Monograph:

Lu, X.Y. and Banks, S.P. (1997) Some New Results in Differential Algebraic Control Theory. Research Report. ACSE Research Repot 699 . Department of Automatic Control and Systems Engineering

Some New Results in Differential Algebraic Control Theory

X.Y.Lu and S.P.Banks
Dep of Engineering, University of Leicester,
Leicester LE1 7RH, U. K. e-mail: xyl@sun.engg.le.ac.uk
Department of Automatic Control and Systems Engineering,
University of Sheffield, Mappin Street,
Sheffield S1 3JD.
e-mail: s.banks@sheffield.ac.uk

ABSTRACT

In this paper, static state and dynamic state feedback linearisation are considered in the framework of differential algebra. The relationship between dynamic feedback and *flatness* defined by Fliess is discussed. The existence of an equivalent *proper differential I-O system* for a given differential I-O system is discussed, which is closely related to the choice of a proper fictitious output in control design. The concept of *flatness* and its relation to dynamic feedback linearisability, controllability, observability, invertibility and minimal realisation are discussed. Finally, it is demonstrated that many fundamental control concepts and their interrelationships can be incorporated into an extended control diagram.

Keywords: feedback linearisability, flatness, proper differential I-O systems, control diagram.
Research Report No 699

1. Introduction

Differential algebra was developed by J. F. Ritt and his students to study nonlinear dynamical systems [27]. In the last ten years it has been demonstrated that a differential algebraic (as distinct from the well established differential geometric) approach can give a clearer understanding of some control concepts and their interrelations, such as controllability, observability, invertibility, decoupling, and controller canonical forms [8, 9, 10] and also observability, the existence and uniqueness of minimal realisations [18], the equivalence of differential I-O control systems [19], and feedback linearisation. Some of these works are discussed in the paper [1].

Let K denote a field of characteristic 0; e.g. $K = \mathbb{R}$, or $\mathbb{C}$ for time invariant systems and $\mathbb{R}\langle t\rangle$ or $\mathbb{C}\langle t\rangle$ for time variant systems. A dynamical system may be considered as a finite differential algebraic extension field over K. Two types of field extensions will be involved in nonlinear control theory, viz. (pure) algebraic extensions and differential algebraic extensions. An algebraic extension may be considered as a special case of a differential algebraic extension when the differential of the algebraic indeterminate is zero. A general differential indeterminate, for example the control variable u, is naturally an algebraic indeterminate.

For a given differential I-O system, there are three important invariants of equivalence: differential dimension, differential co-dimension, and system order. The differential dimension is the number of differentially independent variables (or differential indeterminates) of the system. In control theory, even if we know that the differential dimension of the system (model) is the same as the number of inputs, we are not sure that the inputs are generically differentially independent unless we can practically choose those input variables as differential parametric indeterminates for the system.

Based on the previous work on the differential algebraic approach, we consider the problem of static and dynamic feedback linearisation using the differential algebraic approach. We also consider their relationship with the concept *flatness* defined by Fliess. In fact, flatness is a special case of dynamic feedback linearisability. We will see that the static feedback linearisation problem will not involve any differential field extension. However, dynamic feedback linearisation problem will in general involve a differential algebraic extension. This corresponds to the introduction of extra *pseudo-states*, i.e. the state variables of the dynamic feedback. In this sense, the dynamic feedback linearisation problem is a immersion of the integral manifold in a flat space.

In [30] it is proved that any locally weakly observable nonlinear state space model is equivalent to a differential input-output model with the number of outputs is no less than the number of outputs. If they are the same, we get square systems. Corresponding I-O systems are called *proper* under some mild *regularity* constraints. Proper I-O systems are the basis of dynamic sliding mode control design [20, 21]. We will prove here that algebraically any prime differential I-O system corresponds (up to local equivalence) to a proper differential I-O system.

The relation of flatness with *dynamic feedback linearisability, controllability, observability, invertibility, minimal realisation* are discussed. These control concepts and their relations form a coherent *extended control diagram*, which helps in understanding these concepts in both the differential algebraic and differential geometric frameworks.

In the differential algebra framework, static state and dynamic feedback linearisation is discussed in Section 2. In Section 3, the existence of an equivalent *proper differential I-O system* for a given differential I-O system is discussed. In Section 4, an extended control diagram is presented. Section 5 considers the relation between flatness and other control concepts.

2. Feedback Linearisation

Let $x(t) : \mathbb{R}_+ \longmapsto \mathbb{R}^n$, $u(t) : \mathbb{R}_+ \longmapsto \mathbb{R}^m$, $y(t) : \mathbb{R}_+ \longmapsto \mathbb{R}^p$. Systems under consideration are of two types. Input-state-output (I-S-O) systems and differential input-output (I-O) systems.
(1) Differential I-S-O systems $\sum$:

$$F(\dot{x}, x, u, t) = 0 \tag{2.1}$$

or the following explicit form can be recovered by the Implicit Function Theorem,

$$\dot{x} = f(x, u, t) \tag{2.2}$$

It is noted that a systems of the form (2.2) can always be put into a time invariant affine form as follows. Let $t = x_{n+1}$, $u = (u_1, ..., u_m)^\top = (z_1, ..., z_m)^\top$, and $\dot{u} = \dot{z} = v = (v_1, ..., v_m)^\top$ a new set of controls. Then (2.2) becomes

$$\begin{aligned} \dot{x} &= f(x, u, x_{n+1}) \\ \dot{z} &= v \\ \dot{x}_{n+1} &= 1 \end{aligned}$$

which is clearly time invariant. Thus the theoretical discussion may be restricted to the following time invariant affine systems

$$\dot{x} = f(x) + g(x)u \tag{2.3}$$

In considering feedback linearisation in the differential algebraic approach, there is not much difference in fomulation when system (2.1) or (2.2) is adopted, while there are distinctive fomuations if systems (2.1) or (2.3) are adopted as we will see.
(2) Differential I-O systems:

$$\Gamma : \begin{array}{c} \phi_1(\widetilde{y}, \widehat{u}, t) = 0 \\ \ldots\ldots \\ \phi_p(\widetilde{y}, \widehat{u}, t) = 0 \end{array} \tag{2.4}$$

or locally by the Implicit Function Theorem,

$$\Gamma : \begin{array}{c} y_1^{(n_1)} = \varphi_1(\widetilde{y}, \widehat{u}, t) \\ \ldots\ldots \\ y_p^{(n_p)} = \varphi_p(\widetilde{y}, \widehat{u}, t). \end{array} \tag{2.5}$$

where $\widehat{u} = (u_1, ..., u_1^{(\beta_1)}; ...; u_m, ..., u_m^{(\beta_m)})^\top$, $\widehat{y} = (y_1, ..., y_p; ..., y_1^{(n_1-1)}, ..., y_p^{(n_p-1)})^\top$, $\widetilde{y} = (\widehat{y}, y_1^{(n_1)}, ..., y_p^{(n_p)})^\top$. with $n_1 + ... + n_p = n$. The system (2.5) has the following generalised canonical form realisation (GCCF):

$$\begin{array}{c} \dot{\zeta}_1^{(1)} = \zeta_2^{(1)} \\ \ldots\ldots \\ \dot{\zeta}_{n_1-1}^{(1)} = \zeta_{n_1}^{(1)} \\ \dot{\zeta}_{n_1}^{(1)} = \varphi_1(\zeta. \hat{u}, t) \\ \ldots\ldots \\ \dot{\zeta}_1^{(m)} = \zeta_2^{(m)} \\ \ldots\ldots \\ \dot{\zeta}_{n_m}^{(m)} = \varphi_m(\zeta, \hat{u}, t) \\ y = (\zeta_1^{(1)}, ..., \zeta_1^{(m)})^\top \end{array} \tag{2.6}$$

where $\zeta^{(i)} = (\zeta_1^{(i)}, ..., \zeta_{n_i}^{(i)}), \zeta = (\zeta^{(1)}, ..., \zeta^{(m)})^\top \in \mathbf{R}^n$. With the introduction of some pseudo-state variables as

$$\begin{array}{rcl} z &=& (z^{(1)}, ..., z^{(m)})^\top \\ z^{(i)} &=& (u_i, \dot{u}_i, ..., u_i^{\beta_i - 1}),\ i = 1, ..., m \\ v &=& (v_1, ..., v_m)^\top = \left(u_1^{(\beta_1)}, ..., u_m^{(\beta_m)}\right)^\top \end{array}$$

a new state space form is obtained as

$$\begin{array}{rcl} \dot{\zeta} &=& F(\zeta, z, t) \\ \dot{z} &=& G(z, v) \end{array}$$

which is again in the form of (2.2) with v as new control variable. Thus only (2.3) and (2.1) need to be considered.

