On the number of variables in undecidable superintuitionistic propositional calculi

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Abstract

In this paper, we construct an undecidable 3-variable superintuitionistic propositional calculus, i.e., a finitely axiomatizable extension of the intuitionistic propositional calculus with axioms containing only 3 variables. Since there are no 2-variable superintuitionistic propositional calculi, this is the minimal possible number of variables.

1 Introduction

Decidability is the important property of propositional calculi, it means that the set of their derivable formulas (or theorems) can be effectively determined. A natural question is how to separate classes of decidable and undecidable calculi. On the other hand, since undecidable propositional calculi can be used as a base for obtaining "negative" results to various algorithmic problems, it is of interest to find the simplest possible calculus of that class. There are many possible ways to separate decidable and undecidable calculi. A significant and simplest way is to describe the number of variables in their axioms.

In 1949, Linial and Post [10] found the first undecidable propositional calculus. In 1975, Hughes and Singletary [9] proved that there is an undecidable propositional calculus with axioms containing 3 variables. In 1976, Hughes [8] constructed an undecidable implicational propositional calculus using axioms in 2 variables. Finally, Gladstone in 1979 [7] proved that every 1-variable propositional calculus is decidable.

The first undecidable superintuitionistic propositional calculus was built in 1978 by Shehtman [15, 16]. Axioms of this calculus contain 7 variables. Later Chagrov in 1994 [4] did the same using axioms with only 4 variables. In [5, Sections 16.9] he noted that it is unknown whether there exist undecidable superintuitionistic propositional calculi with axioms in 2 or 3 variables.

In [6] Gladstone proved that the following formula

$$A = (p \to q) \to ((q \to r) \to (p \to r))$$

is not derivable from the set of all 2-variable tautologies by modus ponens and substitution. Since A is an intuitionistic tautology, therefore a 2-variable propositional calculus cannot derive all intuitionistic tautologies. If we combine this with Gladstone's result for 1-variable propositional calculi, we get that there are no undecidable superintuitionistic propositional calculi with axioms containing less than 3 variables. The aim of this paper is to construct an undecidable 3-variable superintuitionistic propositional calculus.

This paper is organized as follows. In the next section we introduce the basic terminology and notation. In Section 3 we state and prove our main result. Finally, in Section 4 we give some concluding remarks and discuss further directions of research.

2 Definitions

In this section, we recall definitions of the intuitionistic propositional calculus and Kripke semantics. For more details we refer the reader to [5].

First, we introduce some notation. Let us consider the language consisting of an infinite set of propositional variables \mathcal{V} , brackets, and the signature $\Sigma = \{\bot, \land, \lor, \rightarrow\}$, where \bot is the constant symbol, \land , \lor and \rightarrow are binary connectives. Letters p, q, x, y, etc., are used to denote propositional variables. We define \neg , \leftrightarrow and \top as the usual abbreviations: $\neg A := A \rightarrow \bot$, $A \leftrightarrow B = (A \rightarrow B) \land (B \rightarrow A)$, and $\top = \neg \bot$.

Propositional formulas or Σ -formulas are built up from the signature Σ , propositional variables from \mathcal{V} , and brackets in the usual way. For example, the following notations

$$x, \neg A, (A \land B), (A \lor B), (A \to B)$$

are formulas if A, B are formulas. Capital letters A, B, C, etc., are used to denote propositional formulas. Throughout the paper, we omit some parentheses in formulas whenever it does not lead to confusion.

By a propositional calculus or a Σ -calculus we mean a finite set P of Σ -formulas referred to as axioms together with two rules of inference:

1) modus ponens

 $A, A \to B \vdash B,$

2) substitution

$$A \vdash \sigma A$$
,

where σA is a substitution instance of A, i.e., the result of applying a substitution σ to the formula A.

Denote by [P] the set of derivable (or provable) formulas of a calculus P. A *derivation* in P is defined from the axioms and the rules of inference in the usual way. The statement that a formula A is derivable from P is denoted by $P \vdash A$.

Let us introduce the following pre-order relation on the set of all propositional calculi. We write $P_1 \leq P_2$ (or, equivalently, $P_2 \geq P_1$) if each derivable formula of P_1 is also derivable from P_2 , i.e., if $[P_1] \subseteq [P_2]$. We write $P_1 \sim P_2$ and say that two calculi P_1 and P_2 are equivalent if $[P_1] = [P_2]$. Finally, we write $P_1 < P_2$ if $[P_1] \subsetneq [P_2]$.

An *intuitionistic Kripke frame* is a pair $\mathfrak{F} = \langle W, R \rangle$ consisting of a nonempty set W and a partial order R on W, which is reflexive, transitive and antisymmetric, i.e., \mathfrak{F} is just a partially ordered set. The elements of W are called the *points* (or *worlds*) of the frame \mathfrak{F} , and the relation R is called the *accessibility relation*. If for some $w, w' \in W$ the relation wRw' holds, we say that w' is *accessible* from w or w sees w'. We write $w \leq_R w'$ (or $w' \geq_R w$) iff wRw'.

A valuation in an intuitionistic frame $\mathfrak{F} = \langle W, R \rangle$ is a map \mathfrak{V} associating with each propositional variable $p \in \mathcal{V}$ some (possibly empty) subset $\mathfrak{V}(p)$ of W such that, for every $w \in \mathfrak{V}(p)$ and every $w' \in W$, $w \leq_R w'$ implies $w' \in \mathfrak{V}(p)$.

An *intuitionistic Kripke model* is a pair $\mathfrak{M} = \langle \mathfrak{F}, \mathfrak{V} \rangle$, where \mathfrak{F} is an intuitionistic frame and \mathfrak{V} is a valuation in \mathfrak{F} .

Let $\mathfrak{M} = \langle \mathfrak{F}, \mathfrak{V} \rangle$ be an intuitionistic Kripke model and w be a point in the frame $\mathfrak{F} = \langle W, R \rangle$. By induction on the construction of a formula A we define a relation $(\mathfrak{M}, w) \models A$, which is read as A is true at w in \mathfrak{M} :

 $\begin{array}{lll} (\mathfrak{M},w) \not\models \bot \\ (\mathfrak{M},w) \models p & \Longleftrightarrow & w \in \mathfrak{V}(p); \\ (\mathfrak{M},w) \models A \wedge B & \Longleftrightarrow & (\mathfrak{M},w) \models A \text{ and } (\mathfrak{M},w) \models B; \\ (\mathfrak{M},w) \models A \vee B & \Longleftrightarrow & (\mathfrak{M},w) \models A \text{ or } (\mathfrak{M},w) \models B; \\ (\mathfrak{M},w) \models A \rightarrow B & \Longleftrightarrow & \text{for all } w' \in W \text{ such that } w \leq_R w', \\ (\mathfrak{M},w') \models A \text{ implies } (\mathfrak{M},w') \models B. \end{array}$

From the definition it follows that

$$(\mathfrak{M}, w) \models \top$$

$$(\mathfrak{M}, w) \models \neg A \quad \Longleftrightarrow \quad \text{for all } w' \in W \text{ such that } w \leq_R w', \ (\mathfrak{M}, w') \not\models A$$

If $(\mathfrak{M}, w) \models A$ does not hold, i.e., $(\mathfrak{M}, w) \not\models A$, we say that A is refuted at the point w in \mathfrak{M} .

We say that A is valid in a model $\mathfrak{M} = \langle \mathfrak{F}, \mathfrak{I} \rangle$ defined on a frame $\mathfrak{F} = \langle W, R \rangle$ if $(\mathfrak{M}, w) \models A$ for all $w \in W$; if A is valid in \mathfrak{M} , we write $\mathfrak{M} \models A$. We say that A is valid in a frame $\mathfrak{F} = \langle W, R \rangle$ if A is valid in every model based on \mathfrak{F} ; if A is valid in \mathfrak{F} , we write $\mathfrak{F} \models A$. We say that A is true at a point w in a frame \mathfrak{F} if $(\mathfrak{M}, w) \models A$ for every model \mathfrak{M} defined on \mathfrak{F} ; if A is true at the point w in frame \mathfrak{F} , we write $(\mathfrak{F}, w) \models A$. If \mathfrak{M} is fixed we write $w \models A$ instead of $(\mathfrak{M}, w) \models A$.

