Finite-region boundedness and stabilization for 2D continuous-discrete systems in Roesser model

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This paper investigates the finite-region boundedness (FRB) and stabilization problems for two-dimensional continuous-discrete linear Roesser models subject to two kinds of disturbances. For two-dimensional continuous-discrete system, we first put forward the concepts of finite-region stability and FRB. Then, by establishing special recursive formulas, sufficient conditions of FRB for two-dimensional continuous-discrete systems with two kinds of disturbances are formulated. Furthermore, we analyze the finite-region stabilization issues for the corresponding two-dimensional continuous-discrete systems and give generic sufficient conditions and sufficient conditions that can be verified by linear matrix inequalities for designing the state feedback controllers which ensure the closed-loop systems FRB. Finally, viable experimental results are demonstrated by illustrative examples.

Keywords: finite-region stability; finite-region stabilization; 2D continuous-discrete Roesser models; linear matrix inequalities.

1. Introduction

The study of two-dimensional systems has a long history, with some meaningful works (Roesser, 1975; Fornasini & Marchesini, 1976; Lin & Bruton, 1989; Lu & Antoniou, 1992; Xu *et al.*, 2005; Hu & Liu, 2006; Singh, 2014; Xie *et al.*, 2015; Ahn *et al.*, 2016) such as systems theory, stability properties and practical applications. Among all research topics of two-dimensional systems, stability as the most fundamental property has obtained fruitful achievements. To list some of these, Bachelier *et al.* (2016) proposed linear matrix inequality (LMI) stability criteria for two-dimensional systems by relaxing the polynomial-based texts of stability into that of LMIs. Bouzidi *et al.* (2015) presented computer algebra-based methods for testing the structural stability of n-dimensional discrete linear systems. Other works dedicated to discrete two-dimensional models are Bliman (2002) and Ebihara *et al.* (2006). Two-dimensional continuous-discrete systems arise naturally in several emerging application areas, for example, in analysis of repetitive processes (Rogers & Owens, 1992; Rogers *et al.*, 2007) and in irrigation channels (Knorn & Middleton, 2013a). As such, the research on two-dimensional continuous-discrete systems has been a hot area in control field. Especially, the stability analysis for two-dimensional continuous-discrete systems has attracted much attention of some researchers over the last few decades, and several interesting findings in linear and non-linear frameworks have been obtained (see Xiao, 2001; Benton *et al.*, 2002; Owens & Rogers, 2002; Rogers & Owens, 2002; Knorn & Middleton, 2013a; Chesi & Middleton, 2014; Knorn & Middleton, 2016; Galkowski *et al.*, 2016; Pakshin *et al.*, 2016; Wang *et al.*, 2017 and references therein). For example, in the linear setting, Xiao (2001) considered three models of two-dimensional continuous-discrete systems and gave sufficient and necessary conditions for their Lyapunov asymptotic stability (LAS)-based two-dimensional characteristic polynomial. In Owens & Rogers (2002), a stability analysis for differential linear repetitive processes, a class of two-dimensional continuous-discrete linear systems, was given in the presence of a general set of boundary conditions. These results further were extended to stability tests based on a one-dimensional Lyapunov equation and strictly bounded real lemma in Benton *et al.* (2002). Chesi & Middleton (2014) proposed necessary and sufficient conditions that can be checked with convex optimization for stability and performance analysis of two-dimensional continuous-discrete systems. For non-linear systems, Knorn & Middleton (2013b) modeled the homogeneous, unidirectional non-linear vehicle strings as a general two-dimensional continuous-discrete non-linear system and presented a sufficient condition for stability of this system. In Galkowski *et al.* (2016), the exponential stability conditions for non-linear differential repetitive processes were established by using a vector Lyapunov function-based approach. However, most of these results related to stability were focused on LAS or exponential stability.

Apart from LAS or exponential stability, finite-time stability (FTS) is also a basic concept in the stability analysis. The concept of FTS was first introduced in Kamenkov (1953) and reintroduced by Dorato in Dorato (1961) which is related to dynamical systems whose state does not exceed some bound during the specified time interval. It is important to note that FTS and LAS are completely independent concepts. FTS aims at analyzing transient behavior of a system within a finite (possibly short) interval rather than the asymptotic behavior within a sufficiently long (in principle, infinite) time interval. In general, the characteristic of FTS does not guarantee stability in the sense of Lyapunov and vice versa. In addition, it should be noted that the FTS considered here is unrelated to the one adopted in some other works (Moulay & Perruquetti, 2008; Nersesov & Haddad, 2008), where the authors focus on the stability analysis of non-linear systems whose trajectories converge to an equilibrium point in finite time. Recent years have witnessed growing interests on FTS (see Amato *et al.*, 2001; Amato & Ariola, 2005; Amato *et al.*, 2010; Jammazi, 2010; Seo *et al.*, 2011; Zhang *et al.*, 2012, 201b, a; Haddad & L'Afflitto, 2016; Tan *et al.*, 2016 and references therein) because it plays vital roles in many practical applications, for example, for many dynamic systems the state trajectories are required to stay within a desirable operative range over a certain time interval to fulfill hardware constraints or to maintain linearity of the system. What these literatures we mentioned here are about one-dimensional systems; little progress related to this problem has been made for two-dimensional systems due to their dynamical and structural complexity. Until recently, the authors in Zhang & Wang (2016a,b) studied the problems of finite-region stability (FRS) and finite-region boundedness (FRB) for two-dimensional discrete Roesser models and Fornasini-Marchesini second models. For two-dimensional continuous-discrete systems, it is worth mentioning that there are some results on the problems of exponential stability, weak stability and asymptotic stability over bounded region for repetitive processes (see Rogers *et al.*, 2007; Galkowski *et al.*, 2016; Pakshin *et al.*, 2016 and references therein), where the variable $x^h(t,k)$ changes on finite interval $[0,T]$. Clearly, these problems are completely different from the finite-region control problem considered in Zhang & Wang (2016a,b) in that their methods can exhibit the transient performance of two-dimensional discrete systems over a given finite region by setting the state variables of the system less than a particular threshold. Therefore, it is necessary and important to consider the FRS and FRB problems for two-dimensional continuous-discrete systems, which motivates our present research. It is worth noting that though the two-dimensional system theory is developed from one-dimensional system theory, it is not a parallel promotion of one-dimensional system theory. There exist deep and substantial differences between one-dimensional case and two-dimensional case (Fan & Wen, 2002; Feng *et al.*, 2012). In the two-dimensional case, the system depends on two variables and the initial conditions consist of infinite vectors, but for a one-dimensional system, the initial condition is a single vector. Moreover, most results obtained for one-dimensional systems cannot be straightforwardly extended to two-dimensional systems. For example, stability texts for one-dimensional systems are based on a simple calculation of the eigenvalues of a matrix or the roots of a polynomial, but this is not the case of two-dimensional systems, as stability conditions are given in terms of multidimensional polynomials. These factors make the analysis of FRS and FRB for two-dimensional continuous-discrete systems much more complicated and difficult than one-dimensional continuous or discrete systems. On the other hand, the existing results on two-dimensional discrete systems (Zhang & Wang, 2016a,b) cannot be immediately extended to two-dimensional continuous-discrete systems. This is because two-dimensional continuous-discrete systems are more complicated and technically more difficult to tackle than two-dimensional discrete systems. To fill this gap, it is necessary for us to further study the finite-region control problems of two-dimensional continuous-discrete systems deeply.

In this paper, we deal with two-dimensional continuous-discrete linear Roesser models subject to two classes of disturbances. We first put forward the definitions of FRS and FRB for two-dimensional continuous-discrete system. Then, by establishing special recursive formulas, sufficient condition of FRB for two-dimensional continuous-discrete system with energy-bounded disturbances and sufficient condition of FRS for two-dimensional continuous-discrete system are given. Furthermore, by using the given sufficient conditions, generic sufficient conditions and sufficient conditions that are solvable by LMIs for the existence of state feedback controllers which ensure the corresponding closed-loop systems FRB or FRS are derived. We also show that, under stronger assumptions, our sufficient conditions for finite-region stabilization recover asymptotic stability. Finally, we address the FRB and finite-region stabilization problems for two-dimensional continuous-discrete system with disturbances generated by an external system and present sufficient condition of FRB and generic sufficient condition and sufficient condition that can be checked with LMIs of FRB via state feedback for the corresponding systems.

The paper is organized as follows. In Section 2, the definitions of FRS and FRB for two-dimensional continuous-discrete linear system are proposed. In Section 3, the FRB and finite-region stabilization issues for two-dimensional continuous-discrete system with the first disturbances are considered, and corresponding sufficient conditions and LMIs conditions are derived. Section 4 presents the results for the two-dimensional continuous-discrete system subject to the second disturbances. Numerical examples are given in Section 5 to show the effectiveness of the proposed approaches. Finally, conclusions are drawn in Section 6.

Notations In this paper, we assume that vectors and matrices are real and have appropriate dimensions. N^+ denotes a set of positive integers, R^n is the n-dimensional space with inner product $x^T y$. $A > 0$ means that the matrix A is symmetric positive definite. A^T denotes the transpose of matrix A, I represents the identity matrix. For a matrix A, its eigenvalue, maximum eigenvalue and minimum eigenvalue are denoted by $\lambda(A)$, $\lambda_{\max}(A)$ and $\lambda_{\min}(A)$, respectively. The symmetric terms in a matrix is represented by $*$. $diag\{\cdot\}$ denotes a black-diagonal matrix.

2. Preliminaries

In this paper, we consider the following two-dimensional continuous-discrete linear system for Roesser model:

$$x^{+}(t,k) = Ax(t,k) + Bu(t,k) + Gw(t,k), \tag{2.1}$$

where $x^{+}(t,k) = \begin{bmatrix} \frac{\partial x^h(t,k)}{\partial t} \\ x^v(t,k+1) \end{bmatrix}$, $x(t,k) = \begin{bmatrix} x^h(t,k) \\ x^v(t,k) \end{bmatrix} \in R^n$ is the state vector, $u(t,k) \in R^p$ is the two-dimensional control input, $w(t,k) \in R^r$ is the exogenous disturbance. t, k are the horizontal continuous variable and vertical discrete variable, respectively. $A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$, $B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}$, $G = \begin{bmatrix} G_1 \\ G_2 \end{bmatrix}$ are real matrices with appropriate dimensions. $x_0(t,k) = \begin{bmatrix} x^h(0,k) \\ x^v(t,0) \end{bmatrix}$ is used to represent the boundary condition.

