Probabilistic logic of quantum observations

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Abstract

A probabilistic propositional logic, endowed with an epistemic component for asserting (non-)compatibility of diagonizable and bounded observables, is presented and illustrated for reasoning about the random results of projective measurements made on a given quantum state. Simultaneous measurements are assumed to imply that the underlying observables are compatible. A sound and weakly complete axiomatization is provided relying on the decidable first-order theory of real closed ordered fields. The proposed logic is proved to be a conservative extension of classical propositional logic.

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1 Introduction

The origin of quantum logic can be traced back to Garrett Birkhoff and John von Neumann's 1936 paper (see [BvN36]). Since then, the question of reasoning in logic about quantum phenomena has been a recurrent research topic.

The introduction of quantum mechanics concepts in formal logic is quite challenging since there is the need to accommodate the continuous nature of quantum mechanics within the discrete setting of symbolic reasoning.

There are two main approaches to quantum logic. In the original approach, proposed in the seminal paper [BvN36], each propositional symbol is an experimental proposition over a set of compatible observables, that is, a subset of the set of possible results of measuring them jointly. The denotation of such a propositional symbol is a closed linear subspace of the Hilbert space at hand. The connectives of the logic reflect the operations in the lattice of all closed subspaces of a Hilbert space. As a consequence, some classical laws like distributivity of $\wedge$ and $\vee$ are no longer valid since the observables associated with the experimental propositions in those laws may not be necessarily compatible. If they were all compatible then the classical laws for $\wedge$ and $\vee$ would be true (see page 831 of [BvN36]). There have been many investigations on quantum logic based on this approach (see, for instance, [Har81, Tak81, DCG02, PM08, Har16, Pav16]), including first-order frames and quantum set theory.

The other approach, assumes *à priori*, that all observables involved are compatible with each other (see [vdMP03, MS06]) and focus on extensions of classical propositional logics with new constructions for reasoning about particular quantum mechanics notions (inspired by the way probabilistic reasoning was introduced in classical logic [Hal03, SRS17]). For an overview of quantum logic according to this approach and quantum structures, see [EGL07, EGL09].

Herein, we address the problem of reasoning about the random results of projective measurements made on a given quantum state. Since the observables involved in the measurements are not necessarily compatible we need also to reason about incompatibility of observables. This was achieved by defining an epistemic-like operator $\odot\,\alpha$ with the intended meaning *it is possible to know the truth value of propositional formula α* (that is, the observables associated with the propositional symbols occurring in α are compatible).

More precisely, we develop a logic (PLQO) where each propositional symbol is an assertion on the result of a measurement (mathematically described by an observable) and a formula is a Boolean assertion on the possible results of the measurements associated with its propositional symbols. The probability of such a formula α being true is expressed in our logic by $\int\alpha$. The present paper generalizes the work in [MS06] namely by presenting a more general semantics as well as a different calculus coping with both the probabilities and incompatibility of observables.

We note that the Boolean connectives inside a formula α have the same meaning of the connectives of the logic in [BvN36]. However, in our logic α only appears in the scope of either $\int\alpha$ or in the scope of $\odot\,\alpha$. When the observables in α are not compatible, the Boolean connectives in that formula do not play any role since we state that $\int\alpha\,@\,r$ is false. The same applies to $\odot\,\alpha$ since the evaluation of this formula corresponds only to checking whether or not the observables in α are compatible.

As a consequence, our logic is a conservative extension of classical propositional logic in the sense that we were able to prove that

$$(\odot\,\alpha) \supset \left(\int\alpha = 1\right) \text{ is valid in } \mathsf{PLQO} \quad \text{iff} \quad \alpha \text{ is classically valid.}$$

That is, α is always true iff α holds in every state of a quantum system whenever the underlying observables are compatible.

As in [BvN36] (page 826), we assume that each finite set of compatible observables is simultaneously diagonalizable, that is, there exists an orthonormal basis of the underlying Hilbert space composed by common eigenvectors of the observables. This is the case, for instance, when the observables are defined by pairwise commuting compact self-adjoint operators (see [Ish95, BEH08, Hal13]).

In Section 2, we briefly review some relevant mathematical concepts underlying quantum mechanics. Moreover, we define our context namely by assuming, for the purpose of this work, that each observable is diagonizable and bounded besides being self-adjoint. With the aim of setting-up the probability context for joint measurements of compatible observables we define the observable resulting from the product of two given compatible observables. In Section 3, we introduce the notion of propositional quantum variable and analyze compatibility of propositional quantum variables.

After that, in Section 4, we introduce the language of PLQO after recalling some notions of classical propositional logic as well as of the theory of real closed ordered fields. This language has two kinds of atomic formulas: $\odot\,\alpha$ as referred above, and $\int\alpha@p$ with the intended meaning of *it is possible to know the truth value of α* and *the probability of α being true is $@p$* where $@$

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is a relational operator. Then, we introduce the notion of quantum interpretation structure which is composed by a Hilbert space representing the envisaged quantum system, a unit vector representing the quantum state and the interpretation of each propositional symbol as a propositional quantum variable. Satisfaction and entailment are then defined. The section ends by presenting an axiomatization for PLQO capitalizing on the decidability of the theory of real closed ordered fields. This axiomatization is shown to be strongly sound in Section 5 after establishing some auxiliary results.

In Section 6, we show that the axiomatization is weakly complete starting by proving a finite-dimensional model existence lemma relating satisfaction of a formula in PLQO with satisfaction of the translation of the formula in the RCOF interpretation structure having as domain the set $\mathbb{R}$. Strong completeness is out of question since PLQO is not a compact logic as we explain in Section 4. Conservativeness of PLQO with respect to classical propositional logic is discussed in Section 7. Finally, in Section 8, we investigate the epistemic nature of observability and provide several examples. We conclude the paper in Section 9 with the global assessment of what was done and future research directions.

2 Preliminaries

Our objective here is to provide a very brief account of what we need to know about the random results of projective measurements made on a quantum state. For a full account see [Rud87, Rud91, Ish95, BEH08, Hal13].

Basic notions

Following the mathematical formulation of quantum mechanics, as introduced by Dirac in 1930, [Dir67], and by von Neumann in 1932, [vN55], a quantum system is defined by a complex separable Hilbert space $\mathcal{H}$. Moreover, a quantum state of the system is represented by a unit vector in $\mathcal{H}$.

We start by reviewing the concept of projection-valued measure. Given a set Ω and a σ-algebra $\mathfrak{B}$ over Ω, a *projection-valued measure* μ is a map from $\mathfrak{B}$ to the class of bounded operators over $\mathcal{H}$ having the following properties.

- For each $S \in \mathfrak{B}$, $\mu(S)$ is an orthogonal projection, that is, $\mu(S)^2 = \mu(S)$ and $\mu(S)$ is a self-ajoint operator.

- $\mu(\emptyset) = 0$ and $\mu(\Omega) = I_{\mathcal{H}}$, where $I_{\mathcal{H}}$ is the identity over $\mathcal{H}$.

- Given a countable family $\{S_j\}_{j \in J}$ of pairwise disjoint elements of $\mathfrak{B}$, let

$$S = \bigcup_{j \in J} S_j.$$

Then, for every $\psi \in \mathcal{H}$,

$$\mu(S)\psi = \sum_{j \in J} \mu(S_j)\psi,$$

where the convergence of the sum is in the norm topology on $\mathcal{H}$, whenever J is countably infinite.

- For every $S_1, S_2 \in \mathfrak{B}$,
$$\mu(S_1 \cap S_2) = \mu(S_1)\mu(S_2).$$

Any projection-valued measure μ and $\psi \in \mathcal{H}$ induce a positive real-valued measure

$$\mu_\psi : S \mapsto \langle \psi, \mu(S)\psi \rangle$$

over $\mathfrak{B}$. Moreover, μ induces a unique linear map

$$f \mapsto \int_\Omega f d\mu$$

that maps each measurable and complex function to an operator on $\mathcal{H}$ such that

$$\left\langle \psi, \left(\int_\Omega f \, d\mu \right) \psi \right\rangle = \int_\Omega f \, d\mu_\psi,$$

for all ψ in

$$\left\{ \psi \in \mathcal{H} : \int_\Omega |f(\lambda)|^2 d\mu_\psi(\lambda) < \infty \right\}.$$

This operator satisfies the following properties:

- $\int_\Omega 1_S \, d\mu = \mu(S)$ for every $S \in \mathfrak{B}$;

- $\int_\Omega fg \, d\mu = \left(\int_\Omega f \, d\mu \right) \left(\int_\Omega g \, d\mu \right)$.

A projective measurement is defined by a self-adjoint operator (aka *observable*) acting on $\mathcal{H}$. The possible results of the measurement are the values in the spectrum Ω of the observable. Clearly, $\Omega \subseteq \mathbb{R}$ since the operator is self-adjoint. From now on, let $\mathfrak{B}$ be the σ-field of the Borel subsets of Ω induced by the usual topology of $\mathbb{R}$.

The spectral theorem tells us that a self-adjoint operator O induces a unique projection-valued measure μ such that

$$\int_\Omega \mathsf{id}_\Omega \, d\mu = O,$$

which is called the *spectral representation* of O, where id_Ω is the identity over Ω. Observe that we are not imposing that O is a bounded operator. In the sequel, we denote by $\mathsf{dom}\, O \subseteq \mathcal{H}$ the domain of O.

