η-conversions of **IPC** implemented in atomic **F**

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Abstract

It is known that the β-conversions of the full intuitionistic propositional calculus (**IPC**) translate into $\beta\eta$-conversions of the atomic polymorphic calculus $\mathbf{F_{at}}$. Since $\mathbf{F_{at}}$ enjoys the property of strong normalization for $\beta\eta$-conversions, an alternative proof of strong normalization for **IPC** considering β-conversions can be derived. In the present paper we improve the previous result by analyzing the translation of the η-conversions of the latter calculus into a technical variant of the former system (the atomic polymorphic calculus $\mathbf{F}^{\wedge}_{\mathbf{at}}$). In fact, from the strong normalization of $\mathbf{F}^{\wedge}_{\mathbf{at}}$ we can derive the strong normalization of the full intuitionistic propositional calculus considering *all the standard (β and η) conversions*.

Keywords. η-conversions, predicative polymorphism, intuitionistic propositional calculus, strong normalization, natural deduction.

1 Introduction

The *atomic polymorphic calculus* $\mathbf{F_{at}}$ [3, 7][1] is the restriction of Jean-Yves Girard's system **F** [9] to atomic universal instantiations. The restriction occurs only in the derivations (terms) allowed, not in the formulas (types) permitted. The formulas in $\mathbf{F_{at}}$ (as in system **F**) are defined as the smallest class of expressions that includes the atomic formulas (propositional constants and second-order variables) and is closed under implication and second-order universal quantification. In the natural deduction calculus proofs in $\mathbf{F_{at}}$ are built using the following introduction rules:

$$\dfrac{\begin{matrix}[A]\\ \vdots \\ B\end{matrix}}{A \to B}\ \to\mathrm{I} \qquad\qquad \dfrac{\begin{matrix}\vdots \\ A\end{matrix}}{\forall X.A}\ \forall\mathrm{I}$$

[1]The system $\mathbf{F_{at}}$ was first introduced by Fernando Ferreira in [3] under the designation of *atomic* $\mathsf{PSOL}^{\mathsf{i}}$.

where, in the second rule, X does not occur free in any undischarged hypothesis; and the following elimination rules:

$$\frac{\begin{array}{cc}\vdots & \vdots \\ A \to B & A\end{array}}{B}\ \to\mathrm{E} \qquad\qquad \frac{\begin{array}{c}\vdots \\ \forall X.A\end{array}}{A[C/X]}\ \forall\mathrm{E}$$

with C an *atomic* formula, free for X in A. It is the restriction to atomic instantiations in the latter rule that distinguishes $\mathbf{F_{at}}$ from $\mathbf{F}$. The (impredicative) system $\mathbf{F}$ allows, in the $\forall$E rule, the instantiation of X by any (not necessarily atomic) formula of the system.

The introduction of $\mathbf{F_{at}}$ may be a possible answer to Girard's dissatisfaction with the natural deduction rules for $\perp$ and $\vee$. From page 80 of [9]:

"One tends to think that natural deduction should be modified to correct such atrocities [referring to the commuting conversions needed to deal with the *bad* connectives $\perp$, $\vee$ and $\exists$]: if a connector has such bad rules, one ignores it (a very common attitude) or one tries to change the very spirit of natural deduction in order to be able to integrate it harmoniously with the others. It does not seem that the ($\perp$, $\vee$, $\exists$) fragment of the calculus is etched on tablets of stone."

$\mathbf{F_{at}}$ is an alternative to full intuitionistic propositional calculus (**IPC**) in the sense that **IPC** can be translated into $\mathbf{F_{at}}$ via a sound [3, 8] and faithful [6, 5] embedding. Thus, any deduction in **IPC** can be performed into $\mathbf{F_{at}}$ - a predicative system with no *bad* connectives, with no commuting conversions, with a simple strong normalization proof [7, 4] and whose normal proofs enjoy the subformula property [3].

The embedding of **IPC** into $\mathbf{F_{at}}$ relies on the well-known Russell-Prawitz's definition of the connectives $\perp$, $\vee$ and $\wedge$ in terms of $\to$ and $\forall$ and on instantiation overflow. To make this paper reasonably self-contained, in the next section we remember these notions.

Since $\wedge$ is not a *bad*[2] connective we can take it as primitive in $\mathbf{F_{at}}$. So, till the end of the present paper we will work with an atomic polymorphic calculus, we denote by $\mathbf{F_{at}^{\wedge}}$, which has the primitive connectives $\wedge$ (for conjunction), $\to$ (for implication) and $\forall$ (for second-order universal quantification) and, in addition to the introduction and elimination rules previously presented for $\mathbf{F_{at}}$, $\mathbf{F_{at}^{\wedge}}$ has also the following rules for $\wedge$:

$$\frac{\begin{array}{cc}\vdots & \vdots \\ A & B\end{array}}{A \wedge B}\ \wedge\mathrm{I} \qquad\qquad \frac{\begin{array}{c}\vdots \\ A \wedge B\end{array}}{A}\ \wedge\mathrm{E} \qquad\qquad \frac{\begin{array}{c}\vdots \\ A \wedge B\end{array}}{B}\ \wedge\mathrm{E}$$

The *standard conversions* of $\mathbf{F_{at}^{\wedge}}$ are the following β-conversions:

$$\dfrac{\dfrac{\begin{array}{cc}\vdots & \vdots \\ A & B\end{array}}{A \wedge B}}{A} \quad \rightsquigarrow \quad \begin{array}{c}\vdots \\ A\end{array} \qquad\qquad \dfrac{\dfrac{\begin{array}{cc}\vdots & \vdots \\ A & B\end{array}}{A \wedge B}}{B} \quad \rightsquigarrow \quad \begin{array}{c}\vdots \\ B\end{array}$$

[2]"*Bad*" in the previous (Girard's) sense. I.e., as opposed to $\perp$, $\vee$, $\exists$ (see [9], pages 73–74), the natural deduction rules for the elimination of $\wedge$ do not introduce formulas without connection with the formulas being eliminated and $\wedge$ is not responsible for the introduction of commuting conversions in **IPC**.

$$\dfrac{\dfrac{\begin{matrix}[A]\\ \vdots\\ B\end{matrix}}{A \to B} \qquad \begin{matrix}\vdots\\ A\end{matrix}}{B} \quad\rightsquigarrow\quad \begin{matrix}\vdots\\ A\\ \vdots\\ B\end{matrix} \qquad\qquad \dfrac{\dfrac{\begin{matrix}\vdots\\ A\end{matrix}}{\forall X.A}}{A[C/X]} \quad\rightsquigarrow\quad \begin{matrix}\vdots\\ A[C/X]\end{matrix}$$

where C is an *atomic* formula, free for X in A, and the following η-conversions:

$$\dfrac{\dfrac{\begin{matrix}\vdots\\ A \wedge B\end{matrix}}{A} \qquad \dfrac{\begin{matrix}\vdots\\ A \wedge B\end{matrix}}{B}}{A \wedge B} \quad\rightsquigarrow\quad \begin{matrix}\vdots\\ A \wedge B\end{matrix}$$

$$\dfrac{\dfrac{\begin{matrix}\vdots\\ A \to B\end{matrix} \qquad [A]}{B}}{A \to B} \quad\rightsquigarrow\quad \begin{matrix}\vdots\\ A \to B\end{matrix} \qquad\qquad \dfrac{\dfrac{\begin{matrix}\vdots\\ \forall X.A\end{matrix}}{A}}{\forall X.A} \quad\rightsquigarrow\quad \begin{matrix}\vdots\\ \forall X.A\end{matrix}$$

The formulas on the left hand-side of the conversion are called *redexes* and on the right hand-side *contractums*.