Define the finite region for two-dimensional continuous-discrete system (2.1) as follows:

$$(T,N) = \{(t,k) | 0 \leqslant t \leqslant T, 0 \leqslant k \leqslant N; T > 0, N \in N^{+}\}. \tag{2.2}$$

The main aim of this note is to analyze the transient performance of system (2.1) over a given finite region by setting the state variables less than a particular threshold. Inspired by the definitions of FTS for one-dimensional systems in Amato *et al.* (2001) and Amato & Ariola (2005) and two-dimensional FRS for Roesser model in Zhang & Wang (2016b), the concept of FRS for the two-dimensional continuous-discrete linear system (2.1) in the uncontrolled case with no disturbances can be formalized.

DEFINITION 2.1 Given positive scalars c_1, c_2, T, N, with $c_1 < c_2$, $N \in N^{+}$ and a matrix $R > 0$, where $R = \operatorname{diag}\{R_1, R_2\}$, the two-dimensional continuous-discrete linear system

$$x^{+}(t,k) = Ax(t,k) \tag{2.3}$$

is said to be finite-region stable with respect to (c_1, c_2, T, N, R), if

$$x_0^T(t,k) R x_0(t,k) \leqslant c_1 \Rightarrow x^T(t,k) R x(t,k) < c_2, \quad \forall\, t \in [0,T], k \in \{1, \cdots, N\}.$$

REMARK 2.1 Similar to one-dimensional continuous and discrete cases, LAS and FRS are complete independent concepts for two-dimensional continuous-discrete systems. A system which is FRS may not be LAS and vice versa. If we limit our attention to what happens within a finite region, we can consider Lyapunov stability as an ‘additional’ requirement. Under stronger assumptions, the conditions presented in this paper (see Remark 3.1) include a special case where a system being both finite-region stable and Lyapunov stable.

Next, we consider the situation when the state is subject to some external signal disturbances $\mathscr{W}(d)$. This leads to the definition of FRB, which covers Definition 2.1 as a special case. In this paper, we will address two kinds of external signal disturbances:

(i) energy-bounded disturbances, $\mathscr{W}(d) = \{w(t,k) | w^T(t,k) w(t,k) \leqslant d\}$;

(ii) disturbances generated by an external system, $\mathscr{W}(d) = \{w(k) | w(k+1) = Fw(k), w^T(0)w(0) \leqslant d\}$, where F is a real matrix with appropriate dimensions.

DEFINITION 2.2 Given positive scalars c_1, c_2, d, T, N, with $c_1 < c_2$, $N \in N^+$ and a matrix $R > 0$, where $R = \text{diag}\,\{R_1, R_2\}$, the system

$$x^+(t,k) = Ax(t,k) + Gw(t,k) \tag{2.4}$$

is said to be finite-region bounded with respect to (c_1, c_2, T, N, R, d), if

$$x_0^T(t,k)Rx_0(t,k) \leqslant c_1 \Rightarrow x^T(t,k)Rx(t,k) < c_2, \quad \forall\, t \in [0,T], k \in \{1, \cdots, N\},$$

for all $w(t,k) \in \mathscr{W}(d)$.

REMARK 2.2 When $w(t,\ k) = 0$, the concept of FRB given in Definition 2.2 is consistent with the definition of FRS in Definition 2.1.

3. FRB and stabilization under the disturbances of the first case

In this section, we focus on FRB and finite-region stabilization issues for two-dimensional continuous-discrete system with the energy-bounded external disturbances $\mathscr{W}(d) = \{w(t,k)|w^T(t,k)w(t,k) \leqslant d\}$.

Firstly, we present a sufficient condition of the system (2.1) in the uncontrolled case.

THEOREM 3.1 System (2.4) is finite-region bounded with respect to (c_1, c_2, T, N, R, d), where $R = \text{diag}\{R_1, R_2\}$, if there exist positive scalars $0 < \eta < 1$, α_l, β_l, γ_l, $\alpha_2 + \beta_1 > 1$ and matrices $P_l > 0$, $S > 0$, where $l = 1, 2$, such that the following conditions hold:

$$\begin{bmatrix} A_{11}^T P_1 + P_1 A_{11} & -\alpha_1 P_1 P_1 A_{12} & P_1 G_1 \\ * & -\beta_1 P_2 & 0 \\ * & * & -\gamma_1 S \end{bmatrix} < 0, \tag{3.1a}$$

$$\begin{bmatrix} A_{21}^T P_2 A_{21} - \beta_2 P_1 & A_{21}^T P_2 A_{22} & A_{21}^T P_2 G_2 \\ * & A_{22}^T P_2 A_{22} - \alpha_2 P_2 & A_{22}^T P_2 G_2 \\ * & * & G_2^T P_2 G_2 - \gamma_2 S \end{bmatrix} < 0, \tag{3.1b}$$

$$x^{h^T}(0,k)R_1 x^h(0,k) \leqslant \eta c_1, \quad x^{v^T}(t,0)R_2 x^v(t,0) \leqslant (1-\eta)c_1, \tag{3.1c}$$

$$\frac{\lambda_{\max}(\widetilde{P}_1)}{\lambda_{\min}(\widetilde{P}_1)}\eta c_1 + \frac{\lambda_{\max}(\widetilde{P}_2)}{\lambda_{\min}(\widetilde{P}_1)}\beta_1(1-\eta)c_2 T + \frac{\lambda_{\max}(S)}{\lambda_{\min}(\widetilde{P}_1)}\gamma_1 dT < \eta c_2 e^{-\alpha_1 T}, \tag{3.1d}$$

$$\frac{\lambda_{\max}(\widetilde{P}_2)}{\lambda_{\min}(\widetilde{P}_2)}(1-\eta)c_1\alpha_0 + \frac{\lambda_{\max}(\widetilde{P}_1)}{\lambda_{\min}(\widetilde{P}_2)}N\alpha_0\beta_2\eta c_2 + \frac{\lambda_{\max}(S)}{\lambda_{\min}(\widetilde{P}_2)}N\alpha_0\gamma_2 d < (1-\eta)c_2, \tag{3.1e}$$

where $\alpha_0 = \max\{1, \alpha_2^N\}$, $\widetilde{P}_l = R_l^{-\frac{1}{2}} P_l R_l^{-\frac{1}{2}}$, $l = 1, 2$.

Proof. For system (2.4) and $P_1 > 0$, $P_2 > 0$, define the following Lyapunov functions

$$V_1(x^h(t,k)) = x^{h^T}(t,k)P_1 x^h(t,k), \;\; V_2(x^v(t,k)) = x^{v^T}(t,k)P_2 x^v(t,k),$$

denote $\psi(t,k) = [x^T(t,k)\ w^T(t,k)]^T$, then, we have

$$
\begin{aligned}
&\frac{\partial V_1(x^h(t,k))}{\partial t} - \alpha_1 V_1(x^h(t,k)) - \beta_1 V_2(x^v(t,k)) - \gamma_1 w^T(t,k)Sw(t,k) \\
&= \psi^T(t,k)\begin{bmatrix} A_{11}^T P_1 + P_1 A_{11} - \alpha_1 P_1 & P_1 A_{12} & P_1 G_1 \\ * & -\beta_1 P_2 & 0 \\ * & * & -\gamma_1 S \end{bmatrix}\psi(t,k), \\
&V_2(x^v(t,k+1)) - \alpha_2 V_2(x^v(t,k)) - \beta_2 V_1(x^h(t,k)) - \gamma_2 w^T(t,k)Sw(t,k) \\
&= \psi^T(t,k)\begin{bmatrix} A_{21}^T P_2 A_{21} - \beta_2 P_1 & A_{21}^T P_2 A_{22} & A_{21}^T P_2 G_2 \\ * & A_{22}^T P_2 A_{22} - \alpha_2 P_2 & A_{22}^T P_2 G_2 \\ * & * & G_2^T P_2 G_2 - \gamma_2 S \end{bmatrix}\psi(t,k).
\end{aligned}
$$

According to conditions (3.1a) and (3.1b), then

$$
\frac{\partial V_1(x^h(t,k))}{\partial t} < \alpha_1 V_1(x^h(t,k)) + \beta_1 V_2(x^v(t,k)) + \gamma_1 w^T(t,k)Sw(t,k), \tag{3.2}
$$

$$
V_2(x^v(t,k+1)) < \alpha_2 V_2(x^v(t,k)) + \beta_2 V_1(x^h(t,k)) + \gamma_2 w^T(t,k)Sw(t,k). \tag{3.3}
$$

Integrating from 0 to t for (3.2), with $t \in [0,T]$, we obtain

$$
\begin{aligned}
V_1(x^h(t,k)) <& V_1(x^h(0,k))e^{\alpha_1 t} + \beta_1 \int_0^t e^{\alpha_1(t-\tau)} x^{vT}(\tau,k)P_2 x^v(\tau,k)\,\mathrm{d}\tau \\
&+ \gamma_1 \int_0^t e^{\alpha_1(t-\tau)} w^T(\tau,k)Sw(\tau,k)\,\mathrm{d}\tau \\
\leqslant& \lambda_{\max}(\widetilde{P}_1) x^{hT}(0,k)R_1 x^h(0,k)e^{\alpha_1 T} + \lambda_{\max}(\widetilde{P}_2)\beta_1 e^{\alpha_1 T}\int_0^T x^{vT}(t,k)R_2 x^v(t,k)\,\mathrm{d}t \\
&+ \lambda_{\max}(S)\gamma_1 e^{\alpha_1 T} dT.
\end{aligned} \tag{3.4}
$$

Fixing $t \in [0,T]$, for $k \in \{1, \cdots, N\}$, by iteration to (3.3), we have

$$
\begin{aligned}
V_2(x^v(t,k)) <& \alpha_2^k V_2(x^v(t,0)) + \sum_{l=0}^{k-1} \alpha_2^{k-l-1}\left[\beta_2 x^{hT}(t,l)P_1 x^h(t,l) + \gamma_2 w^T(t,l)Sw(t,l)\right] \\
\leqslant& \lambda_{\max}(\widetilde{P}_2) x^{vT}(t,0)R_2 x^v(t,0)\alpha_0 \\
&+ \sum_{l=0}^{k-1} \alpha_2^{k-l-1}\left[\lambda_{\max}(\widetilde{P}_1)\beta_2 x^{hT}(t,l)R_1 x^h(t,l) + \lambda_{\max}(S)\gamma_2 d\right],
\end{aligned} \tag{3.5}
$$

where $\alpha_0 = \max\{1, \alpha_2^N\}$. On the other hand,

$$
V_1(x^h(t,k)) = x^{hT}(t,k)P_1 x^h(t,k) \geqslant \lambda_{\min}(\widetilde{P}_1) x^{hT}(t,k)R_1 x^h(t,k), \tag{3.6}
$$

$$
V_2(x^v(t,k)) = x^{vT}(t,k)P_2 x^v(t,k) \geqslant \lambda_{\min}(\widetilde{P}_2) x^{vT}(t,k)R_2 x^v(t,k). \tag{3.7}
$$

