume that a given multigraph G is connected, and has n vertices and m edges.

A simple graph is the graph without multiple edges. By replacing the multiple edges between every pair of vertices with one edge, we can obtain a simple graph, called its *underlying graph*. One of the most basic facts is that a multigraph has some cycle with length 2, while the length of every cycle in a simple graph is at least 3.

For an edge subset $U(\subseteqq E)$, let $G - U$ denote the spanning subgraph obtained by removing all edges of U from G. An edge subset $U(\subseteqq E)$ is said to be an edge-cut if $G - U$ is not connected, and let λ_G be the minimum cardinality of an edge-cut in G. An edge e is said to be a bridge if $G - \{e\}$ is not connected. We denote by $N(G; i)$, which is sometimes briefly denoted by N_i, the number of connected spanning i-edge subgraphs of G. Note that G has exactly $\binom{m}{i}$ spanning i-edge subgraphs each of which is either connected or not. It is clear that whenever $i < n-1$, any spanning i-edge subgraph of G is not connected, and whenever $i > m - \lambda_G$, any spanning i-edge subgraph is connected by the definition of λ_G. Thus,

$$
\begin{aligned}
N_i &= 0, && i < n-1;\\
N_i &\leqq \binom{m}{i}, && n-1 \leqq i \leqq m-\lambda_G;\\
N_i &= \binom{m}{i}, && i > m-\lambda_G.
\end{aligned}
$$

Let $\Phi_G^i = \{G_1^i, G_2^i, \cdots, G_{N_i}^i\}$ denote the set of all connected spanning i-edge subgraphs of G. Given a $G_r^i \in \Phi_G^i$, $N(G_r^i; i-d)$ for an integer $d(i-n+1 \geqq d \geqq 1)$ represents the number of connected spanning $(i-d)$-edge subgraphs of G_r^i. In other words, it is equal to the number of connected spanning subgraphs each of which is obtained by removing d edges from G_r^i. We further define $h(\Phi_G^i; d)$ by

$$h(\Phi_G^i; d) = \frac{\sum_{G_r^i \in \Phi_G^i} N(G_r^i; i-d)}{N_i}, \tag{2}$$

which represents the average of N_i values: $N(G_1^i; i-d)$, $N(G_2^i; i-d)$, $\cdots$, $N(G_{N_i}^i; i-d)$.

Note that every connected spanning $(i-d)$-edge subgraph of G is contained as a subgraph in exactly $\binom{m-(i-d)}{d}$ connected spanning i-edge subgraphs of Φ_G^i, and every $G_r^i \in \Phi_G^i$ contains $N(G_r^i; i-d)$ connected spanning $(i-d)$-edge subgraphs of G. Consequently, we can show the validity of

$$\sum_{G_r^i \in \Phi_G^i} N(G_r^i; i-d) = \binom{m-(i-d)}{d} N_{i-d}. \tag{3}$$

Lemma 1: For a multigraph G and two integers $i, d(m \geqq i \geqq n, i-n+1 \geqq d \geqq 1)$,

$$h(\Phi_G^i; d) = \binom{m-i+d}{d} \frac{N_{i-d}}{N_i}. \tag{4}$$

Proof. It is trivial by (2) and (3). □

Essentially, lemma 1 establishes a relation between $h(\Phi_G^i; d)$ and N_i. This means that the problem of computing $h(\Phi_G^i; d)$'s is also #P-complete as that of computing N_i's, since N_{n-1} represents the number of spanning trees of G and is counted in polynomial time. Moreover, when $d = 1$, (4) is written by

$$(m-i+1)N_{i-1} = h(\Phi_G^i; 1)N_i. \tag{5}$$

From (5) we obtain

$$\frac{h(\Phi_G^{i+1}; 1)}{h(\Phi_G^i; 1)} = \frac{(m-i)N_i^2}{(m-i+1)N_{i-1}N_{i+1}}, \tag{6}$$

which implies that if $h(\Phi_G^{i+1}; 1) \geqq h(\Phi_G^i; 1)$ then $N_i^2 > N_{i-1}N_{i+1}$. Therefore, proving $h(\Phi_G^{i+1}; 1) \geqq h(\Phi_G^i; 1)$ for all $i(m > i \geqq n)$ is more hard than proving log-concavity on the sequence $N_{n-1}, N_n, \cdots, N_m$, in general.

An edge of G is said to be a *non-bridge edge* if it is not a bridge of G. By definition, $N(G_r^i; i-1)$ and $h(\Phi_G^i; 1)$ respectively expresses the number of non-bridge edges of G_r^i and the average value of the numbers of non-bridge edges for N_i connected spanning i-edge subgraphs G_r^i's of G. In the following lemma, we give an inequality on $h(\Phi_G^i; 1)$.

Lemma 2: For a multigraph G and an integer $i(m \geqq i \geqq n)$,

$$i \geqq h(\Phi_G^i; 1) \geqq i-n+2.$$

Proof. It is clear that the number of $(i-1)$-edge subgraphs obtained by removing one edge from an i-edge graph is equal to at most i. Therefore, we obtain $i \geqq N(G_r^i; i-1)$ for every $G_r^i \in \Phi_G^i$, which implies that $i \geqq h(\Phi_G^i; 1)$ by definition.

On the other hand, we can see that the number of connected spanning subgraphs obtained by removing one edge from a connceted i-edge graph is equal to at least $i-n+2$, since $i \geqq n$. Thus, $N(G_r^i; i-1) \geqq i-n+2$ for every $G_r^i \in \Phi_G^i$, equivalently, $h(\Phi_G^i; 1) \geqq i-n+2$ by definition. □

As straightforward results from lemma 2 and (5), we obtain two inequalities:

$$iN_i \geqq (m - i + 1)N_{i-1}, \tag{7}$$

which is well known as Sperner's inequality (see, e.g., [5]), and

$$(m - i + 1)N_{i-1} \geqq (i - n + 2)N_i. \tag{8}$$

For a simple graph, however, it has been shown in [3] that the coefficient of N_i in the right-hand side of (8) is $i - n + \lfloor \frac{3+\sqrt{9+8(i-n)}}{2} \rfloor$, which is strictly greater than $i - n + 2$.