3. Control Systems in Differential Algebra

There are several ways to adopt a differential algebraic framework when considering control systems. For example, a control system may be considered as a differential algebraic manifold determined by a differential system. Alternatively, it can be considered as a differential extension field determined by a *prime* differential system. This is because *a perfect differential ideal* [25] can be expressed as a finite irredundant intersection of *prime differential ideals*. In most cases, a differential extension field will be adopted as a framework in this paper unless otherwise stated. The prime differential system could be an I-O system or an I-S-O system.

3.0.1. Differential Extension Field

Let K denote a field of characteristic 0; e.g. $K = \mathbb{R}$, or $\mathbb{C}$ for time invariant system and $\mathbb{R}\langle t\rangle$ or $\mathbb{C}\langle t\rangle$ for time variant systems.

Suppose $W = (W_1, .., W_r)$ denotes the differential indeterminates over K. Consider the differential algebra $K\{W\}$ which is a commutative algebra and is closed with respect to the differential operator $\frac{d}{dt}$ [24, 27]. Let

$$\mathcal{D} = \{\phi_1, ..., \phi_s\}, \phi_i \in K\{W\}, i = 1, ..., s$$

be a *differential prime* system in $K\{W\}$, i.e. it determines a *prime differential ideal* $\{\mathcal{D}\}$ in $K\{W\}$. Then the localisation of the set of differential cosets $K\{W\}/\{\mathcal{D}\}$ forms a differential extension field over K determined by D :

$$K\langle w\rangle = L\left(K\{W\}/\{\mathcal{D}\}\right).$$

where w is used to denote the image of W under the natural projection

$$\begin{aligned} \tau :\ & K\{W\} \rightarrow\ K\{W\}/\{\mathcal{D}\} \\ \tau :\ & W_i \rightarrow w_i,\ i = 1, ..., r \end{aligned} \tag{3.1}$$

From now on, capital letters denote differential indeterminates in a differential algebra, corresponding lower case letters denote its image under the natural differential homomorphism as in (3.1).

When considering a differential algebraic field K, there are two types of extensions with which we may be concerned. i.e. differential extensions and pure algebraic extensions.

The following notations are used throughout the paper, which are coincide with those used in [1, 18]:

Σ : state space model with input u, state x, and output y of appropriate dimension;

Π : differential I-O system;

$K\{W\}$ is the differential algebra generated by $W = (W_1, ..., W_r)$ over K, where W is a set of differential indeterminates over K.

$K\langle\Sigma\rangle$ (or $K\langle\Pi\rangle$) is the *differential field* determined by *prime differential ideal* $\{\Sigma\}$ (or $\{\Pi\}$) over K. Alternatively, they can be denoted as $K\langle u, x\rangle$ (or $K\langle u, y\rangle$).

$\widehat{W}$ (resp. $\widehat{w}$) denotes the derivatives of W (resp. w) to some finite order.

If F is an (non-differential) *extension field* over *field* N, $Trd^o(F : N)$ and $Al.d^o(F : N)$ are used to denote the *transcendental degree* and *algebraic dimension* of F over N respectively.

If F is a *differential extension field* over *differential field* N, $Diff.Trd^o(F : N)$ and $Diff.d^o(F : N)$ are used to denote the *differential transcendental degree* and *differential algebraic dimension* of F over N respectively.

3.0.2. Description of Nonlinear Control Systems

Let $u = (u_1, ..., u_m)$ be differential transcendental over K, $K\langle u\rangle$ be a differential transcendental extension field of K, and $K\langle y\rangle$ be a differential extension field of output y over K.

Let the I-S-O system $\sum$ and the I-O system Π be differential prime systems over K, then the extension filed $M = K\langle u, x\rangle = K\langle\Sigma\rangle$ and $L = K\langle u, y\rangle = K\langle\Pi\rangle$ are obtained. The composite field $N = (L, M)$ which is the minimal differential field containing L and M can be constructed. Furthermore,

$$\begin{aligned} K &\subset K\langle u\rangle \subset K\langle u, x\rangle \\ K &\subset K\langle u\rangle \subset K\langle u, y\rangle, \end{aligned}$$

$K\langle u, x\rangle$ and $K\langle u, y\rangle$ are finite differential algebraic extension over $K\langle u\rangle$, which are two main differential algebraic frameworks.

Let $\overline{K}$ be the differential algebraic closure of K in N.

A property is said to hold *generically* implies that it holds in an *open dense subset of the region* concerned.

Relativity of the Differential Algebraic Framework: A differential algebraic framework for a nonlinear control system is determined by the system concerned and will not be the same, in general, for a different system.

3.1. Algebraic Equivalence

In differential algebraic control theory, the concept of *local equivalence* of two systems used in geometric control theory has a corresponding concept *algebraic equivalence*.

Definition 3.1. Two prime differential (control) systems

$$F(\dot{x}, x, u) = 0$$

and

$$G(\dot{\zeta}, \zeta, v) = 0$$

are said to be algebraically equivalent if the following two conditions hold.

(1) There exist meromorphic functions $\alpha = (\alpha_1, ..., \alpha_n)$ such that

$$\begin{aligned} \alpha(x, \zeta) &= 0 \\ \dim(x) &= \dim(\zeta) = n \end{aligned}$$

over $K\langle u\rangle$ or over $\bar{K}\langle v\rangle$ and that

$$\begin{aligned} \det\left[\frac{\partial\alpha(x,\zeta)}{\partial x}\right] &\neq 0 \\ \det\left[\frac{\partial\alpha(x,\zeta)}{\partial\zeta}\right] &\neq 0 \end{aligned}$$

hold generically.

(2) There exist meromorphic functions $\beta = (\beta_1, ..., \beta_n)$ such that

$$\begin{aligned}\beta(u, v) &= 0\\ \dim(u) &= \dim(v) = m\end{aligned}$$

over $K\langle x\rangle$ or over $K\langle \zeta\rangle$ such that

$$\begin{aligned}\det\left[\frac{\partial\beta(u,v)}{\partial u}\right] &\neq 0\\ \det\left[\frac{\partial\beta(u,v)}{\partial v}\right] &\neq 0\end{aligned}$$

hold generically. $\diamondsuit$

A similar definition can be given for the algebraic equivalence of two affine systems (2.3).

Theorem 3.1 Two differential systems are algebraically equivalent if and only if one can be obtained from the other by a (pure) algebraic elimination procedure described in [15]. $\diamondsuit$

Definition 3.2 Let $\mathbf{Z}_+^m$ denote that set of vector of dimension m with each entry to be non-negative integer. The following relation is introduced within $\mathbf{Z}_+^m$. Let $n = (n_1, ..., n_m)$, $p = (p_1, ..., p_m) \in \mathbf{Z}_+^m$. n is said to be lower than p or

$$n \prec p$$

if

$$n_1 + ... + n_m < p_1 + ... + p_m.$$

They are of the same order if

$$n_1 + ... + n_m = p_1 + ... + p_m.$$

3.2. Feedback Linearisation

Feedback linearisation in the differential algebraic framework can be described in two ways. This corresponds to using an explicit form or an implicit form. We will discuss both static feedback and dynamic feedback in these two cases.

3.2.1. Static state feedback linearisation

SFBK1: A system (2.3) is said to be *static state feedback linearisable* if there exists transformation

$$\begin{aligned}u &= \alpha(x) + \beta(x)v\\ \zeta &= T(x)\\ x &= P(\zeta)\end{aligned}$$

where v is the new control to be designed, $\alpha(.) : \mathbb{R}^n \to \mathbb{R}^m$; $\beta(.) : \mathbb{R}^n \to \mathbb{R}^{m\times m}$; $T(.), P(.) : \mathbb{R}^n \to \mathbb{R}^n$ are meromorphic function matrices, such that $\alpha(0) = 0$, $\beta(0) \neq 0$, $T(0) = 0$ and

$$\begin{aligned}&\det[\beta(x)] \neq 0\\ &\det\left[\tfrac{\partial T(x)}{\partial x}\right] \neq 0\end{aligned} \tag{3.2}$$

hold generically. Under this transformation, system (2.3) is equivalently transformed into a Brunovski canonical form

$$\begin{aligned} \dot{\zeta}_1^{(1)} &= \zeta_2^{(1)} \\ &\ldots\ldots \\ \dot{\zeta}_{n_1-1}^{(1)} &= \zeta_{n_1}^{(1)} \\ \dot{\zeta}_{n_1}^{(1)} &= v_1 \\ &\ldots\ldots \\ \dot{\zeta}_{n_m-1}^{(m)} &= \zeta_{n_m}^{(m)} \\ \dot{\zeta}_{n_m}^{(m)} &= v_m \end{aligned} \tag{3.3}$$

or equivalently

$$\begin{aligned} y_1^{(n_1)} &= v_1 \\ &\ldots\ldots \\ y_m^{(n_m)} &= v_m \end{aligned} \tag{3.4}$$

by choosing the output $y = (y_1, ...y_m) = (\zeta_1^{(1)}, ..., \zeta_1^{(m)})$, where $n_1 + ... + n_m = n = \dim(x) = \dim(\zeta)$. Naturally

$$\begin{aligned} \zeta_j^{(i)} &= y_i^{(j-1)} \\ j &= 1, ..., n_i;\ i = 1, ..., m. \end{aligned}$$

These notations will be used throughout.