It follows from condition (3.1c) and (3.4)–(3.7) that

$$x^{h^T}(t,k)R_1x^h(t,k) < \frac{\lambda_{\max}(\widetilde{P}_1)}{\lambda_{\min}(\widetilde{P}_1)}\eta c_1 e^{\alpha_1 T} + \frac{\lambda_{\max}(\widetilde{P}_2)}{\lambda_{\min}(\widetilde{P}_1)}\beta_1 e^{\alpha_1 T}\int_0^T x^{v^T}(t,k)R_2x^v(t,k)\,\mathrm{d}t + \frac{\lambda_{\max}(S)}{\lambda_{\min}(\widetilde{P}_1)}\gamma_1 e^{\alpha_1 T}\mathrm{d}T, \tag{3.8}$$

$$x^{v^T}(t,k)R_2x^v(t,k) < \frac{\lambda_{\max}(\widetilde{P}_2)}{\lambda_{\min}(\widetilde{P}_2)}(1-\eta)c_1\alpha_0 + \frac{\lambda_{\max}(\widetilde{P}_1)}{\lambda_{\min}(\widetilde{P}_2)}\sum_{l=0}^{k-1}\alpha_2^{k-l-1}\beta_2 x^{h^T}(t,l)R_1x^h(t,l) + \frac{\lambda_{\max}(S)}{\lambda_{\min}(\widetilde{P}_2)}\sum_{l=0}^{k-1}\alpha_2^{k-l-1}\gamma_2\mathrm{d}. \tag{3.9}$$

Next, we will prove that the following inequalities hold for any $t \in [0, T]$, $k \in \{1, \cdots, N\}$:

$$x^{h^T}(t,k)R_1x^h(t,k) < \eta c_2,\ x^{v^T}(t,k)R_2x^v(t,k) < (1-\eta)c_2. \tag{3.10}$$

Noting that $c_1 < c_2$, from condition (3.1c), we have

$$x^{h^T}(0,k)R_1x^h(0,k) < \eta c_2,\ x^{v^T}(t,0)R_2x^v(t,0) < (1-\eta)c_2. \tag{3.11}$$

Setting $k = 0$ in (3.8) and using (3.11) and condition (3.1d), it is easy to obtain that

$$x^{h^T}(t,0)R_1x^h(t,0) < \eta c_2. \tag{3.12}$$

Now, we do the second mathematical induction for k to prove (3.10) holds for any $t \in [0, T]$. When $k = 1$, from (3.9), (3.12) and condition (3.1e), we get

$$\begin{aligned} x^{v^T}(t,1)R_2x^v(t,1) &< \frac{\lambda_{\max}(\widetilde{P}_2)}{\lambda_{\min}(\widetilde{P}_2)}(1-\eta)c_1\alpha_0 + \frac{\lambda_{\max}(\widetilde{P}_1)}{\lambda_{\min}(\widetilde{P}_2)}\beta_2 x^{h^T}(t,0)R_1x^h(t,0) + \frac{\lambda_{\max}(S)}{\lambda_{\min}(\widetilde{P}_2)}\gamma_2\mathrm{d} \\ &< \frac{\lambda_{\max}(\widetilde{P}_2)}{\lambda_{\min}(\widetilde{P}_2)}(1-\eta)c_1\alpha_0 + \frac{\lambda_{\max}(\widetilde{P}_1)}{\lambda_{\min}(\widetilde{P}_2)}\beta_2\eta c_2 + \frac{\lambda_{\max}(S)}{\lambda_{\min}(\widetilde{P}_2)}\gamma_2\mathrm{d} < (1-\eta)c_2. \end{aligned} \tag{3.13}$$

Similarly, setting $k = 1$ in (3.8), it is easy to obtain from (3.13) and condition (3.1d) that

$$x^{h^T}(t,1)R_1x^h(t,1) < \eta c_2.$$

Suppose that the conclusion (3.10) holds for $k < N$. When $k = N$, it follows from (3.9) and condition (3.1e) that

$$x^{vT}(t,N)R_2x^v(t,N) < \frac{\lambda_{\max}(\widetilde{P}_2)}{\lambda_{\min}(\widetilde{P}_2)}(1-\eta)c_1\alpha_0 + \frac{\lambda_{\max}(\widetilde{P}_1)}{\lambda_{\min}(\widetilde{P}_2)}\sum_{l=0}^{N-1}\alpha_2^{N-l-1}\beta_2\eta c_2 + \frac{\lambda_{\max}(S)}{\lambda_{\min}(\widetilde{P}_2)}\sum_{l=0}^{N-1}\alpha_2^{N-l-1}\gamma_2 \mathrm{d}$$
$$< \frac{\lambda_{\max}(\widetilde{P}_2)}{\lambda_{\min}(\widetilde{P}_2)}(1-\eta)c_1\alpha_0 + \frac{\lambda_{\max}(\widetilde{P}_1)}{\lambda_{\min}(\widetilde{P}_2)}N\alpha_0\beta_2\eta c_2 + \frac{\lambda_{\max}(S)}{\lambda_{\min}(\widetilde{P}_2)}N\alpha_0\gamma_2\mathrm{d} < (1-\eta)c_2. \tag{3.14}$$

Setting $k = N$ in (3.8) and employing (3.14) and condition (3.1d), we have

$$x^{h^T}(t,N)R_1x^h(t,N) < \eta c_2.$$

By induction, the condition (3.10) is established for any $t \in [0,T]$, $k \in \{1,\cdots,N\}$.

Therefore, for any $t \in [0,T]$, $k \in \{1,\cdots,N\}$, we have $x^T(t,k)Rx(t,k) < c_2$, which implies that the system (2.4) is finite-region bounded with respect to (c_1,c_2,T,N,R,d). □

For the simpler case of FRB, from Theorem 3.1, we can obtain the following corollary.

COROLLARY 3.1 System (2.3) is finite-region stable with respect to (c_1,c_2,T,N,R), where $R = \mathrm{diag}\{R_1,R_2\}$, if there exist positive scalars $0 < \eta < 1$, α_l, β_l, where $\alpha_2 + \beta_1 > 1$, and matrices $P_l > 0$, where $l = 1,2$, such that the condition (3.1c) and following inequalities hold:

$$\begin{bmatrix} A_{11}^TP_1 + P_1A_{11} - \alpha_1P_1 & P_1A_{12} \\ * & -\beta_1P_2 \end{bmatrix} < 0, \tag{3.15a}$$

$$\begin{bmatrix} A_{21}^TP_2A_{21} - \beta_2P_1 & A_{21}^TP_2A_{22} \\ * & A_{22}^TP_2A_{22} - \alpha_2P_2 \end{bmatrix} < 0, \tag{3.15b}$$

$$\frac{\lambda_{\max}(\widetilde{P}_1)}{\lambda_{\min}(\widetilde{P}_1)}\eta c_1 + \frac{\lambda_{\max}(\widetilde{P}_2)}{\lambda_{\min}(\widetilde{P}_1)}\beta_1(1-\eta)c_2T < \eta c_2 e^{-\alpha_1 T}, \tag{3.15c}$$

$$\frac{\lambda_{\max}(\widetilde{P}_2)}{\lambda_{\min}(\widetilde{P}_2)}(1-\eta)c_1\alpha_0 + \frac{\lambda_{\max}(\widetilde{P}_1)}{\lambda_{\min}(\widetilde{P}_2)}N\alpha_0\beta_2\eta c_2 < (1-\eta)c_2, \tag{3.15d}$$

where $\alpha_0 = \max\{1,\alpha_2^N\}$, $\widetilde{P}_l = R_l^{-\frac{1}{2}}P_lR_l^{-\frac{1}{2}}$, $l = 1,2$.

Proof. The proof can be obtained along the guidelines of Theorem 3.1. □

REMARK 3.1 If conditions (3.15) in Corollary 3.1 are satisfied with $\alpha_1 = -\beta_2 < 0$, $\alpha_2 + \beta_1 = 1$, then system (2.3) is finite-region stable with respect to (c_1,c_2,T,N,R), and it is also asymptotically stable. Specifically, if $\alpha_1 = -\beta_2$, $\alpha_2 + \beta_1 = 1$, from the conditions(3.15a–3.15b), we derive

$$\Psi = \begin{bmatrix} A_{11}^TP_1 + P_1A_{11} + A_{21}^TP_2A_{21} & P_1A_{12} + A_{21}^TP_2A_{22} \\ * & A_{22}^TP_2A_{22} - P_2 \end{bmatrix} < 0.$$

By Theorem 2 in Bachelier *et al.* (2008), the system (2.3) is asymptotically stable. Similarly, if conditions (3.1) in Theorem 3.1 hold for $\alpha_1 = -\beta_2$, $\alpha_2 + \beta_1 = 1$, then system (2.4) is not only finite-region bounded with respect to (c_1, c_2, T, N, R, d), but also asymptotically stable. It is worth noting that in the analysis of FRS, Ψ does not need to be negative definite but just less than $diag\{(\alpha_1 + \beta_2)P_1, (\alpha_2 + \beta_1 - 1)P_2\}$.

REMARK 3.2 In the derivation of Theorem 3.1, we used constant Lyapunov function. It is well known that the use of constant Lyapunov function will lead to a certain conservatism. Recently, Bachelier *et al.* (2016) gave the solution to reduce this conservatism by relaxing the polynomial-based texts of asymptotic stability into that of LMIs. Chesi & Middleton (2014) provided the solutions to reduce or possibly cancel the conservatism by using the frequency domain method in the analysis of exponential stability. It is worth noting that the problem they considered is asymptotic stability for two-dimensional continuous-discrete systems. Though these methods reduce the conservatism, they do not apply to the analysis of FRS. This is because the matrix A_{11} is not necessary Hurwitz and A_{22} is not necessary Schur in the FRS analysis. The study on the reduction of conservatism will be discussed in the future.

Next, we study the finite-region stabilization issue for two-dimensional continuous-discrete system (2.1) with energy-bounded external disturbances $\mathscr{W}(d) = \{w(t,k)|w^T(t,k)w(t,k) \leqslant d\}$. For given system (2.1), consider the following state feedback controller:

$$u(t,k) = Kx(t,k), \tag{3.16}$$

where $K = [K_1, K_2]$ is the constant real matrix with appropriate dimensions.