The following definition is used in the sequel. If O is a self-adjoint operator and $f : \Omega \to \mathbb{C}$ then the *image* of O by f is the operator

$$(\ddagger) \qquad f(O) = \int_\Omega f \, d\mu$$

where μ is the projection-valued measure induced by O.

Observables

The notion of propositional quantum variable, introduced below, assumes that an observable besides being self-adjoint is also diagonalizable. We say that an operator $O : \mathcal{H} \to \mathcal{H}$ is *diagonalizable* whenever

- O is a self-adjoint operator;

- O has a countable pure point spectrum $\sigma_p(O)$;

- $\{\mathrm{Proj}_\omega\}_{\omega \in \sigma_p(O)}$ is a complete system of eigenprojections of O, that is,

  - $\{\psi \in \mathsf{dom}\, O : \mathrm{Proj}_\omega \psi = \psi\}$ is the eigenspace of O associated with the eigenvalue ω, for each $\omega \in \sigma_p(O)$;

  - $\displaystyle\sum_{\omega \in \sigma_p(O)} \mathrm{Proj}_\omega = I_{\mathcal{H}}.$

Then, as shown in [BEH08],

$$\int_\Omega \mathsf{id}_\Omega\, d\mu = O$$

where μ is the discrete projection-valued measure such that

$$\mu(S) = \sum_{\omega \in S \cap \sigma_p(O)} \chi_S(\omega)\mathrm{Proj}_\omega$$

for every $S \in \mathfrak{B}$. Moreover,

$$O\psi = \sum_{\omega \in \sigma_p(O)} \omega\mathrm{Proj}_\omega \psi$$

for every ψ in the domain of O. That is, O is a suboperator of $\sum_{\omega \in \Omega} \omega\mathrm{Proj}_\omega$. When O is a bounded operator then

$$O = \sum_{\omega \in \sigma_p(O)} \omega\mathrm{Proj}_\omega.$$

For example, all compact operators are diagonizable (see Riesz-Schauder Theorem, see [BEH08], pp 79). Moreover, there are operators that although not compact are still diagonizable (see [Hal13] pp 124) as the Hamiltonian of the simple harmonic oscillator whose eigenvalues are $(n + \frac{1}{2})\hbar\omega$ with $n \in \mathbb{N}$ (see [Ish95] pp 60 and [BEH08] pp 262).

From now on, we assume that all observables are diagonizable. Moreover, in order to avoid the problem of domains in unbounded operators, we also assume that all observables are bounded.

Example 2.1 Consider a spin-$\frac{1}{2}$ particle (see [Dir67, Gri14]). The Hilbert space for the spin is $\mathcal{H} = \mathbb{C}^2$. Let $O_x, O_y, O_z : \mathcal{H} \to \mathcal{H}$ be the observables for the x, the y and the z components of the spin defined by the matrices:

$$O_x = \frac{1}{2}\hbar \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad O_y = \frac{1}{2}\hbar \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \quad O_z = \frac{1}{2}\hbar \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},$$

respectively. The set of eigenvalues is $\{\frac{\hbar}{2}, -\frac{\hbar}{2}\}$ for the three operators. Moreover, the orthonormal bases induced by O_x, O_y and O_z for $\mathbb{C}^2$ are:

$$\left\{ \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\}, \left\{ \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ i \end{bmatrix}, \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -i \end{bmatrix} \right\}, \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\},$$

respectively. So O_x, O_y and O_z are diagonalizable observables. For example, the spectral representation of O_x is as follows:

$$O_x = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \frac{\hbar}{2} \left\langle \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \cdot \right\rangle - \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix} \frac{\hbar}{2} \left\langle \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \cdot \right\rangle.$$

It can be easily seen that no pair of these operators commute. However, the total angular momentum O^2 defined as

$$O^2 = O_x^2 + O_y^2 + O_z^2 = \frac{3}{4}\hbar \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

is such that

$$O^2 O_x = O_x O^2 \quad O^2 O_y = O_y O^2 \quad O^2 O_z = O_z O^2.$$

Thus, O^2 and O_x, O^2 and O_y, and O^2 and O_z commute and so, as we shall see later on, they are compatible observables. Observe that O^2 is diagonizable as well. $\qquad \nabla$

As we shall see later on each observable and unit vector induce a probability space allowing the calculation of the probabilities of obtaining a certain result after a measurement of that observable. When performing simultaneous measurements by a finite set of pairwise commuting observables we need to know the probability space where we can calculate the probabilities of the joint results. This probability space can be induced by the observable resulting from the product of the observables at hand (see [Ish95]).

We now define the product of observables starting by introducing the notion of morphism between observables. Let O_1 and O_2 be observables and $\{\mathrm{Proj}_\omega^1\}_{\omega \in \sigma_p(O_1)}$ be a complete system of eigenprojections of O_1. A morphism

$$f : O_1 \to O_2$$

is a map $f : \sigma_p(O_1) \to \mathbb{C}$ such that $f(O_1) = O_2$, where $f(O_1)$ is such that

$$f(O_1) = \sum_{\omega \in \sigma_p(O_1)} f(\omega)\mathrm{Proj}_\omega.$$

Proposition 2.2 Let O' and O'' be a pair of (diagonizable and bounded) commuting observables such that

$$O' = \sum_{\omega' \in \sigma_p(O')} \omega' \mathrm{Proj}_{\omega'}' \quad \text{and} \quad O'' = \sum_{\omega'' \in \sigma_p(O'')} \omega'' \mathrm{Proj}_{\omega''}'',$$

where $\{\mathrm{Proj}_{\omega'}'\}_{\omega' \in \sigma_p(O')}$ and $\{\mathrm{Proj}_{\omega''}''\}_{\omega'' \in \sigma_p(O'')}$ are complete systems of eigenprojections of O' and O'', respectively. Let

$$O' \times O'' = \sum_{(\omega', \omega'') \in \sigma_p(O') \times \sigma_p(O'')} \lambda_{\omega'\omega''} \, \mathrm{Proj}_{\omega'}' \mathrm{Proj}_{\omega''}''$$

be an operator where $\lambda_{\omega'\omega''} \in (0,1)$ such that $\lambda_{\omega_1'\omega_1''} \neq \lambda_{\omega_2'\omega_2''}$ for every $\omega_1', \omega_2' \in \sigma_p(O')$ and $\omega_1'', \omega_2'' \in \sigma_p(O'')$ with $(\omega_1', \omega_1'') \neq (\omega_2', \omega_2'')$. Then,

$$O' \times O''$$

is a diagonizable and bounded operator such that there are morphisms $f' : O' \times O'' \to O'$ and $f'' : O' \times O'' \to O''$.

Proof:

We start by proving some auxiliary results.

(a) $\text{Proj}'_{\omega'}\text{Proj}''_{\omega''} = \text{Proj}''_{\omega''}\text{Proj}'_{\omega'}$, for every $\omega' \in \sigma_p(O')$ and $\omega'' \in \sigma_p(O'')$. Observe that, for every $\psi \in \mathcal{H}$,

$$O'O''\text{Proj}'_{\omega'}\psi = O''O'\text{Proj}'_{\omega'}\psi = \omega'(O''\text{Proj}'_{\omega'}\psi)$$

since O' and O'' commute. Hence, $O''\text{Proj}'_{\omega'}\psi$ belongs to the eigenspace of O' associated with ω'. Therefore,

$$\text{Proj}'_{\omega'}O''\text{Proj}'_{\omega'} = O''\text{Proj}'_{\omega'}.$$

Moreover,

$$\text{Proj}'_{\omega'}O''\text{Proj}'_{\omega'} = \text{Proj}'_{\omega'}O''.$$

Therefore,

$$O''\text{Proj}'_{\omega'} = \text{Proj}'_{\omega'}O''.$$

Similarly,

$$O'\text{Proj}''_{\omega''} = \text{Proj}''_{\omega''}O'.$$

Thus,

$$O''\text{Proj}'_{\omega'}\text{Proj}''_{\omega''}\psi = \text{Proj}'_{\omega'}O''\text{Proj}''_{\omega''}\psi = \omega''\text{Proj}'_{\omega'}\text{Proj}''_{\omega''}\psi$$

and so $\text{Proj}'_{\omega'}\text{Proj}''_{\omega''}\psi$ belongs to the eigenspace of O'' associated with ω''. Hence,

$$\text{Proj}''_{\omega''}\text{Proj}'_{\omega'}\text{Proj}''_{\omega''}\psi = \text{Proj}'_{\omega'}\text{Proj}''_{\omega''}\psi.$$

Moreover,

$$\text{Proj}''_{\omega''}\text{Proj}'_{\omega'}\text{Proj}''_{\omega''}\psi = \text{Proj}''_{\omega''}\text{Proj}'_{\omega'}\psi.$$

Then,

$$\text{Proj}'_{\omega'}\text{Proj}''_{\omega''} = \text{Proj}''_{\omega''}\text{Proj}'_{\omega'}.$$

(b) $\text{Proj}'_{\omega'}\text{Proj}''_{\omega''}$ is a projection for every $\omega' \in \sigma_p(O')$ and $\omega'' \in \sigma_p(O'')$. In fact,

$$(\text{Proj}'_{\omega'}\text{Proj}''_{\omega''})^2 = \text{Proj}'_{\omega'}\text{Proj}''_{\omega''},$$

using (a) and the fact $(\text{Proj}'_{\omega'})^2 = \text{Proj}'_{\omega'}$ and $(\text{Proj}''_{\omega''})^2 = \text{Proj}''_{\omega''}$. Moreover,

$$(\text{Proj}'_{\omega'}\text{Proj}''_{\omega''})^* = \text{Proj}'_{\omega'}\text{Proj}''_{\omega''}$$

also by (a) and the fact $(\text{Proj}'_{\omega'})^* = \text{Proj}'_{\omega'}$ and $(\text{Proj}''_{\omega''})^* = \text{Proj}''_{\omega''}$.