3. The Characterizations of Multigraphs with $h(\Phi^i_G; 1) = i-n+2$, and with $h(\Phi^i_G; 1)$ Nearly Equal to $i-n+2$, Respectively

In this section, we concentrate on investigating the characterizations of multigraphs for which $h(\Phi^i_G; 1) = i - n + 2$ for all $m \geqq i \geqq n$, and $h(\Phi^i_G; 1)$ nearly equals to $i - n + 2$ for a fixed i, respectively.

A multigraph is said to be *simplified*, if it has one pair of vertices with $m - n + 2$ multiple edges. The multigraph shown in Fig. 1 (a) is simplified, while that shown in Fig. 1 (b) is not simplified. Note that the underlying graph of a simplified multigraph is a spanning tree, since a multigraph considered here is connected.

Lemma 3: If G is a simplified multigraph, then

$$(m - i + 1)N_{i-1} = (i - n + 2)N_i$$

for all $m \geqq i \geqq n$.

Proof. Since G is simplified, it is easy to see that for all $m \geqq i \geqq n - 1$

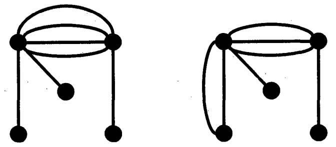
$$N_i = \binom{m - n + 2}{i - n + 2},$$

which shows the validity of this lemma. □

In fact, lemma 3 asserts that $h(\Phi^i_G; 1) = i - n + 2$ for all $m \geqq i \geqq n$ by (5). Indeed, it is easily verified that $N(G^i_r; i - 1) = i - n + 2$ for every $G^i_r \in \Phi^i_G$ when G is simplified. For $i \geqq n$, we can observe that every $G^i_r \in \Phi^i_G$ has at most $n - 2$ bridges. This means that the number of connected spanning $(i - 1)$-edge subgraphs of $G^i_r \in \Phi^i_G$ is at least $i - n + 2$, equivalently,

$$N(G^i_r; i - 1) \geqq i - n + 2.$$

We can also observe that if G is not simplified, then there is at least one connected spanning i-edge subgraph G^i_r so that $N(G^i_r; i - 1) > (i - n + 2)$ for each $i(m \geqq i \geqq n + 1)$, equivalently, $h(\Phi^i_G; 1) > (i - n + 2)$. Hence the following theorem has been obtained by lemma 3 and (5).



(a) Simplified (b) Not simplified

Fig. 1 Two multigraphs whose underlying graphs are trees.

Theorem 1: For a multigraph G, $h(\Phi^i_G; 1) = i - n + 2$ for all $m \geqq i \geqq n + 1$ iff G is simplified. □

When $i = n$, we shall show that $h(\Phi^n_G; 1) = 2$ holds for a multigraph G whose underlying graph is a tree. This means that the condition that G is simplified is not necessary for G to satisfy $h(\Phi^n_G; 1) = 2$, since a multigraph whose underlying graph is a tree may not be simplified, in general. In order to characterize the multigraphs with $h(\Phi^n_G; 1) = 2$, we need new notations.

Recall that Φ^i_G stands for the set of connected spanning i-edge subgraphs of G. Let $\Phi^i_G(k) \subseteqq \Phi^i_G$ for an integer $k(i \geqq k \geqq i - n + 2)$ denote the set of subgraphs with $i - k$ bridges. See Fig. 2.

In particular, $\Phi^i_G(i)$ where $k = i$ is the set of connected spanning i-edge subgraphs with no bridge, and $\Phi^i_G(i-n+2)$ where $k = i - n + 2$ is the set of connected spanning i-edge subgraphs with $n - 2$ bridges. Evidently, Φ^i_G is partitioned into subsets $\Phi^i_G(i - n + 2)$, $\Phi^i_G(i - n + 3)$, $\cdots$, $\Phi^i_G(i)$. Let, further, $N_i(k) = |\Phi^i_G(k)|$ for $i \geqq k \geqq i - n + 2$. Then,

$$N_i = \sum_{k=i-n+2}^{i} N_i(k). \tag{9}$$

Since $N(G^i_r; i - 1) = k$ for every $G^i_r \in \Phi^i_G(k)$,

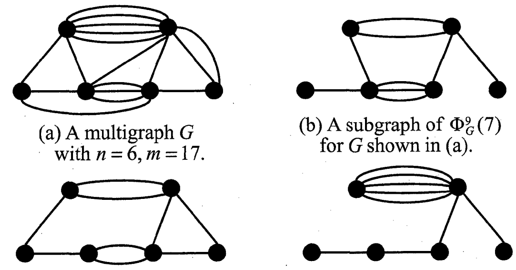
$$\begin{aligned} &\sum_{G^i_r \in \Phi^i_G} N(G^i_r; i - 1) \\ &= \sum_{k=i-n+2}^{i} \left(\sum_{G^i_r \in \Phi^i_G(k)} N(G^i_r; i - 1) \right) \\ &= \sum_{k=i-n+2}^{i} kN_i(k). \end{aligned} \tag{10}$$

Thus, $h(\Phi^i_G; 1)$ is rewritten as follows:

$$\begin{aligned} h(\Phi^i_G; 1) &= \frac{\sum_{k=i-n+2}^{i} kN_i(k)}{\sum_{k=i-n+2}^{i} N_i(k)} \quad \text{(by (2),(9),(10))} \\ &= (i - n + 2) + \beta_i, \end{aligned} \tag{11}$$

where

$$\beta_i = \frac{\sum_{k=i-n+3}^{i} [k - (i - n + 2)]N_i(k)}{\sum_{k=i-n+2}^{i} N_i(k)}. \tag{12}$$



(a) A multigraph G with $n = 6, m = 17$.