Our goal is to find sufficient condition which guarantees the interconnection of (2.1) with the controller (3.16)

$$x^+(t,k) = (A + BK)x(t,k) + Gw(t,k) \tag{3.17}$$

is finite-region bounded with respect to (c_1, c_2, T, N, R, d). The following theorem gives the solution of this problem.

THEOREM 3.2 System (3.17) is finite-region bounded with respect to (c_1, c_2, T, N, R, d), where $R = \text{diag}\{R_1, R_2\}$, if there exist positive scalars $0 < \eta < 1$, α_l, β_l, γ_l, where $\alpha_2 + \beta_1 > 1$, and matrices $H_l > 0$, $M > 0$, L_l, wh ere $l = 1, 2$, such that the condition (3.1c) and following inequalities hold:

$$\begin{bmatrix} \Psi - \alpha_1 \widetilde{H}_1 & A_{12}\widetilde{H}_2 + B_1 L_2 & G_1 M \\ * & -\beta_1 \widetilde{H}_2 & 0 \\ * & * & -\gamma_1 M \end{bmatrix} < 0, \tag{3.18a}$$

$$\begin{bmatrix} -\beta_2 \widetilde{H}_1 & 0 & 0 & \widetilde{H}_1 A_{21}^T + L_1^T B_2^T \\ * & -\alpha_2 \widetilde{H}_2 & 0 & \widetilde{H}_2 A_{22}^T + L_2^T B_2^T \\ * & * & -\gamma_2 M & M G_2^T \\ * & * & * & -\widetilde{H}_2 \end{bmatrix} < 0, \tag{3.18b}$$

$$\frac{\eta c_1}{\lambda_{\min}(H_1)} + \frac{\beta_1(1-\eta)c_2 T}{\lambda_{\min}(H_2)} + \frac{\gamma_1 d T}{\lambda_{\min}(M)} < \frac{\eta c_2 e^{-\alpha_1 T}}{\lambda_{\max}(H_1)}, \tag{3.18c}$$

$$\frac{(1-\eta)c_1\alpha_0}{\lambda_{\min}(H_2)} + \frac{N\alpha_0\beta_2\eta c_2}{\lambda_{\min}(H_1)} + \frac{N\alpha_0\gamma_2 d}{\lambda_{\min}(M)} < \frac{(1-\eta)c_2}{\lambda_{\max}(H_2)}, \tag{3.18d}$$

where $\Psi = \widetilde{H}_1 A_{11}^T + L_1^T B_1^T + A_{11}\widetilde{H}_1 + B_1 L_1$, $\alpha_0 = \max\{1, \alpha_2^N\}$, $\widetilde{H}_l = R_l^{-\frac{1}{2}} H_l R_l^{-\frac{1}{2}}$, $l = 1, 2$. In this case, the controller K is given by $K = [L_1\widetilde{H}_1^{-1}, L_2\widetilde{H}_2^{-1}]$.

Proof. Let $\widetilde{H}_l = P_l^{-1}$, $l = 1, 2$ and $M = S^{-1}$ in Theorem 3.1.

Noting that for symmetric positive definite matrices $H_l = \widetilde{P}_l^{-1}$, $M = S^{-1}$, their eigenvalues satisfy the following equations:

$$\lambda_{\max}(H_l) = \frac{1}{\lambda_{\min}(\widetilde{P}_l)}, \lambda_{\min}(H_l) = \frac{1}{\lambda_{\max}(\widetilde{P}_l)}, \lambda_{\min}(M) = \frac{1}{\lambda_{\max}(S)}.$$

Clearly, conditions (3.1d)–(3.1e) can be rewritten as in (3.18c)–(3.18d).

Now, let $\widehat{A} = A + BK$. If A_{1l} and A_{2l} in conditions (3.1a)–(3.1b) of Theorem 3.1 are replaced by $\widehat{A}_{1l} = A_{1l} + B_1 K_l$ and $\widehat{A}_{2l} = A_{2l} + B_2 K_l$, respectively, and in terms of $Q_l = P_l$, $\widetilde{H}_l = P_l^{-1}$ and $M = S^{-1}$, where $l = 1, 2$, we derive

$$\begin{bmatrix} \widehat{A}_{11}^T\widetilde{H}_1^{-1} + \widetilde{H}_1^{-1}\widehat{A}_{11} - \alpha_1\widetilde{H}_1^{-1} & \widetilde{H}_1^{-1}\widehat{A}_{12} & \widetilde{H}_1^{-1}G_1 \\ * & -\beta_1\widetilde{H}_2^{-1} & 0 \\ * & * & -\gamma_1 M^{-1} \end{bmatrix} < 0, \tag{3.19}$$

$$\begin{bmatrix} \widehat{A}_{21}^T\widetilde{H}_2^{-1}\widehat{A}_{21} - \beta_2\widetilde{H}_1^{-1} & \widehat{A}_{21}^T\widetilde{H}_2^{-1}\widehat{A}_{22} & \widehat{A}_{21}^T\widetilde{H}_2^{-1}G_2 \\ * & \widehat{A}_{22}^T\widetilde{H}_2^{-1}\widehat{A}_{22} - \alpha_2\widetilde{H}_2^{-1} & \widehat{A}_{22}^T\widetilde{H}_2^{-1}G_2 \\ * & * & G_2^T\widetilde{H}_2^{-1}G_2 - \gamma_2 M^{-1} \end{bmatrix} < 0. \tag{3.20}$$

Next, we will prove that the conditions (3.19) and (3.20) are equivalent to (3.18a) and (3.18b), respectively.

Pre- and post-multiplying (3.19) by the symmetric matrix *diag*$\{\widetilde{H}_1, \widetilde{H}_2, M\}$, we obtain the following equivalent condition

$$\begin{bmatrix} \widetilde{H}_1\widehat{A}_{11}^T + \widehat{A}_{11}\widetilde{H}_1 - \alpha_1\widetilde{H}_1 & \widehat{A}_{12}\widetilde{H}_2 & G_1 M \\ * & -\beta_1\widetilde{H}_2 & 0 \\ * & * & -\gamma_1 M \end{bmatrix} < 0. \tag{3.21}$$

Recalling that $\widehat{A}_{1l} = A_{1l} + B_1 K_l$, $l = 1, 2$, and letting $L_l = K_l\widetilde{H}_l$, $l = 1, 2$, we obtain that the condition (3.21) is equivalent to (3.18a).

Applying Schur complement lemma (Boyd *et al.*, 1994) twice to (3.20) produces

$$\begin{bmatrix} -\beta_2\widetilde{H}_1^{-1} & 0 & \widehat{A}_{21}^T\widetilde{H}_2^{-1}G_2 & 0 & \widehat{A}_{21}^T \\ * & -\alpha_2\widetilde{H}_2^{-1} & \widehat{A}_{22}^T\widetilde{H}_2^{-1}G_2 & 0 & \widehat{A}_{22}^T \\ * & * & -\gamma_2 M^{-1} & G_2^T & 0 \\ * & * & * & -\widetilde{H}_2 & 0 \\ * & * & * & * & -\widetilde{H}_2 \end{bmatrix} < 0. \tag{3.22}$$

Pre-multiplying (3.22) by

$$\begin{bmatrix} \widetilde{H}_1 & 0 & 0 & -\widetilde{H}_1\widehat{A}_{21}^T\widetilde{H}_2^{-1} & \widetilde{H}_1\widehat{A}_{21}^T\widetilde{H}_2^{-1} \\ 0 & \widetilde{H}_2 & 0 & -\widetilde{H}_2\widehat{A}_{22}^T\widetilde{H}_2^{-1} & \widetilde{H}_2\widehat{A}_{22}^T\widetilde{H}_2^{-1} \\ 0 & 0 & M & 0 & -MG_2^T\widetilde{H}_2^{-1} \\ 0 & 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 & I \end{bmatrix} \tag{3.23}$$

and post-multiplying it by the transpose of (3.23), we have the following equivalent condition:

$$\begin{bmatrix} -\beta_2\widetilde{H}_1 & 0 & 0 & \widetilde{H}_1\widehat{A}_{21}^T & 0 \\ * & -\alpha_2\widetilde{H}_2 & 0 & \widetilde{H}_2\widehat{A}_{22}^T & 0 \\ * & * & -\gamma_2 M - MG_2^T\widetilde{H}_2^{-1}G_2M & MG_2^T & MG_2^T \\ * & * & * & -\widetilde{H}_2 & 0 \\ * & * & * & * & -\widetilde{H}_2 \end{bmatrix} < 0. \tag{3.24}$$

Applying Schur complement lemma (Boyd *et al.*, 1994) to (3.24) yields the following equivalent condition:

$$\begin{bmatrix} -\beta_2\widetilde{H}_1 & 0 & 0 & \widetilde{H}_1\widehat{A}_{21}^T \\ * & -\alpha_2\widetilde{H}_2 & 0 & \widetilde{H}_2\widehat{A}_{22}^T \\ * & * & -\gamma_2 M & MG_2^T \\ * & * & * & -\widetilde{H}_2 \end{bmatrix} < 0. \tag{3.25}$$

Noting that $\widehat{A}_{2l} = A_{2l} + B_2K_l$, $l = 1, 2$, and letting $L_l = K_l\widetilde{H}_l$, $l = 1, 2$, we obtain that the condition (3.25) is equivalent to (3.18b).

From Theorem 3.1, we obtain that the system (3.17) is finite-region bounded with respect to (c_1, c_2, T, N, R, d). □

The following corollary of Theorem 3.2 allows us to find a state feedback controller K such that closed-loop system

$$x^+(t,k) = (A + BK)x(t,k) \tag{3.26}$$

is finite-region stable with respect to (c_1, c_2, T, N, R).