We are now ready to prove the envisaged properties of $O' \times O''$.

(1) $O' \times O''$ is a self-adjoint operator. Observe that:

$$(O' \times O'')^* = \sum_{(\omega',\omega'')\in\sigma_p(O')\times\sigma_p(O'')} \lambda_{\omega'\omega''}(\text{Proj}'_{\omega'}\text{Proj}''_{\omega''})^*.$$

Note that,

$$(\text{Proj}'_{\omega'}\text{Proj}''_{\omega''})^* = \text{Proj}'_{\omega'}\text{Proj}''_{\omega''}$$

since $\text{Proj}'_{\omega'}\text{Proj}''_{\omega''}$ is a projection by (b). Therefore,

$$(O' \times O'')^* = O' \times O''.$$

(2) $O' \times O''$ is diagonizable.

(a) It is immediate to see that the spectrum of $O' \times O''$ is countable.

(b) $\{\mathrm{Proj}'_{\omega'}\mathrm{Proj}''_{\omega''}\psi : \psi \in \mathcal{H}\} = \{\psi \in \mathcal{H} : O' \times O''\psi = \lambda_{\omega'\omega''}\psi\}$, for each $\omega' \in \sigma_p(O')$ and $\omega'' \in \sigma_p(O'')$ such that $\mathrm{Proj}'_{\omega'}\mathrm{Proj}''_{\omega''} \neq 0$. That is,

$$\{\mathrm{Proj}'_{\omega'}\mathrm{Proj}''_{\omega''}\psi : \psi \in \mathcal{H}\}$$

is an eigenspace of $O' \times O''$ for the eigenvalue $\lambda_{\omega'\omega''}$. Let $\omega'_1 \in \sigma_p(O')$ and $\omega''_1 \in \sigma_p(O'')$ such that $\mathrm{Proj}_{\omega'_1}\mathrm{Proj}_{\omega''_1} \neq 0$.

($\subseteq$) Assume that $\mathrm{Proj}_{\omega'_1}\mathrm{Proj}_{\omega''_1}\psi \in \{\mathrm{Proj}_{\omega'_1}\mathrm{Proj}_{\omega''_1}\psi : \psi \in \mathcal{H}\}$. Observe that,

$$O' \times O''\mathrm{Proj}_{\omega'_1}\mathrm{Proj}_{\omega''_1}\psi =$$

$$= \sum_{(\omega',\omega'')\in\sigma_p(O')\times\sigma_p(O'')} \lambda_{\omega'\omega''}\,\mathrm{Proj}'_{\omega'}\mathrm{Proj}_{\omega'_1}\mathrm{Proj}''_{\omega''}\mathrm{Proj}_{\omega''_1}\psi$$

$$= \lambda_{\omega'_1\omega''_1}\,\mathrm{Proj}_{\omega'_1}\mathrm{Proj}_{\omega''_1}\psi.$$

($\supseteq$) Assume that $\psi \in \{\psi \in \mathcal{H} : O' \times O''\psi = \lambda_{\omega'_1\omega''_1}\psi\}$, that is, ψ is such that

$$O' \times O''\psi = \lambda_{\omega'_1\omega''_1}\psi.$$

That is,

$$\left(\sum_{(\omega',\omega'')\in\sigma_p(O')\times\sigma_p(O'')} \lambda_{\omega'\omega''}\,\mathrm{Proj}'_{\omega'}\mathrm{Proj}''_{\omega''}\right)\psi = \lambda_{\omega'_1\omega''_1}\psi.$$

Thus,

$$\psi = \mathrm{Proj}_{\omega'_1}\mathrm{Proj}_{\omega''_1}\psi.$$

Therefore,

$$\psi \in \{\mathrm{Proj}_{\omega'_1}\mathrm{Proj}_{\omega''_1}\psi : \psi \in \mathcal{H}\}.$$

(c) $\mathrm{Proj}'_{\omega'}\mathrm{Proj}''_{\omega''}\psi$ is a simultaneous eigenvector of O' and O'' with respect to ω' and ω'', for each $\psi \in \mathcal{H}$, $\omega' \in \sigma_p(O')$ and $\omega'' \in \sigma_p(O'')$ such that $\mathrm{Proj}'_{\omega'}\mathrm{Proj}''_{\omega''} \neq 0$. Indeed, let $\psi \in \mathcal{H}$, $\omega'_1 \in \sigma_p(O')$ and $\omega''_1 \in \sigma_p(O'')$ such that $\mathrm{Proj}_{\omega'_1}\mathrm{Proj}_{\omega''_1} \neq 0$. Then,

$$O'\mathrm{Proj}_{\omega'_1}\mathrm{Proj}_{\omega''_1}\psi = \sum_{\omega'\in\sigma_p(O')} \omega'\,\mathrm{Proj}'_{\omega'}\mathrm{Proj}_{\omega'_1}\mathrm{Proj}_{\omega''_1}\psi$$

$$= \omega'_1\,\mathrm{Proj}_{\omega'_1}\mathrm{Proj}_{\omega''_1}\psi.$$

(d) There is a countable orthonormal basis of $\mathcal{H}$ composed by eigenvectors of $O' \times O''$ which are common eigenvectors of both O' and O''. Observe that

$$\dim\mathcal{H} = \sum_{\substack{(\omega',\omega'') \in \sigma_p(O') \times \sigma_p(O'') \\ \mathrm{Proj}'_{\omega'}\mathrm{Proj}''_{\omega''} \neq 0}} \dim\{\psi \in \mathcal{H} : \mathrm{Proj}'_{\omega'}\mathrm{Proj}''_{\omega''}\psi\}$$

by (b) and since $\dim\{\psi \in \mathcal{H} : \mathrm{Proj}'_{\omega'}\mathrm{Proj}''_{\omega''}\psi\} = 0$ whenever $\mathrm{Proj}'_{\omega'}\mathrm{Proj}''_{\omega''} = 0$. Then, an orthonormal basis for $\mathcal{H}$ is obtained by joining the orthonormal bases of each eigenspace of

$O' \times O''$. Moreover, each vector in this basis is also an eigenvector of O' and O'' as shown in (c).

(3) There are morphisms $f' : O' \times O'' \to O'$ and $f'' : O' \times O'' \to O''$. In fact let $f' : \sigma_p(O' \times O'') \to \mathbb{C}$ and $f'' : \sigma_p(O' \times O'') \to \mathbb{C}$ be maps such that $f'(\lambda_{\omega'\omega''}) = \omega'$ and $f''(\lambda_{\omega'\omega''}) = \omega''$. We start by observing that

$$\sum_{\omega'' \in \sigma_p(O'')} \mathrm{Proj}''_{\omega''} = I_{\mathcal{H}}.$$

Hence,

$$
\begin{aligned}
f'(O' \times O'') &= \sum_{(\omega',\omega'') \in \sigma_p(O') \times \sigma_p(O'')} f'(\lambda_{\omega'\omega''}) \mathrm{Proj}'_{\omega'} \mathrm{Proj}''_{\omega''} \\
&= \sum_{\omega' \in \sigma_p(O')} \omega' \mathrm{Proj}'_{\omega'} \sum_{\omega'' \in \sigma_p(O'')} \mathrm{Proj}''_{\omega''} \\
&= \sum_{\omega' \in \sigma_p(O')} \omega' \mathrm{Proj}'_{\omega'} I_{\mathcal{H}} \\
&= O'.
\end{aligned}
$$

Similarly, $f''(O' \times O'') = O''$.

(4) $O' \times O''$ is a bounded operator. Indeed, its domain is $\mathcal{H}$ and it is self-adjoint so it is bounded (see [Hal13] Corollary 9.9 in page 173).   QED

We can say that $O' \times O''$ is the observable where the observables O' and O'' can be simultaneously measured. The notation $O' \times O''$ is justified since $O' \times O''$ is a binary product in the category of observables and their morphisms. Indeed, the universal property also holds. Let $g' : O''' \to O'$ and $g'' : O''' \to O''$ be morphisms. Let $g : \sigma_p(O''') \to \mathbb{C}$ be defined as follows: $g(\omega''') = \lambda_{g'(\omega''')g''(\omega''')}$. Then, it is immediate to see that g is the unique morphism from O''' to $O' \times O''$ such that $f' \circ g = g'$ and $f'' \circ g = g''$.

3   Propositional quantum variables

By a *propositional quantum variable* we mean a triple $Y = (O, \mathcal{D}, \uparrow)$ where O is an observable acting on $\mathcal{H}$ and $\mathcal{D}$ is a finite partition of $\sigma_p(O)$ and $\uparrow \in \mathcal{D}$. We say that Y is defined on $\mathcal{H}$ and takes values in $\mathcal{D}$. For the purpose of this paper, the set $\uparrow$ plays the role of verum and $\sigma_p(O) \setminus \uparrow$, denoted by

$$\downarrow,$$

plays the role of falsum.