Note that since $\mathbf{F}^{\wedge}_{\mathbf{at}}$ has no *bad* connectives there is no need for commuting conversions in the system. The reason why we choose to work within $\mathbf{F}^{\wedge}_{\mathbf{at}}$ instead of $\mathbf{F_{at}}$ will become clear in the last two sections of the paper.

It is an easy exercise to adapt the proof of strong normalization presented in [7] (for $\mathbf{F_{at}}$) to $\mathbf{F}^{\wedge}_{\mathbf{at}}$. We sketch such a proof in the next section.

It was shown in [7] that the β-conversions of **IPC** could be implemented in $\mathbf{F_{at}}$ through $\beta\eta$-conversions and, as an application of strong normalization for $\mathbf{F_{at}}$ considering $\beta\eta$-conversions, the strong normalization for full **IPC** considering *β-conversions* was derived. What about the η-conversions?

In the present paper we show that the η-conversions of **IPC** can be implemented in $\mathbf{F}^{\wedge}_{\mathbf{at}}$ via $\beta\eta$-conversions. As a consequence we are able to improve the previous results: from the strong normalization of $\mathbf{F}^{\wedge}_{\mathbf{at}}$ considering $\beta\eta$-conversions we can derive the strong normalization of full **IPC** considering all the *standard (β and η) conversions*.

The paper is organized as follows: In Section 2 we present the embedding of full **IPC** into $\mathbf{F}^{\wedge}_{\mathbf{at}}$ and the proof of strong normalization for the latter calculus considering $\beta\eta$-conversions. In Section 3 we study the translation of the η-conversions of **IPC** into $\mathbf{F}^{\wedge}_{\mathbf{at}}$ and in Section 4, as an application, we present an alternative proof for the strong normalization of full **IPC** considering the standard (β and η) conversions.

2 Preliminaries

The embedding of full **IPC** into $\mathbf{F}^{\wedge}_{\mathbf{at}}$ uses a well-known translation of the connectives $\perp$ and $\vee$ in terms of $\to$ and $\forall$ due to Bertrand Russell [12] and Dag Prawitz [11]. For every formula A of the full propositional calculus we define its translation A^* into $\mathbf{F}^{\wedge}_{\mathbf{at}}$ inductively as follows:

$(P)^* :\equiv P$, for P a propositional constant

$(\bot)^* :\equiv \forall X.X$

$(A \to B)^* :\equiv A^* \to B^*$

$(A \wedge B)^* :\equiv A^* \wedge B^*$

$(A \vee B)^* :\equiv \forall X\,((A^* \to X) \to ((B^* \to X) \to X))$,

where X is a second-order variable which does not occur in A^* nor in B^*. Note that the Russell-Prawitz translation also allows for the translation of $\wedge$ in terms of $\to$ and $\forall$. In our context we do not need to translate conjunction in such a way because $\wedge$ is a primitive symbol in $\mathbf{F}_{\mathbf{at}}^{\wedge}$.

The previous translation is, in fact, a sound embedding, i.e., denoting by $\vdash_i$ provability in the full intuitionistic propositional calculus and by $\vdash_{\mathbf{F}_{\mathbf{at}}^{\wedge}}$ provability in the atomic polymorphic system $\mathbf{F}_{\mathbf{at}}^{\wedge}$, we have: If $\vdash_i A$ then $\vdash_{\mathbf{F}_{\mathbf{at}}^{\wedge}} A^*$.

The proof can be found in [3, 8][3] and relies in the phenomenon of *instantiation overflow*. Instantiation overflow ensures that from formulas of the form

$\forall X.X$

$\forall X\,((A \to X) \to ((B \to X) \to X))$,

it is possible to deduce in $\mathbf{F}_{\mathbf{at}}^{\wedge}$ (respectively)

F

$(A \to F) \to ((B \to F) \to F)$,

for *any* (not necessarily atomic) formula F. The proof of instantiation overflow is given in [3, 8] and it yields algorithmic methods for obtaining the two kinds of deductions above. For a recent study on instantiation overflow see [1]. Since the (canonical) deductions provided by instantiation overflow are going to be extensively used in sections 3 and 4, we exemplify instantiation overflow with the case of disjunction.

More precisely, by induction on the complexity of F, we show how to deduce in $\mathbf{F}_{\mathbf{at}}^{\wedge}$ the formula $(A \to F) \to ((B \to F) \to F)$, for *arbitrary* F, from $\forall X\,((A \to X) \to ((B \to X) \to X))$. For F atomic there is nothing to argue, it is the application of a single rule: $\forall$E. We just have to analyze the cases in which F is $D_1 \wedge D_2$, $D_1 \to D_2$ and $\forall X D$ admitting (by induction hypothesis) that instantiation overflow is available for D_1, D_2 and D.

For $F :\equiv D_1 \wedge D_2$, we have:

[3] In the present context of $\mathbf{F}_{\mathbf{at}}^{\wedge}$, the proof is even simpler than in the papers cited because the conjunction is primitive in the atomic polymorphic calculus so the translation of the rules $\wedge$I and $\wedge$E becomes trivial.

$$
\begin{prooftree}
\AxiomC{$\forall X((A \to X) \to ((B \to X) \to X))$}
\doubleLine
\RightLabel{I.H.}
\UnaryInfC{$(A \to D_1) \to ((B \to D_1) \to D_1)$}
\AxiomC{$[A \to (D_1 \wedge D_2)]$}
\AxiomC{$[A]$}
\BinaryInfC{$D_1 \wedge D_2$}
\UnaryInfC{D_1}
\UnaryInfC{$A \to D_1$}
\BinaryInfC{$(B \to D_1) \to D_1$}
\AxiomC{$[B \to (D_1 \wedge D_2)]$}
\AxiomC{$[B]$}
\BinaryInfC{$D_1 \wedge D_2$}
\UnaryInfC{D_1}
\UnaryInfC{$B \to D_1$}
\BinaryInfC{D_1}
\AxiomC{$\mathcal{D}$}
\BinaryInfC{$D_1 \wedge D_2$}
\UnaryInfC{$(B \to (D_1 \wedge D_2)) \to (D_1 \wedge D_2)$}
\UnaryInfC{$(A \to (D_1 \wedge D_2)) \to ((B \to (D_1 \wedge D_2)) \to (D_1 \wedge D_2))$}
\end{prooftree}
$$

where $\mathcal{D}$ is the derivation:

$$
\begin{prooftree}
\AxiomC{$\forall X((A \to X) \to ((B \to X) \to X))$}
\doubleLine
\RightLabel{I.H.}
\UnaryInfC{$(A \to D_2) \to ((B \to D_2) \to D_2)$}
\AxiomC{$[A \to (D_1 \wedge D_2)]$}
\AxiomC{$[A]$}
\BinaryInfC{$D_1 \wedge D_2$}
\UnaryInfC{D_2}
\UnaryInfC{$A \to D_2$}
\BinaryInfC{$(B \to D_2) \to D_2$}
\AxiomC{$[B \to (D_1 \wedge D_2)]$}
\AxiomC{$[B]$}
\BinaryInfC{$D_1 \wedge D_2$}
\UnaryInfC{D_2}
\UnaryInfC{$B \to D_2$}
\BinaryInfC{D_2}
\end{prooftree}
$$