Corollary 3.2 System (3.26) is finite-region stable with respect to (c_1, c_2, T, N, R), where $R = \mathrm{diag}\{R_1, R_2\}$, if there exist positive scalars $0 < \eta < 1$, α_l, β_l, where $\alpha_2 + \beta_1 > 1$, and matrices

$H_l > 0$, L_l, where $l = 1, 2$, such that the condition (3.1c) and following inequalities hold:

$$\begin{bmatrix} \Psi - \alpha_1 \widetilde{H}_1 & A_{12}\widetilde{H}_2 + B_1 L_2 \\ * & -\beta_1 \widetilde{H}_2 \end{bmatrix} < 0, \tag{3.27a}$$

$$\begin{bmatrix} -\beta_2 \widetilde{H}_1 & 0 & \widetilde{H}_1 A_{21}^T + L_1^T B_2^T \\ * & -\alpha_2 \widetilde{H}_2 & \widetilde{H}_2 A_{22}^T + L_2^T B_2^T \\ * & * & -\widetilde{H}_2 \end{bmatrix} < 0, \tag{3.27b}$$

$$\frac{\eta c_1}{\lambda_{\min}(H_1)} + \frac{\beta_1(1-\eta)c_2 T}{\lambda_{\min}(H_2)} < \frac{\eta c_2 e^{-\alpha_1 T}}{\lambda_{\max}(H_1)}, \tag{3.27c}$$

$$\frac{(1-\eta)c_1\alpha_0}{\lambda_{\min}(H_2)} + \frac{N\alpha_0\beta_2\eta c_2}{\lambda_{\min}(H_1)} < \frac{(1-\eta)c_2}{\lambda_{\max}(H_2)}, \tag{3.27d}$$

where $\Psi = \widetilde{H}_1 A_{11}^T + L_1^T B_1^T + A_{11}\widetilde{H}_1 + B_1 L_1$, $\alpha_0 = \max\{1, \alpha_2^N\}$, $\widetilde{H}_l = R_l^{-\frac{1}{2}} H_l R_l^{-\frac{1}{2}}$, $l = 1, 2$. In this case, the controller K is given by $K = [L_1\widetilde{H}_1^{-1}, L_2\widetilde{H}_2^{-1}]$.

Proof. The proof can be obtained as in Theorem 3.2, applying the results of Corollary 3.1 to system (3.26). □

From a computational point of view, it is important to point out that the conditions (3.18c), (3.18d) in Theorem 3.2 and the conditions (3.27c), (3.27d) in Corollary 3.2 are difficult to solve. Besides, the conditions in Theorem 3.2 and Corollary 3.2 involve the unknown finite-region scalar $\alpha_l, \beta_l, \gamma_l, l = 1, 2$, which lead to Theorem 3.2 and Corollary 3.2 are difficult to solve by means of LMI Toolbox. In the following, we will show that once we have fixed values of $\alpha_l, \beta_l, \gamma_l, l = 1, 2$, the feasibility of Theorem 3.2 and Corollary 3.2 can be turned into LMIs-based feasibility problems (Boyd *et al.*, 1994) using procedures proposed in Amato *et al.* (2001).

Clearly, the conditions (3.18c) and (3.18d) are guaranteed by imposing additional conditions

$$\lambda_{l1} I_l < H_l < \lambda_{l2} I_l,\ \lambda_{31} I_r < M, \quad l = 1, 2, \tag{3.28a}$$

$$\frac{\eta c_1}{\lambda_{11}} + \frac{\beta_1(1-\eta)c_2 T}{\lambda_{21}} + \frac{\gamma_1 dT}{\lambda_{31}} < \frac{\eta c_2 e^{-\alpha_1 T}}{\lambda_{12}}, \tag{3.28b}$$

$$\frac{(1-\eta)c_1\alpha_0}{\lambda_{21}} + \frac{N\alpha_0\beta_2\eta c_2}{\lambda_{11}} + \frac{N\alpha_0\gamma_2 d}{\lambda_{31}} < \frac{(1-\eta)c_2}{\lambda_{22}} \tag{3.28c}$$

for some positive numbers λ_{l1}, λ_{l2}, λ_{31}. Using Schur complement (Boyd *et al.*, 1994) to inequalities (3.28b) and (3.28c) produces

$$\begin{bmatrix} \lambda_{12}\eta c_2 e^{-\alpha_1 T} & \lambda_{12}\sqrt{\beta_1(1-\eta)c_2 T} & \lambda_{12}\sqrt{\eta c_1} & \lambda_{12}\sqrt{\gamma_1 dT} \\ * & \lambda_{21} & 0 & 0 \\ * & * & \lambda_{11} & 0 \\ * & * & * & \lambda_{31} \end{bmatrix} > 0, \tag{3.29a}$$

$$\begin{bmatrix} \lambda_{22}(1-\eta)c_2 & \lambda_{22}\sqrt{N\alpha_0\beta_2\eta c_2} & \lambda_{22}\sqrt{(1-\eta)c_1\alpha_0} & \lambda_{22}\sqrt{N\alpha_0\gamma_2 d} \\ * & \lambda_{11} & 0 & 0 \\ * & * & \lambda_{21} & 0 \\ * & * & * & \lambda_{31} \end{bmatrix} > 0. \tag{3.29b}$$

The following theorem gives the LMI feasibility problem of Theorem 3.2.

THEOREM 3.3 Given system (3.17) and (c_1, c_2, T, N, R, d), where $R = \text{diag}\{R_1, R_2\}$, fix $\alpha_l > 0$, $\beta_l > 0$, $\gamma_l > 0$, $0 < \eta < 1$, where $\alpha_2 + \beta_1 > 1$, and find matrices $H_l > 0$, $M > 0$, L_l and positive scalars λ_{l1}, λ_{l2}, λ_{31} satisfying (3.1c) and the LMIs (3.18a), (3.18b), (3.28a) and (3.29), where $\Psi = \widetilde{H}_1 A_{11}^T + L_1^T B_1^T + A_{11}\widetilde{H}_1 + B_1 L_1$, $\alpha_0 = \max\{1, \alpha_2^N\}$, $\widetilde{H}_l = R_l^{-\frac{1}{2}} H_l R_l^{-\frac{1}{2}}$, $l = 1, 2$. If the problem is feasible, the controller K given by $K = [L_1\widetilde{H}_1^{-1}, L_2\widetilde{H}_2^{-1}]$ renders system (3.17) finite-region bounded with respect to (c_1, c_2, T, N, R, d).

Similarly, LMI feasibility problem can be derived from Corollary 3.2.

COROLLARY 3.3 Given system (3.26) and (c_1, c_2, T, N, R), where $R = \text{diag}\{R_1, R_2\}$, fix $\alpha_l > 0$, $\beta_l > 0$, $0 < \eta < 1$, where $\alpha_2 + \beta_1 > 1$, and find matrices $H_l > 0$, L_l and positive scalars λ_{l1}, λ_{l2} satisfying (3.1c), the LMIs (3.27a), (3.27b) and

$$\lambda_{l1} I_l < H_l < \lambda_{l2} I_l, \quad l = 1, 2, \tag{3.30a}$$

$$\begin{bmatrix} \lambda_{12}\eta c_2 e^{-\alpha_1 T} & \lambda_{12}\sqrt{\beta_1(1-\eta)c_2 T} & \lambda_{12}\sqrt{\eta c_1} \\ * & \lambda_{21} & 0 \\ * & * & \lambda_{11} \end{bmatrix} > 0, \tag{3.30b}$$

$$\begin{bmatrix} \lambda_{22}(1-\eta)c_2 & \lambda_{22}\sqrt{N\alpha_0\beta_2\eta c_2} & \lambda_{22}\sqrt{(1-\eta)c_1\alpha_0} \\ * & \lambda_{11} & 0 \\ * & * & \lambda_{21} \end{bmatrix} > 0, \tag{3.30c}$$

where $\Psi = \widetilde{H}_1 A_{11}^T + L_1^T B_1^T + A_{11}\widetilde{H}_1 + B_1 L_1$, $\alpha_0 = \max\{1, \alpha_2^N\}$, $\widetilde{H}_l = R_l^{-\frac{1}{2}} H_l R_l^{-\frac{1}{2}}$, $l = 1, 2$. If the problem is feasible, the controller $K = [L_1\widetilde{H}_1^{-1}, L_2\widetilde{H}_2^{-1}]$ renders system (3.26) finite-region stable with respect to (c_1, c_2, T, N, R).

4. FRB and stabilization under the second case of disturbances

In this section, we will study the FRB and finite-region stabilization issues for two-dimensional continuous-discrete system with disturbances generated by an external system $\mathscr{W}(d) = \{w(k) | w(k+1) = Fw(k), w^T(0)w(0) \leqslant d\}$.

Firstly, we consider the FRB issue for the two-dimensional continuous-discrete system in the form

$$x^+(t, k) = Ax(t, k) + Gw(k), \tag{4.1a}$$

$$w(k+1) = Fw(k). \tag{4.1b}$$

THEOREM 4.1 System (4.1) is finite-region bounded with respect to (c_1, c_2, T, N, R, d), where $R = \text{diag}\{R_1, R_2\}$, if there exist positive scalars $0 < \eta < 1$, α_l, β_l, where $\alpha_2 + \beta_1 > 1$, and matrices

$P_l > 0$, $S > 0$, where $l = 1, 2$, such that the condition (3.1c) and following inequalities hold:

$$\begin{bmatrix} A_{11}^T P_1 + P_1 A_{11} - \alpha_1 P_1 & P_1 A_{12} & P_1 G_1 \\ * & -\beta_1 P_2 & 0 \\ * & * & -\alpha_1 S \end{bmatrix} < 0, \tag{4.2a}$$

$$\begin{bmatrix} A_{21}^T P_2 A_{21} - \beta_2 P_1 & A_{21}^T P_2 A_{22} & A_{21}^T P_2 G_2 \\ * & A_{22}^T P_2 A_{22} - \alpha_2 P_2 & A_{22}^T P_2 G_2 \\ * & * & G_2^T P_2 G_2 + F^T S F - \alpha_2 S \end{bmatrix} < 0, \tag{4.2b}$$

$$\frac{\lambda_{\max}(\widetilde{P}_1)\eta c_1 + \lambda_{\max}(S)\gamma d}{\lambda_{\min}(\widetilde{P}_1)} + \frac{\lambda_{\max}(\widetilde{P}_2)}{\lambda_{\min}(\widetilde{P}_1)}\beta_1(1-\eta)c_2 T < \eta c_2 e^{-\alpha_1 T}, \tag{4.2c}$$

$$\frac{\lambda_{\max}(\widetilde{P}_2)(1-\eta)c_1 + \lambda_{\max}(S)d}{\lambda_{\min}(\widetilde{P}_2)}\alpha_0 + \frac{\lambda_{\max}(\widetilde{P}_1)}{\lambda_{\min}(\widetilde{P}_2)}N\alpha_0\beta_2\eta c_2 < (1-\eta)c_2, \tag{4.2d}$$

where $\gamma = \max\{1, \|F^k\|^2_{k=1,2,\cdots,N}\}$, $\alpha_0 = \max\{1, \alpha_2^N\}$, $\widetilde{P}_l = R_l^{-\frac{1}{2}} P_l R_l^{-\frac{1}{2}}$, $l = 1, 2$.