We want to reason about the probability of the measurement of observable O having a result in either $\uparrow$ or $\downarrow$ when the system is in a state ψ. In fact, an observable O and a state ψ induce the probability space $(\sigma_p(O), \wp \sigma_p(O), \mu_\psi)$ where μ_ψ is the probability measure such that

$$\mu_\psi(\omega) = \langle \psi, \mu(S)\psi \rangle = \langle \psi, \mathrm{Proj}_\omega \psi \rangle,$$

where μ is the discrete projection-valued measure induced by O and S is a Borel set such $S \cap \sigma_p(O) = \{\omega\}$. Observe that the value of any Borel set with this property by μ is the same. Moreover, for every $D \subseteq \sigma_p(O)$,

$$\mu_\psi(D) = \sum_{\omega \in D} \mu_\psi(\omega) = \langle \psi, \mu(S)\psi \rangle = \langle \psi, \mathrm{Proj}_D \psi \rangle,$$

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where S is a Borel set such $S \cap \sigma_p(O) = D$ and

$$\text{Proj}_D = \sum_{\omega \in D} \text{Proj}_\omega.$$

Now we are ready, given a quantum variable $Y = (O, \mathcal{D}, \uparrow)$ and a state ψ, to define a random variable

$$Y_\psi = \omega \mapsto D_\omega : \sigma_p(O) \to \mathcal{D},$$

on $(\sigma_p(O), \wp\sigma_p(O), \mu_\psi)$, where D_ω is the element of the partition $\mathcal{D}$ to which ω belongs. Observe that, the probability distribution induced by Y_ψ on $\mathcal{D}$ is such that:

$$D \mapsto \text{Prob}\,(Y_\psi = D) = \mu_\psi(D) : \mathcal{D} \to [0, 1].$$

We next address what happens when one wants to perform several simultaneous measurements on a given state of the quantum system.

Compatibility of propositional quantum variables

The theory of quantum mechanics tells us that it is possible to measure simultaneously a finite number of observables $O_1, \ldots, O_n$ if only if they are *compatible*, that is, O_i and O_j commute for every $i, j = 1, \ldots, n$ with $i \neq j$.

Example 3.1 As seen in Example 2.1, the observables O^2 and O_x, O^2 and O_y, and O^2 and O_z commute and so they are compatible. ∇

Example 3.2 Consider the electron of a hydrogen atom. Then, its Hamiltonian H, total angular momentum L^2, z-component L_z of the angular momentum and z-component S_z of the spin operator are compatible (see [Ish95] pp 119). Denote the state vectors of a hydrogen atom by $\psi_{n\ell m\lambda}$, where $n > 0$, $\ell = 0, \ldots, n-1$ and $m = -\ell, \ldots, \ell + 1$ are natural numbers and $\lambda = \pm\frac{1}{2}$. These are simultaneous eigenvectors of those observables. The eigenvalues are as follows:

- $H\psi_{n\ell m\lambda} = E_n\psi_{n\ell m\lambda}$ where $E_n = -\frac{R}{n^2}$ and R is the Rydberg constant (see [Hal13] pp 8);

- $L^2\psi_{n\ell m\lambda} = \ell(\ell+1)\hbar^2\psi_{n\ell m\lambda}$;

- $L_z\psi_{n\ell m\lambda} = m\hbar\psi_{n\ell m\lambda}$;

- $S_z\psi_{n\ell m\lambda} = \lambda\hbar\psi_{n\ell m\lambda}$. ∇

When performing simultaneous measurements of compatible observables O' and O'' on a state ψ, the probability space where we can calculate the probabilities of the joint results is

$$(\sigma_p(O' \times O''), \wp\sigma_p(O' \times O''), \mu_\psi)$$

induced by $O' \times O''$. Note that,

$$\mu_\psi(\lambda_{\omega'\omega''}) = \langle \psi, \text{Proj}'_{\omega'}\text{Proj}''_{\omega''}\psi \rangle.$$

Moreover, for any
$$D_{D'D''} = \{\lambda_{\omega'\omega''} : \omega' \in D' \text{ and } \omega'' \in D''\}$$
where $D' \subseteq \sigma_p(O')$ and $D'' \subseteq \sigma_p(O'')$, we have

$$\mu_\psi(D_{D'D''}) = \langle \psi, \sum_{\lambda_{\omega'\omega''} \in D_{D'D''}} \mathrm{Proj}'_{\omega'} \mathrm{Proj}''_{\omega''} \psi \rangle$$

$$= \langle \psi, \sum_{\omega' \in D'} \sum_{\omega'' \in D''} \mathrm{Proj}'_{\omega'} \mathrm{Proj}''_{\omega''} \psi \rangle$$

$$= \langle \psi, \sum_{\omega' \in D'} \mathrm{Proj}'_{\omega'} \sum_{\omega'' \in D''} \mathrm{Proj}''_{\omega''} \psi \rangle$$

$$= \langle \psi, \left(\sum_{\omega' \in D'} \mathrm{Proj}'_{\omega'} \right) \mathrm{Proj}''_{D''} \psi \rangle$$

$$= \langle \psi, \mathrm{Proj}'_{D'} \mathrm{Proj}''_{D''} \psi \rangle.$$

We proceed now with the task of bringing the notion of compatibility to the realm of propositional quantum variables. We say that pqvs $Y' = (O', \mathcal{D}', \uparrow')$ and $Y'' = (O'', \mathcal{D}'', \uparrow'')$ are *compatible* if O' and O'' commute. As expected, we say that a finite set of quantum variables is *incompatible* if it contains (at least) an incompatible pair. Otherwise, the set is said to be *compatible*. Clearly, the empty set is compatible and so is each singleton set. Moreover, every superset of an incompatible set of quantum variables is also incompatible.

Given compatible pqvs $Y' = (O', \mathcal{D}', \uparrow')$ and $Y'' = (O'', \mathcal{D}'', \uparrow'')$, we consider the pqv

$$Y = (O' \times O'', \mathcal{D}, \uparrow)$$

where $\mathcal{D}$ is composed by the sets

$$D_{D'D''} = \{\lambda_{\omega'\omega''} : \omega' \in D' \text{ and } \omega'' \in D''\}$$

for each $D' \in \mathcal{D}'$ and $D'' \in \mathcal{D}''$, and $\uparrow$ is $D_{\uparrow'\uparrow''}$.

The joint probability distribution of the random variables induced by Y' and Y'' and quantum state ψ is such that, for each $\langle D', D'' \rangle \in \mathcal{D}' \times \mathcal{D}''$,

$$\mathsf{Prob}\left(\langle Y'_\psi, Y''_\psi \rangle = \langle D', D'' \rangle\right) = \mathsf{Prob}(Y_\psi = D_{D'D''}) = \langle \psi, \mathrm{Proj}'_{D'} \mathrm{Proj}''_{D''} \psi \rangle.$$

If a pair of quantum variables is incompatible, then we cannot, in general, use them simultaneously for observing the quantum system at hand. This fact will have an enormous impact on the nature of the envisaged logic. Namely, it is at the heart of the need for an epistemic component in the logic. Actually, this is the key difference between observing a probabilistic system and observing a quantum system.

4   Language, semantics and calculi

The objective of this section is to define an enrichment of classical propositional logic (CPL) to reason about the random results of simultaneously applying observables to a given quantum state.

The idea is to add as little as possible to the classical propositional language. To this end, we add a constructor for expressing that (the truth value of) a (classical) formula representing an assertion on the results of simultaneous measurements is knowable and a symbolic construct for constraining the probability of such a formula.

Concerning CPL we need to adopt some notation and review some concepts and facts.

Let L_c be the classical language built with the propositional symbols in $B = \{B_j : j \in \mathbb{N}\}$ and with tt (verum), using the connectives $\neg$ (negation) and $\supset$ (implication). The other connectives as well as ff (falsum) are introduced as abbreviations. Concerning non-primitive connectives, we use $\wedge$ (conjunction), $\vee$ (disjunction) and $\equiv$ (equivalence) in the paper. It becomes handy to use B_α for the set of propositional symbols that occur in a formula α.

Recall that a valuation (on A) is a map $v : A \to \{0,1\}$ where $A \subseteq B$. Given a valuation $v : A \to \{0,1\}$ and $A' \subseteq A$ we denote by $v|_{A'}$ the restricton of v to A'. We use $v \Vdash_c \alpha$ for stating that valuation $v : A \to \{0,1\}$, with $B_\alpha \subseteq A$, satisfies formula α and $\Delta \vDash_c \alpha$ for stating that Δ classically entails α. One says that α is a tautology if $\vDash_c \alpha$.

Given a valuation v on $A \subseteq B$ and $B_j \in A$, we denote by

$$v^{B_j \hookleftarrow d}$$

the valuation such that

$$v^{B_j \hookleftarrow d}(B_i) = \begin{cases} v(B_i) & \text{if } i \neq j \\ d & \text{otherwise.} \end{cases}$$

A propositional symbol B_j is said to be *essential* for a formula α whenever $B_j \in B_\alpha$ and there exists a valuation v on B_α such that

$$v^{B_j \hookleftarrow 0} \Vdash_c \alpha \quad \text{iff} \quad v^{B_j \hookleftarrow 1} \nVdash_c \alpha.$$

We denote by

$$eB_\alpha$$

the set of essential propositional symbols for α. Observe that eB_α contains only the propositional symbols that contribute in an essential way to the truth value of α. Clearly, $eB_\alpha \subseteq B_\alpha$. Moreover, α is a tautology if and only if $eB_\alpha = \emptyset$. That is, the truth values of the propositional symbols are irrelevant for determining the truth value of α. The set eB_α can be effectively calculated by finding the algebraic normal form of α and considering its propositional symbols (all of them essential). For more details on algorithms for obtaining the algebraic normal form see [Zhe27, SP07]. So, from now on, we only work with formulas in the algebraic normal form. Hence,

$$eB_\alpha = B_\alpha.$$

Before proceeding with the presentation of the envisaged probabilistic logic of quantum observations PLQO, we need to adopt some notation and review some facts concerning also the first-order theory of real closed ordered fields (RCOF), having in mind the use of its terms for denoting probabilities and other quantities.