For $F :\equiv D_1 \to D_2$, we have:

$$
\begin{prooftree}
\AxiomC{$\forall X((A \to X) \to ((B \to X) \to X))$}
\doubleLine
\RightLabel{I.H.}
\UnaryInfC{$(A \to D_2) \to ((B \to D_2) \to D_2)$}
\AxiomC{$[A \to (D_1 \to D_2)]$}
\AxiomC{$[A]$}
\BinaryInfC{$D_1 \to D_2$}
\AxiomC{$[D_1]$}
\BinaryInfC{D_2}
\UnaryInfC{$A \to D_2$}
\BinaryInfC{$(B \to D_2) \to D_2$}
\AxiomC{$[B \to (D_1 \to D_2)]$}
\AxiomC{$[B]$}
\BinaryInfC{$D_1 \to D_2$}
\AxiomC{$[D_1]$}
\BinaryInfC{D_2}
\UnaryInfC{$B \to D_2$}
\BinaryInfC{D_2}
\UnaryInfC{$D_1 \to D_2$}
\UnaryInfC{$(B \to (D_1 \to D_2)) \to (D_1 \to D_2)$}
\UnaryInfC{$(A \to (D_1 \to D_2)) \to ((B \to (D_1 \to D_2)) \to (D_1 \to D_2))$}
\end{prooftree}
$$

For $F :\equiv \forall XD$ we have:

$$
\begin{prooftree}
\AxiomC{$\forall X((A \to X) \to ((B \to X) \to X))$}
\doubleLine
\RightLabel{I.H.}
\UnaryInfC{$(A \to D) \to ((B \to D) \to D)$}
\AxiomC{$[A \to \forall XD]$}
\AxiomC{$[A]$}
\BinaryInfC{$\forall XD$}
\UnaryInfC{D}
\UnaryInfC{$A \to D$}
\BinaryInfC{$(B \to D) \to D$}
\AxiomC{$[B \to \forall XD]$}
\AxiomC{$[B]$}
\BinaryInfC{$\forall XD$}
\UnaryInfC{D}
\UnaryInfC{$B \to D$}
\BinaryInfC{D}
\UnaryInfC{$\forall XD$}
\UnaryInfC{$(B \to \forall XD) \to \forall XD$}
\UnaryInfC{$(A \to \forall XD) \to ((B \to \forall XD) \to \forall XD)$}
\end{prooftree}
$$

When we refer to the *translation* of a certain derivation of **IPC** into $\mathbf{F}^{\wedge}_{\mathbf{at}}$, we mean the canonical translation (rule-by-rule) provided by the proof of the embedding of **IPC** into $\mathbf{F}^{\wedge}_{\mathbf{at}}$ (see [3, 8]). We exemplify the canonical translation with the elimination rule of disjunction:

$$
\begin{prooftree}
\AxiomC{$\vdots$}
\noLine
\UnaryInfC{$A \vee B$}
\AxiomC{$[A]$}
\noLine
\UnaryInfC{$\vdots$}
\noLine
\UnaryInfC{F}
\AxiomC{$[B]$}
\noLine
\UnaryInfC{$\vdots$}
\noLine
\UnaryInfC{F}
\TrinaryInfC{F}
\end{prooftree}
$$

The translation of the **IPC** rule above into $\mathbf{F}^{\wedge}_{\mathbf{at}}$ is:

$$
\dfrac{\dfrac{\dfrac{\begin{matrix}\vdots\\ \forall X((A^*\to X)\to((B^*\to X)\to X))\end{matrix}}{(A^*\to F^*)\to((B^*\to F^*)\to F^*)} \qquad \dfrac{\begin{matrix}[A^*]\\ \vdots\\ F^*\end{matrix}}{A^*\to F^*}}{(B^*\to F^*)\to F^*} \qquad \dfrac{\begin{matrix}[B^*]\\ \vdots\\ F^*\end{matrix}}{B^*\to F^*}}{F^*}
$$

where the double line hides the instantiation overflow discussed before.

Next we will observe that the strategy to prove strong normalization for $\mathbf{F_{at}}$ presented in [7] also works to prove strong normalization for $\mathbf{F}^{\wedge}_{\mathbf{at}}$.

By the Curry-Howard isomorphism also known as "formulas-as-types paradigm", $\mathbf{F}^{\wedge}_{\mathbf{at}}$ can be presented in the (operational) λ-calculus style. Types in $\mathbf{F}^{\wedge}_{\mathbf{at}}$ are the ones in $\mathbf{F_{at}}$ - see [7] Definition 1, page 261, resulting from the atomic types (propositional constants and type variables) by means of two type-forming operations $\to$ and $\forall$ - with an extra type-forming operation $\wedge$, i.e. if A and B are types then $A \wedge B$ is a type. Terms in $\mathbf{F}^{\wedge}_{\mathbf{at}}$ are defined as the terms in $\mathbf{F_{at}}$ (see [7], Definition 2, page 261-262) adding two clauses:

i) If $t^{A\wedge B}$ is a term of type $A \wedge B$ then $(\pi^1 t)^A$ is a term of type A and $(\pi^2 t)^B$ is a term of type B,

ii) If t^A is a term of type A and s^B is a term of type B then $\langle t, s\rangle^{A\wedge B}$ is a term of type $A \wedge B$.

Note that in $\mathbf{F}^{\wedge}_{\mathbf{at}}$ we have the same conversions we have in $\mathbf{F_{at}}$ plus the conversions for $\wedge$ which, in the λ-calculus style, are the following two β-conversions:

$$
\begin{array}{lcl}
\pi^1\langle t, s\rangle & \rightsquigarrow & t\\
\pi^2\langle t, s\rangle & \rightsquigarrow & s
\end{array}
$$

and the following η-conversion:

$$
\langle \pi^1 t, \pi^2 t\rangle \quad \rightsquigarrow \quad t
$$

Remember that the strategy in [7] to prove that $\mathbf{F_{at}}$ has the strong normalization property (a simple adaptation of Tait's convertibility technique) proceeds as follows: i) we define by induction on the complexity of the types a class Red of terms of $\mathbf{F_{at}}$; ii) we prove that all terms in Red are strongly normalizable considering $\beta\eta$-conversions; iii) we prove that all terms in $\mathbf{F_{at}}$ are in Red.

Remember also that Red was defined in the following way:

For C an atomic type, $t \in \mathsf{Red}_C :\equiv t$ is strongly normalizable.

$t \in \mathsf{Red}_{A\to B} :\equiv$ for all q, if $q \in \mathsf{Red}_A$ then $tq \in \mathsf{Red}_B$.

$t \in \mathsf{Red}_{\forall X.A} :\equiv$ for all atomic types C, $tC \in \mathsf{Red}_{A[C/X]}$.

In the context of $\mathbf{F}^{\wedge}_{\mathbf{at}}$ we only have to add a new clause for conjunction:

$t \in \mathsf{Red}_{A\wedge B} :\equiv \pi^1 t \in \mathsf{Red}_A$ and $\pi^2 t \in \mathsf{Red}_B$.

We say that a term is neutral if it is not of the form $\langle t, s\rangle$ or $\lambda x.t$ or $\Lambda X.t$.[4]

The proof of strong normalization for $\mathbf{F}_{\mathbf{at}}^{\wedge}$, considering $\beta\eta$-conversions, proceeds as in [7]. For the treatment of conjunction see [9] pages 42-46.