We define the intuitionistic propositional calculus **Int** as the smallest propositional calculus containing the following set of axioms:

$$\begin{array}{ll} (\rightarrow_1) & p \rightarrow (q \rightarrow p) \\ (\rightarrow_2) & (p \rightarrow (q \rightarrow r)) \rightarrow ((p \rightarrow q) \rightarrow (q \rightarrow r)) \\ (\wedge_1) & p \wedge q \rightarrow p \\ (\wedge_2) & p \wedge q \rightarrow q \\ (\wedge_3) & p \rightarrow (q \rightarrow p \wedge q) \\ (\vee_1) & p \rightarrow p \vee q \\ (\vee_2) & q \rightarrow p \vee q \\ (\vee_3) & (p \rightarrow r) \rightarrow ((q \rightarrow r) \rightarrow (p \vee q \rightarrow r)) \\ (\neg_1) & (p \rightarrow q) \rightarrow ((p \rightarrow \neg q) \rightarrow \neg p) \\ (\neg_2) & p \rightarrow (\neg p \rightarrow q) \end{array}$$

It is well known that

 $\mathbf{Int} \vdash A \quad \Longleftrightarrow \quad \mathfrak{F} \models A, \text{ for every Kripke frame } \mathfrak{F}.$

By a superintuitionistic propositional calculus we mean a propositional calculus obtained from **Int** by adding a finite set of new axioms. If M is a finite set of propositional formulas, then a propositional calculus obtained from **Int** by adding new axioms M is denoted by **Int** + M. Since

$$\operatorname{Int} + \{A_1, \ldots, A_n\} \sim \operatorname{Int} + A_1 \wedge \ldots \wedge A_n$$

we can assume that a superintuitionistic propositional calculus is a calculus Int + A for some intuitionistic propositional formula A.

3 Main result

Our main result is the following theorem.

Theorem 3.1. There is a 3-variable intuitionistic propositional formula A such that Int + A is undecidable.

First, we recall what a Minsky machine is and encode configurations of a Minsky machine by superintuitionistic propositional formulas. Next, we construct a Kripke model refuting all codes of derivable configurations. Finally, we encode instructions of a Minsky machine \mathcal{M} by a single superintuitionistic formula $A_{\mathcal{M}}$ and formally reduce the configuration problem of \mathcal{M} to the derivation problem of a superintuitionistic propositional calculus $\mathbf{Int} + A_{\mathcal{M}}$.

3.1 Minsky machine

There are many algorithmic formalisms to prove the undecidability of a propositional calculus [3]. For example, the undecidability of a calculus contained in the classical [1], intuitionistic [2] propositional calculus or in another subcalculus [3] can be easily proved by using tag systems. But for extensions of the intuitionistic propositional calculus, this is very hard [12, 17]. For this reason, in order to prove the undecidability of superintuitionistic propositional calculi we will use an algorithmic formalism which is called *Minsky machines* [11]. In [5] Chagrov mentioned that it is the most convenient formalism for being simulated by modal and intuitionistic formulas.

In accordance with [5] we define a *Minsky machine* as a finite set of instructions for transforming triples $\langle s, m, n \rangle$ of natural numbers, called *configurations*, where s is the number of the instruction to be executed at the next step (referred to as the *current machine state*), and $m, n \in \mathbb{N}^{-1}$. Each instruction has one of the following four forms:

$$\begin{array}{rcl} s \ \mapsto \ \langle t,1,0\rangle\,, & s \ \mapsto \ \langle t,-1,0\rangle\,/\,\langle u,0,0\rangle\,, \\ s \ \mapsto \ \langle t,0,1\rangle\,, & s \ \mapsto \ \langle t,0,-1\rangle\,/\,\langle u,0,0\rangle\,, \end{array}$$

where s, t, u are the machine states. Note that all Minsky machines are assumed to be deterministic, i.e., they may not contain distinct instructions with the same numbers.

As an example, let us consider the applying of first two instructions. The instruction

$$s \mapsto \langle t, 1, 0 \rangle$$

¹We assume that $\mathbb{N} = \{0, 1, 2, \ldots\}.$

transforms $\langle s, m, n \rangle$ into $\langle t, m+1, n \rangle$, and the instruction

$$s \mapsto \langle t, -1, 0 \rangle / \langle u, 0, 0 \rangle$$

transforms $\langle s, m, n \rangle$ into $\langle t, m - 1, n \rangle$ if m > 0 and into (u, m, n) if m = 0. The meaning of the others is defined analogously.

Let \mathcal{M} be a Minsky machine, then the notation $\langle s, m, n \rangle \xrightarrow{\mathcal{M}} \langle t, k, l \rangle$ means that the configuration $\langle t, k, l \rangle$ is obtained from $\langle s, m, n \rangle$ by applying an instruction of machine \mathcal{M} once. We write $\langle s, m, n \rangle \xrightarrow{\mathcal{M}} \langle t, k, l \rangle$ if the configuration $\langle t, k, l \rangle$ is obtained from $\langle s, m, n \rangle$ by applying instructions of machine \mathcal{M} in finitely many steps (possibly, in 0 steps). Particularly, we always have $\langle s, m, n \rangle \xrightarrow{\mathcal{M}} \langle s, m, n \rangle$.

The configuration problem for a Minsky machine M and a configuration $\langle s, m, n \rangle$ is, given a configuration $\langle t, k, l \rangle$, to determine whether $\langle s, m, n \rangle \stackrel{\mathcal{M}}{\Longrightarrow} \langle t, k, l \rangle$.

Theorem 3.2 (Minsky, [11]). There exist a Minsky machine \mathcal{M} and a configuration $\langle s, m, n \rangle$ for which the configuration problem is undecidable.

Let \mathcal{M} be a Minsky machine and $\langle s_0, m_0, n_0 \rangle$ a configuration for which the configuration problem is undecidable.

3.2 Encoding of configurations

Let p, q and r be three distinct propositional variables. Now we define some propositional formulas using only variables p, q, r, which encode configurations of Minsky machines. Note that some basic ideas of defining these formulas was found in [5] and [13].

First, let us define the following groups of propositional formulas constructed from variables p, q and r. If

$$S_{-2}[x] = \neg x, \qquad S_{-1}[x] = T_{-2}[x] \to x,$$

$$T_{-2}[x] = \neg \neg x, \qquad T_{-1}[x] = S_{-1}[x] \to S_{-2}[x] \lor T_{-2}[x],$$

$$S_i[x] = T_{i-1}[x] \to S_{i-1}[x] \lor T_{i-2}[x],$$

$$T_i[x] = S_i[x] \to S_{i-1}[x] \lor T_{i-1}[x],$$

for all $i \ge 0$, then we define

Groups (A^0) and (B^0) :

$$A_i^0 = S_{i+3}[r], \ B_i^0 = T_{i+3}[r] \text{ for all } i \ge -5.$$

Let $C_1 = A_0^0$ and $C_2 = B_0^0$, then Groups (A^1) and (B^1) :

$$\begin{aligned} A_i^1 &= S_{i+3}[p], \quad B_i^1 &= T_{i+3}[p] \quad \text{for } i \in \{-3, -4, -5\}, \\ A_{-2}^1 &= B_{-3}^1 \to A_{-3}^1 \lor B_{-4}^1, \qquad A_{-1}^1 &= B_{-2}^1 \to A_{-2}^1 \lor B_{-3}^1, \\ B_{-2}^1 &= A_{-3}^1 \to C_1 \lor B_{-3}^1, \qquad B_{-1}^1 &= A_{-2}^1 \to A_{-3}^1 \lor B_{-2}^1, \\ A_i^1 &= C_2 \land B_{i-1}^1 \to C_1 \lor A_{i-1}^1 \lor B_{i-2}^1, \\ B_i^1 &= C_2 \land A_{i-1}^1 \to C_1 \lor A_{i-2}^1 \lor B_{i-1}^1, \quad \text{for all } i \ge 0; \end{aligned}$$

Groups (A^2) and (B^2) :

$$\begin{aligned} A_i^2 &= S_{i+3}[q], \quad B_i^2 = T_{i+3}[q] \quad \text{for } i \in \{-3, -4, -5\}, \\ A_{-2}^2 &= B_{-3}^2 \to A_{-3}^2 \lor B_{-4}^2, \qquad A_{-1}^2 = B_{-2}^2 \to A_{-2}^2 \lor B_{-3}^2, \\ B_{-2}^2 &= A_{-3}^2 \to C_2 \lor B_{-3}^2, \qquad B_{-1}^2 = A_{-2}^2 \to A_{-3}^2 \lor B_{-2}^2, \\ A_i^2 &= C_1 \land B_{i-1}^2 \to C_2 \lor A_{i-1}^2 \lor B_{i-2}^2, \\ B_i^2 &= C_1 \land A_{i-1}^2 \to C_2 \lor A_{i-2}^2 \lor B_{i-1}^2, \quad \text{for all } i \ge 0. \end{aligned}$$

Note that the groups (A^0) , (B^0) contain only variable r, (A^1) , (B^1) contain only variables r, p, and (A^2) , (B^2) contain only variables r, q. Now we define formulas encoding configurations of the Minsky machine \mathcal{M} .