Recall that the first-order signature of RCOF contains the constants 0 and 1, the unary function symbol $-$, the binary function symbols $+$ and $\times$, and the binary predicate symbols $=$ and $<$. We take the set

$$X = X_\mathbb{N} \cup X_{L_c} \cup \{x_{A'} : A' \subseteq B, |A'| = 2\},$$

where $X_\mathbb{N} = \{x_k : k \in \mathbb{N}\}$ and $X_{L_c} = \{x_\alpha : \alpha \in L_c\}$, as the set of variables. By T_RCOF° we mean the set of terms in RCOF that do not use variables in $X_{L_c} \cup \{x_{A'} : A' \subseteq B, |A'| = 2\}$ while T_RCOF denotes the set of all terms. As we shall see, the variables in X_{L_c} become handy as fresh variables in the proposed axiomatization PLQO, for representing within the language of RCOF the probability of α being true. On the other hand, the variables in $\{x_{A'} : A' \subseteq B, |A'| = 2\}$ are used for representing whether or not the observables associated with the propositional symbols in A' are compatible.

As usual, we may write $t_1 \neq t_2$ for $\neg(t_1 = t_2)$, $t_1 \leq t_2$ for $(t_1 < t_2) \vee (t_1 = t_2)$, $t_1 > t_2$ for $\neg(t_1 \leq t_2)$, $t_1 \geq t_2$ for $\neg(t_1 < t_2)$, $t_1 t_2$ for $t_1 \times t_2$ and t^n for

$$\underbrace{t \times \cdots \times t}_{n \text{ times}}.$$

Furthermore, we also use the following abbreviations for any given $m \in \mathbb{N}^+$ and $n \in \mathbb{N}$:

- m for $\underbrace{1 + \cdots + 1}_{\text{addition of } m \text{ units}}$;

- m^{-1} for the unique z such that $m \times z = 1$;

- $\dfrac{n}{m}$ for $m^{-1} \times n$.

The last two abbreviations might be extended to other terms, but we need them only for numerals. We do not notationally distinguish, for the sake of a lighter style, between a natural number and the corresponding numeral and between each predicate symbol and its denotation.

In order to avoid confusion with the other notions of satisfaction used herein, we adopt $\Vdash_\mathsf{fo}$ for denoting satisfaction in first-order logic (over the language of RCOF).

Recall also that the theory RCOF is decidable [Tar51]. This fact will be put to good use in the axiomatization for PLQO (presented at the end of this section). Furthermore, every model of RCOF satisfies a formula in the RCOF language if and only the formula is a theorem of RCOF (Corollary 3.3.16 in [Mar02]). We shall take advantage of this result in the semantics of PLQO for adopting the ordered field $\mathbb{R}$ of the real numbers as the model of RCOF.

Language

The language L_PLQO of the probabilistic logic of quantum observations PLQO is inductively defined as follows:

- $\odot\, \alpha \in L_\mathsf{PLQO}$ when $\alpha \in L_c$;

- $\int \alpha @ p \in L_\mathsf{PLQO}$ when $\alpha \in L_c$, $p \in T_\mathsf{RCOF}^\circ$ and $@ \in \{=, <, \neq, \leq, >, \geq\}$;

- $\varphi_1 \supset \varphi_2 \in L_\mathsf{PLQO}$ when $\varphi_1, \varphi_2 \in L_\mathsf{PLQO}$.

Propositional abbreviations can be introduced as usual. For instance,

$$\neg\varphi \quad \text{for} \quad \varphi \supset \left(\textstyle\int \mathtt{tt} < 1\right)$$

and similarly for $\wedge$, $\vee$ and $\equiv$.

Semantics

Given a term t in T_{RCOF} and an assignment $\rho : X \to \mathbb{R}$, we write $t^{\mathbb{R}\rho}$ for the denotation of term t in $\mathbb{R}$ for ρ. When t does not contain variables we may use $t^{\mathbb{R}}$ for the denotation of t in $\mathbb{R}$.

By a *quantum interpretation structure* (in short, *quantum structure*) we mean a triple

$$I = (\mathcal{H}, \psi, B_j \mapsto \underline{B}_j)$$

where:

- $\mathcal{H}$ is a complex separable Hilbert space;

- ψ is a unit vector in $\mathcal{H}$;

- $B_j \mapsto \underline{B}_j$ is a map that associates to each propositional symbol B_j a propositional quantum variable $\underline{B}_j = (O_j, \mathcal{D}_j, \uparrow_j)$ defined on $\mathcal{H}$ where $|\mathcal{D}_j| = 2$.

In the sequel, given $A \subseteq B$ and a valuation $v : A \to \{0, 1\}$, let

$$D_j^v = \begin{cases} \uparrow_j & \text{if } v(B_j) = 1 \\ \downarrow_j & \text{otherwise} \end{cases}$$

for each $B_j \in A$. Therefore, we are taking $\uparrow$ as true and $\downarrow$ as false.

Given $\alpha \in L_{\mathsf{c}}$, let $\underline{\alpha} = \{\underline{B}_j : B_j \in B_\alpha\}$. We say that α is an *I-observable formula* if $\underline{\alpha}$ is a compatible set of propositional quantum variables. Observe that only the essential propositional symbols are relevant.

A quantum structure I assigns a probability to any formula $\alpha \in L_{\mathsf{c}}$ such that $\underline{\alpha}$ is a compatible set as follows:

$$\mathsf{Prob}_I(\alpha) = \sum_{\substack{v \,:\, B_\alpha \to \{0,1\} \\ v \Vdash_{\mathsf{c}} \alpha}} \mathsf{Prob}\left(\langle (\underline{B}_{j_1})\psi, \ldots, (\underline{B}_{j_k})\psi \rangle = \langle D_{j_1}^v, \ldots, D_{j_k}^v \rangle \right)$$

$$= \sum_{\substack{v \,:\, B_\alpha \to \{0,1\} \\ v \Vdash_{\mathsf{c}} \alpha}} \langle \psi, \mathrm{Proj}_{D_{j_1}^v}^{j_1} \ldots \mathrm{Proj}_{D_{j_k}^v}^{j_k} \psi \rangle.$$

assuming that $B_\alpha = \{B_{j_1}, \ldots, B_{j_k}\}$. We prove in Section 5 that Prob_I is indeed a *probability assignment* (see [Ada98, SRS17]) when the formula at hand involves only a compatible set of observables according to I, that is, a map P on L_{c} satisfying the following (Adams) principles:

P1 $0 \leq P(\alpha) \leq 1$;

P2 If $\vDash_{\mathsf{c}} \alpha$ then $P(\alpha) = 1$;

P3 If $\alpha \vDash_{\mathsf{c}} \beta$ then $P(\alpha) \leq P(\beta)$;

P4 If $\vDash_{\mathsf{c}} \neg(\beta \wedge \alpha)$ then $P(\beta \vee \alpha) = P(\beta) + P(\alpha)$.

In [SRS17] we showed that P3 follows from P1, P2 and P4.

As we proceed to explain, a quantum structure I tells us, among other things, which are the knowable classical formulas in the sense that they do not involve incompatible sets of quantum variables.

Given a quantum structure I and an assignment $\rho : X \to \mathbb{R}$, *satisfaction of formulas* by I and ρ is inductively defined as follows:

- $I\rho \Vdash \odot\,\alpha$ whenever α is I-observable;

- $I\rho \Vdash \int \alpha \,@\, p$ whenever $I\rho \Vdash \odot\,\alpha$ and $\mathsf{Prob}_I(\alpha) \,@\, p^{\mathbb{R}\rho}$;

- $I\rho \Vdash \varphi_1 \supset \varphi_2$ whenever $I\rho \not\Vdash \varphi_1$ or $I\rho \Vdash \varphi_2$.

We may omit the reference to the assignment ρ whenever the PLQO formula in hand does not include variables. For examples illustrating satisfaction, see Section 8.

Let $\Gamma \subseteq L_{\mathsf{PLQO}}$ and $\varphi \in L_{\mathsf{PLQO}}$. We say that Γ *entails* φ, written $\Gamma \vDash \varphi$, whenever, for every quantum structure I and assignment ρ, if $I\rho \Vdash \gamma$ for each $\gamma \in \Gamma$ then $I\rho \Vdash \varphi$. As expected, φ is said to be *valid* when $\vDash \varphi$.