3 How do the η-conversions of IPC translate into $\mathbf{F}_{\mathbf{at}}^{\wedge}$?

In (full) intuitionistic propositional calculus we have the following η-conversions:

$$\dfrac{\dfrac{\begin{array}{c}\vdots\\ A\wedge B\end{array}}{A} \qquad \dfrac{\begin{array}{c}\vdots\\ A\wedge B\end{array}}{B}}{A\wedge B} \quad\rightsquigarrow\quad \begin{array}{c}\vdots\\ A\wedge B\end{array} \qquad\qquad \dfrac{\dfrac{\begin{array}{c}\vdots\\ A\to B\end{array} \qquad [A]}{B}}{A\to B} \quad\rightsquigarrow\quad \begin{array}{c}\vdots\\ A\to B\end{array}$$

$$\dfrac{\begin{array}{c}\vdots\\ A\vee B\end{array} \qquad \dfrac{[A]}{A\vee B} \qquad \dfrac{[B]}{A\vee B}}{A\vee B} \quad\rightsquigarrow\quad \begin{array}{c}\vdots\\ A\vee B\end{array}$$

Proposition 1. *Consider an η-conversion of (full)* **IPC***. The canonical translation of its* redex *into $\mathbf{F}_{\mathbf{at}}^{\wedge}$ reduces, by means of a finite number (at least one) of $\beta\eta$-conversions, into the canonical translation of its* contractum *into $\mathbf{F}_{\mathbf{at}}^{\wedge}$.*

Proof. The case of the η-conversions for $\wedge$ and $\to$ is trivial. Let us study the η-conversion for $\vee$. In what follows, for ease of notation, we ignore the translations of A and B.

The translation of the *redex* into $\mathbf{F}_{\mathbf{at}}^{\wedge}$, we denote by derivation $\mathcal{D}$, has the form:

$$\dfrac{\dfrac{\begin{array}{c}\vdots\\ \forall X((A\to X)\to((B\to X)\to X)) \\ \hline\hline (A\to(A\vee B))\to((B\to(A\vee B))\to(A\vee B))\end{array} \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[A] \qquad [A\to X]}{X}}{(B\to X)\to X}}{(A\to X)\to((B\to X)\to X)}}{A\vee B}}{A\to(A\vee B)}}{(B\to(A\vee B))\to(A\vee B)} \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[B] \qquad [B\to X]}{X}}{(B\to X)\to X}}{(A\to X)\to((B\to X)\to X)}}{A\vee B}}{B\to(A\vee B)}}{A\vee B}$$

where the double line hides the proof in $\mathbf{F}_{\mathbf{at}}^{\wedge}$ that exists by instantiation overflow, and for reasons of space, we write $A\vee B$ in some points of the derivation to abbreviate $\forall X\ ((A\to X)\to((B\to X)\to X))$. This abbreviation for economy of space will also be used in other parts of this proof. Note that no confusion arises from this abuse of notation since all deductions are in the context of $\mathbf{F}_{\mathbf{at}}^{\wedge}$ where there is no disjunction symbol $\vee$.

We want to prove that from the derivation $\mathcal{D}$ above, applying standard (β and η) conversions of $\mathbf{F}_{\mathbf{at}}^{\wedge}$, we obtain the derivation

[4] $\Lambda X.t$ denotes the universal abstraction: if t^A is a term of type A and the type variable X does not occur free in the type of any free assumption variable of t^A, then $(\Lambda X.t^A)^{\forall X.A}$ is a term of type $\forall X.A$ (see [7], Definition 2).

$$\begin{array}{c}\vdots\\ \forall X((A\to X)\to((B\to X)\to X))\end{array}$$

i.e., the portion of the derivation $\mathcal{D}$ above the double line.

At first is seems that there are no *redexes* where we can apply β or η conversions of $\mathbf{F}^{\wedge}_{\mathbf{at}}$ but indeed there are. They become visible as we start disclosing the portion of the proof hidden in the double line. Abbreviating by $\phi(X)$ the formula $(A\to X)\to((B\to X)\to X)$ (i.e. $A\lor B :\equiv \forall X\,\phi(X)$), and revealing part of the instantiation overflow, what we have above $(A\to(A\lor B))\to((B\to(A\lor B))\to(A\lor B))$ is

$$\begin{array}{c} \begin{array}{c} \begin{array}{c}\vdots\\ \forall X((A\to X)\to((B\to X)\to X)) \\ \hline\hline (A\to\phi(X))\to((B\to\phi(X))\to\phi(X))\end{array} \quad \begin{array}{c}[A\to\forall X\phi(X)]\quad[A] \\ \hline \forall X\phi(X) \\ \hline \phi(X) \\ \hline A\to\phi(X)\end{array} \\ \hline (B\to\phi(X))\to\phi(X)\end{array} \quad \begin{array}{c}[B\to\forall X\phi(X)]\quad[B] \\ \hline \forall X\phi(X) \\ \hline \phi(X) \\ \hline B\to\phi(X)\end{array} \\ \hline \phi(X) \\ \hline \forall X\phi(X) \\ \hdashline A\lor B \\ \hline (B\to(A\lor B))\to(A\lor B) \\ \hline (A\to(A\lor B))\to((B\to(A\lor B))\to(A\lor B))\end{array}$$

where the dashed line means syntactically equal. Note that the last rule above is the introduction of an implication which is going to be (see derivation $\mathcal{D}$) immediately followed by the elimination of that implication. Thus, applying a β-conversion we obtain:

$$\begin{array}{c} \begin{array}{c} \begin{array}{c} \begin{array}{c}\vdots\\ \forall X((A\to X)\to((B\to X)\to X)) \\ \hline\hline (A\to\phi(X))\to((B\to\phi(X))\to\phi(X))\end{array} \quad \begin{array}{c} \begin{array}{c}[A]\quad[A\to X]\\ \hline X \\ \hline (B\to X)\to X \\ \hline (A\to X)\to((B\to X)\to X) \\ \hline A\lor B \\ \hline A\to(A\lor B)\end{array} \quad [A] \\ \hline A\lor B \\ \hline \phi(X) \\ \hline A\to\phi(X)\end{array} \\ \hline (B\to\phi(X))\to\phi(X)\end{array} \quad \begin{array}{c}[B\to\forall X\phi(X)]\quad[B] \\ \hline \forall X\phi(X) \\ \hline \phi(X) \\ \hline B\to\phi(X)\end{array} \\ \hline \phi(X) \\ \hline A\lor B \\ \hline (B\to(A\lor B))\to(A\lor B)\end{array} \quad \begin{array}{c}[B]\quad[B\to X]\\ \hline X \\ \hline (B\to X)\to X \\ \hline (A\to X)\to((B\to X)\to X) \\ \hline A\lor B \\ \hline B\to(A\lor B)\end{array} \\ \hline A\lor B\end{array}$$

With another β-conversion we get:

$$\begin{array}{c} \begin{array}{c} \begin{array}{c}\vdots\\ \forall X((A\to X)\to((B\to X)\to X)) \\ \hline\hline (A\to\phi(X))\to((B\to\phi(X))\to\phi(X))\end{array} \quad \begin{array}{c} \begin{array}{c}[A]\quad[A\to X]\\ \hline X \\ \hline (B\to X)\to X \\ \hline (A\to X)\to((B\to X)\to X) \\ \hline A\lor B \\ \hline A\to(A\lor B)\end{array} \quad [A] \\ \hline A\lor B \\ \hline \phi(X) \\ \hline A\to\phi(X)\end{array} \\ \hline (B\to\phi(X))\to\phi(X)\end{array} \quad \begin{array}{c} \begin{array}{c}[B]\quad[B\to X]\\ \hline X \\ \hline (B\to X)\to X \\ \hline (A\to X)\to((B\to X)\to X) \\ \hline A\lor B \\ \hline B\to(A\lor B)\end{array} \quad [B] \\ \hline A\lor B \\ \hline \phi(X) \\ \hline B\to\phi(X)\end{array} \\ \hline \phi(X) \\ \hline A\lor B\end{array}$$