Group (E):

$$\begin{split} E_{s,m,n} &= A^0_{3s+2} \wedge B^0_{3s+2} \wedge A^1_{m+1} \wedge B^1_{m+1} \wedge A^2_{n+1} \wedge B^2_{n+1} \to \\ &\to A^0_{3s+1} \vee B^0_{3s+1} \vee A^1_m \vee B^1_m \vee A^2_n \vee B^2_n, \end{split}$$

for all $s, m, n \ge 0$. The formula $E_{s,m,n}$ is called the *code* of a configuration $\langle s, m, n \rangle$.

Denote by (A) and (B) the following sets of formulas:

$$\begin{array}{rcl} (A) & = & (A^0) \cup (A^1) \cup (A^2), \\ (B) & = & (B^0) \cup (B^1) \cup (B^2), \end{array}$$

and by M the set of formulas:

$$M = (A) \cup (B) \cup (E).$$

3.3 Kripke model refuting codes of derivable configurations

In this section, we construct a Kripke model $\mathfrak{M} = \langle \mathfrak{F}, \mathfrak{V} \rangle$ refuting all formulas from M, i.e., for every formula from M, there exists a unique maximal point, at which this formula is refuted.

First, let us define the following equivalence relation $\sim_{\mathcal{M}}$ on the set of all configurations $\{\langle s, m, n \rangle \mid s, m, n \geq 0\}$:

$$\langle s, m, n \rangle \sim_{\mathcal{M}} \langle t, k, l \rangle \rightleftharpoons \langle s, m, n \rangle \stackrel{\mathcal{M}}{\longmapsto} \langle t, k, l \rangle \text{ and } \langle t, k, l \rangle \stackrel{\mathcal{M}}{\longmapsto} \langle s, m, n \rangle.$$

Denote by [s, m, n] the equivalence class of a configuration $\langle s, m, n \rangle$:

$$[s, m, n] = \{ \langle t, k, l \rangle \mid \langle s, m, n \rangle \sim_{\mathcal{M}} \langle t, k, l \rangle \}.$$

The set of all equivalence classes of relation $\sim_{\mathcal{M}}$ is denoted by $\mathcal{E}_{\mathcal{M}}$.

Let us define the relation $\stackrel{\mathcal{M}}{\Longrightarrow}$ on the set of equivalence classes $\mathcal{E}_{\mathcal{M}}$:

$$[s,m,n] \stackrel{\mathcal{M}}{\longmapsto} [t,k,l] \leftrightarrows \langle s,m,n \rangle \stackrel{\mathcal{M}}{\longmapsto} \langle t,k,l \rangle \,.$$

Greek letters α, β, γ , etc., are used to denote equivalence classes. Denote by α_0 the equivalence class of the initial configuration $\langle s_0, m_0, n_0 \rangle$, i.e., $\alpha_0 = [s_0, m_0, n_0]$.

Now we define a Kripke frame $\mathfrak{F} = \langle W, R \rangle$ as follows. Let

$$\bigcup_{\substack{i \ge -5, \\ j \in \{0,1,2\}}} \{a_i^j, b_i^j\} \cup \bigcup_{\substack{\alpha \in \mathcal{E}_{\mathcal{M}}: \\ \alpha_0 \stackrel{\mathcal{M}}{\Longrightarrow} \alpha}} \{e_\alpha\}.$$

To define the accessibility relation R on W, we consider the following groups of relations: Group R_i^j , $i \ge -4$, $j \in \{0, 1, 2\}$:

$$\begin{split} R_{-4}^{j} &= \left\{ \left\langle a_{-4}^{j}, a_{-5}^{j} \right\rangle, \ \left\langle b_{-4}^{j}, a_{-5}^{j} \right\rangle, \ \left\langle b_{-4}^{j}, b_{-5}^{j} \right\rangle \right\}, \\ R_{-3}^{j} &= \left\{ \left\langle a_{-3}^{j}, a_{-4}^{j} \right\rangle, \ \left\langle a_{-3}^{j}, b_{-5}^{j} \right\rangle, \ \left\langle b_{-3}^{j}, a_{-4}^{j} \right\rangle, \ \left\langle b_{-3}^{j}, b_{-4}^{j} \right\rangle \right\}, \\ R_{i}^{0} &= \left\{ \left\langle a_{i}^{0}, a_{i-1}^{0} \right\rangle, \ \left\langle a_{i}^{0}, b_{i-2}^{0} \right\rangle, \ \left\langle b_{i}^{0}, a_{i-1}^{0} \right\rangle, \ \left\langle b_{i}^{0}, b_{i-1}^{0} \right\rangle \right\} \quad \text{for all } i \geq -2 \text{ and} \\ R_{-2}^{1} &= \left\{ \left\langle a_{-2}^{1}, a_{-3}^{1} \right\rangle, \ \left\langle a_{-2}^{1}, b_{-4}^{1} \right\rangle, \ \left\langle b_{-2}^{1}, a_{0}^{0} \right\rangle, \ \left\langle b_{-2}^{1}, b_{-3}^{1} \right\rangle \right\}, \\ R_{-2}^{2} &= \left\{ \left\langle a_{-2}^{2}, a_{-3}^{2} \right\rangle, \ \left\langle a_{-2}^{2}, b_{-4}^{2} \right\rangle, \ \left\langle b_{i}^{2}, a_{i-2}^{0} \right\rangle, \ \left\langle b_{i}^{j}, b_{i-1}^{j} \right\rangle \right\} \quad \text{for all } i \geq -1, \ j \in \{1, 2\}; \\ R_{i}^{j} &= \left\{ \left\langle a_{i}^{j}, a_{i-1}^{j} \right\rangle, \ \left\langle a_{i}^{j}, b_{i-2}^{j} \right\rangle, \ \left\langle b_{i}^{j}, a_{i-2}^{j} \right\rangle, \ \left\langle b_{i}^{j}, b_{i-1}^{j} \right\rangle \right\} \quad \text{for all } i \geq -1, \ j \in \{1, 2\}; \end{split}$$

Group $R_{s,m,n}$, $s, m, n \ge 0$, $\alpha_0 \stackrel{\mathcal{M}}{\longmapsto} [s, m, n]$:

$$R_{s,m,n} = \left\{ \left\langle e_{[s,m,n]}, a_{3s+1}^{0} \right\rangle, \left\langle e_{[s,m,n]}, b_{3s+1}^{0} \right\rangle, \left\langle e_{[s,m,n]}, a_{m}^{1} \right\rangle, \\ \left\langle e_{[s,m,n]}, b_{m}^{1} \right\rangle, \left\langle e_{[s,m,n]}, a_{n}^{2} \right\rangle, \left\langle e_{[s,m,n]}, b_{n}^{2} \right\rangle \right\}.$$

Let

$$R' = \bigcup_{\substack{i \ge -4, \\ j \in \{0,1,2\}}} R_i^j \cup \bigcup_{\substack{s,m,n \ge 0: \\ \alpha_0 \stackrel{\mathcal{M}}{\longrightarrow} [s,m,n]}} R_{s,m,n} \cup \bigcup_{\substack{\alpha,\beta \in \mathcal{E}_{\mathcal{M}}: \\ \alpha \stackrel{\mathcal{M}}{\Longrightarrow} \beta}} \{ \langle e_\alpha, e_\beta \rangle \}$$

We take as R the reflexive and transitive closure of R'.

Let us define a valuation \mathfrak{V} of the Kripke model $\mathfrak{M} = \langle \mathfrak{F}, \mathfrak{V} \rangle$ in the following way:

$$(\mathfrak{M}, w) \not\models r \iff w \leq_R a_{-4}^0 \text{ or } w \leq_R b_{-5}^0; (\mathfrak{M}, w) \not\models p \iff w \leq_R a_{-4}^1 \text{ or } w \leq_R b_{-5}^1; (\mathfrak{M}, w) \not\models q \iff w \leq_R a_{-4}^2 \text{ or } w \leq_R b_{-5}^2.$$

The model \mathfrak{M} is depicted on Figure 1. Now we prove some basic semantic properties of the Kripke model \mathfrak{M} .