Observe that entailment in PLQO is not compact. Indeed, since $\mathbb{R}$ is Archimedean and $\odot B_1$ is valid,

$$\left\{ \int B_1 \leq \frac{1}{n} : n \in \mathbb{N} \right\} \vDash \int B_1 = 0.$$

However, there is no finite subset Ψ of $\{ \int B_1 \leq \frac{1}{n} : n \in \mathbb{N} \}$ such that

$$\Psi \vDash \int B_1 = 0.$$

Calculus

The PLQO calculus combines propositional reasoning with RCOF reasoning. We intend to use the RCOF reasoning to a minimum, namely to prove assertions like

$$(\beta_1 \wedge \cdots \wedge \beta_k) \supset \beta$$

where $\beta_1, \ldots, \beta_k, \beta$ are PLQO literals. To this end, we need to introduce some preliminary material. Given $U \subseteq A \subseteq B$ where A is finite, we use the abbreviation

$$\phi_A^U \quad \text{for} \quad \left(\bigwedge_{B_j \in U} B_j \right) \wedge \left(\bigwedge_{B_j \in A \setminus U} \neg B_j \right),$$

when $A \neq \emptyset$, and, for $\mathfrak{t}$, otherwise. Note that each classical formula ϕ_A^U identifies the valuation on A that assigns true to the propositional symbols in U and assigns false to the propositional symbols in $A \setminus U$.

Given a propositional formula α, we denote by

$$Q^{\odot\,\alpha}$$

the RCOF formula

$$\bigwedge_{\substack{A' \subseteq B_\alpha \\ |A'| = 2}} x_{A'} = 0,$$

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and by

$$Q^{\int \alpha}$$

the RCOF formula

$$\left(\bigwedge_{U \subseteq B_\alpha} 0 \le x_{\phi^U_{B_\alpha}} \le 1 \right) \wedge \left(\sum_{U \subseteq B_\alpha} x_{\phi^U_{B_\alpha}} = 1 \right) \wedge \left(\bigwedge_{\substack{A' \subseteq B_\alpha \\ U' \subseteq A'}} x_{\phi^{U'}_{A'}} = \sum_{\substack{U \subseteq B_\alpha \\ U \cap A' = U'}} x_{\phi^U_{B_\alpha}} \right) \wedge$$

$$\left(x_\alpha = \sum_{\substack{v \,:\, B_\alpha \to \{0,1\} \\ v \Vdash_{\mathsf{c}} \alpha}} x_{\phi^{\{B_j \in B_\alpha : v(B_j)=1\}}_{B_\alpha}} \right).$$

We proceed to explain the meaning of each conjunct of formula $Q^{\int \alpha}$. The idea is that each variable x_γ represents the probability of γ being true in a particular quantum structure which, according to the definition, is the sum of the probabilities of obtaining the results induced by the valuations that satisfy γ when performing the measurements associated with the formula (assuming that the underlying observables are compatible). When γ is $\phi^U_{B_\alpha}$ there is a unique valuation on B_α that satisfies γ and so we can look at x_γ as the probability of obtaining the results induced by that valuation. The conjunct

$$\bigwedge_{U \subseteq B_\alpha} 0 \le x_{\phi^U_{B_\alpha}} \le 1$$

imposes that the probability of obtaining the results induced by each possible valuation on A is in the interval $[0,1]$. The conjunct

$$\sum_{U \subseteq B_\alpha} x_{\phi^U_{B_\alpha}} = 1$$

expresses that the sum of those probabilities is one. The conjunct

$$x_{\phi^{U'}_{A'}} = \sum_{\substack{U \subseteq B_\alpha \\ U \cap A' = U'}} x_{\phi^U_{B_\alpha}}$$

is a marginal probability condition, and the conjunct

$$x_\alpha = \sum_{\substack{v \,:\, B_\alpha \to \{0,1\} \\ v \Vdash_{\mathsf{c}} \alpha}} x_{\phi^{\{B_j \in B_\alpha : v(B_j)=1\}}_{B_\alpha}}$$

expresses that the probability of α_i being true in a quantum structure is the sum of the probabilities of obtaining the results induced by each valuation that satisfies the formula.

With respect to $Q^{\odot \alpha}$, each variable $x_{A'}$ is intended to represent whether or not the observables associated with the propositional symbols in A' are compatible.

We are now ready to define an equivalent RCOF formula for each PLQO formula. Given a formula β of PLQO of the form $\int \alpha \,@\, p$ or $\odot \alpha$ or $\neg \odot \alpha$, let β^{RCOF} be the RCOF formula

$$\begin{cases} Q^{\odot \alpha} \wedge Q^{\int \alpha} \wedge (x_\alpha \,@\, p) & \text{if } \beta \text{ is } \int \alpha \,@\, p \\ Q^{\odot \alpha} \wedge Q^{\int \alpha} & \text{if } \beta \text{ is } \odot \alpha \\ \neg\, Q^{\odot \alpha} & \text{if } \beta \text{ is } \neg \odot \alpha. \end{cases}$$

Moreover, we define φ^{RCOF}, where φ is a formula of PLQO, inductively as follows:

- φ^{RCOF} is β^{RCOF} whenever φ is either $\int \alpha \,@\, p$ or $\odot \alpha$ or $\neg \odot \alpha$;

- φ^{RCOF} is $\varphi_1^{\mathsf{RCOF}} \supset \varphi_2^{\mathsf{RCOF}}$ whenever φ is $\varphi_1 \supset \varphi_2$ and is not of the form $\neg \odot \alpha$.

The calculus for PLQO embodies classical reasoning and makes use of RCOF theorems. It contains the following axioms and rules, assuming that φ_1, φ_2 are formulas and $\beta_1, \cdots, \beta_k, \beta$ are formulas of the form $\odot \alpha$, or $\neg \odot \alpha$ or $\int \alpha \,@\, p$:

TT $\dfrac{}{\varphi_1}$

        provided that φ_1 is a tautological formula;

MP $\dfrac{\begin{array}{c} \varphi_1 \\ \varphi_1 \supset \varphi_2 \end{array}}{\varphi_2}$;

RR $\dfrac{}{(\beta_1 \wedge \cdots \wedge \beta_k) \supset \beta}$ provided that

$$\forall \left(\left(\bigwedge_{j=1}^{k} \beta_j^{\mathsf{RCOF}} \right) \supset \beta^{\mathsf{RCOF}} \right)$$

        is a theorem of RCOF.

Axioms TT and rule MP extend classical reasoning to PLQO formulas. Rule RR import all we need from RCOF for deriving PLQO assertions on quantum observability.

Examples

We start by showing that
$$\vdash (\odot \alpha_1) \equiv (\odot \alpha_2)$$

provided that $\vDash_{\mathsf{c}} \alpha_1 \equiv \alpha_2$, that is, observability is preserved by classical equivalence. Observe that $eB_{\alpha_1} = eB_{\alpha_2}$ since $\vDash_{\mathsf{c}} \alpha_1 \equiv \alpha_2$. Thus, $B_{\alpha_1} = B_{\alpha_2}$. Hence,

$$Q^{\odot \alpha_1} \text{ is } Q^{\odot \alpha_2} \text{ and } Q^{\int \alpha_1} \text{ is } Q^{\int \alpha_2}$$

and so,

$$\forall \left((Q^{\odot \alpha_1} \wedge Q^{\int \alpha_1}) \supset (Q^{\odot \alpha_2} \wedge Q^{\int \alpha_2}) \right)$$

$$1 \quad \forall \Big((Q^{\odot \alpha_1} \wedge Q^{\int \alpha_1}) \supset (Q^{\odot \alpha_2} \wedge Q^{\int \alpha_2}) \Big) \qquad\qquad\qquad \text{RCOF}$$

$$2 \quad (\odot\, \alpha_1) \supset (\odot\, \alpha_2) \qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{RR } 1$$

$$3 \quad \forall \Big((Q^{\odot \alpha_2} \wedge Q^{\int \alpha_2}) \supset (Q^{\odot \alpha_1} \wedge Q^{\int \alpha_1}) \Big) \qquad\qquad\qquad \text{RCOF}$$

$$4 \quad (\odot\, \alpha_2) \supset (\odot\, \alpha_1) \qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{RR } 3$$

$$5 \quad ((\odot\, \alpha_1) \supset (\odot\, \alpha_2)) \supset (((\odot\, \alpha_2) \supset (\odot\, \alpha_1)) \supset ((\odot\, \alpha_1) \equiv (\odot\, \alpha_2))) \qquad \text{TT}$$

$$6 \quad ((\odot\, \alpha_2) \supset (\odot\, \alpha_1)) \supset ((\odot\, \alpha_1) \equiv (\odot\, \alpha_2)) \qquad\qquad\qquad \text{MP } 2,5$$

$$7 \quad (\odot\, \alpha_1) \equiv (\odot\, \alpha_2) \qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{MP } 4,6$$

Figure 1: $\;\vdash (\odot\, \alpha_1) \equiv (\odot\, \alpha_2)$ provided that $\vDash_{\mathsf{c}} \alpha_1 \equiv \alpha_2$.

and

$$\forall \Big((Q^{\odot \alpha_2} \wedge Q^{\int \alpha_2}) \supset (Q^{\odot \alpha_1} \wedge Q^{\int \alpha_1}) \Big)$$

are theorems of RCOF. Thus, we have the derivation in Figure 1.

As another example, we now prove that

$$\vdash \odot\, \mathord{t\!\!t},$$

that is, that $\mathord{t\!\!t}$ is always observable. Indeed,

$$\forall \Big(\mathord{t\!\!t} \supset (\odot\, \mathord{t\!\!t})^{\mathsf{RCOF}} \Big)$$

is a theorem of RCOF since $(\odot\, \mathord{t\!\!t})^{\mathsf{RCOF}}$ is $Q^{\odot \mathord{t\!\!t}} \wedge Q^{\int \mathord{t\!\!t}}$ which is $\mathord{t\!\!t}$. Therefore, by RR, we have that $\odot\, \mathord{t\!\!t}$ is a theorem.