SFBK2: System (2.1) is said to be static state feedback linearisable if there exsist the following relations

$$\begin{aligned} \alpha(x, u, v) &= 0 \\ T(\zeta, x) &= 0 \end{aligned} \tag{3.5}$$

where α and T are meromorphic function matrices, such that $\alpha(0,0,0) = 0$, $T(0,0) = 0$ and

$$\begin{aligned} \det \left[\frac{\partial \alpha}{\partial u}\right] &\neq 0 \\ \det \left[\frac{\partial \alpha}{\partial v}\right] &\neq 0 \\ \det \left[\frac{\partial T}{\partial x}\right] &\neq 0 \\ \det \left[\frac{\partial T}{\partial \zeta}\right] &\neq 0 \end{aligned} \tag{3.6}$$

hold generically and that (2.1) can be algebraically equivalently transformed into (3.4) with dimension $\sum_{i=1}^{m} n_i = n = \dim(x) = \dim(\zeta)$.

3.2.2. Dynamic state feedback linearisation

DFBK1: System (2.3) is said to be *dynamic feedback linearisable* if there exist the following relations

$$\begin{aligned}\dot{z} &= \alpha(x,z) + \beta(x,z)v \\ u &= \gamma(x,z) + \eta(x,z)v \\ \zeta &= T(x,z) \\ \begin{bmatrix} x \\ z \end{bmatrix} &= P(\zeta)\end{aligned} \tag{3.7}$$

where $z \in \mathbb{R}^p, \zeta \in \mathbb{R}^{n+p}$ and $\alpha(.,.) : \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}^p$; $\beta(.,.) : \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}^{p\times m}$; $\gamma(.,.) : \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}^m$; $\eta(.,.) : \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}^{m\times m}$; $T(.,.), P(.,.) : \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}^{n+p}$ are meromorphic function matrices such that $\alpha(0,0) = 0,\ \gamma(0,0) = 0,\ T(0,0) = P(0,0) = 0$, and

$$\begin{aligned}\det\left[\eta(x,z)\right] &\neq 0 \\ \det\left[\frac{\partial T}{\partial(x,z)}\right] &\neq 0\end{aligned}$$

are satisfied generically, such that (2.3) is equivalently transformed into the form (3.4) with dimension $\sum_{i=1}^{m} n_i = n + p = \dim(x) + \dim(z) = \dim(\zeta)$.

DFBK2: System (2.1) is said to be *dynamic feedback linearisable* if there exist the following relations

$$\begin{aligned}\alpha(\dot{z}, z, x, v) &= 0 \\ \gamma(z, x, u, v) &= 0 \\ T(\zeta, x, z) &= 0\end{aligned} \tag{3.8}$$

where $\alpha = (\alpha_1, ..., \alpha_p)^\top, \gamma = (\gamma_1, ..., \gamma_m)^\top$, and T are meromorphic function matrices such that $\alpha(0,0,0,0) = 0,\ \ \gamma(0,0,0,0) = 0,\ T(0,0,0) = 0$ and

$$\begin{aligned}\det\left[\frac{\partial \alpha}{\partial \dot{z}}\right] &\neq 0 \\ \det\left[\frac{\partial \alpha}{\partial v}\right] &\neq 0 \\ \det\left[\frac{\partial \gamma}{\partial v}\right] &\neq 0 \\ \det\left[\frac{\partial \gamma}{\partial u}\right] &\neq 0 \\ \det\left[\frac{\partial T}{\partial \zeta}\right] &\neq 0 \\ \det\left[\frac{\partial T}{\partial(x,z)}\right] &\neq 0\end{aligned} \tag{3.9}$$

hold generically and that (2.1) is algebraically equivalently transformed into (3.3) with dimension $\sum_{i=1}^{m} n_i = n + p = \dim(x) + \dim(z) = \dim(\zeta)$.

3.2.3. Results in the Differential Geometric Framework

Theorem 3.2 A SISO system (2.3), is locally static feedback linearisable in N_δ, which is a neighbourhood of the origin, if and only if the following condition hold in N_δ :

(i) rank $span\{g,\ ad_f g, ...,\ ad_f^{n-1}\} = n$;

(ii) $span\{g,\ ad_f g, ...,\ ad_f^{n-2}\}$ is involutive.◇

The following theorem links state feedback linearisability with dynamic feedback linearisability in the SISO case. This naturally implies some necessary and sufficient conditions for dynamic feedback linearisability although these are not known for MIMO systems.

Theorem 3.3 For an SISO system (2.3), static feedback linearisability is equivalent to dynamic feedback linearisability ([2]) . ◇

3.3. Feedback Linearisability in Differential Algebra

Linearisation via static state feedback and coordinate transformation is equivalent to the linearisation by the feedback of some outputs and its finite order of derivatives. Linearisation via dynamic state feedback and coordinate transformation is equivalent to the linearisation with dynamic feedback of some outputs and its finite order of derivatives. This is because local weak observability means that there is local diffeomorphism

$$\sigma : (x_1, ..., x_n) \rightarrow (y, \dot{y}, ..., y^{(n-1)}).$$

This is also related to the choice of output. If a system is feedback linearisable, the corresponding output is called a *linearizing* output. The following discussion will show that, in the differential algebraic framework, static state feedback linearisation will not involve a differential field extension, while dynamic feedback linearisation, on the contrary, will involve a differential algebraic field extension. This corresponds to the immersion of a maximal integral manifold in a differential geometric framework.

To study static feedback linearisation in a differential algebraic framework, consider nonlinear systems of the form (2.1) and (2.3) respectively.

Proposition 3.1 System (2.3) is *static feedback linearisable* in the sense of **SFBK1** if and only if there exists a set of outputs $y = (y_1, ..., y_m) \subset K\langle x, u\rangle$ such that

(1)

$$K\langle y\rangle = K\langle x, u\rangle.$$

(2) $K(x) = K(\bar{y})$ as two (pure) algebraic field, where $y = (y_1, ..., y_p; ..., y_1^{(n_1-1)}, ..., y_p^{(n_p-1)})$.

Proof. *Necessity:* From (3.4), it is clear that $v \in K\langle y\rangle$. By choosing

$$y_i = \zeta_1^{(i)},\ i = 1, ..., m\ ,$$

$y = (y_1, \cdots, y_m)$ is a set of observable outputs such that $y \in K\langle x, u\rangle$ beacuse $\zeta = T(x) \in K\langle x, u\rangle$. Thus $K\langle y\rangle \subset K\langle x, u\rangle$. To obtain $x = P(\zeta)$ algebraically within $K\langle x, u\rangle$, it is necessary that $x \in K\langle y\rangle$. On the other hand, $u \in K\langle x, y\rangle$, so $K\langle x, u\rangle \subset K\langle y\rangle$. It is concluded that

$$K\langle y\rangle = K\langle x, u\rangle.$$

(2) is clear from the relation

$$\zeta = T(x)$$
$$x = P(\zeta).$$

Sufficiency: By condition (2),

$$\begin{aligned} x &= T(\hat{y}) \\ \hat{y} &= T(x) \\ \hat{y} &= (y_1, ..., y_1^{(n_1-1)}, ..., y_m, ..., y_m^{(n_m-1)}). \end{aligned}$$

This also implies that $\hat{y} = T(P(\hat{y}))$ and $x = P(T(x))$. Thus

$$n_1 + ... + n_m = \dim(x).$$

Besides,

$$\begin{aligned} \dot{x} &= H(\hat{y}) + \frac{\partial P}{\partial y^{(n-1)}} y^{(n)} \\ y^{(n-1)} &= (y_1^{(n_1-1)}, ..., y_m^{(n_m-1)}) \\ y^{(n)} &= (y_1^{(n_1)}, ..., y_m^{(n_m)}) \end{aligned}$$

where $H(\hat{y})$ is a meromorphic function of $\hat{y}$.