(b) A subgraph of $\Phi^9_G(7)$ for G shown in (a).

(c) A subgraph of $\Phi^9_G(9)$ for G shown in (a).

(d) A subgraph of $\Phi^9_G(5)$ for G shown in (a).

Fig. 2 Illustrating subgraphs of $\Phi^i_G(k)$.

Clearly, $h(\Phi^i_G; 1)$ is completely determined by β_i. From lemma 2, we immediately obtain $n-2 \geqq \beta_i \geqq 0$. When $i = n$, by (11),(12),

$$h(\Phi^n_G; 1) = 2 + \frac{\sum_{k=3}^{n}(k-2)N_n(k)}{\sum_{k=2}^{n} N_n(k)}. \tag{13}$$

It is not hard to verify that, whenever G has at least one simple cycle with length $k \geqq 3$, it contains at least one connected spanning n-edge subgraph with at most $n-k$ bridges. This means that $N_n(k) > 0$ for some $k \geqq 3$, equivalently, $h(\Phi^n_G; 1) > 2$ by (13). Hence the following lemma 4 holds.

Lemma 4: If G contains at least one simple cycle with length at least 3, then $h(\Phi^n_G; 1) > 2$, equivalently, $\frac{N_n}{(m-n+1)N_{n-1}} < \frac{1}{2}$ by (5) with $i = n$. □

Theorem 2: For a multigraph G, $h(\Phi^n_G; 1) = 2$ iff the underlying graph of G is a tree.

Proof. Necessity. It is trivial by lemma 4.
Sufficiency. Let G be a multigraph whose underlying graph is a tree. Then the length k of every cycle in G must be equal to 2. Therefore, $\sum_{k=3}^{n}(k-2)N_n(k) = 0$, which is equivalent to $h(\Phi^n_G; 1) = 2$ by (13). □

In the following, we shall discuss the characterization of multigraphs for which $h(\Phi^i_G; 1)$ nearly equals to $i-n+2$, namely, β_i nearly equals to 0 for a fixed i.

Let E_{uv} denote the set of multiple edges between a pair $e = (u, v)$ of vertices in G. Let $G - E_{uv}$ and, for short, $G_{\check{e}}$ denote the graph obtained by deleting all edges of E_{uv}. See Fig. 3 (a).

We easily see that every $G^i_r \in \Phi^i_G$ contains at most $i-n+2$ edges in E_{uv}, but may contain no edge in E_{uv}. Clearly, if $G^i_r \in \Phi^i_G(k)$ has $i-n+2$ edges in E_{uv}, then it must contain $n-2$ bridges, which implies that $k = i-n+2$. In other words, if $k > i-n+2$, then $G^i_r \in \Phi^i_G(k)$ contains at most $i-n+1$ edges in E_{uv}.

For an integer $t(i-n+2 \geqq t \geqq 1)$, we denote by $\Phi^{i-t}_{G_{\check{e}}}(k; e^t)$ the set of spanning $(i-t)$-edge subgraphs of $G_{\check{e}}$, from each of which at least one connected spanning i-edge subgraph with $i-k$ bridges is obtained by adding t edges of E_{uv}. See Fig. 3 (b).

It is clear by definition that exactly $\binom{|E_{uv}|}{t}$ connected spanning i-edge subgraphs of G are obtained from every $G^{i-t}_{\check{e}} \in \Phi^{i-t}_{G_{\check{e}}}(k; e^t)$.

Let $\Phi^{i-0}_{G_{\check{e}}}(k; e^0)$ be the set of subgraphs of $\Phi^i_G(k)$ with no edge in E_{uv}, and let $N^{G_{\check{e}}}_{i-t}(k; e^t) = |\Phi^{i-t}_{G_{\check{e}}}(k; e^t)|$ for $i-n+2 \geqq t \geqq 0$.

When $i \geqq k \geqq i-n+3$, each subgraph of $\Phi^i_G(k)$ has at most $n-3$ bridges by definition. This means that each subgraph of $\Phi^i_G(k)$ contains at least $n-1$ edges not in E_{uv}, equivalently, at most $i-n+1$ edges in E_{uv}. Then, for $i \geqq k \geqq i-n+3$,

$$N_i(k) = \sum_{t=0}^{i-n+1} \binom{|E_{uv}|}{t} N^{G_{\check{e}}}_{i-t}(k; e^t). \tag{14}$$

When $k = i-n+2$, each subgraph of $\Phi^i_G(i-n+2)$ has exactly $n-2$ bridges by definition. This means that if each subgraph of $\Phi^i_G(i-n+2)$ has some edges of E_{uv} then it contains either only one edge, or exactly $i-n+2$ edges of E_{uv}. Therefore, we obtain $N^{G_{\check{e}}}_{i-t}(i-n+2; e^t) = 0$ for $i-n+3 \geqq t \geqq 2$. Note that each subgraph of $\Phi^i_G(i-n+2)$ might contain no edge of E_{uv}. Then, for $k = i-n+2$

$$\begin{aligned} &N_i(i-n+2) \\ &= N^{G_{\check{e}}}_i(i-n+2; e^0) + \binom{|E_{uv}|}{1} N^{G_{\check{e}}}_{i-1}(i-n+2; e^1) \\ &\quad + \binom{|E_{uv}|}{i-n+2} N^{G_{\check{e}}}_{n-2}(i-n+2; e^{i-n+2}). \end{aligned} \tag{15}$$

By definition, the value of $N^{G_{\check{e}}}_{i-t}(k; e^t)$ is independent of $|E_{uv}|$. Then, $\frac{\sum_{k=i-n+3}^{i}[k-(i-n+2)]N_i(k)}{\binom{|E_{uv}|}{i-n+2}}$ for a fixed i is a decreasing function in $|E_{uv}|$ by (14), which implies that

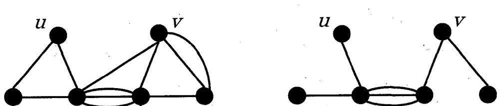
$$\lim_{|E_{uv}|\to\infty} \frac{\sum_{k=i-n+3}^{i}[k-(i-n+2)]N_i(k)}{\binom{|E_{uv}|}{i-n+2}} = 0.$$

Moreover, it is verified that $\frac{\sum_{k=i-n+2}^{i} N_i(k)}{\binom{|E_{uv}|}{i-n+2}}$ is at least $N^{G_{\check{e}}}_{n-2}(i-n+2; e^{i-n+2})$ by (14), (15). Consequently, we obtain an interesting fact that the value of β_i is reduced by adding a number of edges into a fixed pair $e = (u, v)$ of vertices.