With two β-conversions we obtain:

$$
\begin{prooftree}
\AxiomC{$\vdots$}
\noLine
\UnaryInfC{$\forall X((A \to X) \to ((B \to X) \to X))$}
\doubleLine
\UnaryInfC{$(A \to \phi(X)) \to ((B \to \phi(X)) \to \phi(X))$}
\AxiomC{$[A]$}
\AxiomC{$[A \to X]$}
\BinaryInfC{X}
\UnaryInfC{$(B \to X) \to X$}
\UnaryInfC{$(A \to X) \to ((B \to X) \to X)$}
\UnaryInfC{$A \vee B$}
\UnaryInfC{$\phi(X)$}
\UnaryInfC{$A \to \phi(X)$}
\BinaryInfC{$(B \to \phi(X)) \to \phi(X)$}
\AxiomC{$[B]$}
\AxiomC{$[B \to X]$}
\BinaryInfC{X}
\UnaryInfC{$(B \to X) \to X$}
\UnaryInfC{$(A \to X) \to ((B \to X) \to X)$}
\UnaryInfC{$A \vee B$}
\UnaryInfC{$\phi(X)$}
\UnaryInfC{$B \to \phi(X)$}
\BinaryInfC{$\phi(X)$}
\UnaryInfC{$A \vee B$}
\end{prooftree}
$$

Since $\phi(X) :\equiv (A \to X) \to ((B \to X) \to X)$, applying two β-conversions we obtain:

$$
\begin{prooftree}
\AxiomC{$\vdots$}
\noLine
\UnaryInfC{$\forall X((A \to X) \to ((B \to X) \to X))$}
\doubleLine
\UnaryInfC{$(A \to \phi(X)) \to ((B \to \phi(X)) \to \phi(X))$}
\AxiomC{$[A]$}
\AxiomC{$[A \to X]$}
\BinaryInfC{X}
\UnaryInfC{$(B \to X) \to X$}
\UnaryInfC{$(A \to X) \to ((B \to X) \to X)$}
\noLine
\UnaryInfC{$A \to \phi(X)$}
\BinaryInfC{$(B \to \phi(X)) \to \phi(X)$}
\AxiomC{$[B]$}
\AxiomC{$[B \to X]$}
\BinaryInfC{X}
\UnaryInfC{$(B \to X) \to X$}
\UnaryInfC{$(A \to X) \to ((B \to X) \to X)$}
\noLine
\UnaryInfC{$B \to \phi(X)$}
\BinaryInfC{$\phi(X)$}
\UnaryInfC{$A \vee B$}
\end{prooftree}
$$

To reveal other redexes and proceed with the reduction process, we need to disclose a bit more the portion of the proof hidden in the double line. Let $\psi(X)$ abbreviate $(B \to X) \to X$, i.e., $\phi(X) \equiv (A \to X) \to \psi(X)$. The derivation above - denoting by $\mathcal{D}_A$ the portion of the proof above $A \to \phi(X)$ and by $\mathcal{D}_B$ the portion above $B \to \phi(X)$ - disclosing part of the proof has the form:

$$
\begin{prooftree}
\AxiomC{$\vdots$}
\noLine
\UnaryInfC{$\forall X((A \to X) \to ((B \to X) \to X))$}
\doubleLine
\UnaryInfC{$(A \to \psi(X)) \to ((B \to \psi(X)) \to \psi(X))$}
\AxiomC{$[A \to \phi(X)]$}
\AxiomC{$[A]$}
\BinaryInfC{$\phi(X)$}
\AxiomC{$[A \to X]$}
\BinaryInfC{$\psi(X)$}
\UnaryInfC{$A \to \psi(X)$}
\BinaryInfC{$(B \to \psi(X)) \to \psi(X)$}
\AxiomC{$[B \to \phi(X)]$}
\AxiomC{$[B]$}
\BinaryInfC{$\phi(X)$}
\AxiomC{$[A \to X]$}
\BinaryInfC{$\psi(X)$}
\UnaryInfC{$B \to \psi(X)$}
\BinaryInfC{$\psi(X)$}
\UnaryInfC{$(A \to X) \to \psi(X)$}
\dashedLine
\UnaryInfC{$\phi(X)$}
\UnaryInfC{$(B \to \phi(X)) \to \phi(X)$}
\UnaryInfC{$(A \to \phi(X)) \to ((B \to \phi(X)) \to \phi(X))$}
\AxiomC{$\mathcal{D}_A$}
\BinaryInfC{$(B \to \phi(X)) \to \phi(X)$}
\AxiomC{$\mathcal{D}_B$}
\BinaryInfC{$\phi(X)$}
\UnaryInfC{$A \vee B$}
\end{prooftree}
$$

Applying one β-conversion and disclosing $\mathcal{D}_A$ we get:

$$
\AxiomC{$\vdots$}
\noLine
\UnaryInfC{$\forall X((A\to X)\to((B\to X)\to X))$}
\doubleLine
\UnaryInfC{$(A\to\psi(X))\to((B\to\psi(X))\to\psi(X))$}
\AxiomC{$[A]$}
\AxiomC{$[A\to X]$}
\BinaryInfC{X}
\UnaryInfC{$(B\to X)\to X$}
\UnaryInfC{$\phi(X)$}
\UnaryInfC{$A\to\phi(X)$}
\AxiomC{$[A]$}
\BinaryInfC{$\phi(X)$}
\AxiomC{$[A\to X]$}
\BinaryInfC{$\psi(X)$}
\UnaryInfC{$A\to\psi(X)$}
\BinaryInfC{$(B\to\psi(X))\to\psi(X)$}
\AxiomC{$[B\to\phi(X)]$}
\AxiomC{$[B]$}
\BinaryInfC{$\phi(X)$}
\AxiomC{$[A\to X]$}
\BinaryInfC{$\psi(X)$}
\UnaryInfC{$B\to\psi(X)$}
\BinaryInfC{$\psi(X)$}
\UnaryInfC{$\phi(X)$}
\UnaryInfC{$(B\to\phi(X))\to\phi(X)$}
\AxiomC{$\mathcal{D}_B$}
\BinaryInfC{$\phi(X)$}
\UnaryInfC{$A\vee B$}
\DisplayProof
$$

With two β-conversions and disclosing $\mathcal{D}_B$ we obtain

$$
\AxiomC{$\vdots$}
\noLine
\UnaryInfC{$\forall X((A\to X)\to((B\to X)\to X))$}
\doubleLine
\UnaryInfC{$(A\to\psi(X))\to((B\to\psi(X))\to\psi(X))$}
\AxiomC{$[A]$}
\AxiomC{$[A\to X]$}
\BinaryInfC{X}
\UnaryInfC{$(B\to X)\to X$}
\UnaryInfC{$\phi(X)$}
\AxiomC{$[A\to X]$}
\BinaryInfC{$\psi(X)$}
\UnaryInfC{$A\to\psi(X)$}
\BinaryInfC{$(B\to\psi(X))\to\psi(X)$}
\AxiomC{$[B]$}
\AxiomC{$[B\to X]$}
\BinaryInfC{X}
\UnaryInfC{$(B\to X)\to X$}
\UnaryInfC{$\phi(X)$}
\UnaryInfC{$B\to\phi(X)$}
\AxiomC{$[B]$}
\BinaryInfC{$\phi(X)$}
\AxiomC{$[A\to X]$}
\BinaryInfC{$\psi(X)$}
\UnaryInfC{$B\to\psi(X)$}
\BinaryInfC{$\psi(X)$}
\UnaryInfC{$\phi(X)$}
\UnaryInfC{$A\vee B$}
\DisplayProof
$$