Now let

$$\begin{aligned} \zeta_j^{(i)} &= y_i^{(j-1)} \\ y_i^{(n_i)} &= v_i \\ j &= 1, ..., n_i;\ i = 1, ..., m \end{aligned} \tag{3.10}$$

Then

$$\begin{aligned} x &= P(\zeta) \\ f(x) + g(x)u &= H(\zeta) + \frac{\partial P}{\partial y^{(n-1)}} v \end{aligned}$$

Or equivalently

$$\begin{aligned} u &= \alpha(x) + \beta(x)v \\ x &= P(\zeta) \\ \alpha(x) &= g^{-1}(x)\left(-f(x) + H\left(T(x)\right)\right) \\ \beta(x) &= g^{-1}(x)\frac{\partial P}{\partial y^{(n-1)}}\left(T(x)\right) \\ \det\left[\beta(x)\right] &\neq 0. \end{aligned}$$

Under this transformation, (2.3) is of the form (3.10) which is exactly (3.4). $\diamond$

Proposition 3.2 System (2.1) is *static feedback linearisable* in the sense of **SFBK2** if and only if

(1) there exists a set of outputs $y = (y_1, ..., y_m) \subset \Upsilon$ where Υ is some (pure) algebraic extension field of $K\langle x, u\rangle$ such that

(2) $K\langle y\rangle$ and $K\langle x, u\rangle$ are algebraically equivalent;

(3) $K(x)$ and $K(\widehat{y})$ are algebrically equivalent when considered as (pure) algebraic fields, where $y = (y_1, ..., y_1^{(n_1-1)}, ..., y_p ..., y_p^{(n_p-1)})$.

Proof. *Necessity:* (a) (3.8) determines a pure algebraic extension Υ over $K\langle x, u\rangle$.

(b) By (3.4), it is clear that $v \in K\langle y\rangle$. By (3.4) and (3.5) it is concluded that y and v are (pure) algebraic over $K\langle x, u\rangle$. Similarly, x and u are (pure) algebraic over $K\langle y\rangle$. Furthermore, (3.6) implies that the conditions (1) and (2) in Definition 2.1 hold. Thus condition (2) is necessary.

(c) Condition (3) follows directly from the relation

$$T(\zeta, x) = 0.$$

Sufficiency: Suppose $y = (y_1, ..., y_m) \in K\langle x, u\rangle$ satisfies (1)-(3).

By condition (1), u_i is algebraic over $K\langle y\rangle$ and differentially transcendental over K. i.e. there exist non-trivial meromorphic functions ω_i over K such that

$$\begin{aligned} \omega_i(u_i, \widehat{y}) &= 0, i = 1, ..., m \\ \det\left[\frac{\partial\omega}{\partial u}\right] &\neq 0 \\ \omega &= (\omega_1, ..., \omega_m) \end{aligned}$$

Suppose that $n_j \geq 0$ is the lowest order derivative of y_j, $j = 1, ..., m$, such that

$$\det\left[\frac{\partial\omega}{\partial\left(y_1^{(n_1)}, ..., y_m^{(n_m)}\right)}\right] \neq 0$$

holds generically. Let

$$v_i = y^{(n_i)}, i = 1, ..., m. \tag{3.11}$$

Then

$$\begin{aligned} \omega_i(u_i, \zeta, v) &= 0,\ i = 1, ..., m \\ \det\left[\frac{\partial\omega}{\partial v}\right] &\neq 0 \\ \det\left[\frac{\partial\omega}{\partial\zeta}\right] &\neq 0 \\ (\zeta_1, ..., \zeta_n) &= (y_1, ..., y_p; ..., y_1^{(n_1-1)}, ..., y_p^{(n_p-1)}) \end{aligned}$$

Because $u = (u_1, ..., u_m)$ is a set of differential independent elements over $K\langle x\rangle$, $v = (v_1, ..., v_m)$ has the same property . Thus the first equation in (3.5) and the first two equalities in (3.6) are recovered.

Condition (3) implies that there exists meromorphic function vectors $T = (T_1, ..., T_n)$ such that $T(0,0) = 0$ and that

$$\begin{aligned} \det\left[\frac{\partial T}{\partial x}\right] &\neq 0 \\ \det\left[\frac{\partial T}{\partial\zeta}\right] &\neq 0 \end{aligned}$$

hold generically.

Meanwhile, $(v_1, ..., v_m)$ is a set of new but equivalent control variables. With this set of control variables and such a set of outputs y, the system (2.1) has the Brunovski canonical form (3.11).

This completes the proof. $\diamondsuit$

Intuitively, in a differential algebraic framework, static state feedback linearisation does not involve a differential extension of $K\langle x, u\rangle$. It may cause some (pure) algebraic extension. However, the following discussion will show that dynamic feedback linearisation involves the introduction of some new state variable z to represent the dynamic compensator beyond the original state variable. The variable z belongs to some proper differential extension field of $K\langle x, u\rangle$.

Consider differential algebras $K\{X,U\}$ and $K\{X,Z,V\}$ Suppose ι is a differential inclusion (injective homomorphism):

$$K\{X,U\} \xrightarrow{\iota} K\{X,Z,V\}.$$

Thus $K\{X,Y\}$ can be identified with its image in $K\{X,Z,V\}$. Thus $K\{X,U\}$ can be considered as a differential sub-algebra of $K\{X,Z,V\}$. Under this inclusion

$$\begin{gathered}
\phi_i(X,U) \xrightarrow{\iota} \psi_{j_i}(X,Z,V), \; j_i \in \{1,...,q\} \\
\sum : \{\phi_i(X,U), i = 1,...,p\} \subset K\{X,U\} \\
\widetilde{\sum} : \{\psi_j(X,Z,V), j = 1,...,q\} \in K\{X,Z,V\}
\end{gathered}$$

where both $\sum$ and $\widetilde{\sum}$ are prime differential systems. This induces the following commutative diagram

$$\begin{array}{ccc}
K\{X,U\} & \xrightarrow{\iota} & K\{X,Z,V\} \\
\tau_1 \downarrow & & \downarrow \tau_2 \\
K\langle x,u\rangle & \xrightarrow{\iota_0} & K\langle x,z,v\rangle
\end{array}$$

where τ_1, τ_2 are natural projections

$$\begin{aligned}
K\{X,U\} \xrightarrow{\tau_1} K\langle x,u\rangle &= L\left(K\{X,U\}/\left\{\sum\right\}\right) \\
K\{X,Z,U\} \xrightarrow{\tau_2} K\langle x,z,v\rangle &= L\left(K\{X,Z,V\}/\left\{\widetilde{\sum}\right\}\right)
\end{aligned}$$

which are surjective . This implies that $K\langle x,u\rangle$ can be considered as a differential algebraic sub-field of $K\langle x,z,u\rangle$. ι_0 is a differential inclusion (injective homomorphism).

Proposition 3.3 System (2.3) is *dynamic feedback linearisable* in the sense of **DFBK1** if and only if the following conditions hold.

(1) There exists a differential algebraic extension field $K\langle x,z,u\rangle$ of $K\langle x,u\rangle$ determined by

$$\begin{gathered}
\dot{z} = \alpha(x,z) + \beta(x,z)\, v \\
v = \eta^{-1}(x,z)\,(u - \gamma(x,z)).
\end{gathered} \tag{3.12}$$

(2) There exists a set of outputs $y = (y_1, ..., y_m) \subset K\langle x,z,u\rangle$, such that (3) and (4) hold;

(3) $K\langle y\rangle = K\langle x,z,u\rangle$.

(4) $K(x,z) = K(\widehat{y})$ as two (pure) algebraic fileds, where $y = (y_1, ..., y_1^{(n_1-1)}, ..., y_p ..., y_p^{(n_p-1)})$.