Lemma 3.3. Let w be a world of \mathfrak{M} , then

$$w \not\models A_i^j \iff w \leq_R a_i^j,$$
$$w \not\models B_i^j \iff w \leq_R b_i^j$$

for all $i \ge -4$ and $j \in \{0, 1, 2\}$.

Proof. By induction on $i \geq -4$.

Induction base consists of the following cases:

1) i = -4. Let $x_0 = r$, $x_1 = p$, and $x_2 = q$.

Since $w \not\models x_j$ iff $w \leq_R a_{-4}^j$ or $w \leq_R b_{-5}^j$, we have that A_{-4}^j is refuted at a_{-4}^j and B_{-4}^j is refuted at b_{-4}^j . Therefore, $w \not\models A_{-4}^j$ if $w \leq_R a_{-4}^j$ and $w \not\models B_{-4}^j$ if $w \leq_R b_{-4}^j$.

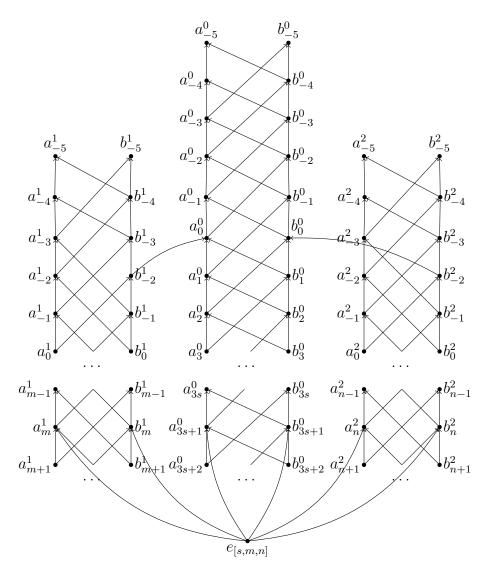


Figure 1: Kripke model \mathfrak{M} .

If $w \not\models A_{-4}^j$, then there exists a point $w' \ge_R w$ such that $w' \models \neg \neg x_j$ and $w' \not\models x_j$. By definition of the valuation \mathfrak{V} , we have either $w' \le_R a_{-4}^j$ or $w' \le_R b_{-5}^j$. Since $w' \models \neg \neg x_j$, therefore for all point $w'' \ge_R w'$ there is a point $w''' \ge_R w''$ such that $w''' \models x_j$. It is clear that $w' \not\leq_R b_{-5}^j$. Hence, $w \le_R a_{-4}^j$.

If $w \not\models B_{-4}^{j}$, then there exist points $w' \geq_{R} w$ and $w'' \geq_{R} w$ such that $w' \models x_{j}$ and $w'' \models \neg x_{j}$. By definition of the valuation \mathfrak{V} , we have $w' \not\leq_{R} a_{-4}^{j}$, $w' \not\leq_{R} b_{-5}^{j}$, and $w'' = b_{-5}^{j}$. If $w' \leq_{R} a_{-5}^{j'}$ or $w' \leq_{R} b_{-5}^{j'}$ for some $j' \in \{0, 1, 2\} \setminus \{j\}$, then $w \leq_{R} e_{[s,m,n]}$ for some $s, m, n \geq 0$ and therefore $w \leq_{R} b_{-4}^{j}$. Otherwise, $w' = a_{-5}^{j}$. Note that there is a unique point $w''' \geq_{R} w$ such that $w''' \leq_{R} a_{-5}^{j}$, $w''' \leq_{R} b_{-5}^{j}$, and $w''' \models A_{-4}^{j}$. It is easily seen that $w''' = b_{-4}^{j}$ and therefore $w \leq_{R} b_{-4}^{j}$.

2) i = -3.

Note that $w \not\models A_{-3}^j$ if $w \leq_R a_{-4}^j$, $w \leq_R b_{-5}^j$, $w \not\leq_R b_{-4}^j$ and $w \not\models B_{-3}^j$ if $w \leq_R a_{-4}^j$, $w \leq_R b_{-4}^j$, $w \not\leq_R a_{-3}^j$. Since a_{-3}^j and b_{-3}^j are unique maximal points satisfying this condition, we have that $w \not\models A_{-3}^j$ if $w \leq_R a_{-3}^j$ and $w \not\models B_{-3}^j$ if $w \leq_R b_{-3}^j$.

If $w \not\models A_{-3}^j$, then there exists a point $w' \ge_R w$ such that $w' \not\models A_{-4}^j$, $\neg \neg x_j$ and $w' \models B_{-4}^j$. So, $w' \le_R a_{-4}^j$ and $w' \not\leq_R b_{-4}^j$. Since $w' \not\models \neg \neg x_j$, there is a point $w'' \ge_R w'$ such that $w'' \models \neg x_j$. It is clear that $w'' = b_{-5}^j$. Evidently, the model \mathfrak{M} contains only one point a_{-3}^j satisfying the following condition: $w' \le_R a_{-4}^j$, $w' \le_R b_{-5}^j$ and $w' \not\leq_R b_{-4}^j$. Hence, $w \le_R a_{-3}^j$.

If $w \not\models B_{-3}^j$, then there exists a point $w' \geq_R w$ such that $w' \not\models A_{-4}^j$, B_{-4}^j and $w' \models A_{-3}^j$. Then $w' \leq_R a_{-4}^j$, $w' \leq_R b_{-4}^j$ and $w' \not\leq_R a_{-3}^j$. Evidently, the model \mathfrak{M} contains only one point b_{-3}^j satisfying this condition. Therefore, $w \leq_R b_{-3}^j$.

3) i = -2 and $j \in \{1, 2\}$.

We have that $w \not\models A_{-2}^j$ if $w \leq_R a_{-3}^j$, $w \leq_R b_{-4}^j$, $w \not\leq_R b_{-3}^j$ and $w \not\models B_{-2}^j$ if $w \leq_R c$, $w \leq_R b_{-3}^j$, $w \not\leq_R a_{-2}^j$, where $c = a_0^0$ for j = 1 and $c = b_0^0$ for j = 2. Since a_{-2}^j and b_{-2}^j are unique maximal points satisfying this condition, we have that $w \not\models A_{-2}^j$ if $w \leq_R a_{-2}^j$ and $w \not\models B_{-2}^j$ if $w \leq_R b_{-2}^j$.

If $w \not\models A_{-2}^j$, then there exists a point $w' \ge_R w$ such that $w' \not\models A_{-3}^j$, B_{-4}^j and $w' \models B_{-3}^j$. So, $w' \le_R a_{-3}^j$, $w' \le_R b_{-4}^j$, and $w' \not\le_R b_{-3}^j$. It is clear that the model \mathfrak{M} contains only one point a_{-2}^j satisfying this condition. Hence, $w \le_R a_{-2}^j$.

If $w \not\models B_{-2}^j$, then there exists a point $w' \geq_R w$ such that $w' \not\models C_j$, B_{-3}^j and $w' \models A_{-3}^j$. Then $w' \leq_R c$, $w' \leq_R b_{-3}^j$ and $w' \not\leq_R a_{-3}^j$, where $c = a_0^0$ if j = 1 and $c = b_0^0$ if j = 2. Evidently, the model \mathfrak{M} contains only one point b_{-2}^j satisfying this condition. Therefore, $w \leq_R b_{-2}^j$.

4) i = -1 and $j \in \{1, 2\}$. This case easily follows by analogy.

Induction step: assume that $i \ge -2$ if j = 0 and $i \ge 0$ if $j \in \{1, 2\}$. Without loss of generality, we can consider the case j = 1. The cases j = 0 and j = 2 are proved by analogy.

By induction assumption, we have that $w \not\models A_i^1$ if $w \leq_R a_0^0, a_{i-1}^1, b_{i-2}^1, w \not\leq_R b_0^0, b_{i-1}^1$ and $w \not\models B_i^1$ if $w \leq_R a_0^0, a_{i-2}^1, b_{i-1}^1, w \not\leq_R b_0^0, a_{i-1}^1$. Since a_i^1 and b_i^1 are unique maximal points satisfying this condition, we have that $w \not\models A_i^1$ if $w \leq_R a_i^1$ and $w \not\models B_i^1$ if $w \leq_R b_i^1$.