With two β-conversions we have:

$$
\AxiomC{$\vdots$}
\noLine
\UnaryInfC{$\forall X((A\to X)\to((B\to X)\to X))$}
\doubleLine
\UnaryInfC{$(A\to\psi(X))\to((B\to\psi(X))\to\psi(X))$}
\AxiomC{$[A]$}
\AxiomC{$[A\to X]$}
\BinaryInfC{X}
\UnaryInfC{$\psi(X)$}
\UnaryInfC{$A\to\psi(X)$}
\BinaryInfC{$(B\to\psi(X))\to\psi(X)$}
\AxiomC{$[B]$}
\AxiomC{$[B\to X]$}
\BinaryInfC{X}
\UnaryInfC{$(B\to X)\to X$}
\UnaryInfC{$\phi(X)$}
\AxiomC{$[A\to X]$}
\BinaryInfC{$\psi(X)$}
\UnaryInfC{$B\to\psi(X)$}
\BinaryInfC{$\psi(X)$}
\UnaryInfC{$\phi(X)$}
\UnaryInfC{$A\vee B$}
\DisplayProof
$$

With one more β-conversion we obtain:

$$
\AxiomC{$\vdots$}
\noLine
\UnaryInfC{$\forall X((A\to X)\to((B\to X)\to X))$}
\doubleLine
\UnaryInfC{$(A\to\psi(X))\to((B\to\psi(X))\to\psi(X))$}
\AxiomC{$[A]$}
\AxiomC{$[A\to X]$}
\BinaryInfC{X}
\UnaryInfC{$\psi(X)$}
\UnaryInfC{$A\to\psi(X)$}
\BinaryInfC{$(B\to\psi(X))\to\psi(X)$}
\AxiomC{$[B]$}
\AxiomC{$[B\to X]$}
\BinaryInfC{X}
\UnaryInfC{$\psi(X)$}
\UnaryInfC{$B\to\psi(X)$}
\BinaryInfC{$\psi(X)$}
\UnaryInfC{$\phi(X)$}
\UnaryInfC{$A\vee B$}
\DisplayProof
$$

Revealing completely the instantiation overflow hidden in the double line, the previous derivation is in fact:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\begin{array}{c}\vdots\\ \forall X((A\to X)\to((B\to X)\to X))\end{array}}{(A\to X)\to((B\to X)\to X)}\quad \dfrac{\dfrac{\dfrac{[A\to\psi(X)]\quad [A]}{\psi(X)}\quad [B\to X]}{X}}{A\to X}}{(B\to X)\to X}\quad \dfrac{\dfrac{\dfrac{[B\to\psi(X)]\quad [B]}{\psi(X)}\quad [B\to X]}{X}}{B\to X}}{X}}{\psi(X)}}{(B\to\psi(X))\to\psi(X)}}{(A\to\psi(X))\to((B\to\psi(X))\to\psi(X))}\quad \mathcal{P}_A}{(B\to\psi(X))\to\psi(X)}\quad \mathcal{P}_B}{\dfrac{\dfrac{\psi(X)}{\phi(X)}}{A\vee B}}$$

where $\mathcal{P}_A$ is the derivation $\dfrac{\dfrac{\dfrac{[A]\quad [A\to X]}{X}}{\psi(X)}}{A\to\psi(X)}$ and $\mathcal{P}_B$ the homologous derivation in B.

Applying one β-conversion and disclosing $\mathcal{P}_A$ we obtain:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\begin{array}{c}\vdots\\ \forall X((A\to X)\to((B\to X)\to X))\end{array}}{(A\to X)\to((B\to X)\to X)}\quad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[A]\quad [A\to X]}{X}}{\psi(X)}}{A\to\psi(X)}\quad [A]}{\psi(X)}\quad [B\to X]}{X}}{A\to X}}{(B\to X)\to X}\quad \dfrac{\dfrac{\dfrac{[B\to\psi(X)]\quad [B]}{\psi(X)}\quad [B\to X]}{X}}{B\to X}}{\dfrac{\dfrac{X}{\psi(X)}}{(B\to\psi(X))\to\psi(X)}}\quad \mathcal{P}_B}{\dfrac{\dfrac{\psi(X)}{\phi(X)}}{A\vee B}}$$

With two β-conversions and disclosing $\mathcal{P}_B$ we have:

$$\dfrac{\dfrac{\dfrac{\dfrac{\begin{array}{c}\vdots\\ \forall X((A\to X)\to((B\to X)\to X))\end{array}}{(A\to X)\to((B\to X)\to X)}\quad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[A]\quad [A\to X]}{X}}{\psi(X)}\quad [B\to X]}{X}}{A\to X}}{(B\to X)\to X}\quad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[B]\quad [B\to X]}{X}}{\psi(X)}}{B\to\psi(X)}\quad [B]}{\psi(X)}\quad [B\to X]}{X}}{B\to X}}{X}}{\dfrac{\dfrac{\psi(X)}{\phi(X)}}{A\vee B}}$$

With two more β-conversions we have:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\begin{array}{c}\vdots\\ \forall X((A \to X) \to ((B \to X) \to X))\end{array}}{(A \to X) \to ((B \to X) \to X)} \qquad \dfrac{\dfrac{[A] \qquad [A \to X]}{X}}{A \to X}}{(B \to X) \to X} \qquad \dfrac{\dfrac{\dfrac{\dfrac{[B] \qquad [B \to X]}{X}}{\psi(X)} \qquad [B \to X]}{X}}{B \to X}}{X}}{\psi(X)}}{\phi(X)}}{A \vee B}$$

Applying one η-conversion and one β-conversion we obtain:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\begin{array}{c}\vdots\\ \forall X((A \to X) \to ((B \to X) \to X))\end{array}}{(A \to X) \to ((B \to X) \to X)} \qquad [A \to X]}{(B \to X) \to X} \qquad \dfrac{\dfrac{[B] \qquad [B \to X]}{X}}{B \to X}}{X}}{\psi(X)}}{\phi(X)}}{A \vee B}$$

With another η-conversion (and without making use of abbreviations) we have:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\begin{array}{c}\vdots\\ \forall X((A \to X) \to ((B \to X) \to X))\end{array}}{(A \to X) \to ((B \to X) \to X)} \qquad [A \to X]}{(B \to X) \to X} \qquad [B \to X]}{X}}{(B \to X) \to X}}{(A \to X) \to ((B \to X) \to X)}}{\forall X((A \to X) \to ((B \to X) \to X))}$$

Thus, with three η-conversion we get $\begin{array}{c}\vdots\\ \forall X((A \to X) \to ((B \to X) \to X))\end{array}$

□

4 Alternative strong normalization proof for IPC considering $\beta\eta$-conversions

In [7] it was proved that the β-conversions of **IPC** translate into $\beta\eta$-conversions of $\mathbf{F_{at}}$. In this section we start by arguing that the result remains valid for $\mathbf{F_{at}^{\wedge}}$.