Lemma 5: Suppose that G is a multigraph with $\frac{(m-n)N_n}{2N_{n+1}} \geqq 2$. Then we can obtain a multigraph G' by adding some multiple edges between a fixed pair $e = (u, v)$ of vertices in G so that $\frac{(m-n)N'_n}{2N'_{n+1}} < 2$, where N'_n, N'_{n+1} are respectively defined to correspond to G'.

Proof. Let E'_{uv} be the set of edges between the pair $e = (u, v)$ of vertices in G'. See Fig. 4.

By (5), (11), (12) with $i = n+1$, it is clear that if $\beta'_{n+1} = \frac{\sum_{k=4}^{n+1}(k-3)N'_{n+1}(k)}{\sum_{k=3}^{n+1} N'_{n+1}(k)} < 1$ then $\frac{(m-n)N'_n}{2N'_{n+1}} < 2$, where $N'_{n+1}(k)$ is defined to correspond to G'.



(a) $G_{\check{e}}$ of G shown in Fig.2 (a).
(b) A subgraph of $\Phi^{9-2}_{G_{\check{e}}}(7; e^2)$ for G shown in Fig.2 (a).

Fig. 3 Illustration of $G_{\check{e}}$ and a subgraph of $\Phi^{i-t}_{G_{\check{e}}}(k; e^t)$.

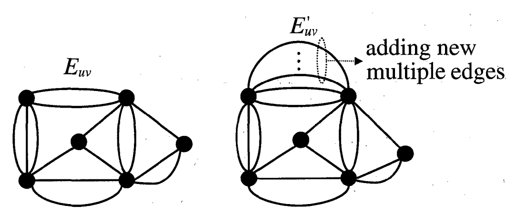


Fig. 4 Adding new multiple edges in a fixed pair of vertices.

$N'_{n+1}(3)$ is the number of connected spanning $(n+1)$-edge subgraphs of G', each of which contains $n-2$ bridges. A connected spanning $(n+1)$-edge subgraph with $n-2$ bridges contains t edges of E'_{uv}, where $t = 3, 1, 0$. By (15) with $i = n+1$,

$$N'_{n+1}(3) = N^{G_{\bar{e}}}_{n+1}(3; e^0) + \binom{|E'_{uv}|}{1} N^{G_{\bar{e}}}_{n}(3; e^1) + \binom{|E'_{uv}|}{3} N^{G_{\bar{e}}}_{n-2}(3; e^3).$$

By (14) with $i = n+1$,

$$N'_{n+1}(k) = \sum_{t=0}^{2} \binom{|E'_{uv}|}{t} N^{G_{\bar{e}}}_{n+1-t}(k; e^t).$$

By solving $\beta_{n+1} = \frac{\sum_{k=4}^{n+1}(k-3)N'_{n+1}(k)}{\sum_{k=3}^{n+1} N'_{n+1}(k)} < 1$, we obtain

$$|E'_{uv}| > 2 + \frac{3}{\binom{|E'_{uv}|}{2} N^{G_{\bar{e}}}_{n-2}(3; e^3)} \Bigg[\sum_{k=4}^{n+1}(k-4) \sum_{t=0}^{2} \binom{|E'_{uv}|}{t} N^{G_{\bar{e}}}_{n+1-t}(k; e^t) - \binom{|E'_{uv}|}{1} N^{G_{\bar{e}}}_{n}(3; e^1) - N^{G_{\bar{e}}}_{n+1}(3; e^0) \Bigg].$$

Since the value of $N^{G_{\bar{e}}}_{n+1-t}(k; e^t)$ is independent of $|E'_{uv}|$, the right-hand side of the above formula is a decreasing function in $|E'_{uv}|$. Consequently, we can obtain some integer r so that if $|E'_{uv}| \geqq r$ then the above formula holds. This means that by adding $|E'_{uv}| - |E_{uv}|$ new multiple edges between the pair $e = (u, v)$ of vertices we can obtain G' so that $\frac{(m-n)N'_n}{2N'_{n+1}} < 2$. □

More generally, we can similarily prove the following interesting result by employing the same method as that of proving lemma 5.

Theorem 3: Given a multigraph G with $\beta_i > 1$ for a fixed $i (\geqq n)$. Then we can obtain a multigraph G' by adding a number of multiple edges into a fixed pair $e = (u, v)$ of vertices in G so that $\beta'_i \leqq 1$, where β'_i corresponds to that of G'. □

4. Inequalities on N_{n-1}, N_n, N_{n+1}

Lemmas 4 and 5 tell us that there are some multigraphs so that $\frac{N_n}{(m-n+1)N_{n-1}} < \frac{1}{2}$ and $\frac{(m-n)N_n}{2N_{n+1}} < 2$, which implies that the validity of the following inequality (16) is not necessarily obvious. In this section, we shall prove it to be true for all multigraphs as well.

$$\frac{(m-n)N_n}{2N_{n+1}} + \frac{N_n}{(m-n+1)N_{n-1}} \geqq 2. \tag{16}$$

From (11) with $i = n, n+1$ we have

$$h(\Phi^n_G; 1) = 2 + \beta_n, \tag{17}$$
$$h(\Phi^{n+1}_G; 1) = 3 + \beta_{n+1}, \tag{18}$$

which implies that lemma 6 holds.