If $w \not\models A_i^1$, then there exists a point $w' \ge_R w$ such that $w' \not\models C_1$, A_{i-1}^1 , B_{i-2}^1 and $w' \models C_2, B_{i-1}^1$. By induction hypothesis, we obtain that $w' \le_R a_{i-1}^1$, $w' \le_R b_{i-2}^1$, and $w' \not\le_R b_{i-1}^1$. So, $w' = a_i^1$ and $w \le_R a_i^1$ by definition of the accessibility relation R. Analogously, if $w \not\models B_i^1$, then $w \le_R b_i^1$. The lemma is proved.

Lemma 3.4. Let w be a world of \mathfrak{M} , then

 $w \not\models E_{s,m,n} \iff w \leq_R e_{[s,m,n]}$

for all $s, m, n \ge 0$ such that $\alpha_0 \stackrel{\mathcal{M}}{\Longrightarrow} [s, m, n]$.

The proof is trivial by definition of the accessibility relation R. Finally, we prove the key lemma of this section.

Lemma 3.5. If $(\mathfrak{F}, w) \not\models E_{s,m,n}$, then $e_{[s,m,n]} \in W$ and $w \leq_R e_{[s,m,n]}$ for all $s, m, n \geq 0$.

Proof. Let $\mathfrak{M}' = \langle \mathfrak{F}, \mathfrak{V}' \rangle$ be a Kripke model such that $(\mathfrak{M}', w) \not\models E_{s,m,n}$. Since $w \not\models E_{s,m,n}$, there is a point $w' \geq_R w$ such that the formulas $A_{3s+1}^0, B_{3s+1}^0, A_m^1, B_m^1, A_n^2, B_n^2$ are refuted at w', and the formulas $A_{3s+2}^0, B_{3s+2}^0, A_{m+1}^1, B_{m+1}^1, A_{n+1}^2, B_{n+1}^2$ are true at w'.

Denote by w_s^a and w_s^b points of the frame \mathfrak{F} such that

1. $w_s^a \ge_R w', w_s^a \models B_{3s}^0$, and $w_s^a \not\models A_{3s}^0, B_{3s-1}^0$; 2. $w_s^b \ge_R w', w_s^b \models A_{3s+1}^0$, and $w_s^b \not\models A_{3s}^0, B_{3s}^0$.

It is clear that these points exist.

If w_s^a or w_s^b are in $\{a_n^0, b_n^0\}$ for some $n \ge -5$, then $a_{-5}^0 \in \mathfrak{V}'(r)$, $a_{-4}^0, b_{-5}^0 \notin \mathfrak{V}'(r)$. By analogy with Lemma 3.3, it is not hard to prove that $w_s^a = a_{3s+1}^0$, $w_s^b = b_{3s+1}^0$ by induction on $s \ge 0$.

Let w_s^a and w_s^b are not in $\{a_n^0, b_n^0\}$ for all $n \ge -5$. Then there are $n \ge -5$ and $j \in \{1, 2\}$ such that w_s^a or w_s^b are in $\{a_n^j, b_n^j\}$. Evidently, either $a_{-5}^0 \notin \mathfrak{V}'(r)$, $a_{-4}^0 \in \mathfrak{V}'(r)$, or $b_{-5}^0 \notin \mathfrak{V}'(r)$. We need to consider the following cases:

- 1. $a_{-5}^{j} \notin \mathfrak{V}'(r)$. In this case, A_{-4}^{0} is refuted at a_{0}^{0} if j = 1 and b_{0}^{0} if j = 2. Then it can easily be seen that $a_{-5}^{0}, b_{-5}^{0} \in \mathfrak{V}'(r)$ and $a_{0}^{0} \notin \mathfrak{V}'(r)$. If $b_{-5}^{j} \notin \mathfrak{V}'(r)$, then B_{-4}^{0} is true at w_{s}^{a}, w_{s}^{b} , which is impossible. If $b_{-5}^{j} \in \mathfrak{V}'(r)$, then B_{-4}^{0} is refuted at a_{-3}^{j}, b_{-4}^{j} and therefore A_{-3}^{0} is true at w_{s}^{a}, w_{s}^{b} , which is impossible. Hence $a_{-5}^{j} \in \mathfrak{V}'(r)$.
- 2. $b_{-5}^j \in \mathfrak{V}'(r)$. In this case, B_{-4}^0 is refuted at b_{-2}^j . Then A_{-3}^0 is true at w_s^a , w_s^b , which is impossible. Hence $b_{-5}^j \notin \mathfrak{V}'(r)$.
- 3. $a_{-4}^j \in \mathfrak{V}'(r)$. In this case, A_{-4}^0 is refuted at a_0^0 if j = 1 and b_0^0 if j = 2. As the above, we have that $a_{-5}^0, b_{-5}^0 \in \mathfrak{V}'(r)$ and $a_0^0 \notin \mathfrak{V}'(r)$. Then B_{-4}^0 is refuted at a_{-3}^j, b_{-4}^j and therefore A_{-3}^0 is true at w_s^a, w_s^b , which is impossible. Hence $a_{-4}^j \notin \mathfrak{V}'(r)$.

So, we have that $a_{-5}^j \in \mathfrak{V}'(r)$ and $a_{-4}^j, b_{-5}^j \notin \mathfrak{V}'(r)$. Then A_{-4}^0 is refuted at a_{-4}^j and B_{-4}^0 is refuted at b_{-4}^j . It can easily be proved by induction on $s \ge 0$ that $w_s^a = a_{4s+2}^j$ and $w_s^b = b_{4s+3}^j$. Therefore, if $w_s^a = a_{4s+2}^{j_1}$ and $w_s^b = b_{4s+3}^{j_2}$ for some $j_1, j_2 \in \{1, 2\}$, then $w' \le_R a_{4s+3}^{j_2}$ and therefore A_{3s+2}^0 is refuted at w', which is impossible. Hence w_s^a or w_s^b are in $\{a_n^0, b_n^0\}$ for some $n \ge -5$ and therefore $a_{-5}^0 \in \mathfrak{V}'(r), a_{-4}^0, b_{-5}^0 \notin \mathfrak{V}'(r)$.

Since C_1 is refuted at w_1 if $w_1 \leq_R a_0^0$ and C_2 is refuted at w_2 if $w_2 \leq_R b_0^0$, we have that, for a given $i \geq 0$ and $j \in \{1, 2\}$, the formulas A_i^j , B_i^j are refuted at a_k^j , b_k^j for some $k \geq 0$ and true at $a_k^{j'}$, $b_k^{j'}$ for all $k \geq 0$, $j' \in \{0, 1, 2\} \setminus \{j\}$ by analogy with the above. Therefore, $a_{-5}^1 \in \mathfrak{V}'(p)$, a_{-4}^1 , $b_{-5}^1 \notin \mathfrak{V}'(p)$ and $a_{-5}^2 \in \mathfrak{V}'(q)$, a_{-4}^2 , $b_{-5}^2 \notin \mathfrak{V}'(q)$. Now if we recall the proof of Lemma 3.3, then we obtain that $w' \leq_R a_{3s+1}^0$, $w' \leq_R b_{3s+1}^0$, $w' \leq_R a_{3s+1}^n$, $w' \leq_R a_{3s+1}^n$, $w' \leq_R a_{3s+2}^n$, $w' \leq_R a_{3s+2}^0$, $w' \leq_R b_{3s+2}^0$, $w' \leq_R a_{3s+2}^n$, $w' \leq_R a_$

3.4 Key formulas

In this section, we consider the key formulas depending on variables p, q, r. First, let us define the following formulas $F_k = F_k[p, q, x, y]$ and $G_k = G_k[p, q, x, y]$ in variables p, q, x and y:

$$F_0 = p,$$

$$G_0 = q,$$

$$F_1 = y \land q \to x \lor p,$$

$$G_1 = y \land p \to x \lor q, \text{ and}$$

$$F_k = y \land G_{k-1} \to x \lor F_{k-1} \lor G_{k-2},$$

$$G_k = y \land F_{k-1} \to x \lor G_{k-1} \lor F_{k-2}, \text{ for all } k \ge 2.$$

Now we introduce the following key formulas:

$$\begin{array}{rcl} F_k^1[p,q] &=& F_k[p,q,C_1,C_2],\\ G_k^1[p,q] &=& G_k[p,q,C_1,C_2],\\ F_k^2[p,q] &=& F_k[p,q,C_2,C_1],\\ G_k^2[p,q] &=& G_k[p,q,C_2,C_1]. \end{array}$$

Note that the formulas F_k^m and G_k^m are depending on three variables p, q, and r, for all $k \ge 0$ and $m \in \{1, 2\}$.