As a consequence of these two examples, every tautology is observable taking into account the two given examples.

Finally, we prove that

$$\odot(B_1 \wedge B_2) \vdash \textstyle\int(B_1 \wedge B_2) \geq 0.$$

We start by verifying that

$$\forall \Big((\odot(B_1 \wedge B_2))^{\mathsf{RCOF}} \supset (\textstyle\int(B_1 \wedge B_2) \geq 0)^{\mathsf{RCOF}} \Big)$$

is a theorem of RCOF. Indeed,

- $(\odot(B_1 \wedge B_2))^{\mathsf{RCOF}}$ is $Q^{\odot(B_1 \wedge B_2)} \wedge Q^{\int B_1 \wedge B_2}$;

- $(\int(B_1 \wedge B_2) \geq 0)^{\mathsf{RCOF}}$ is $Q^{\odot(B_1 \wedge B_2)} \wedge Q^{\int B_1 \wedge B_2} \wedge (x_{B_1 \wedge B_2} \geq 0)$.

Thus, we have the derivation in Figure 2.

1	$\odot(B_1 \wedge B_2)$	HYP
2	$\forall\left((\odot(B_1 \wedge B_2))^{\mathsf{RCOF}} \supset (\int(B_1 \wedge B_2) \geq 0)^{\mathsf{RCOF}}\right)$	RCOF
3	$(\odot(B_1 \wedge B_2)) \supset (\int(B_1 \wedge B_2) \geq 0)$	RR 2
4	$\int(B_1 \wedge B_2) \geq 0$	MP 1,3

Figure 2: $\odot(B_1 \wedge B_2) \vdash \int(B_1 \wedge B_2) \geq 0$.

5 Strong soundness

In this section we show that the calculus for PLQO is strongly sound, starting by proving some auxiliary results namely that Prob_I is a probability assignment.

Let I be a quantum structure, $A \subseteq B$ a set such that $\{\underline{B}_j : B_j \in A\}$ is a compatible set of propositional quantum variables and v a valuation on A. Given an orthonormal basis U of $\mathcal{H}$ composed of common eigenvectors of the observables in $\{\underline{B}_j : B_j \in A\}$ (observe that we assume that observables are diagonizable and bounded), we denote by

$$U_v$$

the set $\{u \in U : O_j u = \lambda_j u, \lambda_j \in D^v_{B_j}, B_j \in A\}$.

Proposition 5.1 Given a quantum structure I, sets $A \subseteq A' \subseteq B$ such that $\{\underline{B}_j : B_j \in A'\}$ is a compatible set of propositional quantum variables, an orthonormal basis U of $\mathcal{H}$ composed of common eigenvectors of the observables in $\{\underline{B}_j : B_j \in A'\}$, and $v : A \rightarrow \{0,1\}$, then

$$\{U_{v'} : v' : A' \rightarrow \{0,1\} \text{ and } v'|_A = v\}$$

is a partition of U_v.

Proof:
(1) $U_{v'_1} \cap U_{v'_2} = \emptyset$ for every $v'_1, v'_2 : A' \rightarrow \{0,1\}$ such that $v'_1|_A = v$ and $v'_2|_A = v$ and $v'_1 \neq v'_2$. Indeed, suppose, by contradiction that $U_{v'_1} \cap U_{v'_2} \neq \emptyset$. Let $u' \in U_{v'_1} \cap U_{v'_2}$. Since $v'_1 \neq v'_2$ then there is $B_{j'} \in A'$ such that $v'_1(B_{j'}) \neq v'_2(B_{j'})$. Hence $D^{v'_1}_{B_{j'}} \neq D^{v'_2}_{B_{j'}}$ and so $D^{v'_1}_{B_{j'}} \cap D^{v'_2}_{B_{j'}} = \emptyset$. Thus, there is no $\lambda_{j'}$ such that $O_{j'} u' = \lambda_{j'} u'$ and $\lambda_{j'} \in D^{v'_1}_{B_{j'}}$ and $\lambda_{j'} \in D^{v'_2}_{B_{j'}}$ contradicting the assumption that $u' \in U_{v'_1}$ and $u' \in U_{v'_2}$.

(2) $U_v \subseteq \bigcup_{v':A'\rightarrow\{0,1\}, v'|_A=v} U_{v'}$.
Indeed, let $u \in U_v$. Then, for every $B_j \in A$ we have $O_j u = \lambda_j u$ and $\lambda_j \in D^v_{B_j}$. Let $v' : A' \rightarrow \{0,1\}$ be the valuation such that

$$v'(B_{j'}) = \begin{cases} 1 & \text{if } \lambda_{j'} \in \uparrow_{j'} \\ 0 & \text{otherwise} \end{cases}$$

for each $B_{j'} \in A'$ and $\lambda_{j'}$ such that $O_{j'} u = \lambda_{j'} u$. Then, it is immediate to see that $u \in U_{v'}$ by definition of v'. It remains to show that $v'|_A = v$. Indeed, let $B_j \in A$. There are two cases.
(a) $v'|_A(B_j) = 1$. Then, $O_j u = \lambda_j u$ and $\lambda_j \in \uparrow_j$. Hence, $D^v_{B_j} = \uparrow_j$ since $u \in U_v$. So,

$v(B_j) = 1$.

(b) $v'|_A(B_j) = 0$. We omit the proof since it is similar to (1).

(3) $U_{v'} \subseteq U_v$ for every $v' : A' \to \{0,1\}$ such that $v'|_A = v$. Let $v' : A' \to \{0,1\}$ be an arbitrary valuation such that $v'|_A = v$. Let $u' \in U_{v'}$. Then, for every $B_{j'} \in A'$ we have $O_{j'} u' = \lambda_{j'} u'$ and $\lambda_{j'} \in D_{B_{j'}}^{v'}$. Let $B_j \in A$. Hence, $O_j u' = \lambda_j u'$ and $\lambda_j \in D_{B_j}^{v'}$. Observe that $v'(B_j) = v(B_j)$. So $D_{B_j}^{v'} = D_{B_j}^v$. Therefore, $u' \in U_v$. $\hfill$ QED

Note that, given a quantum structure $I = (\mathcal{H}, \psi, B_j \mapsto \underline{B}_j)$, a set $\{B_{j_1}, \ldots, B_{j_k}\}$ such that $\{\underline{B}_{j_i} : 1 \leq i \leq k\}$ is a compatible set, an orthonormal basis U of $\mathcal{H}$ composed of common eigenvectors of the observables in $\{\underline{B}_{j_i} : 1 \leq i \leq k\}$ and a valuation $v : \{B_{j_1}, \ldots, B_{j_k}\} \to \{0,1\}$, then

$$\mathrm{Proj}_{D_{j_k}^v}^{j_k} \ldots \mathrm{Proj}_{D_{j_1}^v}^{j_1} \psi \quad \text{is} \quad \sum_{u \in U_v} \langle \psi, u \rangle u.$$

Proposition 5.2 Given a quantum structure $I = (\mathcal{H}, \psi, B_j \mapsto \underline{B}_j)$, $\alpha \in L_{\mathsf{c}}$ such that $\underline{\alpha}$ is a compatible set, an orthonormal basis U of $\mathcal{H}$ composed of common eigenvectors of the observables in $\underline{\alpha}$, then

$$\mathsf{Prob}_I(\alpha) = \sum_{\substack{v \,:\, B_\alpha \to \{0,1\} \\ v \Vdash_{\mathsf{c}} \alpha}} \left(\sum_{u \in U_v} \langle \psi, u \rangle^2 \right).$$

Proof:
Assume without loss of generality that $B_\alpha = \{B_1, \ldots, B_k\}$. Then,

$$\mathsf{Prob}_I(\alpha) = \sum_{\substack{v \,:\, B_\alpha \to \{0,1\} \\ v \Vdash_{\mathsf{c}} \alpha}} \mathsf{Prob}\left(\langle (\underline{B}_1)_\psi, \ldots, (\underline{B}_k)_\psi \rangle = \langle D_1^v, \ldots, D_k^v \rangle \right)$$

$$= \sum_{\substack{v \,:\, B_\alpha \to \{0,1\} \\ v \Vdash_{\mathsf{c}} \alpha}} \langle \psi, \mathrm{Proj}_{D_k^v} \ldots \mathrm{Proj}_{D_1^v} \psi \rangle$$

$$= \sum_{\substack{v \,:\, B_\alpha \to \{0,1\} \\ v \Vdash_{\mathsf{c}} \alpha}} \langle \psi, \sum_{u \in U_v} \langle \psi, u \rangle u \rangle$$

$$= \sum_{\substack{v \,:\, B_\alpha \to \{0,1\} \\ v \Vdash_{\mathsf{c}} \alpha}} \left(\sum_{u \in U_v} \langle \psi, u \rangle^2 \right).$$

$\hfill$ QED

Proposition 5.3 Given a quantum structure $I = (\mathcal{H}, \psi, B_j \mapsto \underline{B}_j)$ and $\alpha, \gamma \in L_{\mathsf{c}}$ such that $B_\gamma \subseteq B_\alpha$ and $\underline{\alpha}$ is a compatible set, then $\alpha \vDash_{\mathsf{c}} \gamma$ implies $\mathsf{Prob}_I(\alpha) \leq \mathsf{Prob}_I(\gamma)$. Moreover, $\gamma \vDash_{\mathsf{c}} \alpha$ implies $\mathsf{Prob}_I(\gamma) \leq \mathsf{Prob}_I(\alpha)$.

