Justifying induction on modal μ-formulae

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Abstract

We define a rank function for formulae of the propositional modal μ-calculus such that the rank of a fixed point is strictly bigger than the rank of any of its finite approximations. A rank function of this kind is needed, for instance, to establish the collapse of the modal μ-hierarchy over transitive transition systems. We show that the range of the rank function is ω^ω. Further we establish that the rank is computable by primitive recursion, which gives us a uniform method to generate formulae of arbitrary rank below ω^ω.

1 Introduction

The propositional modal μ-calculus, introduced by Kozen [11], is an extension of modal logic with least and greatest fixed points for positive formulae. It subsumes many dynamic and temporal logics like PDL, PLTL, CTL, and CTL*, cf. [8, 14, 6, 7].

The least fixed point $\mu x.\varphi$ of a formula φ positive in x can be approximated from below by the formulae $\varphi^n_x(\bot)$ where

$$\varphi^0_x(\psi) := \psi \qquad \text{and} \qquad \varphi^{n+1}_x(\psi) := \varphi[\varphi^n_x(\psi)/x].$$

Dually, the greatest fixed point $\nu x.\varphi$ can be approximated from above by the formulae $\varphi^n_x(\top)$.

From this perspective, the approximations $\varphi^n_x(\bot)$ and $\varphi^n_x(\top)$ are simpler than the fixed points $\mu x.\varphi$ and $\nu x.\varphi$. However, so far there is no rank function f known such that f maps formulae of the μ-calculus to ordinals with

1. $f(\psi) < f(\varphi)$ if ψ is a proper subformula of φ,
2. $f(\varphi^n_x(\bot)) < f(\mu x.\varphi)$ for all natural numbers n,
3. $f(\varphi^n_x(\top)) < f(\nu x.\varphi)$ for all natural numbers n.

In this paper, we present a rank function for the modal μ-calculus and establish that its range is ω^ω. We also introduce a method to compute the rank of a formula by primitive recursion, which makes it possible to uniformly generate formulae of arbitrary rank below ω^ω.

Our rank function has several applications. For instance, it is used

1. to show that the modal μ-calculus hierarchy collapses over transitive transition systems [2];

2. to prove without using the de Jong-Sambin theorem that the μ-calculus over GL collapses, which explains why provability fixed points are explicitly definable in the modal language [3];

3. to develop analytical sequent calculi for the propositional modal μ-calculus over $\mathsf{S5}$ [1];

4. to establish a completeness theorem for the hybrid μ-calculus [15].

Moreover, employing this rank function would simplify the canonical model construction for the modal μ-calculus presented in [9]. Rank functions are also needed to study syntactic cut-elimination procedures. So far, results of this kind are only available for fragments of the modal μ-calculus [4, 5, 13]. The rank function we present here is a step towards a general syntactic cut-elimination result for the modal μ-calculus.

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2 Preliminaries

The language of the propositional modal μ-calculus results from adding least and greatest fixed points for positive formulae to the basic language of modal logic. More precisely, given a countable set of *propositional variables* Var, the collection $\mathcal{L}_\mu$ of *μ-formulae* is given by the following grammar

$$\varphi ::= x \mid {\sim}x \mid \top \mid \bot \mid (\varphi \wedge \varphi) \mid (\varphi \vee \varphi) \mid \Diamond\varphi \mid \Box\varphi \mid \mu x.\varphi \mid \nu x.\varphi,$$

where $x \in \mathsf{Var}$ and where we require for formulae of the form $\mu x.\varphi$ and $\nu x.\varphi$ that x occurs only positively in φ, i.e. ${\sim}x$ does not occur in φ. We set

$$\mathsf{Atm} := \mathsf{Var} \cup \{\top, \bot\} \quad \text{and} \quad \mathsf{Lit} := \mathsf{Atm} \cup \{{\sim}x \mid x \in \mathsf{Var}\}.$$

We use the usual notion of *subformula* where literals do not have proper subformulae. Hence x is not a subformula of ${\sim}x$. We denote the set of all subformulae of a formula φ by $\mathsf{sub}(\varphi)$.

The *negation* $\overline{\varphi}$ of a formula φ is defined in the usual way by using De Morgan's laws, the law of double negation, and the duality laws for modal and fixed point operators.

The fixed point operators μx and νx bind the variable x in the same way as quantifiers in predicate logic bind variables. Hence we use the standard terminology of *bound* and *free* occurrences of variables. By $\mathsf{free}(\varphi)$ we denote the set of all variables that occur free in φ, and $\mathsf{bound}(\varphi)$ denotes the set of all variables that have bound occurrences in φ. Further we set

$$\mathsf{var}(\varphi) := \mathsf{free}(\varphi) \cup \mathsf{bound}(\varphi)$$

and

$$\mathsf{atm}(\varphi) := \mathsf{var}(\varphi) \cup (\mathsf{sub}(\varphi) \cap \{\top, \bot\}).$$

Substitution is defined as usual. We write $\varphi[\psi/x]$ for the result of simultaneously replacing all free occurrences of x in φ with ψ. Two formulae φ and ψ are equal up to *renaming* of a bound variable, $\varphi \sim_1 \psi$, if there are formulae $\alpha(z)$, $\beta(z')$ and variables $x, y \notin \mathsf{var}(\alpha)$ such that $\varphi \equiv \beta[\sigma x.\alpha[x/z]/z']$ and $\psi \equiv \beta[\sigma y.\alpha[y/z]/z']$ for $\sigma \in \{\mu, \nu\}$. The relation $\sim_\infty$ is the transitive closure of $\sim_1$, that is $\varphi \sim_\infty \psi$ holds if φ and ψ are equal up to renaming of bound variables.

We call a formula φ *safe* if $\mathsf{bound}(\varphi) \cap \mathsf{free}(\varphi) = \emptyset$. Further, we call a formula φ *well-bound* if

1. φ is safe and
2. for each $x \in \mathsf{bound}(\varphi)$, there is only one single occurrence of either μx or νx in φ.

Note that any formula can be turned into an equivalent well-bound formula by renaming bound variables. Moreover, subformulae of well-bound formulae are well-bound. This does not hold for safe formulae: $x \wedge \mu x.x$ is an *un*safe subformula of the safe formula $\mu x.(x \wedge \mu x.x)$.