Proof. Firstly, we derive the recursive relations of the weights of state variables of system (4.1a).

For simple mark, let $z^+(t,k) = \begin{bmatrix} x^+(t,k) \\ w(k+1) \end{bmatrix}$, $z(t,k) = \begin{bmatrix} x(t,k) \\ w(k) \end{bmatrix}$, then the system (4.1) is equivalent to

$$z^+(t,k) = \begin{bmatrix} A & G \\ 0 & F \end{bmatrix} z(t,k). \tag{4.3}$$

We define the Lyapunov functions of system (4.3) as follows

$$V_1(z^h(t,k)) = z^{hT}(t,k)\begin{bmatrix} P_1 & 0 \\ * & S \end{bmatrix} z^h(t,k), V_2\left(z^v(t,k)\right) = z^{vT}(t,k)\begin{bmatrix} P_2 & 0 \\ * & S \end{bmatrix} z^v(t,k),$$

where $z^h(t,k) = \begin{bmatrix} x^h(t,k) \\ w(k) \end{bmatrix}$, $z^v(t,k) = \begin{bmatrix} x^v(t,k) \\ w(k) \end{bmatrix}$, then it follows that

$$\begin{aligned} &\frac{\partial V_1(z^h(t,k))}{\partial t} - \alpha_1 V_1(z^h(t,k)) - \beta_1 x^{vT}(t,k) P_2 x^v(t,k) \\ &\qquad = z^T(t,k) \begin{bmatrix} A_{11}^T P_1 + P_1 A_{11} - \alpha_1 P_1 & P_1 A_{12} & P_1 G_1 \\ * & -\beta_1 P_2 & 0 \\ * & * & -\alpha_1 S \end{bmatrix} z(t,k). \end{aligned} \tag{4.4}$$

Similarly, we can obtain the following equation

$$
\begin{aligned}
&V_2(z^v(t,k+1)) - \alpha_2 V_2(z^v(t,k)) - \beta_2 x^{h^T}(t,k)P_1x^h(t,k) \\
&\qquad = z^T(t,k)\begin{bmatrix} A_{21}^TP_2A_{21}-\beta_2P_1 & A_{21}^TP_2A_{22} & A_{21}^TP_2G_2 \\ * & A_{22}^TP_2A_{22}-\alpha_2P_2 & A_{22}^TP_2A_{22} \\ * & * & G_2^TP_2G_2+F^TSF-\alpha_2S \end{bmatrix} z(t,k).
\end{aligned} \tag{4.5}
$$

Using conditions (4.2a) and (4.2b) for (4.4) and (4.5), we derive

$$
\frac{\partial V_1(z^h(t,k))}{\partial t} < \alpha_1 V_1(z^h(t,k)) + \beta_1 x^{vT}(t,k)P_2x^v(t,k), \tag{4.6}
$$

$$
V_2(z^v(t,k+1)) < \alpha_2 V_2(z^v(t,k)) + \beta_2 x^{h^T}(t,k)P_1x^h(t,k). \tag{4.7}
$$

When $k=0$, integrating from 0 to t for (4.6), with $t \in [0, T]$, we derive

$$
\begin{aligned}
V_1(z^h(t,0)) &< V_1(z^h(0,0))e^{\alpha_1 t} + \beta_1\int_0^t e^{\alpha_1(t-\tau)}x^{vT}(\tau,0)P_2x^v(\tau,0)\,\mathrm{d}\tau \\
&\leqslant \left[x^{h^T}(0,0)P_1x^h(0,0) + w^T(0)Sw(0)\right]e^{\alpha_1 T} + \beta_1 e^{\alpha_1 T}\int_0^T x^{vT}(t,0)P_2x^v(t,0)\,\mathrm{d}t \\
&\leqslant \left[\lambda_{\max}(\widetilde{P}_1)\eta c_1 + \lambda_{\max}(S)d\right]e^{\alpha_1 T} + \lambda_{\max}(\widetilde{P}_2)\beta_1 e^{\alpha_1 T}\int_0^T x^{vT}(t,0)R_2x^v(t,0)\,\mathrm{d}t,
\end{aligned} \tag{4.8}
$$

for fixed $k \in \{1, \cdots, N\}$, integrating from 0 to t for (4.6), with $t \in [0, T]$, we obtain

$$
\begin{aligned}
V_1(z^h(t,k)) &< V_1(z^h(0,k))e^{\alpha_1 t} + \beta_1\int_0^t e^{\alpha_1(t-\tau)}x^{vT}(\tau,k)P_2x^v(\tau,k)\,\mathrm{d}\tau \\
&\leqslant \left[\lambda_{\max}(\widetilde{P}_1)\eta c_1 + \lambda_{\max}(S)\|F^k\|^2 d\right]e^{\alpha_1 T} + \lambda_{\max}(\widetilde{P}_2)\beta_1 e^{\alpha_1 T}\int_0^T x^{vT}(t,k)R_2x^v(t,k)\,\mathrm{d}t.
\end{aligned} \tag{4.9}
$$

Fixing $t \in [0, T]$, by iteration to (4.7), we have

$$
\begin{aligned}
V_2(z^v(t,k)) &< \alpha_2^k V_2(x^v(t,0)) + \sum_{l=0}^{k-1}\alpha_2^{k-l-1}\beta_2 x^{h^T}(t,l)P_1x^h(t,l) \\
&\leqslant \left[x^{vT}(t,0)P_2x^v(t,0) + w^T(0)Sw(0)\right]\alpha_0 + \sum_{l=0}^{k-1}\alpha_2^{k-l-1}\beta_2 x^{h^T}(t,l)P_1x^h(t,l) \\
&\leqslant \left[\lambda_{\max}(\widetilde{P}_2)(1-\eta)c_1 + \lambda_{\max}(S)d\right]\alpha_0 + \lambda_{\max}(\widetilde{P}_1)\sum_{l=0}^{k-1}\alpha_2^{k-l-1}\beta_2 x^{h^T}(t,l)R_1x^h(t,l),
\end{aligned} \tag{4.10}
$$

where $\alpha_0 = \max\{1, \alpha_2^N\}$. On the other hand,

$$V_1(z^h(t,k)) = x^{h^T}(t,k)P_1x^h(t,k) + w^T(k)Sw(k) \geqslant \lambda_{\min}(\widetilde{P}_1)x^{h^T}(t,k)R_1x^h(t,k), \tag{4.11}$$

$$V_2(z^v(t,k)) = x^{v^T}(t,k)P_2x^v(t,k) + w^T(k)Sw(k) \geqslant \lambda_{\min}(\widetilde{P}_2)x^{v^T}(t,k)R_2x^v(t,k). \tag{4.12}$$

Putting together (4.8), (4.9) with (4.11) and (4.10) with (4.12), respectively, we obtain

$$\begin{aligned} x^{h^T}(t,k)R_1x^h(t,k) < {} & \frac{\lambda_{\max}(\widetilde{P}_1)\eta c_1 + \lambda_{\max}(S)\gamma d}{\lambda_{\min}(\widetilde{P}_1)} e^{\alpha_1 T} \\ & + \frac{\lambda_{\max}(\widetilde{P}_2)}{\lambda_{\min}(\widetilde{P}_1)}\beta_1 e^{\alpha_1 T}\int_0^T x^{v^T}(t,k)R_2x^v(t,k)\,\mathrm{d}t, \end{aligned} \tag{4.13}$$

$$\begin{aligned} x^{v^T}(t,k)R_2x^v(t,k) < {} & \frac{\lambda_{\max}(\widetilde{P}_2)(1-\eta)c_1 + \lambda_{\max}(S)d}{\lambda_{\min}(\widetilde{P}_2)}\alpha_0 \\ & + \frac{\lambda_{\max}(\widetilde{P}_1)}{\lambda_{\min}(\widetilde{P}_2)}\sum_{l=0}^{k-1}\alpha_2^{k-l-1}\beta_2 x^{h^T}(t,l)R_1x^h(t,l), \end{aligned} \tag{4.14}$$

where $\gamma = \max\{1, \|F^k\|^2_{k=1,2,\cdots,N}\}$.

Similar to the proof of Theorem 3.1, we can get that for any $t \in [0, T]$, $k \in \{1, \cdots, N\}$, $x^T(t,k)Rx(t,k) < c_2$ holds. This implies that the two-dimensional continuous-discrete system (4.1) is finite-region bounded with respect to (c_1, c_2, T, N, R, d). □

REMARK 4.1 Let $F = I$ and $\gamma = 1$ in Theorem 4.1, the sufficient condition of the FRB for system (4.1a) subject to exogenous unknown constant disturbances can be derived. In Theorem 4.1, when the external system (4.1b) is a one-dimensional continuous system or a two-dimensional continuous-discrete system, the corresponding results can also be obtained.

REMARK 4.2 The sufficient condition which ensures system (2.3) finite-region stable with respect to (c_1, c_2, T, N, R) by Theorem 4.1 is consistent with Corollary 3.1.

Next, we investigate the finite-region stabilization issue for the system (2.1) with disturbances generated by an external system $\mathscr{W}(d) = \{w(k)|w(k+1) = Fw(k), w^T(0)w(0) \leqslant d\}$, with the aim to find sufficient condition that ensures the system

$$x^+(t,k) = (A+BK)x(t,k) + Gw(k), \tag{4.15a}$$

$$w(k+1) = Fw(k) \tag{4.15b}$$

is finite-region bounded with respect to (c_1, c_2, T, N, R, d). The solution of this issue is given by the following theorem.