The β-conversions of **IPC** are the ones for conjunction and implication (see pages 2 and 3 in the context of $\mathbf{F_{at}^{\wedge}}$) plus the following ones for disjunction:

$$\dfrac{\dfrac{\begin{array}{c}\vdots\\ A\end{array}}{A \vee B} \qquad \begin{array}{c}[A]\\ \vdots\\ F\end{array} \qquad \begin{array}{c}[B]\\ \vdots\\ F\end{array}}{F} \quad \rightsquigarrow \quad \begin{array}{c}\vdots\\ A\\ \vdots\\ F\end{array} \qquad\qquad \dfrac{\dfrac{\begin{array}{c}\vdots\\ B\end{array}}{A \vee B} \qquad \begin{array}{c}[A]\\ \vdots\\ F\end{array} \qquad \begin{array}{c}[B]\\ \vdots\\ F\end{array}}{F} \quad \rightsquigarrow \quad \begin{array}{c}\vdots\\ B\\ \vdots\\ F\end{array}$$

Proposition 2. *Consider a β-conversion of (full)* **IPC**. *The canonical translation of its* redex *into* $\mathbf{F}^{\wedge}_{\mathbf{at}}$ *reduces, by means of a finite number (at least one) of $\beta\eta$-conversions, into the canonical translation of its* contractum *into* $\mathbf{F}^{\wedge}_{\mathbf{at}}$.

Proof. The only case non trivial is disjunction. It is possible to prove that the derivation

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\begin{array}{c}\vdots\\ A\end{array}\quad [A\to X]}{X}}{(B\to X)\to X}}{(A\to X)\to((B\to X)\to X)}}{\forall X((A\to X)\to((B\to X)\to X))}}{(A\to F)\to((B\to F)\to F)}\qquad \dfrac{\begin{array}{c}[A]\\ \vdots\\ F\end{array}}{A\to F}}{(B\to F)\to F}\qquad \dfrac{\begin{array}{c}[B]\\ \vdots\\ F\end{array}}{B\to F}}{F}$$

reduces in $\mathbf{F}^{\wedge}_{\mathbf{at}}$ to $\begin{array}{c}\vdots\\ A\\ \vdots\\ F\end{array}$ by induction on the complexity of the formula F exactly as in [7], Lemma 4, pp. 268-271 (for $\mathbf{F}_{\mathbf{at}}$). In the present context of $\mathbf{F}^{\wedge}_{\mathbf{at}}$, we only need to analyze a new case ($F :\equiv D_1 \wedge D_2$). Take the derivation

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\begin{array}{c}\vdots\\ A\end{array}\quad [A\to X]}{X}}{(B\to X)\to X}}{(A\to X)\to((B\to X)\to X)}}{\forall X((A\to X)\to((B\to X)\to X))}}{(A\to(D_1\wedge D_2))\to((B\to(D_1\wedge D_2))\to(D_1\wedge D_2))}\qquad \dfrac{\begin{array}{c}[A]\\ \vdots\\ D_1\wedge D_2\end{array}}{A\to(D_1\wedge D_2)}}{(B\to(D_1\wedge D_2))\to(D_1\wedge D_2)}\qquad \dfrac{\begin{array}{c}[B]\\ \vdots\\ D_1\wedge D_2\end{array}}{B\to(D_1\wedge D_2)}}{D_1\wedge D_2}$$

Disclosing a bit the double line we have:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\begin{array}{c}\vdots\\ A\end{array}\quad [A\to X]}{X}}{(B\to X)\to X}}{(A\to X)\to((B\to X)\to X)}}{\forall X((A\to X)\to((B\to X)\to X))}}{(A\to D_1)\to((B\to D_1)\to D_1)}\qquad \dfrac{\dfrac{\dfrac{[A\to(D_1\wedge D_2)]\quad [A]}{D_1\wedge D_2}}{D_1}}{A\to D_1}}{(B\to D_1)\to D_1}\qquad \dfrac{\dfrac{\dfrac{[B\to(D_1\wedge D_2)]\quad [B]}{D_1\wedge D_2}}{D_1}}{B\to D_1}}{D_1}\qquad \mathcal{D}}{D_1\wedge D_2}}{(B\to(D_1\wedge D_2))\to(D_1\wedge D_2)}}{(A\to(D_1\wedge D_2))\to((B\to(D_1\wedge D_2))\to(D_1\wedge D_2))}$$

above the formula $(A \to (D_1 \wedge D_2)) \to ((B \to (D_1 \wedge D_2)) \to (D_1 \wedge D_2))$, where $\mathcal{D}$ is the derivation:

$$
\begin{prooftree}
\AxiomC{$\vdots$}
\noLine
\UnaryInfC{A}
\AxiomC{$[A \to X]$}
\BinaryInfC{X}
\UnaryInfC{$(B \to X) \to X$}
\UnaryInfC{$(A \to X) \to ((B \to X) \to X)$}
\UnaryInfC{$\forall X((A \to X) \to ((B \to X) \to X))$}
\doubleLine
\UnaryInfC{$(A \to D_2) \to ((B \to D_2) \to D_2)$}
\AxiomC{$[A \to (D_1 \wedge D_2)]$}
\AxiomC{$[A]$}
\BinaryInfC{$D_1 \wedge D_2$}
\UnaryInfC{D_2}
\UnaryInfC{$A \to D_2$}
\BinaryInfC{$(B \to D_2) \to D_2$}
\AxiomC{$[B \to (D_1 \wedge D_2)]$}
\AxiomC{$[B]$}
\BinaryInfC{$D_1 \wedge D_2$}
\UnaryInfC{D_2}
\UnaryInfC{$B \to D_2$}
\BinaryInfC{D_2}
\end{prooftree}
$$

Applying the induction hypothesis twice, the derivation reduces to

$$
\begin{prooftree}
\AxiomC{$\vdots$}
\noLine
\UnaryInfC{A}
\AxiomC{$[A \to (D_1 \wedge D_2)]$}
\BinaryInfC{$D_1 \wedge D_2$}
\UnaryInfC{D_1}
\AxiomC{$\vdots$}
\noLine
\UnaryInfC{A}
\AxiomC{$[A \to (D_1 \wedge D_2)]$}
\BinaryInfC{$D_1 \wedge D_2$}
\UnaryInfC{D_2}
\BinaryInfC{$D_1 \wedge D_2$}
\UnaryInfC{$(B \to (D_1 \wedge D_2)) \to (D_1 \wedge D_2)$}
\UnaryInfC{$(A \to (D_1 \wedge D_2)) \to ((B \to (D_1 \wedge D_2)) \to (D_1 \wedge D_2))$}
\AxiomC{$[A]$}
\noLine
\UnaryInfC{$\vdots$}
\noLine
\UnaryInfC{$D_1 \wedge D_2$}
\UnaryInfC{$A \to (D_1 \wedge D_2)$}
\BinaryInfC{$(B \to (D_1 \wedge D_2)) \to (D_1 \wedge D_2)$}
\AxiomC{$[B]$}
\noLine
\UnaryInfC{$\vdots$}
\noLine
\UnaryInfC{$D_1 \wedge D_2$}
\UnaryInfC{$B \to (D_1 \wedge D_2)$}
\BinaryInfC{$D_1 \wedge D_2$}
\end{prooftree}
$$

by means of $\beta\eta$-conversions. Note that we have only changed the portion of derivation above D_1 and D_2.