We define *iterations* by

$$\varphi^0_x(\psi) := \psi \qquad \text{and} \qquad \varphi^{n+1}_x(\psi) := \varphi[\varphi^n_x(\psi)/x].$$

Note that for any safe formula φ and any natural number n, the iteration $\varphi^n_x(x)$ is safe, too.

We denote the first uncountable ordinal by Ω. For any set X there is the set Ω^X of all functions $f : X \to \Omega$, that is, the set of all sequences of ordinals from Ω indexed by elements of X. $\mathbf{0} \in \Omega^X$ is the function which maps every argument to 0.

A *μ-rank* is a mapping $|\cdot| : \mathcal{L}_\mu \to \Omega$ such that

- if ψ is a proper subformula of φ, then $|\psi| < |\varphi|$;
- if φ is safe, then $|\varphi^n_x(\bot)| < |\sigma x.\varphi|$ and $|\varphi^n_x(\top)| < |\sigma x.\varphi|$ for all natural numbers n and $\sigma \in \{\mu, \nu\}$.

3 Existence of a μ-rank with range ω^ω

Before we can introduce our rank function for $\mathcal{L}_\mu$-formulae, we need some preparatory definitions.

Given a sequence $s \in \Omega^{\mathsf{Var}}$, a variable x, and $\xi \in \Omega$, then we define the sequence $s[x{:}\xi] \in \Omega^{\mathsf{Var}}$ by

$$s[x{:}\xi](y) := \begin{cases} \xi & \text{if } x \equiv y, \\ s(y) & \text{otherwise.} \end{cases}$$

The *composition in x* of $f, g : \Omega^{\mathsf{Var}} \to \Omega$ is given by

$$(f \circ_x g)(s) := f(s[x{:}g(s)])$$

and the *iterations of f in x* are given by

$$f_x^0 := \mathbf{0} \qquad \text{and} \qquad f_x^{n+1} := f \circ_x f_x^n.$$

Definition 1. For every $\varphi \in \mathcal{L}_\mu$, we define a function $[\![\varphi]\!] : \Omega^{\mathsf{Var}} \to \Omega$ by

$$[\![\varphi]\!](s) := \begin{cases} 0 & \varphi \equiv \bot, \top \\ s(x) & \varphi \equiv x, {\sim}x \\ [\![\alpha]\!](s) + 1 & \varphi \equiv \Diamond\alpha, \Box\alpha \\ \max\{[\![\alpha]\!](s), [\![\beta]\!](s)\} + 1 & \varphi \equiv \alpha \wedge \beta, \alpha \vee \beta \\ \sup_{n<\omega}\{[\![\alpha]\!]_x^n(s) + 1\} & \varphi \equiv \mu x.\alpha, \nu x.\alpha. \end{cases}$$

The function $\mathsf{rk} : \mathcal{L}_\mu \to \Omega$ is now given by

$$\mathsf{rk}(\varphi) := [\![\varphi]\!](\mathbf{0}).$$

Now we are going to show that the mapping rk is indeed a μ-rank. We start with the following lemma.

Lemma 2. *For all $\varphi, \psi \in \mathcal{L}_\mu$, $x, y \in \mathsf{Var}$, $\xi \in \Omega$, and natural numbers n, we have the following:*

1. $[\![\varphi]\!] = [\![\overline{\varphi}]\!]$
2. $x \notin \mathsf{free}(\varphi) \quad \Rightarrow \quad [\![\varphi]\!](s[x{:}\xi]) = [\![\varphi]\!](s)$
3. $x \not\equiv y,\ y \notin \mathsf{free}(\psi) \quad \Rightarrow \quad ([\![\varphi]\!] \circ_x [\![\psi]\!])_y^n = [\![\varphi]\!]_y^n \circ_x [\![\psi]\!]$
4. $\mathsf{bound}(\varphi) \cap \mathsf{free}(\psi) = \emptyset \quad \Rightarrow \quad [\![\varphi[\psi/x]]\!] = [\![\varphi]\!] \circ_x [\![\psi]\!]$
5. *φ safe* $\quad \Rightarrow \quad [\![\varphi]\!]_x^n = [\![\varphi_x^n(\bot)]\!] = [\![\varphi_x^n(\top)]\!]$

Proof. 1. By induction on the length of φ. This is left to the reader.

2. By induction on the length of φ and a case distinction on the outermost connective. We show only the case $\varphi \equiv \mu y.\psi$.

   By induction on n, we show

$$[\![\psi]\!]_y^n(s[x{:}\xi]) = [\![\psi]\!]_y^n(s), \tag{1}$$

   which implies $[\![\varphi]\!](s[x{:}\xi]) = [\![\varphi]\!](s)$. Because of $x \notin \mathsf{free}(\varphi)$ we either have $x \equiv y$ or $x \notin \mathsf{free}(\psi)$. If $n = 0$, then $[\![\psi]\!]_y^n = \mathbf{0}$ by definition and (1) trivially holds. For the induction step we find in the case $x \not\equiv y$ that

$$\begin{aligned}
[\![\psi]\!]_y^{n+1}(s[x{:}\xi]) &= [\![\psi]\!] \circ_y [\![\psi]\!]_y^n(s[x{:}\xi]) = [\![\psi]\!](s[x{:}\xi][y{:}[\![\psi]\!]_y^n(s[x{:}\xi])]) \\
&= [\![\psi]\!](s[x{:}\xi][y{:}[\![\psi]\!]_y^n(s)]) \quad \text{by i.h. for } n \\
&= [\![\psi]\!](s[y{:}[\![\psi]\!]_y^n(s)][x{:}\xi]) \quad \text{because } x \not\equiv y \text{ and } x \notin \mathsf{free}(\psi) \\
&= [\![\psi]\!](s[y{:}[\![\psi]\!]_y^n(s)]) \quad \text{by i.h. for } l(\psi) \\
&= [\![\psi]\!]_y^{n+1}(s).
\end{aligned}$$

   The induction step in the case $x \equiv y$ is similar.