THEOREM 4.2 System (4.15) is finite-region bounded with respect to (c_1, c_2, T, N, R, d), where $R = \mathrm{diag}\{R_1, R_2\}$, if there exist positive scalars $0 < \eta < 1$, α_l, β_l, where $\alpha_2 + \beta_1 > 1$, and matrices $H_l > 0$,

$M > 0$, L_l, where $l = 1, 2$, such that the condition (3.1c) and following inequalities hold:

$$\begin{bmatrix} \Psi - \alpha_1 \widetilde{H}_1 & A_{12}\widetilde{H}_2 + B_1 L_2 & G_1 M \\ * & -\beta_1 \widetilde{H}_2 & 0 \\ * & * & -\alpha_1 M \end{bmatrix} < 0, \tag{4.16a}$$

$$\begin{bmatrix} -\beta_2 \widetilde{H}_1 & 0 & 0 & 0 & \widetilde{H}_1 A_{21}^T + L_1^T B_2^T \\ * & -\alpha_2 \widetilde{H}_2 & 0 & 0 & \widetilde{H}_2 A_{22}^T + L_2^T B_2^T \\ * & * & -\alpha_2 M & MF^T & MG_2^T \\ * & * & * & -M & 0 \\ * & * & * & * & -\widetilde{H}_2 \end{bmatrix} < 0, \tag{4.16b}$$

$$\frac{\eta c_1}{\lambda_{\min}(H_1)} + \frac{\gamma d}{\lambda_{\min}(M)} + \frac{\beta_1(1-\eta)c_2 T}{\lambda_{\min}(H_2)} < \frac{\eta c_2 e^{-\alpha_1 T}}{\lambda_{\max}(H_1)}, \tag{4.16c}$$

$$\frac{(1-\eta)c_1\alpha_0}{\lambda_{\min}(H_2)} + \frac{d\alpha_0}{\lambda_{\min}(M)} + \frac{N\alpha_0\beta_2\eta c_2}{\lambda_{\min}(H_1)} < \frac{(1-\eta)c_2}{\lambda_{\max}(H_2)}, \tag{4.16d}$$

where $\Psi = \widetilde{H}_1 A_{11}^T + L_1^T B_1^T + A_{11}\widetilde{H}_1 + B_1 L_1$, $\gamma = \max\{1, \|F^k\|^2_{k=1,2,\cdots,N}\}$, $\alpha_0 = \max\{1, \alpha_2^N\}$, $\widetilde{H}_l = R_l^{-\frac{1}{2}} H_l R_l^{-\frac{1}{2}}$, $l = 1, 2$. In this case, the controller K is given by $K = [L_1\widetilde{H}_1^{-1}, L_2\widetilde{H}_2^{-1}]$.

Proof. Let $\widetilde{H}_l = P_l^{-1}$, $l = 1, 2$ and $M = S^{-1}$ in Theorem 4.1. Similar to the proof of Theorem 3.2, applying the results of Theorem 4.1 to system (4.15), the proof can be obtained.

Here, we only give the proof of condition (4.16b). If A_{2l} in condition (4.2b) of Theorem 4.1 is replaced by $\widehat{A}_{2l} = A_{2l} + B_2 K_l$, $l = 1, 2$, we derive

$$\begin{bmatrix} \widehat{A}_{21}^T \widetilde{H}_2^{-1} \widehat{A}_{21} - \beta_2 \widetilde{H}_1^{-1} & \widehat{A}_{21}^T \widetilde{H}_2^{-1} \widehat{A}_{22} & \widehat{A}_{21}^T \widetilde{H}_2^{-1} G_2 \\ * & \widehat{A}_{22}^T \widetilde{H}_2^{-1} \widehat{A}_{22} - \alpha_2 \widetilde{H}_2^{-1} & \widehat{A}_{22}^T \widetilde{H}_2^{-1} G_2 \\ * & * & G_2^T \widetilde{H}_2^{-1} G_2 + F^T M^{-1} F - \alpha_2 M^{-1} \end{bmatrix} < 0. \tag{4.17}$$

Applying Schur complement lemma (Boyd *et al.*, 1994) to (4.17) produces

$$\begin{bmatrix} -\beta_2 \widetilde{H}_1^{-1} & 0 & 0 & 0 & \widehat{A}_{21}^T \\ * & -\alpha_2 \widetilde{H}_2^{-1} & 0 & 0 & \widehat{A}_{22}^T \\ * & * & -\alpha_2 M^{-1} & F^T & G_2^T \\ * & * & * & -M & 0 \\ * & * & * & * & -\widetilde{H}_2 \end{bmatrix} < 0. \tag{4.18}$$

Pre- and post-multiplying (4.18) by $diag\{\widetilde{H}_1, \widetilde{H}_2, M, I, I\}$, we have the following equivalent condition:

$$\begin{bmatrix} -\beta_2\widetilde{H}_1 & 0 & 0 & 0 & \widetilde{H}_1\widehat{A}_{21}^T \\ * & -\alpha_2\widetilde{H}_2 & 0 & 0 & \widetilde{H}_2\widehat{A}_{22}^T \\ * & * & -\alpha_2 M & MF^T & MG_2^T \\ * & * & * & -M & 0 \\ * & * & * & * & -\widetilde{H}_2 \end{bmatrix} < 0. \tag{4.19}$$

Letting $L_l = K_l\widetilde{H}_l$, $l = 1, 2$, we finally obtain that the condition (4.19) is equivalent to (4.16b). □

REMARK 4.3 Let $F = I$ and $\gamma = 1$ in Theorem 4.2, the sufficient condition of the FRB via state feedback for system (4.15a) subject to unknown constant disturbances can be derived. Similarly, when the external system is a one-dimensional continuous-discrete system or a two-dimensional continuous-discrete system, the corresponding conclusions can also be obtained.

REMARK 4.4 When setting $\eta = 0$, $\eta = 1$ in Theorems 4.1 and 4.2, respectively, we can obtain the corresponding conclusions for one-dimensional discrete linear system (Amato & Ariola, 2005) and continuous linear system (Amato *et al.*, 2001), respectively.

Similarly, Theorem 4.2 can be reducible to the following LMIs-based feasibility problem.

THEOREM 4.3 Given system (4.15) and (c_1, c_2, T, N, R, d), where $R = \text{diag}\{R_1, R_2\}$, fix $\alpha_l > 0$, $\beta_l > 0$, $0 < \eta < 1$, where $\alpha_2 + \beta_1 > 1$, and find matrices $H_l > 0$, $M > 0$, L_l and positive scalars λ_{l1}, λ_{l2}, λ_{31} satisfying (3.1c) and the LMIs (4.16a), (4.16b), (3.28a) and

$$\begin{bmatrix} \lambda_{12}\eta c_2 e^{-\alpha_1 T} & \lambda_{12}\sqrt{\beta_1(1-\eta)c_2 T} & \lambda_{12}\sqrt{\eta c_1} & \lambda_{12}\sqrt{\gamma d} \\ * & \lambda_{21} & 0 & 0 \\ * & * & \lambda_{11} & 0 \\ * & * & * & \lambda_{31} \end{bmatrix} > 0, \tag{4.20a}$$

$$\begin{bmatrix} \lambda_{22}(1-\eta)c_2 & \lambda_{22}\sqrt{N\alpha_0\beta_2\eta c_2} & \lambda_{22}\sqrt{(1-\eta)c_1\alpha_0} & \lambda_{22}\sqrt{d\alpha_0} \\ * & \lambda_{11} & 0 & 0 \\ * & * & \lambda_{21} & 0 \\ * & * & * & \lambda_{31} \end{bmatrix} > 0, \tag{4.20b}$$

where $\Psi = \widetilde{H}_1 A_{11}^T + L_1^T B_1^T + A_{11}\widetilde{H}_1 + B_1 L_1$, $\widetilde{H}_l = R_l^{-\frac{1}{2}} H_l R_l^{-\frac{1}{2}}$, $l = 1, 2$. If the problem is feasible, the controller $K = [L_1\widetilde{H}_1^{-1}, L_2\widetilde{H}_2^{-1}]$ renders system (4.15) finite-region bounded with respect to (c_1, c_2, T, N, R, d).

5. Numerical examples

In this section, numerical examples are used to illustrate the effectiveness of the proposed methods.

EXAMPLE 5.1 It is well known that some dynamical processes in gas absorption, water stream heating and air drying can be described by the Darboux equation (Marszalek, 1984):

$$\frac{\partial^2 s(t,\tau)}{\partial t \partial \tau} = a_1 \frac{\partial s(t,\tau)}{\partial \tau} + a_2 \frac{\partial s(t,\tau)}{\partial t} + a_0 s(t,\tau) + bf(t,\tau), \tag{5.1}$$

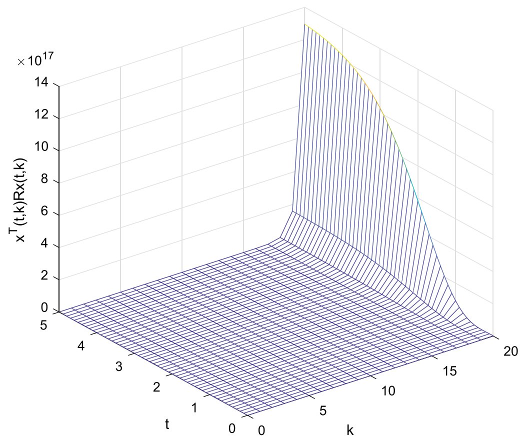

FIG. 1. $x^T(t,k)Rx(t,k)$ of system (2.4).

where $s(t,\tau)$ is an unknown vector function at $[0,t_f]$ and $\tau \in [0,\infty]$, a_0, a_1, a_2, b are real constants and $f(t,\tau)$ is the input function.

Taking $r(t,\tau) = \partial s(t,\tau)/\partial\tau - a_2 s(t,\tau)$, $x^h(t,k) = r(t,k) = r(t,k\Delta\tau)$ and $x^v(t,k) = s(t,k) = s(t,k\Delta\tau)$, we can write the Equation (5.1) in the two-dimensional continuous-discrete system of the form (2.1). Via appropriate selection of the parameters a_0, a_1, a_2, b, we consider the system (2.1) subject to the energy-bounded external disturbances with

$$A = \begin{bmatrix} -2.1 & 0.5 \\ 1.5 & 2.5 \end{bmatrix},\ B = \begin{bmatrix} -0.2 \\ 1.5 \end{bmatrix}, G = \begin{bmatrix} 0.3 \\ 0 \end{bmatrix},$$

where $c_1 = 2.5$, $c_2 = 10$, $T = 5$, $N = 20$, $d = 1$, $R = I$, $\eta = 0.7$.

Given above positive constants, the positive definite matrix R and the initial condition $x^h(0,k) = -1.3$, $x^v(t,0) = 0.85$. When external disturbances satisfy $w^T(t,k)w(t,k) \leqslant 1$, we have considered the problem of FRB via state feedback and designed the state feedback controller by solving the feasibility problem in Theorem 3.3. When control input $u(t,k) = 0$, Fig. 1 shows that the open-loop system (2.4) is not finite-region bounded with respect to (2.5, 10, 5, 20, I, 1). Using LMI toolbox of MATLAB and Theorem 3.3, the LMIs (3.18a), (3.18b), (3.28a) and (3.29) are feasible with $\alpha_1 = 0.03$, $\beta_1 = 0.05$, $\gamma_1 = 0.1$, $\alpha_2 = 1.03$, $\beta_2 = 0.01$, $\gamma_2 = 0.15$, the solution is given below

$$\begin{bmatrix} \widetilde{H}_1 & 0 \\ * & \widetilde{H}_2 \end{bmatrix} = \begin{bmatrix} 11.6622 & 0 \\ * & 1.8365 \end{bmatrix},\ \begin{bmatrix} L_1 \\ L_2 \end{bmatrix} = \begin{bmatrix} -11.7054 \\ -3.0609 \end{bmatrix},\ M = 14.5931.$$

Then, we find the state feedback controller

$$K = [-1.0037, -1.6667].$$

The weighted-state values $x^T(t,k)Rx(t,k)$ are limited by the given bound 10 for the closed-loop system (3.17) obtained after stabilization (see Fig. 2).