Proof. *Necessity*:

(a) From (2.3) and (3.7), it is necessary that z satisfies (3.12), which determines a differential extension field over $K\langle x, z, v\rangle$ over $K\langle x, u\rangle$ with $v = \eta^{-1}(x, z)\,(u - \gamma(x, z))$. This proves (1).
(b) From

$$\begin{aligned}
\widehat{y} &= \zeta = T(x, z) \\
(x, z) &= P(\zeta) \\
\widehat{y} &= (y_1, ..., y_1^{(n_1-1)}, ..., y_m, ..., y_m^{(n_m-1)})^\top
\end{aligned}$$

it is deduced that

$$\begin{aligned}
\zeta &= T(P(\zeta)) \\
(x, z) &= P(T(x, z)).
\end{aligned}$$

Then

$$n_1 + ... + n_m = n + p.$$

$$\begin{aligned}
v &= y^{(n)} = H(x, z) + \frac{\partial T}{\partial x}\,(f(x) + g(x)u) \\
u &= \left[\frac{\partial T}{\partial x}\,(P(\zeta))\right](v - H\,(P(\zeta))) + f\,(P(\zeta))
\end{aligned}$$

where $y^{(n)} = (y^{(n_1)}, ..., y^{(n_m)})$ and $H(x, z)$ is a vector of meromorphic function of (x, z). Thus

$$\begin{aligned}
v &\in K\langle x, z, u\rangle \\
u &\in K\langle y\rangle.
\end{aligned}$$

Besides, $(x, z) = P(\zeta)$ implies $(x, z) \in K\langle y\rangle$. Thus $K\langle y\rangle = K\langle x, z, u\rangle$.
(c) Condition (4) follows directly from the relation

$$\zeta = T(x, z)$$
$$\begin{bmatrix} x \\ z \end{bmatrix} = P(\zeta).$$

Sufficiency:
From (4), there exists a relation

$$\begin{aligned}
\varphi_i(x_i, \widehat{y}) &= 0, \; i = 1, ..., n \\
\psi_j(z_j, \widehat{y}) &= 0, \; j = 1, ..., p \\
\dim(\widehat{y}) &= n + p
\end{aligned}$$

which satisfies

$$\det\left[\frac{\partial(\varphi, \psi)}{\partial(x, z)}\right] \neq 0$$

which implies the regulaity requirement in DFBK1. Now

$$\begin{aligned}
\frac{\partial \varphi_i}{\partial x_i}(f(x) + g(x)u) + H_i(x_i, \widehat{y}) + \sum_{j=1}^{m} \frac{\partial \varphi_i}{\partial y_j^{(n_j)}} y_j^{(n_j+1)} &= 0 \\
i &= 1, ...n.
\end{aligned}$$

where $H_i(x_i, \widehat{y})$ is a meoromorphic function of $(x_i, \widehat{y})$. Let

$$v_i = y_j^{(n_j+1)},\ i = 1, ..., m.$$

Then

$$u = -g^{-1}(x)f(x) - g^{-1}(x)\left[\frac{\partial\varphi}{\partial x}\right]^{-1}\left\{H + \frac{\partial\varphi}{\partial y^{(n)}}v\right\}$$

where $H = [H_1(x_1, \widehat{y}), ..., H_n(x_n, \widehat{y})]^{\top},\ v = [v_1, ..., v_m]^{\top}$.

$$\begin{aligned} \frac{\partial\psi_j(z_j, \widehat{y})}{\partial z_j}\dot{z}_j + H'_j(z_j, \widehat{y}) + \sum_{k=1}^{m}\frac{\partial\psi_j}{\partial y_k^{(n_k)}}y_k^{(n_k+1)} &= 0 \\ j &= 1, ..., p \end{aligned}$$

where H'_j is a meromorphic function of $(z_j, \widehat{y})$. Thus

$$\dot{z} = -\left[\frac{\partial\psi}{\partial z}\right]^{-1}\left\{H' + \frac{\partial\psi}{\partial y^{(n)}}v\right\}$$

where $H' = [H'_1(z_1, \widehat{y}), ..., H'_n(z_p, \widehat{y})]^{\top}$.

Thus all the conditions in DFBK1 are satisfied. $\diamondsuit$

Proposition 3.4 System (2.1) is *dynamic feedback linearisable* in the sense of **DFBK2** if and only if the following conditions hold.

(1) There exists a differential algebraic extension field $K\langle x, z, v\rangle$

$$\begin{aligned} \gamma(z, x, u, v) &= 0 \\ \alpha(\dot{z}, z, x, v) &= 0 \end{aligned}$$

where $v = (v_1, ..., v_m)$. $K\langle x, u\rangle$ can be identified with a differential subfield of $K\langle x, z, v\rangle$.

(2) There exists a finite differential transcendental extension field $K\langle y\rangle$, $y = (y_1, ..., y_m)$, over K such that (3) and (4) hold;

(3) $K\langle y\rangle$ is algebraically equivalent to $K\langle x, z, v\rangle$;

(4) $K(x)$ and $K(\widehat{y})$ are algebrically equivalent when considered as (pure) algebraic fields, where $y = (y_1, ..., y_1^{(n_1-1)}, ..., y_p..., y_p^{(n_p-1)})$.

Proof. *Necessity*:

(a) Condition (1) is required by the definition of **DFBK2**.

(b) From the Brunovski canonical form (3.4), $v \in K\langle y\rangle$ and

$$\begin{aligned} T(x, z, \widehat{y}) &= 0 \\ \gamma(x, z, u, v) &= 0. \end{aligned}$$

Thus (x, z, u) is (pure) algebraic over $K\langle y\rangle$. From (3.9), v is algebraic over $K\langle x, z, u\rangle$ and so are all the differentials of v. Thus y and all its derivatives are algebraic over $K\langle x, z, u\rangle$. So $K\langle x, z, u\rangle$ and $K\langle y\rangle$ are algebraically equivalent.

(c) From (3.9),

$$T(x, z, \zeta) = 0$$

so (4) is obtained.

Sufficiency: If (1)-(3) hold, the first two relations in (3.9) are obvious. Let

$$\begin{aligned} y_i^{(j)} &= \zeta_{j+1}^{(i)} \\ j &= 1, ..., n_i;\ i = 1, ..., m \end{aligned}$$

Condition (3) implies that the following two compatibility conditions hold;

(a)

$$\begin{aligned} \gamma(x, z, u, v) &= 0 \\ \det\left[\frac{\partial \gamma(x, z, u, v)}{\partial u}\right] &\neq 0 \\ \det\left[\frac{\partial \gamma(x, z, u, v)}{\partial v}\right] &\neq 0 \end{aligned}$$

holds over K, which is just the third relation in (3.9).

(b)

$$\begin{aligned} \beta(x, z, \zeta, u) &= 0 \\ \det\left[\frac{\partial \beta(x, z, \zeta, u)}{\partial (x, z)}\right] &\neq 0 \end{aligned}$$

or

$$\begin{aligned} \beta(x, z, \zeta, v) &= 0 \\ \det\left[\frac{\partial \beta(x, z, \zeta, v)}{\partial \zeta}\right] &\neq 0 \end{aligned}$$

holds over K . If $\beta(x, z, \zeta, u) = 0$, eliminate u from

$$\begin{aligned} \beta(x, z, \zeta, u) &= 0 \\ \gamma(x, z, \zeta, u) &= 0, \end{aligned}$$

A compatibility condition

$$T(x, z, \zeta) = 0$$

holds.

Similarly, if $\beta(x, z, \zeta, v) = 0$, then $\beta(x, z, \zeta, \gamma(x, z, u)) = 0$. Eliminate u by the method in [15] from

$$\begin{aligned} \beta(x, z, \zeta, \gamma(x, z, u)) &= 0 \\ \gamma(x, z, u) &= 0, \end{aligned}$$

A compatibility condition

$$T(x, z, \zeta) = 0$$

will result. $\diamondsuit$

Corollary 3.1 The following results are true:

(1) $Diff.Trd^o\left(K\langle x,z,u\rangle : K\right) = Diff.Trd^o\left(K\langle x,z,v\rangle : K\right)$;

(2) there exists another set of differential parameter indeterminates $v = (v_1, \ldots, v_m) \subset K\langle x,z,u\rangle$ such that

$$A\lg.Trd^o\left(K\langle x,z,u\rangle : K\langle u\rangle\right) = A\lg.Trd^o\left(K\langle x,z,v\rangle : K\langle v\rangle\right) = n + p.$$

Proof. From the results above, $u \in K\langle x,u,v\rangle$ and $v \in K\langle x,z,u\rangle$. It is thus true that

$$Diff.Trd^o\left(K\langle x,z,u\rangle : K\right) \leq Diff.Trd^o\left(K\langle x,z,v\rangle : K\right)$$

and

$$Diff.Trd^o\left(K\langle x,z,u\rangle : K\right) \geq Diff.Trd^o\left(K\langle x,z,v\rangle : K\right).$$

So (1) is true.

$K\langle x,z,u\rangle$ is differential algebraic over $K\langle u\rangle$ and $K\langle x,z,v\rangle$ is differential algebraic over $K\langle v\rangle$. This is also the case for $K\langle y\rangle$ over $K\langle u\rangle$ and $K\langle v\rangle$. Besides, (x,z) and $\widehat{y}$ are equivalent algebraic transcendence basis over both $K\langle u\rangle$ and $K\langle v\rangle$. While

$$A\lg.Trd^o\left(K\langle\widehat{y}\rangle : K\langle u\rangle\right) = A\lg.Trd^o\left(K\langle\widehat{y}\rangle : K\langle v\rangle\right) = n + p.$$

So (2) is true. $\diamondsuit$

3.4. Flatness

Proposed by Fliess et al (1992). Consider dynamic feedback (3.7)

Let

$$Y = (y_1, \ldots, y_1^{(n_1-1)}; \ldots, y_m, \ldots, y_m^{(n_m-1)}).$$

$$\begin{bmatrix} x \\ z \end{bmatrix} = P(Y).$$

Thus from (3.7) x and u can be expressed as smooth function of $(y_1, \ldots, y_m)$ and its finite number of derivatives:

$$\begin{aligned} x &= \mathcal{A}(y, \dot{y}, \ldots, y^{(\alpha)}) \\ u &= \mathcal{B}(y, \dot{y}, \ldots, y^{(\beta)}) \end{aligned} \tag{3.13}$$

Such a dynamic feedback is *endogenous* if and only if the converse holds. i.e.

$$y = \mathcal{C}(x, u, \dot{u}, \ldots, u^{(\gamma)}). \tag{3.14}$$

Such an output y is called *flat output.* Thus a system (2.2) is *flat* if and only if such a *flat output* exists.