Lemma 6: Proving (16) is equivalent to showing

$$2\beta_{n+1} + \beta_n\beta_{n+1} \geqq \beta_n. \tag{19}$$

Proof. By (5),(17),(18), both $\frac{(m-n)N_n}{2N_{n+1}} = \frac{3+\beta_{n+1}}{2}$ and $\frac{N_n}{(m-n+1)N_{n-1}} = \frac{1}{2+\beta_n}$ hold. We rewrite (16) as follows:

$$\frac{3+\beta_{n+1}}{2} + \frac{1}{2+\beta_n} \geqq 2,$$

which is equivalent to (19), as required. □

In order to show the validity of (19), we further introduce notations. Let $m(e)$ denote the number of multiple edges between a pair $e = (u, v)$ of vertices, and $e_{max} = (u, v)$ denote a pair of vertices with the maximum number of multiple edges among all pairs of vertices in G. Clearly, $N_{n+1}(3) > 0$ iff $m(e_{max}) \geqq 3$. We give lemma 7 to express a basic relation between $N_n(2)$ and $N_{n+1}(3)$.

Lemma 7: Let $e_{max} = (u, v)$ be a vertex pair with the maximum number of multiple edges in G. Then

$$\frac{m(e_{max}) - 2}{3} N_n(2) \geqq N_{n+1}(3). \tag{20}$$

In addition, (20) holds with equality iff either of the following conditions holds.

(i) $m(e_{max}) \leqq 2$;
(ii) $m(e_{max}) > 2$, and the number of multiple edges between every pair of vertices, except for the vertex pairs having no edge, is identical.

Proof. When $m(e_{max}) = 1$, both $N_n(2) = 0$ and $N_{n+1}(3) = 0$ hold by definition. Clearly, (20) holds with equality. In this case, in fact, G is a simple graph.

When $m(e_{max}) = 2$, $N_{n+1}(3) = 0$ by definition. Thus, (20) also holds with equality.

When $m(e_{max}) \geqq 3$, we have $N_{n+1}(3) > 0$ by definition. Since every subgraph of $\Phi^{n+1}_G(3)$ must be a tree with three multiple edges between only one pair of vertices, we obtain

$$N_{n+1}(3) = \sum_{e \in \hat{E}} t(G; e) \binom{m(e)}{3},$$

where $t(G; e)$ denotes the number of subgraphs, from each of which a spanning tree of G is obtained by adding an edge between a pair $e = (u, v)$ of vertices u, v, and $\hat{E}$ denotes the edge set of underlying graph of G. Analogously, since every subgraph of $\Phi^n_G(2)$ must be a tree with two multiple edges between only one pair of vertices, we also have

$$N_n(2) = \sum_{e \in \hat{E}} t(G; e) \binom{m(e)}{2}.$$

Therefore, (20) is derived by the definition of e_{max}. In addition, it is not difficult to see that (20) holds with equality iff G has the same number of multiple edges between every pair of vertices, except for the vertex pairs having no edge. □

Given a pair $e = (u, v)$ of vertices in G, we define new notations for an integer $l (n \geqq l \geqq 2)$ as follows:

$\Delta_l(\overline{e})$**:** the subset of $\Phi_G^n(l)$, each of which has at least one edge of E_{uv} but not as its bridge.

$\Delta_l(\underline{e})$**:** the subset of $\Phi_G^n(l)$, each of which has exactly one edge of E_{uv} as its bridge. Note that $|\Delta_n(\underline{e})| = 0$

$\Delta_l(\check{e})$**:** the subset of $\Phi_G^n(l)$, each of which has no edge in E_{uv}.

$\Phi_G^n(l)$ is partitioned into three subsets $\Delta_l(\overline{e})$, $\Delta_l(\underline{e})$, $\Delta_l(\check{e})$. Let $\delta_l(e) = |\Delta_l(e)|$ where $e = \overline{e}, \underline{e}, \check{e}$, then

$$N_n(l) = \delta_l(\overline{e}) + \delta_l(\underline{e}) + \delta_l(\check{e}). \tag{21}$$

Recall that $\Phi_G^{n+1}(t)$ $(n+1 \geqq t \geqq 3)$ stands for the set of connected spanning $(n+1)$-edge subgraphs with $n+1-t$ bridges. For a pair $e = (u, v)$ of vertices and an integer $t(n+1 \geqq t \geqq 4)$, we define

$\Theta_t(\overline{\overline{e}})$**:** the subset of $\Phi_G^{n+1}(t)$, each of which has two edges of E_{uv} but not as its edge-cut.

$\Theta_t(\underline{\underline{e}})$**:** the subset of $\Phi_G^{n+1}(t)$, each of which has two edges of E_{uv} as its edge-cut.

$\Theta_t(\overline{e})$**:** the subset of $\Phi_G^{n+1}(t)$, each of which has one edge of E_{uv} but not as its bridge.

$\Theta_t(\underline{e})$**:** the subset of $\Phi_G^{n+1}(t)$, each of which has one edge of E_{uv} as its bridge.

$\Theta_t(\check{e})$**:** the subset of $\Phi_G^{n+1}(t)$, each of which has no edge in E_{uv}.

$\Phi_G^{n+1}(t)$ is also partitioned into five subsets $\Theta_t(\overline{\overline{e}})$, $\Theta_t(\underline{\underline{e}})$, $\Theta_t(\overline{e})$, $\Theta_t(\underline{e})$, $\Theta_t(\check{e})$. Let $\theta_t(e) = |\Theta_t(e)|$ where $e = \overline{\overline{e}}, \underline{\underline{e}}, \overline{e}, \underline{e}, \check{e}$, then

$$N_{n+1}(t) = \theta_t(\overline{\overline{e}}) + \theta_t(\underline{\underline{e}}) + \theta_t(\overline{e}) + \theta_t(\underline{e}) + \theta_t(\check{e}). \tag{22}$$

The following lemmas state relations between $\delta_l()$ and $\theta_t()$, which are also applied to prove (19).