Besides, we define the following auxiliary formulas:

$$\begin{array}{rcl} P_{i,j} & = & (C_2 \to C_1 \lor A_i^1 \lor B_{i-1}^1) \land (C_1 \to C_2 \lor A_i^2 \lor B_{i-1}^2), \\ Q_{i,j} & = & (C_2 \to C_1 \lor A_{i-1}^1 \lor B_i^1) \land (C_1 \to C_2 \lor A_{i-1}^2 \lor B_i^2), \end{array}$$

for all $i, j \geq -1$. The following lemma is describing the basic properties of the key formulas.

Lemma 3.6. For all $i, j \ge -1$, $k \ge 1$ and $m \in \{1, 2\}$,

$$\begin{array}{rcl} \textit{Int} & \vdash & F_k^m[P_{i,j},Q_{i,j}] \leftrightarrow A_{n+k}^m, \\ \textit{Int} & \vdash & G_k^m[P_{i,j},Q_{i,j}] \leftrightarrow B_{n+k}^m, \end{array}$$

where

$$n = \begin{cases} i, & m = 1; \\ j, & m = 2. \end{cases}$$

Proof. By induction on $k \ge 1$. Without loss of generality, we can assume that m = 1. The basis of induction consists of two cases: k = 1 and k = 2.

Induction base: k = 1. In this case we have

$$\begin{array}{lll} F_1^1[P_{i,j},Q_{i,j}] &=& C_2 \wedge Q_{i,j} \to C_1 \vee P_{i,j}, \\ G_1^1[P_{i,j},Q_{i,j}] &=& C_2 \wedge P_{i,j} \to C_1 \vee Q_{i,j}. \end{array}$$

It can easily be checked that the following derivations holds in Int:

$$\begin{array}{rcl} \mathbf{Int} & \vdash & C_2 \wedge B_i^1 \to C_2 \wedge Q_{i,j}, & \mathbf{Int} & \vdash & C_1 \vee P_{i,j} \to (C_2 \to C_1 \vee A_i^1 \vee B_{i-1}^1), \\ \mathbf{Int} & \vdash & C_2 \wedge A_i^1 \to C_2 \wedge P_{i,j}, & \mathbf{Int} & \vdash & C_1 \vee Q_{i,j} \to (C_2 \to C_1 \vee A_{i-1}^1 \vee B_i^1). \end{array}$$

Hence,

$$\begin{array}{rcl} \mathbf{Int} & \vdash & F_1^1[P_{i,j},Q_{i,j}] \to A_{i+1}^1, \\ \mathbf{Int} & \vdash & G_1^1[P_{i,j},Q_{i,j}] \to B_{i+1}^1. \end{array}$$

Conversely, since the formulas $A_{i-1}^1 \to A_i^1$ and $B_{i-1}^1 \to B_i^1$ are derivable from **Int**, we have

and therefore the following derivations holds in Int:

Hence,

$$\begin{array}{rrr} \mathbf{Int} & \vdash & A_{i+1}^1 \to F_1^1[P_{i,j},Q_{i,j}], \\ \mathbf{Int} & \vdash & B_{i+1}^1 \to G_1^1[P_{i,j},Q_{i,j}]. \end{array}$$

Induction base: k = 2. In this case we have

$$\begin{array}{rcl} \mathbf{Int} & \vdash & F_2^1[P_{i,j},Q_{i,j}] \leftrightarrow (C_2 \wedge B_{i+1}^1 \to C_1 \vee A_{i+1}^1 \vee Q_{i,j}), \\ \mathbf{Int} & \vdash & G_2^1[P_{i,j},Q_{i,j}] \leftrightarrow (C_2 \wedge A_{i+1}^1 \to C_1 \vee B_{i+1}^1 \vee P_{i,j}). \end{array}$$

Furthermore, it follows easily that:

$$\begin{array}{rcl} \mathbf{Int} & \vdash & C_2 \wedge B_i^1 \to Q_{i,j}, & \mathbf{Int} & \vdash & Q_{i,j} \to (C_2 \to C_1 \vee A_{i+1}^1 \vee B_i^1), \\ \mathbf{Int} & \vdash & C_2 \wedge A_i^1 \to P_{i,j}, & \mathbf{Int} & \vdash & P_{i,j} \to (C_2 \to C_1 \vee A_i^1 \vee B_{i+1}^1). \end{array}$$

Hence,

$$\begin{array}{rcl} \mathbf{Int} & \vdash & F_2^1[P_{i,j},Q_{i,j}] \leftrightarrow A_{i+2}^1, \\ \mathbf{Int} & \vdash & G_2^1[P_{i,j},Q_{i,j}] \leftrightarrow B_{i+2}^1. \end{array}$$

Induction step is straightforward and left to the reader. The lemma is proved.

3.5 Encoding of the Minsky machine

Now we encode instructions of the Minsky machine \mathcal{M} as superintuitionistic formulas such that derivations from **Int** and these formulas are simulate transformations of \mathcal{M} .

First, let us define the following formulas containing only tree variables p, q, r:

$$\begin{split} \hat{E}_{s,i,j} &= A^0_{3s+2} \wedge B^0_{3s+2} \wedge F^1_{i+1} \wedge G^1_{i+1} \wedge F^2_{j+1} \wedge G^2_{j+1} \to \\ &\to A^0_{3s+1} \vee B^0_{3s+1} \vee F^1_i \vee G^1_i \vee F^2_j \vee G^2_j, \\ \hat{E}_{s,0,*} &= A^0_{3s+2} \wedge B^0_{3s+2} \wedge A^1_1 \wedge B^1_1 \to A^0_{3s+1} \vee B^0_{3s+1} \vee A^1_0 \vee B^1_0 \vee q, \\ \hat{E}_{s,*,0} &= A^0_{3s+2} \wedge B^0_{3s+2} \wedge A^2_1 \wedge B^2_1 \to A^0_{3s+1} \vee B^0_{3s+1} \vee p \vee A^2_0 \vee B^2_0, \\ \hat{E}_{s,0,0} &= E_{s,0,0}, \end{split}$$

where $s \ge 0, i, j \ge 1$. By Lemma 3.6, we have the following evident lemma.

Lemma 3.7. For all $s, m, n \ge 0$,

$$Int \vdash E_{s,m,n} \leftrightarrow \begin{cases} \hat{E}_{s,i,j}[P_{m-i,n-j}, Q_{m-i,n-j}], & 1 \le i \le m+1, \\ 1 \le j \le n+1; \\ A_{n+1}^2 \land B_{n+1}^2 \to \hat{E}_{s,0,*}[p, A_n^2 \lor B_n^2], & m = 0, n \ge 1; \\ A_{m+1}^1 \land B_{m+1}^1 \to \hat{E}_{s,*,0}[A_m^1 \lor B_m^1, q], & m \ge 1, n = 0; \\ \hat{E}_{s,0,0}, & m = 0, n = 0. \end{cases}$$

Let

$$\varphi(x) = \begin{cases} x - 1, & x \ge 1; \\ 0, & x = 0; \\ 0, & x = *. \end{cases}$$

Now we prove that if the Kripke frame \mathfrak{F} refutes $\hat{E}_{s,i,j}$ then it refutes $\hat{E}_{s,i,j}$ at a point $e_{[s,m,n]}$ for some $m \ge \varphi(i), n \ge \varphi(j)$ such that $\alpha_0 \stackrel{\mathcal{M}}{\Longrightarrow} [s,m,n]$.

Lemma 3.8. If $(\mathfrak{F}, w) \not\models \hat{E}_{s,i,j}$, then $w \leq_R e_{[s,m,n]}$ for some $m \geq \varphi(i)$, $n \geq \varphi(j)$ such that $\alpha_0 \stackrel{\mathcal{M}}{\Longrightarrow} [s, m, n]$.