The nonlinear control system (2.2) is called *flat* if some fictitious output

$$w = w(x, \hat{u}),$$

exists such that the state x and control variable u can be expressed, without integrating any differential equation, in terms of the flat output and its associated finite order of derivatives [9]. This can be interpreted in differential algebra as follows.

In differential algebra, (3.4) and (3.13) imply that

$$\begin{aligned} v &\in K\langle y\rangle \\ x &\in K\langle y\rangle \\ z &\in K\langle y\rangle \\ K\langle u,x\rangle &\subset K\langle v,y\rangle = K\langle y\rangle . \end{aligned}$$

While (3.14) implies that

$$\begin{aligned} y &\in K\langle u,x\rangle \\ K\langle y\rangle &\subset K\langle u,x\rangle . \end{aligned}$$

Thus *flatness* in the sense of [7] means that

$$K\langle u,x\rangle = K\langle v,\xi\rangle = K\langle v,y\rangle = K\langle y\rangle .$$

Although the differential dimensions are the same

$$Diff.Trd^{o}\left(K\langle u,x\rangle : K\right) = Diff.Trd^{o}\left(K\langle v,y\rangle : K\right)$$

the algebraic dimensions

$$\begin{aligned} Al.Trd^{o}\left(K\langle u,x\rangle : K\langle u\rangle\right) &= n \\ Al.Trd^{o}\left(K\langle v,\xi\rangle : K\langle v\rangle\right) &= n+p \end{aligned}$$

are different in general.

An important feature here is that the dynamic feedback (3.7) does not cause differential field extension. From the discussion above, it can be seen that

$$K\langle y\rangle \neq K\langle u,x\rangle$$

in general. From this point of view, it is concluded that

$$\textit{flatness} \Rightarrow \ \textit{dynamic feedback linearisability}$$

Corollary 3.2 Flatness is a special case of dynamic feedback linearisability. $\diamondsuit$

4. Output Choice in Control Design

In practical control problems, the choice of output largely depends on physical availability or measurement of state variables. Theoretically, proper output choice may simplify controller design by bringing a state space system (2.2) into a kind of canonical form [14]. In this sense, output choice is in fact a choice of coordinate transformation. From the control design point of view, to achieve (static or dynamic) state feedback, all *observable outputs* can be used in principle. Based on this idea, [11] chooses proper outputs to render a non-minimum phase nonlinear system to have acceptable zero dynamics. A system's *relative order* is also an output dependent concept. For the convenience of design, one wishes to choose outputs with as large relative order as possible. If the relative order with respect to an output is n which is the systems order, the system is exactly linearisable.

Another problem is that, for a given state space model, how many outputs are needed to observe the system, and how to find such a set of outputs? For a given set of differential I-O systems (2.4), it is possible that $p \neq m$. Here arise naturally the following problems:

(p-1) Is it possible that $m > p$?

(p-2) If $p > m$ and the set of output is observable, does there exist another set of outputs $\overline{y} = (\overline{y}_1, ..., \overline{y}_m)$ such that it is a set of observable outputs.

It is noted that **(p-2)** is meaningful from the control design point of view. If such a set of outputs exists, then the controller design will be for a square system. The corresponding differential I-O systems, under some mild regularity condition, are called *proper differential I-O systems* which are considered in *Dynamic Sliding Mode Control* [20, 21, 22, 28]. These problems are now considered in a differential algebraic approach.

Suppose the system Γ (2.4) is differentially prime. $y = (y_1, ..., y_p)$ is a set of observable outputs which means that y is an differential algebraic transcendence basis of the differential field $K\langle\Gamma\rangle$. Thus the differential transcendence degree of $K\langle\Gamma\rangle$ over K cannot be bigger than p. This can be seen by eliminating $y_1, ..., y_p$ from the system (2.4). It is then deduced that there is a set of compatibility conditions among the controls $(u_1, ..., u_m)$, or equivalently, there exists a set of non-trivial differential polynomials in $K\{w_1, ..., w_m\}$, say $\psi(., ..., .)$ such that

$$\psi(\widehat{u}_1, ..., \widehat{u}_m) = 0.$$

This answers **(p-1)**.

Proposition 4.1 If controls form a set of (differentially) independent elements, then $m \leq p$.

Proposition 4.2 If the system (2.4) is differentially prime and $y = (y_1, ..., y_p)$ is a set of observable outputs with $p > m$, then another set of observable outputs $\overline{y} = (\overline{y}_1, ..., \overline{y}_m)$ always exists. Besides, any two such two sets of observable outputs differ by a differential algebraic transformation over K.

Proof. The proof amounts to finding a set of m-outputs which is observable. First, it is noted that $(y_1, ..., y_p, \dot{y}_1, ..., \dot{y}_p, ..., y_1^{(n_1-1)}, ..., y_p^{(n_p-1)})$ is a set of algebraic transcendence basis over the differential field $K\langle u\rangle$. Now consider the differential sub-field

$$K\langle u\rangle\langle y_1, ..., y_m\rangle \subset K\langle u\rangle\langle y\rangle = K\langle u, y\rangle.$$

$K\langle u\rangle\langle y_1, ..., y_m\rangle$ is a finite differential algebraic extension field over $K\langle u\rangle$. If

$$Al.Trd^o\left(K\langle u, y_1, ..., y_m\rangle : K\langle u\rangle\right) = Al.Trd^o\left(K\langle u, y\rangle : K\langle u\rangle\right)$$

then $(y_1, ..., y_m, \dot{y}_1, ..., \dot{y}_m, ..., y_1^{(l_1-1)}, ..., y_m^{(l_m-1)})$ is an algebraic transcendence base of $K\langle u, y\rangle$ over $K\langle u\rangle$ for some $l_1, ..., l_m$. Thus $(y_1, ..., y_m)$ is a set of observable outputs.

Otherwise, suppose

$$Al.Trd^o\left(K\langle u, y_1, ..., y_m\rangle : K\langle u\rangle\right) < Al.Trd^o\left(K\langle u, y\rangle : K\langle u\rangle\right).$$

In this case $K\langle u, y_1, ..., y_{m-1}\rangle$ is a proper differential sub-field of $K\langle u, y\rangle$. The according to [25], there exists a differential primitive element $\xi \in K\langle u, y\rangle$ such that

$$K\langle u, y_1, ..., y_{m-1}\rangle\langle\xi\rangle = K\langle u, y\rangle.$$

This means that $(y_1, ..., y_{m-1}, \xi)$ is a set of m-outputs which is observable. $\diamondsuit$

Remark 4.1 For a given observable state space system (2.2) with output $y = (y_1, ..., y_p)$ which is not proper but $p > m$, to find a set of m observable outputs, one can make it so by finding a sub-matrix of

$$\left[\frac{\partial \hat{y}}{\partial x}\right]$$

such that the following two conditions are satisfied:

(i) there are exactly m outputs appearing in the sub-matrix. By renumbering the subscripts, let them be $(y_1, y_2, ..., y_m)$;

(ii) Let $\hat{Y} = (y_1, \dot{y}_1, ..., y_1^{(\alpha_1 - 1)}; ...; y_m, \dot{y}_m, ..., y^{(\alpha_m - 1)}), \alpha_i \geq 1$. Then

$$rank\left[\frac{\partial \hat{Y}}{\partial x}\right] = n$$

where $\alpha_1 + ... + \alpha_m = n_1 + ... + n_p = n$.