3. By induction on n. For $n = 0$ we have

$$([\![\varphi]\!] \circ_x [\![\psi]\!])_y^n = \mathbf{0} = \mathbf{0} \circ_x [\![\psi]\!] = [\![\varphi]\!]_y^n \circ_x [\![\psi]\!].$$

   For the induction step we have

$$\begin{aligned}
&([\![\varphi]\!] \circ_x [\![\psi]\!])_y^{n+1}(s) \\
&= ([\![\varphi]\!] \circ_x [\![\psi]\!]) \circ_y ([\![\varphi]\!] \circ_x [\![\psi]\!])_y^n(s) \\
&= ([\![\varphi]\!] \circ_x [\![\psi]\!]) \circ_y ([\![\varphi]\!]_y^n \circ_x [\![\psi]\!])(s) \quad \text{by i.h.} \\
&= ([\![\varphi]\!] \circ_x [\![\psi]\!])(s[y{:}\xi]) \quad \text{with } \xi = ([\![\varphi]\!]_y^n \circ_x [\![\psi]\!])(s) \\
&= [\![\varphi]\!](s[y{:}\xi][x{:}[\![\psi]\!](s[y{:}\xi])]) \\
&= [\![\varphi]\!](s[y{:}\xi][x{:}[\![\psi]\!](s)]) \quad \text{by Part 2, } y \notin \mathsf{free}(\psi) \\
&= [\![\varphi]\!](s[x{:}[\![\psi]\!](s)][y{:}\xi]) \quad \text{because } x \not\equiv y \\
&= ([\![\varphi]\!] \circ_y [\![\varphi]\!]_y^n)(s[x{:}[\![\psi]\!](s)]) \quad \text{because } \xi = [\![\varphi]\!]_y^n(s[x{:}[\![\psi]\!](s)]) \\
&= ([\![\varphi]\!]_y^{n+1} \circ_x [\![\psi]\!])(s).
\end{aligned}$$

4. By induction on the length of φ and a case distinction on the outermost connective. We show only two cases.

   Case $\varphi \equiv {\sim}x$. We have $\varphi[\psi/x] = \overline{\psi}$ and thus $[\![\varphi[\psi/x]]\!] = [\![\overline{\psi}]\!]$. Moreover

$$([\![{\sim}x]\!] \circ_x [\![\psi]\!])(s) = [\![{\sim}x]\!](s[x : [\![\psi]\!](s)]) = [\![\psi]\!](s)$$

   and thus $[\![\varphi]\!] \circ_x [\![\psi]\!] = [\![\psi]\!]$. By Part 1 we conclude $[\![\varphi[\psi/x]]\!] = [\![\varphi]\!] \circ_x [\![\psi]\!]$.

Case $\varphi \equiv \mu y.\alpha$, subcase $x \not\equiv y$. We have

$$\begin{aligned}
&[\![\varphi[\psi/x]]\!](s) \\
&= \sup_{n<\omega}\{[\![\alpha[\psi/x]]\!]_y^n(s) + 1\} \\
&= \sup_{n<\omega}\{([\![\alpha]\!] \circ_x [\![\psi]\!])_y^n(s) + 1\} \quad \text{by i.h.} \\
&= \sup_{n<\omega}\{([\![\alpha]\!]_y^n \circ_x [\![\psi]\!])(s) + 1\} \quad \text{by Part 3, } x \not\equiv y,\ y \notin \mathsf{free}(\psi) \\
&= \sup_{n<\omega}\{[\![\alpha]\!]_y^n(s[x{:}[\![\psi]\!](s)]) + 1\} \\
&= [\![\varphi]\!](s[x{:}[\![\psi]\!](s)]) = ([\![\varphi]\!] \circ_x [\![\psi]\!])(s).
\end{aligned}$$

Case $\varphi \equiv \mu y.\alpha$, subcase $x \equiv y$. We have $x \notin \mathsf{free}(\varphi)$, hence using Part 2 we conclude

$$[\![\varphi[\psi/x]]\!](s) = [\![\varphi]\!](s) = [\![\varphi]\!](s[x{:}[\![\psi]\!](s)]) = ([\![\varphi]\!] \circ_x [\![\psi]\!])(s).$$

5. We assume $\mathsf{bound}(\varphi) \cap \mathsf{free}(\varphi) = \emptyset$ and show $[\![\varphi]\!]_x^n = [\![\varphi_x^n(\bot)]\!]$ by induction on n.

   Case $n = 0$. We have $[\![\bot]\!]_x^0 = \mathbf{0}$ by definition. Moreover, also by definition, $\varphi_x^0(\bot) = \bot$ and thus $[\![\varphi_x^0(\bot)]\!] = \mathbf{0}$.

   Case $n + 1$. We find

$$\begin{aligned}
[\![\varphi]\!]_x^{n+1} &= [\![\varphi]\!] \circ_x [\![\varphi]\!]_x^n = [\![\varphi]\!] \circ_x [\![\varphi_x^n(\bot)]\!] \quad \text{by i.h.} \\
&= [\![\varphi[\varphi_x^n(\bot)/x]]\!] \quad \text{by Part 4, } \mathsf{bound}(\varphi) \cap \mathsf{free}(\varphi_x^n(\bot)) = \emptyset \\
&= [\![\varphi_x^{n+1}(\bot)]\!].
\end{aligned}$$

   $[\![\varphi]\!]_x^n = [\![\varphi_x^n(\top)]\!]$ is shown similarly. □

Corollary 3. *The mapping* rk *is a* μ*-rank.*

Proof. First observe that if ψ is a proper subformula of φ, then $\mathsf{rk}(\psi) < \mathsf{rk}(\varphi)$ follows easily from Definition 1. It remains to show $\mathsf{rk}(\varphi_x^n(\bot)) < \mathsf{rk}(\sigma x.\varphi)$ for safe formulae φ, which we obtain as follows.

$$\begin{aligned}
\mathsf{rk}(\varphi_x^n(\bot)) &= [\![\varphi_x^n(\bot)]\!](\mathbf{0}) \\
&= [\![\varphi]\!]_x^n(\mathbf{0}) \\
&< \sup_{m<\omega}\{[\![\varphi]\!]_x^m(\mathbf{0}) + 1\} \\
&= [\![\sigma x.\varphi]\!](\mathbf{0}) = \mathsf{rk}(\sigma x.\varphi).
\end{aligned}$$

$\mathsf{rk}(\varphi_x^n(\top)) < \mathsf{rk}(\sigma x.\varphi)$ is established similarly. □

Next we show $\mathsf{rk}(\xi) < \omega^\omega$ for any $\mathcal{L}_\mu$-formula ξ, that means ω^ω is an upper bound for the range of rk. We first need to establish that renaming bound variables does not change the rank of a formula.