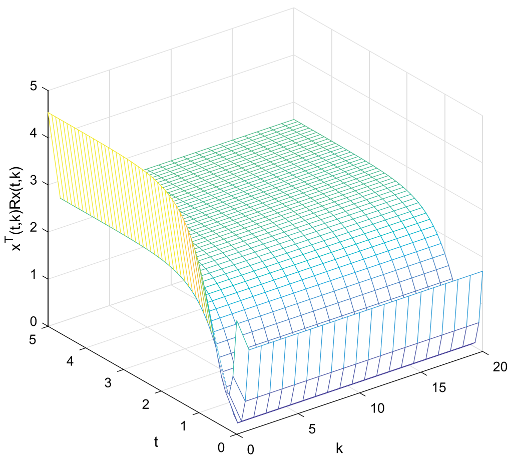

FIG. 2. $x^T(t,k)Rx(t,k)$ of system (3.17).

Let us take the dynamical process in gas absorption as an example to explain the practical significance of state feedback. In the process of absorption of a gas, $s(t,\tau)$ in Darboux equation (5.1) denotes the quantity of gas absorbed by unit volume of the absorbent (Tichonov & Samarsky, 1963). The kinetics of absorption is represented by $\partial s(t,\tau)/\partial\tau$. Let $a_2 = -\gamma$, where $\frac{1}{\gamma}$ is Henry's coefficient, then $\partial s(t,\tau)/\partial\tau - a_2 s(t,\tau)$ denotes the concentration of gas in the pores of the absorbent in the layer t. In practice, due to the material constraints, the concentration of gas in the pores of the absorbent in the layer t and the quantity of gas absorbed by unit volume of the absorbent are required to stay within a desirable threshold range during the specified absorbing layer and time interval, that is, the states in system (2.1) are required to stay within a particular threshold range over a given finite-region. Therefore, when the established system is not finite-region stable, we need to use state feedback to make the system states not exceed the particular threshold.

EXAMPLE 5.2 Consider the system (2.1) with $w(t,k) = 0$, where

$$A = \begin{bmatrix} -0.5 & 2.5 \\ 1.1 & 2.5 \end{bmatrix},\ B = \begin{bmatrix} -0.2 \\ 1.2 \end{bmatrix}.$$

Assume that $c_1 = 2$, $c_2 = 10$, $T = 5$, $N = 10$, $R = I$, $\eta = 0.7$ and $x^h(0,k) = 1.1$, $x^v(t,0) = -0.2$. We have considered the FRS via state feedback and designed the state feedback controller by solving the feasibility problem in Corollary 3.3. When the system has control input $u(t,k) = 0$, the weighted-state values $x^T(t,k)Rx(t,k)$ of system (2.3) are as shown in Fig. 3, obviously, the open-loop system (2.3) is not finite-region stable with respect to $(2, 10, 5, 10, I)$ before stabilization. Using LMI control toolbox and Corollary 3.3, the LMIs (3.27a), (3.27b) and (3.30) are feasible with $\alpha_1 = 0.03$, $\beta_1 = 0.1$, $\alpha_2 = 1.1$, $\beta_2 = 0.01$. The solution is given below

$$\begin{bmatrix} \widetilde{H}_1 & 0 \\ * & \widetilde{H}_2 \end{bmatrix} = \begin{bmatrix} 1.3719 & 0 \\ * & 0.9058 \end{bmatrix},\ \begin{bmatrix} L_1 \\ L_2 \end{bmatrix} = \begin{bmatrix} -1.1670 \\ -1.8870 \end{bmatrix}.$$

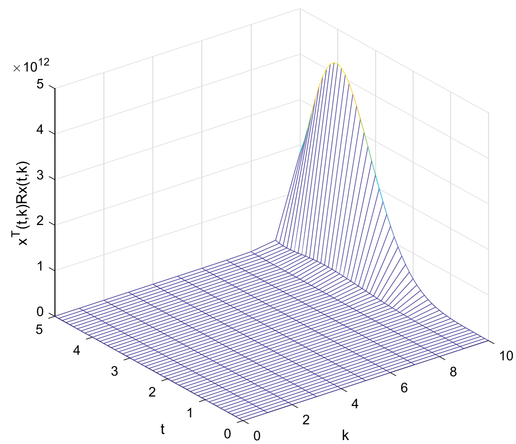

FIG. 3. $x^T(t,k)Rx(t,k)$ of system (2.3).

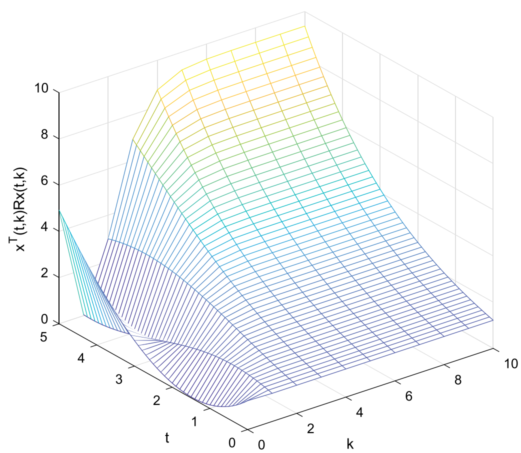

FIG. 4. $x^T(t,k)Rx(t,k)$ of system (3.26).

Then, we obtain the state feedback controller

$$K = [-0.8506, -2.0833].$$

The weighted-state values $x^T(t,k)Rx(t,k)$ of the closed-loop system (3.26) are depicted in Fig. 4. It can be seen that the closed-loop system (3.26) is finite-region stable with respect to (2, 10, 5, 10, I) but not asymptotically stable.

EXAMPLE 5.3 Let us consider the system (2.1) with disturbances generated by an external system $\mathscr{W}(d) = \{w(k)|w(k+1) = Fw(k), w^T(0)w(0) \leqslant d\}$, where

$$A = \begin{bmatrix} -2.1 & 0.5 \\ 1 & 1.2 \end{bmatrix}, \; B = \begin{bmatrix} -0.2 \\ 1.5 \end{bmatrix}, \; G = \begin{bmatrix} 0.6 \\ 0.1 \end{bmatrix}, \; F = 0.5.$$

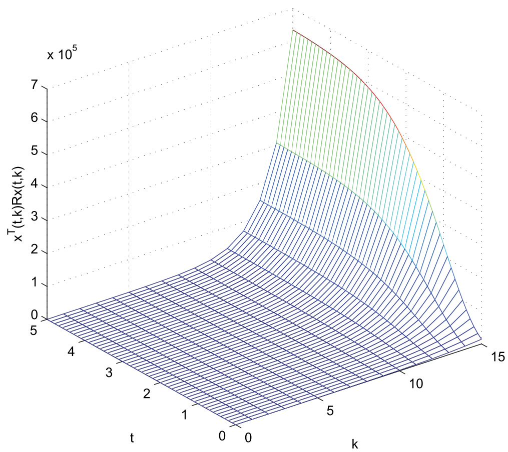

FIG. 5. $x^T(t,k)Rx(t,k)$ of system (4.1).

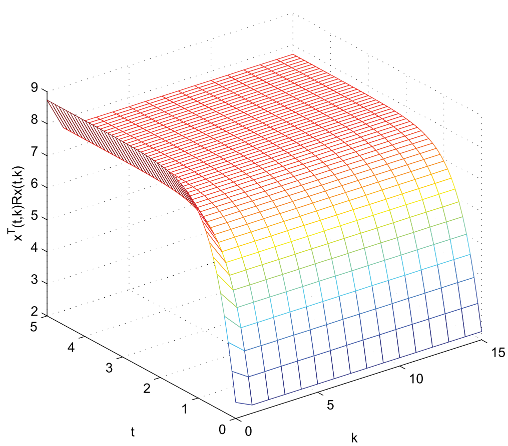

FIG. 6. $x^T(t,k)Rx(t,k)$ of system (4.15).

Given $c_1 = 2.5$, $c_2 = 10$, $T = 5$, $N = 15$, $d = 1$, $R = I$, $\eta = 0.9$, $\alpha_1 = 0.01$, $\beta_1 = 0.05$, $\alpha_2 = 1.1$, $\beta_2 = 0.01$ and the initial condition $x^h(0,k) = 1.5$, $x^v(t,0) = 0.5$. Using LMI toolbox of MATLAB and Theorem 4.3, a feasible solution of the LMIs (4.16a), (4.16b), (3.28a) and (4.20) can be derived as follows:

$$\begin{bmatrix} \widetilde{H}_1 & 0 \\ * & \widetilde{H}_2 \end{bmatrix} = \begin{bmatrix} 23.4079 & 0 \\ * & 2.2412 \end{bmatrix}, \begin{bmatrix} L_1 \\ L_2 \end{bmatrix} = \begin{bmatrix} -15.6388 \\ -1.7929 \end{bmatrix}, M = 1.7078.$$

Moreover, the state feedback controller is given by $K = [-0.6681, -0.8000]$. Figures 5 and 6 show the weighted-state values $x^T(t,k)Rx(t,k)$ of systems (4.1) and (4.15) with the same initial condition, respectively.

6. Conclusions

In this paper, we have investigated the FRB and finite-region stabilization problems for two-dimensional continuous-discrete linear Roesser models subject to two kinds of disturbances. First, the definitions of FRS and FRB for two-dimensional continuous-discrete linear system were put forward. Next, sufficient condition of FRB for two-dimensional continuous-discrete system subject to energy-bounded disturbances and sufficient condition of FRS for two-dimensional continuous-discrete system were established. By employing the given conditions, the sufficient conditions for the finite-region stabilization via state feedback were obtained. The conditions then were turned into optimization problems involving LMIs. Moreover, the sufficient conditions of FRB and finite-region stabilization for two-dimensional continuous-discrete system with disturbances generated by an external system were presented. Finally, numerical examples were provided to illustrate the proposed results. It should be pointed out that the future research topics may include the robust finite-region control synthesis of two-dimensional systems subject to uncertain time-varying parameters.

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