The corresponding set of outputs are just as required. $\diamondsuit$

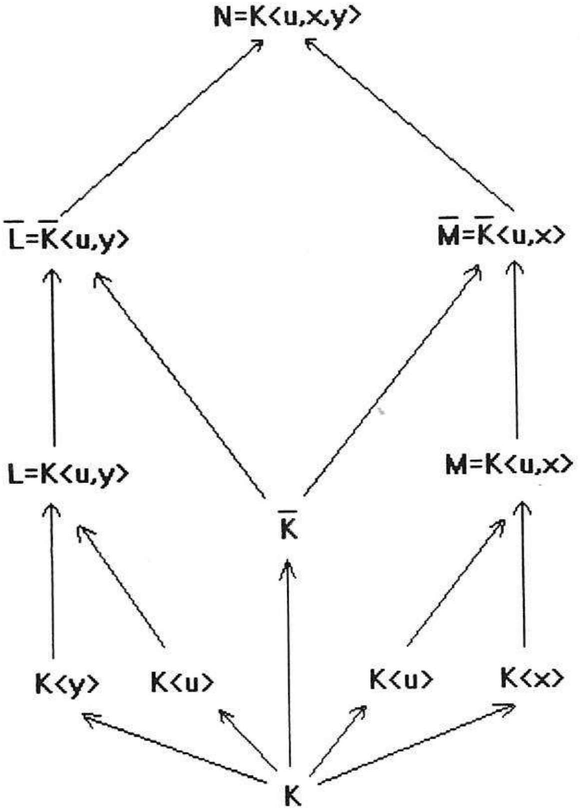
5. Control Diagram

5.1. Control Diagram

[26]tries to describe some of the fundamental concepts of control theory with a single control diagram:

$$\begin{tikzcd} & & N & & \\ \overline{L} \arrow[urr] & & & & \overline{M} \arrow[ull] \\ L \arrow[u] & & \overline{K} \arrow[ull] \arrow[urr] & & M \arrow[u] \\ & & K \arrow[ull] \arrow[u] \arrow[urr] & & \end{tikzcd}$$

where K is the basic field of charactoristic zero, $\overline{K}$ is the differential algebraic closure field of K in N, $L = K\langle u, y\rangle$, $M = K\langle u, x\rangle$, $\overline{L} = (L, \overline{K})$ is the composite field of L and $\overline{K}$, $\overline{M}$ is the composite field of M and $\overline{K}$, and $N = K\langle u, x, y\rangle$. However, only few concepts such as controllability and observability in nonlinear control theory may be described within this figure. To embody most control concepts, we generalise it to the following *control diagram:*



Control Diagram

5.2. Controllability

The proper definition of controllability is given in [25] which is equivalent to the strong accessibility in [3, 13, 14, 23] if the system (2.2) is affine or (2.5) has an affine realisation [12].
Definition 5.1 Controllability: $K = \overline{K}$. i.e. K is differentially algebraically closed in N. Thus $L = \overline{L}$ and $M = \overline{M}$. In this case the control diagram collapsed as :
Interpretation: All element $\vartheta \in K\langle u, x\rangle$ but not in K are affected by u because and element ϑ in $K\langle u, x\rangle$ other than in K is differentially transcendental over K.

5.3. Observability

In the work of [6, 25] the following is proposed:
Definition 5.2 Observability: L is algebraically equivalent to M.
Interpretation: x and $\widehat{y}$ can be represented in terms of each other as differential polynomial functions, from which we can conclude that output feedback control and state feedback control are essentially equivalent. This is true for both static feedback and dynamic feedback.

If adopting a realisation point of view and considering $K\langle y, u\rangle$ as a *jet space* on input u and output y as in [18], the following definition is obtained:
Definition 5.3 Observability: $x \in K\langle y, u\rangle$ and x is an algebraic transcendence basis over $K\langle u\rangle$.

Theorem 5.1 This definition is equivalent to the local observability in [13, 14, 23].

5.4. Elimination (I-S-O $\rightarrow$ I-O)

Elimination is a projection: $N \rightarrow L$.

A differential geometric elimination method is given by [30, 23]. This method is a local result based on local observability and the Implicit Function Theorem.

There are two methods for differential algebraic elimination: one is due to the work of [5] and the other is due to [10]. Both methods use some special ranking systems and have their roots in [27]. The difference between these two methods is that [10] introduces the method of characteristic set and works generically whilst [5] tries to work precisely giving rise to certain inequalities at each step of the elimination procedure.

5.5. Invertibility

Left and right invertibility of nonlinear control systems, in the differential algebraic framework, have been discussed in [9] and [1]. A proper statement of the result should be:

Theorem 5.2 System (2.5) (or (2.2)) is left (right) invertible if and only if $y_1, ..., y_m$ (respectively, $u_1, ..., u_m$) is a differential transcendence basis of $L = K\langle u, y\rangle$.

Note that this is equivalent to saying that the output (input) channels are differentially independent.

Interpretation: (1) right invertibility: y_i is differentially algebraic over $K\langle u\rangle$; i.e. there exist relations $\varphi_i(y_i, \widehat{u}) = 0, i = 1, ..., p$, over K, where $\varphi_i(y_i, \widehat{u})$ is a differential polynomial of y_i with coefficients in $K\langle u\rangle$.

(2) left invertibility: u_j is differential algebraic over $K\langle y\rangle$; i.e. there exist relations $\psi_j(u_j, \widehat{y}) = 0, j = 1, ..., m$, over K, where $\psi_j(u_j, \widehat{y})$ is a differential polynomial of u_j with coefficients in $K\langle y\rangle$.

5.6. Realisation (I-O$\rightarrow$I-S-O)

Realisation of nonlinear systems in the differential algebraic framework was carried out in [16, 17, 29, 4]. Differential algebraic realisation theory is developed in [18].

The realisation process is just a matter of proper choice of state variables x. A faithful realisation is such that $M = K\langle u, x\rangle$ is algebraically equivalent to $L = K\langle u, y\rangle$. A *minimal realisation* should simultaneously achieve both *controllability* and *observability*.

Definition 5.4 A realisation $M = K\langle u, x\rangle$ is minimal if there exists M such that

$$K \subset K\langle u\rangle \subset M \subset L$$

and that

(1) $x = (x_1, ..., x_n)$ is an algebraic transcendence basis of M over $K\langle u\rangle$ (*observability*);

(2) K is differentially algebraically closed in M (*controllability*); and

(3) F is a maximal differential sub-field of L satisfying (1) and (2).

Theorem 5.3 (Existence and Uniqueness of Minimal Differential Algebraic Realisation) For a given prime differential I-O system (2.5) there exists a differential algebraic minimal realisation. Any two minimal realisations of a given prime I-O system are algebraically equivalent (i.e. they differ by an algebraic transformation over $K\langle u\rangle$).

5.7. Equivalence Problem

The equivalence problem of differential algebraic control systems is discussed in [19]. Now these relations can be restated with the help of the control diagram.

Definition 5.5 Let $L_i = K\langle \Pi_i \rangle$, $i = 1, 2$. Two prime differential I-O systems Π_1 and Π_2 are said to be equivalent if L_1 is algbraically equivalent to L_2.

Definition 5.6 Π_1 and Π_2 are said to be *minimal equivalent* if there exists M such that

$$\begin{aligned} K &\subset K\langle u\rangle \subset M \subset L_i \\ i &= 1,2 \end{aligned}$$

and that

(1) $x = (x_1, ..., x_n)$ is an algebraic transcendence basis of M over $K\langle u\rangle$;

(2) K is differentially algebraically closed in M ; and

(3) F is a maximal differential sub-field of both L_1 and L_2 satisfying (1) and (2).

Some necessary conditions for minimal equivalence of SISO systems have been found in [19]. Sufficient conditions are not yet known. Minimal equivalence is useful in realisation, system identification, and system design.

6. Flatness and Other Control Concepts

6.1. Flatness and Controllability and observability

Controllability: K is differential algebraically closed in $K\langle x, u\rangle$. However, for any artificial output

$$y = (y_1, ..., y_p)^\top = h(x, \hat{u}) \in K\langle u, x\rangle$$

such that $(y_1, ..., y_p)$ is an algebraic transcendence basis of $K\langle u, x\rangle$ over $K\langle u\rangle$ (*observability*), i.e.

$$K\langle u, y\rangle \cong K\langle u, x\rangle$$

they are algebraically equivalent. It might happen that

$$K\langle y\rangle \subset K\langle u, x\rangle$$

i.e. $K\langle u, y\rangle$ and $K\langle u, x\rangle$ are not algebraically equivalent.

$$Flatness \Rightarrow Controllability.$$

Now the following concept of *defects* measures the difference between controllability and dynamic feedback linearisability.

Definition 6.1 For a given output $y = (y_1, ..., y_p)$ such that

(1) y is an algebraic transcendence basis of $K\langle u, x\rangle$ over $K\langle u\rangle$;

(2) y is a purely differential transcendental basis of $K < x, u >$ over K.

The differential dimension of $K\langle u, x\rangle$ over $K\langle y\rangle$ is called the *defects of the system with respect to output y*. The minimal defect of $K\langle u, x\rangle$ with respect to all possible artificial output $y \in K\langle u, x\rangle$ is called the *defects of the system.*

Clearly, a system is flat if and only if its defect is zero.