Lemma 4. *For all $\varphi, \psi \in \mathcal{L}_\mu$ we have*

$$\varphi \sim_\infty \psi \quad \Rightarrow \quad [\![\varphi]\!] = [\![\psi]\!]. \tag{2}$$

Proof. We first show $([\![\alpha]\!] \circ_z [\![x]\!])^n_x = [\![\alpha]\!]^n_z$ for $x \notin \mathsf{free}(\alpha)$ by induction on n. For $n = 0$ this is $\mathbf{0} = \mathbf{0}$, and for the induction step we have

$$\begin{aligned}
([\![\alpha]\!] \circ_z [\![x]\!])^{n+1}_x(s) &= ([\![\alpha]\!] \circ_z [\![x]\!]) \circ_x ([\![\alpha]\!] \circ_z [\![x]\!])^n_x(s) \\
&= ([\![\alpha]\!] \circ_z [\![x]\!]) \circ_x [\![\alpha]\!]^n_z(s) \quad \text{by i.h.} \\
&= ([\![\alpha]\!] \circ_z [\![x]\!])(s[x{:}\xi]) \quad \text{with } \xi = [\![\alpha]\!]^n_z(s) \\
&= [\![\alpha]\!](s[x{:}\xi][z{:}[\![x]\!](s[x{:}\xi])]) \\
&= [\![\alpha]\!](s[x{:}\xi][z{:}\xi]) \\
&= [\![\alpha]\!](s[z{:}\xi][x{:}\xi]) \\
&= [\![\alpha]\!](s[z{:}\xi]) \quad \text{by Lemma 2 part 2, } x \notin \mathsf{free}(\alpha) \\
&= [\![\alpha]\!] \circ_z [\![\alpha]\!]^n_z(s) = [\![\alpha]\!]^{n+1}_z(s).
\end{aligned}$$

From this we get $[\![\mu x.\alpha[x/z]]\!] = [\![\mu z.\alpha]\!]$ for $x \notin \mathsf{var}(\alpha)$ as follows:

$$\begin{aligned}
&[\![\mu x.\alpha[x/z]]\!](s) \\
&= \sup_{n<\omega}\{[\![\alpha[x/z]]\!]^n_x(s) + 1\} \\
&= \sup_{n<\omega}\{([\![\alpha]\!] \circ_z [\![x]\!])^n_x(s) + 1\} \quad \text{by Lemma 2 part 4, } z \notin \mathsf{bound}(\alpha) \\
&= \sup_{n<\omega}\{[\![\alpha]\!]^n_z(s) + 1\} \quad \text{because } x \notin \mathsf{free}(\alpha) \\
&= [\![\mu z.\alpha]\!].
\end{aligned}$$

For formulae $\varphi \sim_1 \psi$ such that $\varphi \equiv \beta[\mu x.\alpha[x/z]/z']$ and $\psi \equiv \beta[\mu y.\alpha[y/z]/z']$ and $x, y \notin \mathsf{var}(\alpha)$, we can easily show $[\![\varphi]\!] = [\![\psi]\!]$ by induction on the length of β. Now (2) immediately follows since $\sim_\infty$ is the transitive closure of $\sim_1$. $\square$

Theorem 5. *For all $\varphi, \psi \in \mathcal{L}_\mu$, $x \in \mathsf{Var}$ and $n < \omega$ we have:*

1. $\mathsf{bound}(\varphi) \cap \mathsf{free}(\psi) = \emptyset$, $x \notin \mathsf{free}(\psi)$ *implies*

$$[\![\varphi[\psi/x]]\!](s) \leq [\![\psi]\!](s) + [\![\varphi]\!](s)$$

2. $[\![\varphi]\!]^n_x(s) \leq [\![\varphi]\!](s) \cdot n$

3. $\mathsf{rk}(\varphi) < \omega^\omega$

Proof. 1. By induction on the μ-rank $\mathsf{rk}(\varphi)$. We only show the case $\varphi \equiv \mu y.\alpha$ and $x \not\equiv y$. We distinguish two cases. If φ is well-bound, then α is safe and we have

$$\begin{aligned}
&[\![\varphi[\psi/x]]\!](s)\\
&= \sup_{n<\omega}\{[\![\alpha[\psi/x]]\!]^n_y(s)+1\}\\
&= \sup_{n<\omega}\{([\![\alpha]\!]\circ_x[\![\psi]\!])^n_y(s)+1\} \quad \text{by 2.4, } \mathsf{bound}(\alpha)\cap\mathsf{free}(\psi)=\emptyset\\
&= \sup_{n<\omega}\{([\![\alpha]\!]^n_y\circ_x[\![\psi]\!])(s)+1\} \quad \text{by 2.3, } x\not\equiv y,\ x\notin\mathsf{free}(\psi)\\
&= \sup_{n<\omega}\{([\![\alpha^n_y(\bot)]\!]\circ_x[\![\psi]\!])(s)+1\} \quad \text{by 2.5, } \alpha \text{ safe}\\
&= \sup_{n<\omega}\{[\![\alpha^n_y(\bot)[\psi/x]]\!](s)+1\} \quad \text{by 2.4}\\
&\le \sup_{n<\omega}\{[\![\psi]\!](s)+[\![\alpha^n_y(\bot)]\!](s)+1\} \quad \text{i.h. for } \mathsf{rk}(\alpha^n_y(\bot))\\
&= [\![\psi]\!](s)+\sup_{n<\omega}\{[\![\alpha]\!]^n_y(s)+1\} = [\![\psi]\!](s)+[\![\varphi]\!](s) \quad \text{by 2.5, } \alpha \text{ safe.}
\end{aligned}$$