Lemma 8: Let $e = (u, v)$ be a pair of vertices with multiple edges in G. Then,

$$\theta_{l+1}(\overline{\overline{e}}) = \frac{m(e)-1}{2}\delta_l(\overline{e}) \quad \text{for } n \geqq l \geqq 3; \tag{23}$$

$$\theta_{l+2}(\underline{\underline{e}}) = \frac{m(e)-1}{2}\delta_l(\underline{e}) \quad \text{for } n-1 \geqq l \geqq 3. \tag{24}$$

Proof. By definition, it is clear that every subgraph of $\Theta_{l+1}(\overline{\overline{e}})$ is obtained from some $G' \in \Delta_l(\overline{e})$ by adding one edge of E_{uv} not in G', and that every subgraph of $\Delta_l(\overline{e})$ is also obtained from some $G'' \in \Theta_{l+1}(\overline{\overline{e}})$ by deleting one edge of E_{uv} in G''. See Fig. 5.

Let $G' \in \Delta_l(\overline{e})$ and $G'' \in \Theta_{l+1}(\overline{\overline{e}})$, where G' and G'' are obtained from each other by deleting and adding one edge of E_{uv}. Clearly, the two subgraphs, respectively, obtained from G' and G'' by deleting edges of E_{uv} in G' and G'', are the same tree. Consequently,

$$\theta_{l+1}(\overline{\overline{e}}) = \frac{m(e)-1}{2}\delta_l(\overline{e}).$$

Similarly, (24) is proved by the same method. □

Lemma 9: Given a pair $e = (u, v)$ of vertices with multiple edges in G, and given a sequence a_l's where $n \geqq l \geqq 3$ and $0 < a_3 \leqq a_4 \leqq \cdots \leqq a_n$, then

$$\sum_{l=4}^{n+1} a_{l-1}\theta_l(\overline{e}) \geqq m(e)\sum_{l=3}^{n} a_l\delta_l(\check{e}).$$

Proof. By definition, the subgraph obtained by adding one edge of E_{uv} to any subgraph of $\Delta_l(\check{e})$ must be in $\Theta_{l+k}(\overline{e})$ where $k \geqq 1$. Furthermore, the two subgraphs, respectively, obtained from different two subgraphs of $\Delta_l(\check{e})$ by adding one edge of E_{uv} to them, is different. See Fig. 6.

Thus, for $n \geqq k \geqq 3$, we obtain

$$\sum_{l=k+1}^{n+1} \theta_l(\overline{e}) \geqq m(e)\sum_{l=k}^{n} \delta_l(\check{e}).$$

Note that $0 < a_3 \leqq a_4 \leqq \cdots \leqq a_n$. Let $a_{l+1} = a_l + \epsilon_{l+1}$ for $n-1 \geqq l \geqq 3$, where $\epsilon_{l+1} \geqq 0$. For convenience, let $\epsilon_3 = a_3$. Then $a_l = \sum_{k=3}^{l}\epsilon_l$. As $\epsilon_l \geqq 0$, from the above inequality we obtain

$$\epsilon_k \sum_{l=k+1}^{n+1} \theta_l(\overline{e}) \geqq m(e)\epsilon_k \sum_{l=k}^{n} \delta_l(\check{e}).$$

By getting together the above inequalities obtained by putting $k = 3, 4, \cdots, n$, this lemma is valid. □

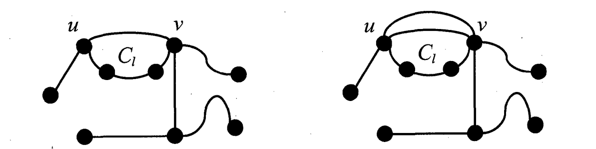
Let $a_l = l - 2$ for $l = 3, 4, \cdots, n$, then, the inequality of lemma 9 is rewritten as follows:

$$\sum_{l=4}^{n+1}(l-3)\theta_l(\overline{e}) \geqq m(e)\sum_{l=3}^{n}(l-2)\delta_l(\check{e}), \tag{25}$$

which is employed to prove the following lemma.

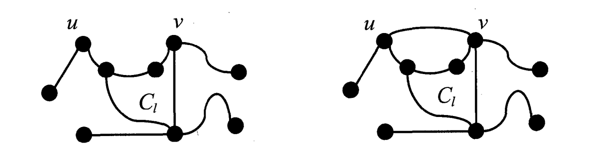
Lemma 10: Let $e = (u, v)$ be a pair of vertices in a multigraph G. Then,

$$\sum_{t=4}^{n+1}(t-3)N_{n+1}(t) \geqq \frac{m(e)-1}{2}\sum_{l=3}^{n}(l-2)N_n(l).$$



(a) A subgraph G' of $\Delta_l(\overline{e})$ with a cycle C_l of length l.
(b) A subgraph G'' of $\Theta_{l+1}(\overline{\overline{e}})$ corresponding to G' of (a).

Fig. 5 Illustration of a relation between $\Theta_{l+1}(\overline{\overline{e}})$ and $\Delta_l(\overline{e})$.



(a) A subgraph of $\Delta_l(\check{e})$ with a cycle C_l of length l.
(b) A subgraph of $\Theta_{l+k}(\overline{e})$ corresponding to that of (a).

Fig. 6 Illustration of a relation between $\Theta_{l+k}(\overline{e})$ and $\Delta_l(\check{e})$.