Proof. Let $\mathfrak{M}' = \langle \mathfrak{F}, \mathfrak{V}' \rangle$ be a Kripke model such that $(\mathfrak{M}', w) \not\models \hat{E}_{s,i,j}$. Since $w \not\models \hat{E}_{s,i,j}$, there is a point $w' \geq_R w$ such that the formulas A^0_{3s+1} and B^0_{3s+1} are refuted at w', and the formulas A^0_{3s+2} and B^0_{3s+2} are true at w'. By the proof of Lemma 3.5, we have that $w \leq_R w' \leq_R e_{[s,m,n]}$ for some $m, n \geq 0$ such that $\alpha_0 \stackrel{\mathcal{M}}{\Longrightarrow} [s,m,n]$. It is clear that m = 0 if i = 0 and n = 0 if j = 0. Hence, in order to prove the lemma it is sufficient to show that $m \geq i - 1, n \geq j - 1$ for some $i \geq 1, j \geq 1$.

If $i \ge 1$, then the formulas F_i^1 , G_i^1 are refuted at w' and the formulas F_{i+1}^1 , G_{i+1}^1 are true at w'. Now we prove that if F_k^1 is refuted at a point f_k^1 and G_k^1 is refuted at a point g_k^1 , then $f_k^1 \le_R c_{k+l-1}^1$ and $g_k^1 \le_R d_{k+l-1}^1$ for some $l \ge 0$ and $\{c, d\} = \{a, b\}$. By induction on $k \ge 1$.

Induction base: k = 1. In this case, there are points $w_f \ge_R f_1^1$ and $w_g \ge_R g_1^1$ such that

- 1. C_1 is refuted at w_f , w_g , then the Kripke frame \mathfrak{F} contains pathes of length 5 from w_f , w_g to maximal points and therefore $w_f \leq_R c_0^{j_1}$ and $w_g \leq_R d_0^{j_2}$ for some $j_1, j_2 \in \{0, 1, 2\}$ and $c, d \in \{a, b\}$;
- 2. C_2 is true at w_f , w_g , therefore $w_f \not\leq_R b_{-1}^0$, $w_g \not\leq_R b_{-1}^0$ by the proof of Lemma 3.5;
- 3. $w_f \in \mathfrak{V}'(q) \setminus \mathfrak{V}'(p)$ and $w_g \in \mathfrak{V}'(p) \setminus \mathfrak{V}'(q)$, therefore w_f, w_g are incomparable points.

Thus, $w_f = c_{i'}^1$ and $w_g = d_{j'}^1$ for some $i', j' \ge 0$ such that |i' - j'| < 2, and $\{c, d\} = \{a, b\}$. **Induction base:** k = 2. In this case, there are points $w_f \ge_R f_2^1$ and $w_g \ge_R g_2^1$ such that

- 1. F_1^1 is refuted at w_f and G_1^1 is refuted at w_g , therefore $w_f \leq_R c_{i'}^1$, $w_g \leq_R d_{j'}^1$;
- 2. F_1^1 is true at w_g and G_1^1 is true at w_f , therefore $w_f \not\leq_R d_{i'}^1$, $w_g \not\leq_R c_{i'}^1$;
- 3. $w_f, w_g \notin \mathfrak{V}'(p) \cup \mathfrak{V}'(q)$, therefore $w_f \neq c_{i'}^1, w_g \neq d_{i'}^1$.

Thus, $w_f = c_{i''}^1$, $w_g = d_{j''}^1$ and $(w_f, d_{j'}^1)$, $(c_{i'}^1, w_g)$ and (w_f, w_g) are pairs of incomparable points. So, we have

$$i' < i'' < j' + 2,$$

 $j' < j'' < i' + 2.$

Since |i'-j'| < 2, it can easily be checked that i' = j' = l and i'' = j'' = l+1 for some $l \ge 0$.

Induction step: k > 2. Let the induction assumption be satisfied for all $2 \le k' < k$, then there are points $w_f \geq_R f_k^1$ and $w_g \geq_R g_k^1$ such that

- 1. F_{k-1}^1, G_{k-2}^1 are refuted at w_f , therefore $w_f \leq_R c_{k+l-2}^1, w_f \leq_R d_{k+l-3}^1;$
- 2. G_{k-1}^1 , F_{k-2}^1 are refuted at w_g , therefore $w_g \leq_R d_{k+l-2}^1$, $w_g \leq_R c_{k+l-3}^1$;
- 3. G_{k-1}^1 is true at w_f and F_{k-1}^1 is true at w_g , therefore $w_f \not\leq_R d_{k+l-2}^1$ and $w_g \not\leq_R c_{k+l-2}^1$.

Thus, $w_f = c_{k+l-1}^1$ and $w_g = d_{k+l-1}^1$. Since F_i^1 , G_i^1 are refuted at w' and the formulas F_{i+1}^1 , G_{i+1}^1 are true at w', we have $w' \leq_R c_{i+l-1}^1$, $w' \leq_R d_{i+l-1}^1$ and $w' \leq_R c_{i+l}^1$, $w' \leq_R d_{i+l}^1$. Therefore, $m = i + l - 1 \geq i - 1$. If $j \geq 1$, then the proof are similar. Hence, $n \geq j - 1$. The lemma is proved.

Next, we define the formula Ax(I) simulating the instruction I of the Minsky machine \mathcal{M} :

1. If I is an instruction of the form $s \mapsto \langle t, 1, 0 \rangle$, then Ax(I) is the following formula

$$\tilde{E}_{t,2,1} \rightarrow \tilde{E}_{s,1,1};$$

2. If I is $s \mapsto \langle t, 0, 1 \rangle$, then Ax(I) is

$$\hat{E}_{t,1,2} \to \hat{E}_{s,1,1};$$

3. If I is $s \mapsto \langle t, -1, 0 \rangle / \langle u, 0, 0 \rangle$, then Ax(I) is

$$(\hat{E}_{t,1,1} \to \hat{E}_{s,2,1}) \land (\hat{E}_{u,0,*} \to \hat{E}_{s,0,*});$$

4. If I is $s \mapsto \langle t, 0, -1 \rangle / \langle u, 0, 0 \rangle$, then Ax(I) is

$$(\hat{E}_{t,1,1} \to \hat{E}_{s,1,2}) \land (\hat{E}_{u,*,0} \to \hat{E}_{s,*,0}),$$

and the formula $Ax(\mathcal{M})$ simulating the behavior of \mathcal{M} itself:

$$Ax(\mathcal{M}) = \bigwedge_{I \in \mathcal{M}} Ax(I).$$

Lemma 3.9. $\mathfrak{F} \models Ax(\mathcal{M})$.

Proof. In order to prove the lemma it is sufficient to show that

$$\mathfrak{F} \models Ax(I)$$

for each instruction I. We need to consider the following 4 cases.

Case 1: I is an instruction of the form $s \mapsto \langle t, 1, 0 \rangle$, i.e.,

$$Ax(I) = \hat{E}_{t,2,1} \to \hat{E}_{s,1,1}.$$

If $(\mathfrak{F}, w) \not\models Ax(I)$, then there is a Kripke model $\mathfrak{M}' = \langle \mathfrak{F}, \mathfrak{V}' \rangle$ such that $(\mathfrak{M}', w) \models \hat{E}_{t,2,1}$ and $(\mathfrak{M}', w) \not\models \hat{E}_{s,1,1}$. By Lemma 3.8, $w \leq_R e_{[s,m,n]}$ for some $m \geq 0$ and $n \geq 0$ such that $\alpha_0 \stackrel{\mathcal{M}}{\Longrightarrow} [s, m, n]$. If we recall the proofs of Lemmas 3.5 and 3.8, we obtain that the following statements hold in \mathfrak{M}'

- 1. A_{3t+1}^0 , B_{3t+1}^0 are refuted at a_{3t+1}^0 , b_{3t+1}^0 and A_{3t+2}^0 , B_{3t+2}^0 are true at them;
- 2. F_2^1 , G_2^1 are refuted at c_{m+1}^1 , d_{m+1}^1 and F_3^1 , G_3^1 are true at them, where $\{c, d\} = \{a, b\}$;
- 3. F_1^2 , G_1^2 are refuted at c_n^2 , d_n^2 and F_2^2 , G_2^2 are true at them, where $\{c, d\} = \{a, b\}$.

Since

$$\langle s, m, n \rangle \xrightarrow{\mathcal{M}} \langle t, m+1, n \rangle,$$

we have that $e_{[t,m+1,n]} \in W$ and $e_{[s,m,n]} \leq_R e_{[t,m+1,n]}$. Hence $\hat{E}_{t,2,1}$ is refuted at w, which contradicts to that $(\mathfrak{M}', w) \models \hat{E}_{t,2,1}$. Therefore, $(\mathfrak{F}, w) \models Ax(I)$.