Otherwise, φ is not well-bound but we can find a well-bound formula φ^* with $\varphi^*\sim_\infty\varphi$ and $\mathsf{bound}(\varphi^*)\cap\mathsf{free}(\psi)=\emptyset$. Hence we have $\varphi^*[\psi/x]\sim_\infty\varphi[\psi/x]$. Using Lemma 4 twice, we conclude

$$[\![\varphi[\psi/x]]\!](s)=[\![\varphi^*[\psi/x]]\!](s)\le[\![\psi]\!](s)+[\![\varphi^*]\!](s)=[\![\psi]\!](s)+[\![\varphi]\!](s).$$

2. By induction on n. Again, we assume that φ is well-bound. For $n=0$ we trivially have $\mathbf{0}(s)\le 0$. For the induction step we have:

$$\begin{aligned}
[\![\varphi]\!]^{n+1}_x(s) &= [\![\varphi^{n+1}_x(\bot)]\!](s) \quad \text{by 2.5}\\
&= [\![\varphi[\varphi^n_x(\bot)/x]]\!](s)\\
&\le [\![\varphi^n_x(\bot)]\!](s)+[\![\varphi]\!](s) \quad \text{by Part 1, } \begin{array}{l} x\notin\mathsf{free}(\varphi^n_x(\bot)) \text{ and}\\ \mathsf{bound}(\varphi)\cap\mathsf{free}(\varphi^n_x)=\emptyset\end{array}\\
&= [\![\varphi]\!]^n_x(s)+[\![\varphi]\!](s)\le[\![\varphi]\!](s)\cdot(n+1). \quad \text{by i.h.}
\end{aligned}$$

For any formula φ there is a well-bound formula φ^* with $\varphi^*\sim_\infty\varphi$. By Lemma 4 we have $[\![\varphi^*]\!]=[\![\varphi]\!]$ and the full claim easily follows.

3. By induction on the length of φ. We only show the case for $\varphi\equiv\mu x.\alpha$. By part 2 we find

$$\mathsf{rk}(\mu x.\alpha)=\sup_{n<\omega}\{[\![\alpha]\!]^n_x(\mathbf{0})+1\}\le\mathsf{rk}(\alpha)\cdot\omega+1.$$

By i.h. we get $\mathsf{rk}(\alpha)<\omega^\omega$. Hence $\mathsf{rk}(\alpha)\cdot\omega+1<\omega^\omega$, which finishes the proof. □

4 Effective computation of the μ-rank

In this section, we show that the rank of a modal μ-formula can be computed by primitive recursion.

Definition 6. 1. For each $\varphi \in \mathcal{L}_\mu$ we define $\langle\varphi\rangle \in \Omega^{\mathsf{Atm}}$ by $\langle\varphi\rangle_u := 0$ if $u \notin \mathsf{atm}(\varphi)$ and otherwise

$$\langle\varphi\rangle_u := \begin{cases} 0 & \varphi \in \mathsf{Lit}, \\ \langle\alpha\rangle_u + 1 & \varphi \equiv \Box\alpha, \Diamond\alpha, \\ \max\{\langle\alpha\rangle_u, \langle\beta\rangle_u\} + 1 & \varphi \equiv \alpha \wedge \beta, \alpha \vee \beta, \\ \langle\alpha\rangle_u + 1 + \langle\alpha\rangle_x \cdot \omega & \varphi \equiv \mu x.\alpha, \nu x.\alpha. \end{cases}$$

2. We fix a mapping $\varphi \mapsto \varphi^*$ on $\mathcal{L}_\mu$ such that

$$\varphi^* \text{ is well-bound} \quad \text{with} \quad \varphi^* \sim_\infty \varphi$$

and

$$\varphi^* \equiv \varphi \text{ if } \varphi \text{ is well-bound.}$$

Now we define the mappings $\mathsf{f}^e, \mathsf{rk}^e : \mathcal{L}_\mu \to \Omega$ by

$$\mathsf{f}^e(\varphi) := \max_{u \in \mathsf{Atm}} \{\langle\varphi\rangle_u\} \quad \text{and} \quad \mathsf{rk}^e(\varphi) := \mathsf{f}^e(\varphi^*).$$

Remark 7. We have

$$\mathsf{f}^e(\varphi) = \max_{u \in \mathsf{atm}(\varphi)} \{\langle\varphi\rangle_u\}$$

because of $\langle\varphi\rangle_u = 0$ for $u \notin \mathsf{atm}(\varphi)$.

The following lemmas can be shown by simple but longish calculations, which we omit here. We refer to Krähenbühl's thesis [12] for more details about the proofs.

Lemma 8. *Let φ be well-bound and $\mathsf{bound}(\varphi) \cap \mathsf{var}(\psi) = \emptyset$ then*

$$x \in \mathsf{free}(\varphi) \quad \Rightarrow \quad \mathsf{f}^e(\varphi[\psi/x]) = \max\{\mathsf{f}^e(\varphi), \mathsf{f}^e(\psi) + \langle\varphi\rangle_x\}.$$

Lemma 9. *Let $x_0, \ldots, x_n \in \mathsf{free}(\varphi)$ be pairwise distinct variables.*

1. *If φ is well-bound, $y \notin \mathsf{bound}(\varphi)$ and $x_i \not\equiv y$ for $i \leq n$ then*

$$\langle\varphi[y/x_0]\ldots[y/x_n]\rangle_y = \max\{\langle\varphi\rangle_y, \max_{i\leq n}\{\langle\varphi\rangle_{x_i}\}\}.$$

2. *If $\varphi[\psi_0/x_0]\ldots[\psi_n/x_n]$ is well-bound, $x_j \notin \mathsf{var}(\psi_i)$ for $i < j \leq n$ and $\mathsf{bound}(\varphi) \cap \mathsf{var}(\psi_i) = \mathsf{bound}(\psi_i) \cap \mathsf{var}(\psi_j) = \emptyset$ for $i < j \leq n$ then*

$$\mathsf{f}^e(\varphi[\psi_0/x_0]\ldots[\psi_n/x_n]) = \max\{\mathsf{f}^e(\varphi), \max_{i\leq n}\{\mathsf{f}^e(\psi_i) + \langle\varphi\rangle_{x_i}\}\}.$$

Lemma 10. *Assume that φ, ψ are well-bound formulae with $\varphi \sim_\infty \psi$ and $x \in \mathsf{free}(\varphi)$. Then we have $\langle\varphi\rangle_x = \langle\psi\rangle_x$.*

The next theorem shows the equivalence of rk and rk^e. Therefore, it provides a method to compute the μ-rank rk by primitive recursion.