Proof. The above inequality is derived as follows:

$$\begin{aligned}
&\frac{m(e)-1}{2}\sum_{i=3}^{n}(l-2)N_n(l)\\
&= \sum_{l=3}^{n}(l-2)\frac{m(e)-1}{2}\left[\delta_l(\overline{e})+\delta_l(\underline{e})+\delta_l(\check{e})\right]\\
&\quad \text{(by (21))}\\
&\leqq \sum_{l=3}^{n}(l-2)\theta_{l+1}(\overline{e})+\sum_{l=3}^{n-1}(l-2)\theta_{l+2}(\underline{e})\\
&\quad +\sum_{l=4}^{n+1}(l-3)\theta_l(\overline{e}) \ \text{(by (23), (24), (25))}\\
&\leqq \sum_{t=4}^{n+1}(t-3)\theta_t(\overline{e})+\sum_{t=5}^{n+1}(t-4)\theta_t(\underline{e})\\
&\quad +\sum_{t=4}^{n+1}(t-3)\theta_t(\overline{e})\\
&\quad \text{(by setting } t=l+1, l+2, l, \text{ respectively)}\\
&\leqq \sum_{t=4}^{n+1}(t-3)N_{n+1}(t), \ \text{(by (22))}
\end{aligned}$$

as required. □

Lemma 11: For a multigraph G,

$$\frac{2}{3}\sum_{t=4}^{n+1}(t-3)N_{n+1}(t) \geqq \beta_n N_{n+1}(3).$$

Proof. Let $e_{max} = (u, v)$ be a pair of vertices with the maximum number of multiple edges. If $m(e_{max}) < 3$ then $N_{n+1}(3) = 0$. Clearly, the assertation is true.

Now, we prove the case of $m(e_{max}) \geqq 3$. Note that $\beta_n \geqq 0$ by definiton. Thus,

$$\begin{aligned}
\beta_n N_n(2) &= \sum_{l=3}^{n}(l-2-\beta_n)N_n(l) \ \text{(by (12))}\\
&\leqq \sum_{l=3}^{n}(l-2)N_n(l). \ (\text{by } \beta_n \geqq 0)
\end{aligned} \tag{26}$$

Therefore, we have

$$\begin{aligned}
&\beta_n N_{n+1}(3)\\
&\leqq \beta_n\frac{m(e_{max})-2}{3}N_n(2) \ \text{(by lemma 7)}\\
&\leqq \frac{m(e_{max})-2}{3}\sum_{l=3}^{n}(l-2)N_n(l) \ \text{(by (26))}\\
&\leqq \frac{2}{3}\sum_{t=4}^{n+1}(t-3)N_{n+1}(t), \ \text{(by lemma 10)}
\end{aligned}$$

which completes the proof of this lemma. □

Now we can prove the following desired result.

Lemma 12: For a multigraph G,

$$2\beta_{n+1}+\beta_n\beta_{n+1} \geqq \beta_n.$$

In addition, it holds with equality iff G is simplified.

Proof. When G is simplified, both $\beta_n = 0$ and $\beta_{n+1} = 0$ by definition, which means that $2\beta_{n+1}+\beta_n\beta_{n+1} = \beta_n$.

When G is not simplified, $\sum_{t=4}^{n+1}(t-3)N_{n+1}(t) > 0$ by definition. From lemma 11,

$$\frac{2}{3}\sum_{t=4}^{n+1}(t-3)N_{n+1}(t) \geqq \beta_n N_{n+1}(3),$$

which implies that we obtain

$$\begin{aligned}
&2\sum_{t=4}^{n+1}(t-3)N_{n+1}(t)+\beta_n\sum_{t=4}^{n+1}(t-4)N_{n+1}(t)\\
&> \beta_n N_{n+1}(3).
\end{aligned}$$

By definition, $2\beta_{n+1}+\beta_n\beta_{n+1} > \beta_n$ is equivalent to the above inequality. Hence, $2\beta_{n+1}+\beta_n\beta_{n+1} > \beta_n$. □

Theorem 4: For a multigraph G,

$$\frac{(m-n)N_n}{2N_{n+1}}+\frac{N_n}{(m-n+1)N_{n-1}} \geqq 2$$

In addition, this formula holds with equality iff G is a simplified multigraph.

Proof. It is trivial by lemmas 6 and 12. □

Before closing this section, we show a sufficient condition for a multigraph with $N_n^2 > N_{n-1}N_{n+1}$.

By (6), it is clear that if $\frac{(m-n+1)h(\Phi_G^{n+1};1)}{(m-n)h(\Phi_G^n;1)} \geqq 1$ then $\frac{N_n^2}{N_{n-1}N_{n+1}} \geqq 1$. Since $h(\Phi_G^{n+1};1) = 3+\beta_{n+1}$ and $h(\Phi_G^n;1) = 2+\beta_n$, it is obvious that if $\beta_n \leqq 1$ then $N_n^2 > N_{n-1}N_{n+1}$. It is clear by definition that if G has simple cycles with length at most 3, then $h(\Phi_G^n;1) \leqq 3$, equivalently, $\beta_n \leqq 1$. Figure 7 (a) illustrates an instance of multigraphs with $\beta_n \leqq 1$. The following theorem gives a sufficient condition stronger than $\beta_n \leqq 1$.

Theorem 5: Let $e = (u, v)$ be a vertex pair having multiple edges in G. If $m(e) \geqq \left\lceil \frac{2(\beta_n-1)(m-n)}{\beta_n(3+\beta_{n+1})} \right\rceil + 1$ then $N_n^2 > N_{n-1}N_{n+1}$.

Proof. When $\beta_n \leqq 1$, it is true by the above argument. Nextly, assume that $\beta_n > 1$, and prove this lemma.