Algebraic Interpretation: defect is the number of times in total needed to integrate y to get u.

Example (Controllable but non-flat) The Kapitsa pendulum

$$\begin{aligned}\dot{\alpha} &= p + \frac{u}{l}\sin\alpha \\ \dot{p} &= (\frac{g}{l} - \frac{u^2}{l^2}\cos\alpha)\sin\alpha - \frac{u}{l}p\cos\alpha \\ \dot{z} &= u\end{aligned}$$

Vertical speed is the control.

To prove that it is not dynamic feedback linearisable, it suffice to prove that it is not static feedback linearisable. To achieve this, let $\dot{u}= v$ as artificial control. Then an affine system is obtained

$$\dot{x}= f(x) + g(x)v$$

with $x = (x_1, ..., x_4) = (\alpha, p, z, u)^\top$ and let $l = 1$ for convenience

$$f(x) = \begin{bmatrix} x_2 + x_4\sin x_1 \\ (g - x_4^2\cos x_1)\sin x_1 - x_4 x_2\cos x_1 \\ x_4 \\ 0 \end{bmatrix}, g(x) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

Now calculate

$$ad_f g = \begin{bmatrix} 1\sin x_1 \\ x_2\cos x_1 + x_4\sin 2x_1 \\ -1 \\ 0 \end{bmatrix},$$

$$ad_f^2 g = \begin{bmatrix} -2\cos x_1(x_2 + x_4\sin x_1 \\ g\sin 2x_1 - x_2^2\sin x_1 + 2x_2x_4\cos 2x_1 - \frac{x_4^2\sin x_1}{2} + \frac{3x_4^2\sin 3x_1}{4} \\ 0 \\ 0 \end{bmatrix},$$

$$[g, ad_f g] = \begin{bmatrix} 0 \\ \sin 2x_1 \\ 0 \\ 0 \end{bmatrix}.$$

It is easy to check that

$$rank\left[g, ad_f g,\ ad_f^2 g, [g, ad_f g]\right] = 4.$$

Thus

$$span\{g, ad_f g,\ ad_f^2 g\}$$

is not involutive. Thus the system is not static feedback linearisable and so not dynamic feedback linearisable.

Suppose y is a flat output. Then

$$K\langle y\rangle \cong K\langle u, x\rangle \Rightarrow K\langle u, y\rangle \cong K\langle u, x\rangle$$

It is clear that y is an observable output. Thus

$$Flatness \Rightarrow Observability.$$

However the reverse is not true as discussed above since observable output may have non-zero defects.

Corollary 6.1 *Flatness* implies minimality in realisation.

6.2. Flatness and Invertibility

If y is a flat output. Then necessarily, y is a purely differential transcendental basis of $K\langle u, x\rangle$ over K. Thus y and u have the same number of elements which is the differential transcendental dimension of $K\langle u, x\rangle$ over K. Thus the system is invertible [9]. Thus

$$Flatness \Rightarrow Invertibility.$$

7. Conclusion

In a differential algebra framework, static state and dynamic state feedback linearisation are considered for affine systems and general nonlinear systems in an implicit form. *Flatness* defined by Fliess is a special case of dynamic feedback linearisability. It can be seen that dynamic feedback linearisability requires very restrictive conditions. The existence of an equivalent *proper differential I-O system* of a given differential I-O system is discussed, which is closely related to he choice of proper outputs in control design. The concept *flatness* and its relation with dynamic feedback linearisability, controllability, observability, invertibility, minimal realisation are discussed. Finally, many fundamental control concepts and their relationships are shown to fit in a control diagram.

References

[1] D. J. Bell and X. Y. Lu. 'Differential algebraic control theory, IMA J. Math. Contr. & Info., **9**,(1992), 361-383.

[2] B.Charlet, J.Levine and R.Marino. 'On dynamic feedback linearization', Systems and Control Lett., **13**, (1989),143-151.

[3] D.Chen. 'Geometric Theory of Nonlinear Systems', Beijing:Science Publishers, (1988).

[4] P. E. Crouch and F. Lamnabhi-Lagarrigue. 'State space realizations of nonlinear systems defined by input-output differential equations', Lecture Notes in Control and Information Science, Berlin:Springer-Verlag, (1988), 138-149.

[5] S. Diop. 'Elimination in control theory',Math. Control Signals Systems, **4,** (1991), 17-32.

[6] S. Diop and M. Fliess. 'Nonlinear observability, identifiability and persistent trajectories',Proc. IEEE CDC.,Brighton, U. K., (1992).

[7] M. Fliess. 'A note on the invertibility of nonlinear input output differential systems', Syst. & Contr. Letts., **8,** (1986), 147-151.

[8] M. Fliess. 'What the Kalman state variable representation is good for', Proc. IEEE CDC, Honolulu, Hawaii, (1990), 1282-1287.

[9] M. Fliess and J. Levine and Ph. Martin and P. Rouchon. 'Differential flatness and defects: an overview, in Workshop on Geometry in Nonlinear Control', Workshop on Geometry in Nonlinear Control, Warsaw:Banach Centre Publications, (1993).

[10] S. T. Glad. 'Nonlinear state space and input-output descriptions using differential polynomials', LNCIS 122, Berlin:Springer-Verlag, (1989), 182-189.

[11] S. Gopalswamy and J. K. Hedrick. 'Tracking nonlinear non-minimum phase systems using sliding control', Int. J. Contr., **57**,No.. 5, (1993), 1141-1158.

[12] A. Haddak. 'Differential algebra and controllability', Nonlinear Control Systems, IFAC Symposia Series 2, (1990), 13-16.

[13] R. Hermann and A. J. Krener. 'Nonlinear controllability and observability, IEEE. Trans. Auto. Contr., **22,** (1977), 728-740.

[14] A. Isidori. 'Nonlinear Control Systems', Berlin:Springer-Verlag, (1989).

[15] N. Jacobson. 'Basic Algebra', San Francisco:W.H. Freeman & Co., (1974).

[16] B. Jakubczyk. 'Local realizations of nonlinear control operators', Local realizations of nonlinear control operators, SIAM. J. Control, **24,** (1986), 230-42.

[17] B. Jakubczyk. 'Realization theory for nonlinear systems : three approaches', Algebraic and Geometric Methods in Nonlinear Control Theory, (Fliess, M. & Hazewinkel, M. Eds.), (1986), 3-31.

[18] X. Y. Lu and D. J. Bell. 'Realization theory for differential algebraic input-output systems', IAM J. Math. Control & Information, **10**,No. 1, (1993), 33-47.

[19] X. Y. Lu and D. J. Bell. 'Equivalence of Input-Output Systems', Proc. of 31st IEEE Conference on Decision and Control, Proc. of 31st IEEE Conference on Decision and Control, Tucson, Arizona, USA, 2492-2497.

[20] X. Y. Lu and S. K. Spurgeon. 'Asymptotic feedback linearization and control of non-flat systems via sliding mode', Proc. 3rd European Control Conference, Sept. 5-8, Rome, Italy, (1995), 693-698.

[21] X. Y. Lu and S. K. Spurgeon. 'Asymptotic feedback linearisation of multiple input systems via sliding modes', Proc. 13th IFAC World Congress, San Francisco, USA, Vol F,(1996), 211-216.

[22] X. Y. Lu and S. K. Spurgeon. 'Robust sliding mode control of uncertain nonlinear systems', Syst. & Contr. Letts., (1997), to appear.

[23] H. Nijmeijer and A. J. van der Schaft. 'Nonlinear Dynamical Control systems', New York:Springer-Verlag, (1990).

[24] J. F. Pommaret. 'Systems of Partial Differential Equations and Lie Pseudogroups', Gordon and Breach Science Publishers , (1978).

[25] J. F. Pommaret. 'Differential Galois Theory', Gordon and Breach Science Publishers, (1983).

[26] J. F. Pommaret. 'Lie Pseudogroups and Mechanics', Gordon and Breach Science Publishers. (1987).

[27] J. F. Ritt. 'Differential Algebra',New York:Dover, (1966).

[28] H. Sira-Ramirez. 'A dynamical variable structure control strategy in asymptotic output tracking problems', IEEE Trans. Auto. Contr., **38**, (1993), 615-620.

[29] H. J. Sussmann. 'Existence and uniqueness of minimal realizations of nonlinear systems', Math. Systems Theory, **10**, (1977), 263-284.

[30] A. J. van der Schaft. 'Representing a nonlinear state space system as a set of higher-order differential equations in the inputs and outputs', Syst. & Contr. Letts., **12**, (1989), 151-160.