Theorem 11. *For all $\varphi \in \mathcal{L}_\mu$ we have $\mathsf{rk}(\varphi) = \mathsf{rk}^e(\varphi)$.*

Proof. We show

$$\mathsf{rk}(\varphi) = \mathsf{f}^e(\varphi) \tag{3}$$

for all well-bound formulae φ. The full claim of the theorem then follows by Lemma 4 because for any $\varphi \in \mathcal{L}_\mu$ we have that

$$\mathsf{rk}(\varphi) = \mathsf{rk}(\varphi^*) = \mathsf{f}^e(\varphi^*) = \mathsf{rk}^e(\varphi)$$

where * is the mapping introduced in Definition 6.

We establish (3) by induction on $\mathsf{rk}(\varphi)$. Let us only show the case $\varphi \equiv \mu x.\alpha$. By Lemma 2 part 5 and because α is well-bound we get

$$\mathsf{rk}(\varphi) = \sup_{n<\omega}\{[\![\alpha]\!]_x^n(\mathbf{0}) + 1\} = \sup_{n<\omega}\{\mathsf{rk}(\alpha_x^n(\bot)) + 1\}.$$

For each natural number n the formula $\alpha_x^n(\bot)^*$ is well-bound and thus $\alpha_x^n(\bot)^* \sim_\infty \alpha_x^n(\bot)$. By Lemma 4 and i.h. we get

$$\mathsf{rk}(\varphi) = \sup_{n<\omega}\{\mathsf{rk}(\alpha_x^n(\bot)^*) + 1\} = \sup_{n<\omega}\{\mathsf{f}^e(\alpha_x^n(\bot)^*) + 1\}.$$

In order to compute $\mathsf{f}^e(\alpha_x^n(\bot)^*)$ we distinguish two cases. In the first case we assume $\langle\alpha\rangle_x = 0$. Thus we have $x \notin \mathsf{free}(\alpha)$ or $\alpha \equiv x$, both of which imply $\alpha_x^n(\bot) \equiv \alpha$ for $n > 0$. Hence we find

$$\begin{aligned}\mathsf{rk}(\varphi) &= \sup_{n<\omega}\{\mathsf{f}^e(\alpha_x^n(\bot)^*) + 1\} = \mathsf{f}^e(\alpha^*) + 1 = \mathsf{f}^e(\alpha) + 1 \quad \text{since } \alpha^* \equiv \alpha \\ &= \max_{u\in\mathsf{Atm}}\{\langle\alpha\rangle_u\} + 1 = \max_{u\in\mathsf{Atm}}\{\langle\alpha\rangle_u + 1 + \langle\alpha\rangle_x \cdot \omega\} = \mathsf{f}^e(\varphi).\end{aligned}$$

In the second case we assume $\langle\alpha\rangle_x > 0$, which implies $x \in \mathsf{free}(\alpha)$. First, we show by induction on n that for $n > 0$

$$\mathsf{f}^e(\alpha_x^n(\bot)^*) = \mathsf{f}^e(\alpha) + \langle\alpha\rangle_x \cdot (n-1). \tag{4}$$

For $n = 1$ we have $\langle\alpha_x^n(\bot)^*\rangle_u = \langle\alpha[\bot/x]^*\rangle_u = \langle\alpha^*\rangle_u = \langle\alpha\rangle_u$ for each u as well as $n - 1 = 0$. Thus we get (4) for $n = 1$.

For $n > 1$ we have $\alpha_x^n(\bot) \equiv \alpha[\alpha_x^{n-1}(\bot)/x]$. Moreover, there are distinct variables $x_0, \ldots, x_k$ and well-bound formulae $\hat\alpha$ and $\psi_0, \ldots, \psi_k$ such that

1. $\alpha \sim_\infty \hat\alpha[x/x_0]\ldots[x/x_k]$ and $\hat\alpha[x/x_0]\ldots[x/x_k]$ is well-bound,
2. $\alpha_x^{n-1}(\bot)^* \sim_\infty \psi_i$ for each $i \le k$,
3. $\alpha_x^n(\bot)^* \sim_\infty \hat\alpha[\psi_0/x_0]\ldots[\psi_k/x_k]$ and $\hat\alpha[\psi_0/x_0]\ldots[\psi_k/x_k]$ is well-bound,
4. $x_i \in \mathsf{free}(\hat\alpha)$ and $x_j \notin \mathsf{var}(\psi_i)$ and $x_i \not\equiv x$ for $i < j \le k$.

Hence we have $x \notin \mathsf{var}(\hat{\alpha})$ and $\mathsf{bound}(\hat{\alpha}) \cap \mathsf{var}(\psi_i) = \mathsf{bound}(\psi_i) \cap \mathsf{var}(\psi_j) = \emptyset$ for $i < j \leq k$. We obtain

$$\begin{aligned} \mathsf{f}^e(\alpha) &= \mathsf{f}^e(\hat{\alpha}[x/x_0] \dots [x/x_k]) \quad \text{by i.h. for } \mathsf{rk}(\alpha) \text{ and L. 4} \\ &= \max\{\mathsf{f}^e(\hat{\alpha}), \max_{i \leq k}\{\mathsf{f}^e(x) + \langle\hat{\alpha}\rangle_{x_i}\}\} \quad \text{by L. 9 part 2} \\ &= \max\{\mathsf{f}^e(\hat{\alpha}), \max_{i \leq k}\{\langle\hat{\alpha}\rangle_{x_i}\}\} = \mathsf{f}^e(\hat{\alpha}). \end{aligned} \tag{5}$$

Now we can establish (4) for $n > 1$ as follows.