By lemma 10, and formulas (9), (12) with $i = n, n+1$, we obtain

$$N_{n+1}\beta_{n+1} \geqq \frac{m(e)-1}{2}N_n\beta_n,$$

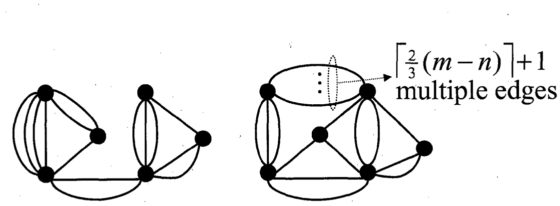


Fig. 7 Two instances of multigraphs with $N_n^2 > N_{n-1}N_{n+1}$.

which leads to the following formula by (5), (11) with $i = n+1$.

$$\beta_{n+1} \geqq \frac{m(e)-1}{2} \cdot \frac{3+\beta_{n+1}}{m-n} \cdot \beta_n.$$

As $\beta_n > 1$, from the above inequality, we have

$$\frac{\beta_{n+1}}{\beta_n - 1} \geqq \frac{m(e)-1}{2} \cdot \frac{3+\beta_{n+1}}{m-n} \cdot \frac{\beta_n}{\beta_n - 1}.$$

By (6), (17), (18), if $\frac{\beta_{n+1}}{\beta_n-1} \geqq 1$ then $N_n^2 > N_{n-1}N_{n+1}$. Thus, from $\frac{m(e)-1}{2} \cdot \frac{3+\beta_{n+1}}{m-n} \cdot \frac{\beta_n}{\beta_n-1} \geqq 1$, we obtain $m(e) \geqq \frac{2(\beta_n-1)(m-n)}{\beta_n(3+\beta_{n+1})}+1$. This means that if $m(e) \geqq \left\lceil \frac{2(\beta_n-1)(m-n)}{\beta_n(3+\beta_{n+1})} \right\rceil + 1$ then $\frac{\beta_{n+1}}{\beta_n-1} \geqq 1$. □

Since $\frac{2}{3}(m-n) \geqq \frac{2(\beta_n-1)(m-n)}{\beta_n(3+\beta_{n+1})}$, we can say by theorem 5 that $N_n^2 > N_{n-1}N_{n+1}$ for such a multigraph with at least $\lceil \frac{2}{3}(m-n) \rceil + 1$ multiple edges between some pair of vertices. Figure 7 (b) illustrates an instance of multigraphs with at least $\lceil \frac{2}{3}(m-n) \rceil + 1$ multiple edges between a pair of vertices.

5. Concluding Remarks

In this paper, for an n-vertex m-edge multigraph G and an integer $i(m \geqq i \geqq n)$, by introducing the notation $h(\Phi_G^i; 1)$ to represent the average value of the numbers of non-bridge edges for N_i connected spanning i-edge subgraphs of G, we have established $(m-i+1)N_{i-1} = h(\Phi_G^i; 1)N_i$ to exploit a relation between $h(\Phi_G^i; 1)$ and N_i. This means that proving log-concavity on $N_{n-1}, N_n, \cdots, N_m$ is reducible to proving $h(\Phi_G^{i+1}; 1) \geqq h(\Phi_G^i; 1)$ for all $i(m > i \geqq n)$.

We have further obtained $h(\Phi_G^i; 1) \geqq i - n + 2$, equivalently, $(m-i+1)N_{i-1} \geqq (i-n+2)N_i$ for all $i(m \geqq i \geqq n)$. In particular, we have shown the characterizations of multigraphs, respectively, where $h(\Phi_G^i; 1) = i - n + 2$ for each $i(m \geqq i \geqq n)$, and $h(\Phi_G^i; 1)$ nearly equals to $i-n+2$ for a fixed i. Since there are multigraphs where $h(\Phi_G^i; 1) = i - n + 2$, equivalently, $(m-i+1)N_{i-1} = (i-n+2)N_i$ for each $i(m \geqq i \geqq n)$, the inequalities are said to be *fundamental*. Moreover, we have shown that $(m-i+1)N_{i-1} > (i-n+\left\lfloor \frac{3+\sqrt{9+8(i-n)}}{2} \right\rfloor)N_i$ for all multigraphs does not hold, in general, which essentially points out a difference between simple graphs and multigraphs for inequalities of N_{i-1}, N_i.

The inequality $\frac{(m-n)N_n}{2N_{n+1}} + \frac{N_n}{(m-n+1)N_{n-1}} \geqq 2$ for all multigraphs has been proved. It has been shown that $\frac{(m-n)N_n}{2N_{n+1}} + \frac{N_n}{(m-n+1)N_{n-1}} = 2$ iff G is simplified. Hence we can also call it a fundamental inequality on N_{n-1}, N_n, N_{n+1}. In fact, the inequality is rewritten as follows:

$$(m-n)N_n \geqq \frac{4}{1 + \frac{2N_{n+1}}{(n-m)(m-n+1)N_{n-1}}} N_{n+1},$$

which implies that $(m-n)N_n \geqq 3N_{n+1}$, namely, $(m-i+1)N_{i-1} \geqq (i-n+2)N_i$ of the case $i = n+1$, has been improved since $\frac{4}{1+\frac{2N_{n+1}}{(n-m)(m-n+1)N_{n-1}}} \geqq 3$ by $(m-i+1)N_{i-1} \geqq (i-n+2)N_i$ where $i = n$ and $i = n+1$, respectively.

Moreover, by proving a sufficient condition by which $h(\Phi_G^{n+1}; 1) \geqq h(\Phi_G^n; 1)$, we have shown that $N_n^2 > N_{n-1}N_{n+1}$ if G contains at least $\left\lceil \frac{2(\beta_n-1)(m-n)}{\beta_n(3+\beta_{n+1})} \right\rceil + 1$ multiple edges in a pair of vertices. In particular, it is easy to verify that if a multigraph G contains $\left\lceil \frac{2}{3}(m-n) \right\rceil + 1$ multiple edges between some pair of vertices, or, no simple cycle with length more than 4, it satisfies the sufficient condition. Note that, in general, proving $N_n^2 \geqq N_{n-1}N_{n+1}$, however, is also remained as an interesting subject.

Since there exits a relation between $h(\Phi_G^i; 1)$ and N_i, by further investigating properties on $h(\Phi_G^i; 1)$, we may get more useful information to solve some open problems such as the log-concavity conjecture on $N_{n-1}, N_n, \cdots, N_m$, or, to find an efficient algorithm for approximately computing $N_{n-1}, N_n, \cdots, N_m$.

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