Applying one η and one β-conversions we obtain

$$
\begin{prooftree}
\AxiomC{$\vdots$}
\noLine
\UnaryInfC{A}
\AxiomC{$[A]$}
\noLine
\UnaryInfC{$\vdots$}
\noLine
\UnaryInfC{$D_1 \wedge D_2$}
\UnaryInfC{$A \to (D_1 \wedge D_2)$}
\BinaryInfC{$D_1 \wedge D_2$}
\UnaryInfC{$(B \to (D_1 \wedge D_2)) \to (D_1 \wedge D_2)$}
\AxiomC{$[B]$}
\noLine
\UnaryInfC{$\vdots$}
\noLine
\UnaryInfC{$D_1 \wedge D_2$}
\UnaryInfC{$B \to (D_1 \wedge D_2)$}
\BinaryInfC{$D_1 \wedge D_2$}
\end{prooftree}
$$

Applying a β-conversion we get

$$
\begin{prooftree}
\AxiomC{$\vdots$}
\noLine
\UnaryInfC{A}
\AxiomC{$[A]$}
\noLine
\UnaryInfC{$\vdots$}
\noLine
\UnaryInfC{$D_1 \wedge D_2$}
\UnaryInfC{$A \to (D_1 \wedge D_2)$}
\BinaryInfC{$D_1 \wedge D_2$}
\end{prooftree}
$$

and, with one more β-conversion we obtain

$$
\begin{array}{c}
\vdots \\
A \\
\vdots \\
D_1 \wedge D_2
\end{array}
$$

□

Finally we are able to present the alternative proof of strong normalization for (full) **IPC** considering $\beta\eta$-conversions.

Theorem 1. *The intuitionistic natural deduction calculus of $\bot$, $\wedge$, $\vee$, $\to$ with the standard (β and η) conversions is strongly normalizable.*

Proof. From propositions 2 and 1 we know that the standard (β and η) conversions of **IPC** translate into a finite (at least one) number of $\beta\eta$-conversions of $\mathbf{F}_{\mathbf{at}}^{\wedge}$. Suppose, in order to obtain a contradiction, that **IPC** is not strongly normalizable for standard conversions. Then, there is a derivation $\mathcal{P}$ in **IPC** and an infinite path of reductions (successive application of $\beta\eta$-conversions) starting from $\mathcal{P}$. Thus, applying the propositions above, the translation of $\mathcal{P}$ into $\mathbf{F}_{\mathbf{at}}^{\wedge}$ also has an infinite path of ($\beta\eta$) reductions. That contradicts the fact that $\mathbf{F}_{\mathbf{at}}^{\wedge}$ is strongly normalizable considering $\beta\eta$-conversions (see the end of Section 2). $\square$

Let us make some final remarks.

1) Note that the study of the η-conversion for disjunction in Section 3 could have been carried out in $\mathbf{F_{at}}$ instead of $\mathbf{F}_{\mathbf{at}}^{\wedge}$. I.e., the η-conversions of **IPC** for disjunction translate into $\beta\eta$-conversions of $\mathbf{F_{at}}$ (no standard conversions for $\wedge$ were needed). Since from [7] we also know that β-conversions of **IPC** translate into $\beta\eta$-conversions of $\mathbf{F_{at}}$, we could ask if standard conversions of **IPC** translate into $\beta\eta$-conversions of $\mathbf{F_{at}}$ and so the last result of the paper - the alternative proof of strong normalization for **IPC** considering standard conversions - could have been obtained using $\mathbf{F_{at}}$ instead of $\mathbf{F}_{\mathbf{at}}^{\wedge}$. The answer is no. When considering the translation into $\mathbf{F_{at}}$ of the η-conversion for conjunction - translating as usual $A \wedge B$ as $\forall X\,((A^* \to (B^* \to X)) \to X)$ - a very simple example with A and B propositional constants is enough to convince ourselves that the canonical translation of the *redex* does not reduce (using $\beta\eta$-conversions of $\mathbf{F_{at}}$) into the canonical translation of the *contractum*.

   In fact, for A and B propositional constants, the derivation

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\forall X((A\to(B\to X))\to X)}{(A\to(B\to A))\to A} \quad \dfrac{\dfrac{[A]}{B\to A}}{A\to(B\to A)}}{A} \quad [A\to(B\to X)]}{B\to X} \quad \dfrac{\dfrac{\forall X((A\to(B\to X))\to X)}{(A\to(B\to B))\to B} \quad \dfrac{\dfrac{[B]}{B\to B}}{A\to(B\to B)}}{B}}{X}}{(A\to(B\to X))\to X}}{\forall X((A\to(B\to X))\to X)}$$

   does not permit the application of any $\beta\eta$-conversion of $\mathbf{F_{at}}$ and therefore does not reduce to $\forall X((A \to (B \to X)) \to X)$.

   Considering the conjunction as primitive in the atomic polymorphic calculus allow us to circumvent the problem and, as mentioned in the introduction section, the calculus keeps the good proof-theoretical properties and philosophical motivations decisive in its genesis, i.e., no *bad* connectives.

2) The present paper deals with *standard* conversions. What about the *commuting* (also known as *permutative*) *conversions*? In [8] it was proved that the commuting conversions of **IPC** could be translated in $\mathbf{F_{at}}$ via *bidirectional* applications of β-conversions, i.e., we can go from the translation of the *redex* to the translation of the *contractum* by means of β-conversions in *both* direction. Note that, because the direction of the reductions is not unique, the argument in the proof

of Theorem 1 can no longer be used when considering the **IPC** commuting conversions. Nevertheless, it remains an open question if the atomic polymorphic framework is able to produce an alternative proof of strong normalization for the full intuitionistic propositional calculus with commuting conversions.

3) The reason we work in the atomic polymorphic calculus (at the price of having to consider instantiation overflow) instead of working directly in system **F** is twofold. Firstly, note that (as opposed to $\mathbf{F}_{\mathbf{at}}^{\wedge}$) the canonical translation of proofs (rule-by-rule) of **IPC** into system **F** (via the Russell-Prawitz translation) does not preserve standard conversions. Take, for instance, the η-conversion for $\vee$ analysed in the proof of Proposition 1. The canonical translation of its redex into system **F** is the derivation $\mathcal{D}$ (see page 7) with the double line (for instantiation overflow) replaced by a single line. [In system **F** from $\forall X((A \to X) \to ((B \to X) \to X))$ we can deduce through a single rule ($\forall$E) the formula $(A \to (A \vee B)) \to ((B \to (A \vee B)) \to (A \vee B))$, where $A \vee B$ abbreviates the formula $\forall X((A \to X) \to ((B \to X) \to X))$.] Since $\mathcal{D}$ with the modification above has no redexes it can not be $\beta\eta$-reduced to the canonical translation (into system **F**) of the contractum of the η-conversion for $\vee$.

   Since proofs in $\mathbf{F}_{\mathbf{at}}^{\wedge}$ are, in particular, proofs in system **F** we can argue that (although not canonical) the simulation of η-conversions in this paper can be seen as having Girard's system **F** as our target system. This leads us to our second point.

   This paper is part of a line of research that intends to develop an alternative to full intuitionistic propositional calculus free from the defects (bad connectives/commuting conversions) pointed by Girard et al. in [9]. Such alternative is the atomic polymorphic framework. As opposed to system **F**, system $\mathbf{F}_{\mathbf{at}}^{\wedge}$ is predicative, enjoys the subformula property and allows for an *elementary* proof of strong normalization (see [4]). Properties in **IPC** (see [5] for the disjunction property, [7] for the strong β-normalization property and the present paper for the strong $\beta\eta$-normalization property) can be reduced to properties elegantly proved in $\mathbf{F}_{\mathbf{at}}^{\wedge}$ with no bad connectives nor permutative conversions.

4) We think that the algorithms involved in the proof-theoretical analysis via the Russell-Prawitz translation – e.g. the one that underlies the embedding of **IPC** into the atomic polymorphic system (relying on instantiation overflow) and the algorithms for the simulation of β, η (and commuting) conversions of **IPC** into $\beta\eta$-conversions of $\mathbf{F}_{\mathbf{at}}^{\wedge}$ – which are in the base of alternative proofs of strong normalization for **IPC**, deserve further investigation and comparison with related studies via CPS transformations (see [10] and [2]).

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