$$\begin{aligned} &\mathsf{f}^e(\alpha^n_x(\bot)^*) \\ &= \mathsf{f}^e(\hat{\alpha}[\psi_0/x_0] \dots [\psi_k/x_k]) \quad \text{by i.h. for } \mathsf{rk}(\alpha^n_x(\bot)^*) \text{ and L. 4} \\ &= \max\{\mathsf{f}^e(\hat{\alpha}), \max_{i \leq k}\{\mathsf{f}^e(\psi_i) + \langle\hat{\alpha}\rangle_{x_i}\}\} \quad \text{by L. 9 part 2} \\ &= \max\{\mathsf{f}^e(\hat{\alpha}), \mathsf{f}^e(\alpha^{n-1}_x(\bot)^*) + \max_{i \leq k}\{\langle\hat{\alpha}\rangle_{x_i}\}\} \quad \text{i.h. for } \mathsf{rk}(\alpha^{n-1}_x(\bot)) \\ &= \max\{\mathsf{f}^e(\hat{\alpha}), \mathsf{f}^e(\alpha^{n-1}_x(\bot)^*) + \langle\hat{\alpha}[x/x_0] \dots [x/x_k]\rangle_x\} \quad \text{by L. 9 part 1} \\ &= \max\{\mathsf{f}^e(\hat{\alpha}), \mathsf{f}^e(\alpha^{n-1}_x(\bot)^*) + \langle\alpha\rangle_x\} \quad \text{by L. 10} \\ &= \max\{\mathsf{f}^e(\hat{\alpha}), \mathsf{f}^e(\alpha) + \langle\alpha\rangle_x \cdot (n-2) + \langle\alpha\rangle_x\} \quad \text{by i.h. for } n-1 \\ &= \mathsf{f}^e(\alpha) + \langle\alpha\rangle_x \cdot (n-1) \quad \text{by (5).} \end{aligned}$$

Because of (4) and our assumption that $\langle\alpha\rangle_x > 0$, we have for $n > 1$

$$\mathsf{f}^e(\alpha^n_x(\bot)^*) + 1 \leq \mathsf{f}^e(\alpha^{n+1}_x(\bot)^*).$$

Therefore, we conclude for $\langle\alpha\rangle_x > 0$

$$\begin{aligned} \mathsf{rk}(\varphi) &= \sup_{n<\omega}\{\mathsf{f}^e(\alpha^n_x(\bot)^*) + 1\} = \sup_{n<\omega}\{\mathsf{f}^e(\alpha^n_x(\bot)^*)\} \\ &= \mathsf{f}^e(\alpha) + \langle\alpha\rangle_x \cdot \omega = \mathsf{f}^e(\alpha) + 1 + \langle\alpha\rangle_x \cdot \omega = \mathsf{f}^e(\varphi). \end{aligned} \qquad \square$$

5 Generating modal μ-formulae of any complexity

We present a uniform method to generate modal μ-formulae of arbitrary rank below ω^ω. This establishes ω^ω as lower bound for the range of the μ-rank. We start with some auxiliary definitions.

Definition 12. We fix an infinite sequence of propositional variables $p_0, p_1, \dots$ such that $p_i \not\equiv p_j$ for $i \neq j$. We set

$$\Psi^k_n :\equiv (p_{n+k} \wedge \dots \wedge (p_n \wedge p_0))$$

and define formulae Φ^k_n by

$$\Phi^k_n :\equiv \begin{cases} \bot \wedge p_0 & k = 0, \\ \mu p_{(n+k-1)} \dots \mu p_n . \Psi^{k-1}_n & k > 0. \end{cases}$$

Lemma 13. *For all natural numbers n and k we have*

$$u \in \mathsf{atm}(\Phi^k_n) \quad \Rightarrow \quad \langle \Phi^k_n \rangle_u = \omega^k.$$

Proof. By induction on k. If $k = 0$ and $u \in \mathsf{atm}(\Phi^k_n)$ we have

$$\langle \Phi^k_n \rangle_u = \langle \bot \wedge p_0 \rangle_u = 1 = \omega^0.$$

If $k > 0$, then for any $k > i \geq 0$ we set $\varphi_i :\equiv \mu p_{n+i} \dots \mu p_n . \Psi^{k-1}_n$. We show $u \in \mathsf{atm}(\Phi^k_n) \Rightarrow \langle \varphi_i \rangle_u = \omega^{i+1}$ by induction on i.

- If $i = 0$ then

  $$\langle \varphi_0 \rangle_u = \langle \Psi^{k-1}_n \rangle_u + 1 + \langle \Psi^{k-1}_n \rangle_{p_n} \cdot \omega = \omega$$

  because of $0 < \langle \Psi^{k-1}_n \rangle_u \leq \langle \Psi^{k-1}_n \rangle_{p_n} < \omega$.

- For $i > 0$ we have $\langle \varphi_{i-1} \rangle_u = \langle \varphi_{i-1} \rangle_{p_{n+i}} = \omega^i$ by i.h. Hence

  $$\begin{aligned}\langle \varphi_i \rangle_u = \langle \mu p_{n+i} . \varphi_{i-1} \rangle_u &= \langle \varphi_{i-1} \rangle_u + 1 + \langle \varphi_{i-1} \rangle_{p_{n+i}} \cdot \omega \\ &= \omega^i + 1 + \omega^i \cdot \omega = \omega^{i+1}.\end{aligned}$$

Observing $\langle \Phi^k_n \rangle_u = \langle \varphi_{k-1} \rangle_u = \omega^k$ finishes the proof. □

For ordinals ξ with $0 < \xi < \omega^\omega$ there is a unique representation in *Cantor normal form* (see, e.g., [10]), which is

$$\xi =_{\mathit{CNF}} \omega^{k_0} + \dots + \omega^{k_n} \quad \text{with} \quad \omega > k_0 \geq \dots \geq k_n \geq 0.$$