Case 2: I is an instruction of the form $s \mapsto \langle t, 0, 1 \rangle$. The proof is analogous.

Case 3: I is an instruction of the form $s \mapsto \langle t, -1, 0 \rangle / \langle u, 0, 0 \rangle$, i.e.,

$$(\hat{E}_{t,1,1} \to \hat{E}_{s,2,1}) \land (\hat{E}_{u,0,*} \to \hat{E}_{s,0,*}).$$

Let $(\mathfrak{F}, w) \not\models Ax(I)$. Then there is a Kripke model $\mathfrak{M}' = \langle \mathfrak{F}, \mathfrak{V}' \rangle$ such that

$$(\mathfrak{M}', w) \not\models (\hat{E}_{t,1,1} \to \hat{E}_{s,2,1}), (\hat{E}_{u,0,*} \to \hat{E}_{s,0,*}).$$

It is clear that if $(\mathfrak{M}', w) \not\models \hat{E}_{s,2,1}$, then $(\mathfrak{M}', w) \not\models \hat{E}_{t,1,1}$. Let $(\mathfrak{M}', w) \not\models \hat{E}_{s,0,*}$ for some point $w \in W$, then by Lemma 3.8 $w \leq_R e_{[s,0,n]}$ for some $n \geq 0$ such that $\alpha_0 \stackrel{\mathcal{M}}{\Longrightarrow} [s, 0, n]$. If we recall the proofs of Lemmas 3.5 and 3.8 again, we obtain that

- 1. A_{3u+1}^0 , B_{3u+1}^0 are refuted at a_{3u+1}^0 , b_{3u+1}^0 and A_{3u+2}^0 , B_{3u+2}^0 are true at them;
- 2. A_0^1 , B_0^1 are refuted at a_0^1 , b_0^1 and A_1^1 , B_1^1 are true at them.

Since

$$\langle s, 0, n \rangle \xrightarrow{\mathcal{M}} \langle u, 0, n \rangle,$$

we have that $e_{[u,0,n]} \in W$ and $e_{[s,0,n]} \leq_R e_{[u,0,n]}$. Hence $(\mathfrak{M}', w) \not\models \hat{E}_{u,0,*}$ and therefore $(\mathfrak{F}, w) \models Ax(I)$.

Case 4: *I* is an instruction of the form $s \mapsto \langle t, 0, -1 \rangle / \langle u, 0, 0 \rangle$. The proof is similar. Thus, $\mathfrak{F} \models Ax(I)$ for each instruction $I \in \mathcal{M}$. The lemma is proved.

3.6 Reduction of configuration problem

In this section we formally reduce the configuration problem of the Minsky machine \mathcal{M} to the derivation problem of the superintuitionistic propositional calculus $\mathbf{Int} + Ax(\mathcal{M})$.

Lemma 3.10. Int + $Ax(\mathcal{M}) \vdash E_{t,k,l} \to E_{s_0,m_0,n_0}$ iff $\langle s_0, m_0, n_0 \rangle \stackrel{\mathcal{M}}{\Longrightarrow} \langle t, k, l \rangle$.

Proof. If Int $+ Ax(\mathcal{M}) \vdash E_{t,k,l} \rightarrow E_{s_0,m_0,n_0}$, then

$$\mathfrak{F}\models E_{t,k,l}\to E_{s_0,m_0,n_0}$$

by Lemma 3.9. If we recall that E_{s_0,m_0,n_0} is refuted at $e_{[s_0,m_0,n_0]}$, then we obtain that $E_{t,k,l}$ is also refuted at $e_{[s_0,m_0,n_0]}$. By Lemma 3.5, $e_{[t,k,l]} \in W$ and

$$e_{[s_0,m_0,n_0]} \leq_R e_{[t,k,l]}$$

Therefore, $\langle s_0, m_0, n_0 \rangle \stackrel{\mathcal{M}}{\Longrightarrow} \langle t, k, l \rangle$ by definition of Kripke frame \mathfrak{F} .

Conversely, if $\langle s_0, m_0, n_0 \rangle \xrightarrow{\mathcal{M}} \langle t, k, l \rangle$, then there exists a finite sequence $\langle s_i, m_i, n_i \rangle$, $0 \leq i \leq \mu$, such that $\langle s_\mu, m_\mu, n_\mu \rangle = \langle t, k, l \rangle$ and

$$\langle s_i, m_i, n_i \rangle \xrightarrow{\mathcal{M}} \langle s_{i+1}, m_{i+1}, n_{i+1} \rangle$$

for all $i, 0 \leq i < \mu$. Let $\langle s_{i+1}, m_{i+1}, n_{i+1} \rangle$ be a result of applying of an instruction $I \in \mathcal{M}$. We need to consider the following 4 cases.

Case 1: I is an instruction of the form $s \mapsto \langle t, 1, 0 \rangle$. Then $m_{i+1} = m_i + 1$ and $n_{i+1} = n_i$. By Lemma 3.7, we have

Therefore $\mathbf{Int} + Ax(\mathcal{M}) \vdash E_{s_{i+1}, m_{i+1}, n_{i+1}} \rightarrow E_{s_i, m_i, n_i}$

Case 2: I is an instruction of the form $s \mapsto \langle t, 0, 1 \rangle$. The proof is analogous.

Case 3: I is an instruction of the form $s \mapsto \langle t, -1, 0 \rangle / \langle u, 0, 0 \rangle$. If $m_{i+1} = m_i - 1 \ge 0$ and $n_{i+1} = n_i$. By Lemma 3.7, we have

$$\begin{aligned}
\mathbf{Int} &\vdash E_{s_{i+1},m_{i+1},n_{i+1}} \leftrightarrow \hat{E}_{s_{i+1},1,1}[P_{m_i-2,n_i-1},Q_{m_i-2,n_i-1}], \\
\mathbf{Int} &\vdash E_{s_i,m_i,n_i} \leftrightarrow \hat{E}_{s_i,2,1}[P_{m_i-2,n_i-1},Q_{m_i-2,n_i-1}].
\end{aligned}$$

If $m_{i+1} = m_i = 0$ and $n_{i+1} = n_i$. By Lemma 3.7, we have

$$\mathbf{Int} \vdash E_{s_{i+1},m_{i+1},n_{i+1}} \leftrightarrow \left(A_{n_i+1}^2 \wedge B_{n_i+1}^2 \to \hat{E}_{s_{i+1},0,*}[p, A_{n_i}^2 \vee B_{n_i}^2]\right),$$
$$\mathbf{Int} \vdash E_{s_i,m_i,n_i} \leftrightarrow \left(A_{n_i+1}^2 \wedge B_{n_i+1}^2 \to \hat{E}_{s_i,0,*}[p, A_{n_i}^2 \vee B_{n_i}^2]\right).$$

Therefore $\operatorname{Int} + Ax(\mathcal{M}) \vdash E_{s_{i+1}, m_{i+1}, n_{i+1}} \to E_{s_i, m_i, n_i}.$

Case 4: *I* is an instruction of the form $s \mapsto \langle t, 0, -1 \rangle / \langle u, 0, 0 \rangle$. The proof is similar. Thus, $\mathbf{Int} + Ax(\mathcal{M}) \vdash E_{s_{i+1}, m_{i+1}, n_{i+1}} \to E_{s_i, m_i, n_i}$ for all $i, 0 \leq i < \mu$. The lemma is proved.

Since the configuration problem for the Minsky machine \mathcal{M} and the initial configuration $\langle s_0, m_0, n_0 \rangle$ is undecidable by Theorem 3.2, we have that the derivation problem for the superintuitionistic propositional calculus $\mathbf{Int} + Ax(\mathcal{M})$ is also undecidable. This completes the proof of Theorem 3.1.

4 Conclusion and further research

In this paper, we established that there is an undecidable superintuitionistic propositional calculus using axioms in only 3 variables. Since there are no undecidable superintuitionistic propositional calculi with axioms containing less than 3 variables, therefore a natural and interesting question is there an intuitionistic propositional formula A containing less than 3 variables for which the superintuitionistic propositional calculus Int + A is undecidable. In this respect, we note that every intermediate logic axiomatised by a 1-variable formula has the finite model property [18] and therefore decidable, but there exists an intermediate logic axiomatised by a 2-variable formula, which is Kripke incomplete [14].

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