Definition 14. We define a mapping $\Theta : \omega^\omega \to \mathcal{L}_\mu$ by

$$\Theta_\xi :\equiv \begin{cases} \bot & \xi = 0, \\ \Phi^k_1[\Theta_0/p_0] & \xi =_{\mathit{CNF}} \omega^k, \\ \Phi^{k_n}_{1+k_0+\dots+k_{n-1}}[\Theta_{\omega^{k_0}+\dots+\omega^{k_{n-1}}}/p_0] & \xi =_{\mathit{CNF}} \omega^{k_0} + \dots + \omega^{k_n}. \end{cases}$$

Example 15. We give some examples to illustrate the structure of the formulae Θ_ξ.

$$\begin{aligned} \Theta_{\omega^2} &\equiv \Phi^2_1[\bot/p_0] \equiv \mu p_2 \mu p_1 (p_2 \wedge (p_1 \wedge \bot)), \\ \Theta_{\omega^2 \cdot 2} &\equiv \Phi^2_3[\Theta_{\omega^2}/p_0] \equiv \mu p_4 \mu p_3 (p_4 \wedge (p_3 \wedge \mu p_2 \mu p_1 (p_2 \wedge (p_1 \wedge \bot)))), \\ \Theta_{\omega^2 \cdot 2 + \omega + 2} &\equiv \bot \wedge (\bot \wedge \mu p_5 (p_5 \wedge \mu p_4 \mu p_3 (p_4 \wedge (p_3 \wedge \mu p_2 \mu p_1 (p_2 \wedge (p_1 \wedge \bot)))))). \end{aligned}$$

Theorem 16. *For each $\xi < \omega^\omega$ we have $\mathsf{rk}(\Theta_\xi) = \mathsf{rk}^e(\Theta_\xi) = \xi$.*

Proof. This is proved by induction on ξ. We simultaneously show the following:

(i) $\mathsf{atm}(\Theta_\xi) = \{\bot, p_0, \ldots, p_{k_0+\ldots+k_n}\} \backslash \{p_0\}$ for $\xi =_{CNF} \omega^{k_0} + \ldots + \omega^{k_n}$, $\mathsf{atm}(\Theta_0) = \{\bot\}$,

(ii) Θ_ξ is well-bound,

(iii) $\mathsf{rk}^e(\Theta_\xi) = \xi$.

If $\xi = 0$, then $\Theta_0 \equiv \bot$ is well-bound, $\mathsf{atm}(\bot) = \{\bot\}$, and

$$\mathsf{rk}^e(\bot) = \max_{u \in \mathsf{Atm}} \{0\} = 0.$$

If $\xi =_{CNF} \omega^{k_0} + \ldots + \omega^{k_n}$ and $\zeta = \omega^{k_0} + \ldots + \omega^{k_{n-1}} < \xi$ and $s = k_0 + \ldots + k_{n-1}$ (for $n = 0$ let $\zeta = 0$ and $s = 0$), then $\Theta_\xi \equiv \Phi^{k_n}_{1+s}[\Theta_\zeta/p_0]$. By the definition of $\Phi^{k_n}_{1+s}$ we have that $\Phi^{k_n}_{1+s}$ is well-bound and

$$\mathsf{bound}(\Phi^{k_n}_{1+s}) = \mathsf{atm}(\Phi^{k_n}_{1+s}) \backslash \{\bot, p_0\} = \{p_{1+s}, \ldots, p_{s+k_n}\}.$$

By i.h. we get that Θ_ζ is well-bound, and that $\mathsf{atm}(\Theta_\zeta) = \{\bot, p_1, \ldots, p_s\}$. Thus, because there is only one occurrence of p_0 in $\Phi^{k_n}_{1+s}$ and $\mathsf{bound}(\Phi^{k_n}_{1+s}) \cap \mathsf{var}(\Theta_\zeta) = \emptyset$, we have that

$$\mathsf{atm}(\Theta_\xi) = \{\bot, p_1, \ldots, p_{s+k_n}\} \text{ and } \Theta_\xi \text{ is well-bound.}$$

Now because Θ_ξ, Θ_ζ and $\Phi^{k_n}_{1+s}$ are well-bound and because $p_0 \in \mathsf{free}(\Phi^{k_n}_{1+s})$ and $\mathsf{bound}(\Phi^{k_n}_{1+s}) \cap \mathsf{var}(\Theta_\zeta) = \emptyset$ the following holds by Lemma 8:

$$\begin{aligned}
\mathsf{rk}^e(\Theta_\xi) &= \mathsf{rk}^e(\Phi^{k_n}_{1+s}[\Theta_\zeta/p_0]) = \max\{\mathsf{rk}^e(\Phi^{k_n}_{1+s}), \mathsf{rk}^e(\Theta_\zeta) + \langle \Phi^{k_n}_{1+s} \rangle_{p_0}\} \\
&= \max\{\omega^{k_n}, \mathsf{rk}^e(\Theta_\zeta) + \omega^{k_n}\} = \mathsf{rk}^e(\Theta_\zeta) + \omega^{k_n} \quad \text{by L. 13} \\
&= \zeta + \omega^{k_n} = \xi \quad \text{by i.h.}
\end{aligned}$$

We conclude $\mathsf{rk}(\Theta_\xi) = \mathsf{rk}^e(\Theta_\xi) = \xi$ for $\xi < \omega^\omega$ by Theorem 11. ☐

Corollary 17.

$$\mathsf{rk}[\mathcal{L}_\mu] = \omega^\omega$$

6 Conclusion

We have introduced a rank function rk for the propositional modal μ-calculus and established that its range is ω^ω. We have also shown that this ordinal is the least upper bound on the ranks of $\mathcal{L}_\mu$-formulae, that is for each $\xi < \omega^\omega$ there is a formula φ with $\mathsf{rk}(\varphi) = \xi$.

We can even prove more. Namely, the mapping rk is a *minimal* μ-rank with respect to well-bound formulae, that is we have the following theorem.

Theorem 18. *For any μ-rank $|.|$ we have*

$$\mathsf{rk}(\varphi) \leq |\varphi| \textit{ for all well-bound formulae } \varphi.$$

The proof of this theorem, however, requires a detour via a more general rank function that is minimal with respect to all $\mathcal{L}_\mu$-formulae. A full definition of this general rank function and a detailed proof of the above theorem are given in Krähenbühl's thesis [12].

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