Proof:
Let U be an orthonormal basis of $\mathcal{H}$ composed of common eigenvectors of the observables in

$\underline{\alpha}$.

We start by showing that $\alpha \vDash_{\mathsf{c}} \gamma$ implies $\mathsf{Prob}_I(\alpha) \leq \mathsf{Prob}_I(\gamma)$. Assume that $\alpha \vDash_{\mathsf{c}} \gamma$. Then,

$$\mathsf{Prob}_I(\alpha) = \sum_{\substack{v\, :\, B_\alpha\, \to\, \{0,1\} \\ v\, \Vdash_{\mathsf{c}}\, \alpha}} \left(\sum_{u \in U_v} \langle \psi, u \rangle^2 \right)$$

$$\leq \sum_{\substack{v\, :\, B_\alpha\, \to\, \{0,1\} \\ v\, \Vdash_{\mathsf{c}}\, \gamma}} \left(\sum_{u \in U_v} \langle \psi, u \rangle^2 \right)$$

$$= \sum_{\substack{v\, :\, B_\gamma\, \to\, \{0,1\} \\ v\, \Vdash_{\mathsf{c}}\, \gamma}} \left(\sum_{\substack{v'\, :\, B_\alpha\, \to\, \{0,1\} \\ v'|_{B_\gamma}\, =\, v}} \sum_{|u'\rangle \in U_{v'}} \langle \psi, u' \rangle^2 \right)$$

$$= \sum_{\substack{v\, :\, B_\gamma\, \to\, \{0,1\} \\ v\, \Vdash_{\mathsf{c}}\, \gamma}} \left(\sum_{u \in U_v} \langle \psi, u \rangle^2 \right) \quad (*)$$

$$= \mathsf{Prob}_I(\gamma)$$

where $(*)$ follows from Proposition 5.1.

We now show that $\gamma \vDash_{\mathsf{c}} \alpha$ implies $\mathsf{Prob}_I(\gamma) \leq \mathsf{Prob}_I(\alpha)$. Assume that $\gamma \vDash_{\mathsf{c}} \alpha$. Then,

$$\mathsf{Prob}_I(\gamma) = \sum_{\substack{v\, :\, B_\gamma\, \to\, \{0,1\} \\ v\, \Vdash_{\mathsf{c}}\, \gamma}} \left(\sum_{u \in U_v} \langle \psi, u \rangle^2 \right)$$

$$= \sum_{\substack{v\, :\, B_\gamma\, \to\, \{0,1\} \\ v\, \Vdash_{\mathsf{c}}\, \gamma}} \left(\sum_{\substack{v'\, :\, B_\alpha\, \to\, \{0,1\} \\ v'|_{B_\gamma}\, =\, v}} \sum_{|u'\rangle \in U_{v'}} \langle \psi, u' \rangle^2 \right) \quad (*)$$

$$= \sum_{\substack{v\, :\, B_\alpha\, \to\, \{0,1\} \\ v\, \Vdash_{\mathsf{c}}\, \gamma}} \left(\sum_{u \in U_v} \langle \psi, u \rangle^2 \right)$$

$$\leq \sum_{\substack{v\, :\, B_\alpha\, \to\, \{0,1\} \\ v\, \Vdash_{\mathsf{c}}\, \alpha}} \left(\sum_{u \in U_v} \langle \psi, u \rangle^2 \right)$$

$$= \mathsf{Prob}_I(\alpha)$$

where $(*)$ follows from Proposition 5.1. QED

Proposition 5.4 Given a quantum structure $I = (\mathcal{H}, \psi, B_j \mapsto \underline{B}_j)$, Prob_I is a probability assignment for every formula $\alpha \in L_{\mathsf{c}}$ such that $\underline{\alpha}$ is a compatible set.

Proof:

(1) $0 \leq \mathsf{Prob}_I(\alpha) \leq 1$, for every formula α such that $\underline{\alpha}$ is a compatible set. Let U be an orthonormal basis of $\mathcal{H}$ composed of common eigenvectors of the observables in $\underline{\alpha}$. Then,

$$\mathsf{Prob}_I(\alpha) = \sum_{\substack{v \,:\, B_\alpha \to \{0,1\} \\ v \Vdash_{\mathsf{c}} \alpha}} \left(\sum_{u \in U_v} \langle \psi, u \rangle^2 \right)$$

$$\geq 0$$

and

$$\mathsf{Prob}_I(\alpha) = \sum_{\substack{v \,:\, B_\alpha \to \{0,1\} \\ v \Vdash_{\mathsf{c}} \alpha}} \left(\sum_{u \in U_v} \langle \psi, u \rangle^2 \right)$$

$$\leq \sum_{v : B_\alpha \to \{0,1\}} \left(\sum_{u \in U_v} \langle \psi, u \rangle^2 \right)$$

$$= \sum_{u \in U} \langle \psi, u \rangle^2 \qquad (*)$$

$$= 1,$$

where $(*)$ holds, by Proposition 5.1 since $\{U_v : v : B_\alpha \to \{0,1\}$ is a partition of U.

(2) If $\vDash_{\mathsf{c}} \alpha$ then $\mathsf{Prob}_I(\alpha) = 1$, for every formula α such that $\underline{\alpha}$ is a compatible set. Let U be an orthonormal basis of $\mathcal{H}$ composed of common eigenvectors of the observables in $\underline{\alpha}$. Assuming that α is a tautology, then

$$\mathsf{Prob}_I(\alpha) = \sum_{\substack{v \,:\, B_\alpha \to \{0,1\} \\ v \Vdash_{\mathsf{c}} \alpha}} \left(\sum_{u \in U_v} \langle \psi, u \rangle^2 \right)$$

$$= \sum_{v : B_\alpha \to \{0,1\}} \left(\sum_{u \in U_v} \langle \psi, u \rangle^2 \right)$$

$$= \sum_{u \in U} \langle \psi, u \rangle^2 \qquad (*)$$

$$= 1,$$

where $(*)$ holds, by Proposition 5.1 since $\{U_v : v : B_\alpha \to \{0,1\}$ is a partition of U.

(3) If $\alpha \vDash_{\mathsf{c}} \beta$ then $\mathsf{Prob}_I(\alpha) \leq \mathsf{Prob}_I(\beta)$, for every formulas α and β such that $\underline{\alpha}$ and $\underline{\beta}$ are compatible sets. Assume that $\alpha \vDash_{\mathsf{c}} \beta$. Then, since propositional logic has the Craig interpolation property there is γ such $B_\gamma \subseteq B_\alpha \cap B_\beta$ and $\alpha \vDash_{\mathsf{c}} \gamma$ and $\gamma \vDash_{\mathsf{c}} \beta$. Then, by Proposition 5.3, $\mathsf{Prob}_I(\alpha) \leq \mathsf{Prob}_I(\gamma)$ and $\mathsf{Prob}_I(\gamma) \leq \mathsf{Prob}_I(\beta)$.

(4) If $\vDash_{\mathsf{c}} \neg(\beta \wedge \alpha)$ then $\mathsf{Prob}_I(\beta \vee \alpha) = \mathsf{Prob}_I(\beta) + \mathsf{Prob}_I(\alpha)$, for every formulas α and β such

that $\underline{\alpha} \cup \underline{\beta}$ is a compatible set. Assume that $\vDash_{\mathsf{c}} \neg(\beta \wedge \alpha)$. Then,

$$\mathsf{Prob}_I(\beta \vee \alpha) = \sum_{\substack{v \,:\, B_{\beta \vee \alpha} \to \{0,1\} \\ v \,\Vdash_{\mathsf{c}}\, \beta \vee \alpha}} \left(\sum_{u \in U_v} \langle \psi, u \rangle^2 \right)$$

$$= \sum_{\substack{v' \,:\, B_\alpha \cup B_\beta \to \{0,1\} \\ v' \,\Vdash_{\mathsf{c}}\, \beta \vee \alpha}} \left(\sum_{|u'\rangle \in U_{v'}} \langle \psi, u' \rangle^2 \right)$$

$$= \sum_{\substack{v' \,:\, B_\alpha \cup B_\beta \to \{0,1\} \\ v' \,\Vdash_{\mathsf{c}}\, \beta}} \left(\sum_{|u'\rangle \in U_{v'}} \langle \psi, u' \rangle^2 \right) +$$

$$\sum_{\substack{v' \,:\, B_\alpha \cup B_\beta \to \{0,1\} \\ v' \,\Vdash_{\mathsf{c}}\, \alpha}} \left(\sum_{|u'\rangle \in U_{v'}} \langle \psi, u' \rangle^2 \right) \qquad (*)$$

$$= \left(\sum_{\substack{v \,:\, B_\beta \to \{0,1\} \\ v \,\Vdash_{\mathsf{c}}\, \beta}} \sum_{u \in U_v} \langle \psi, u \rangle^2 \right) + \left(\sum_{\substack{v \,:\, B_\alpha \to \{0,1\} \\ v \,\Vdash_{\mathsf{c}}\, \alpha}} \sum_{u \in U_v} \langle \psi, u \rangle^2 \right) \quad (**)$$

$$= \mathsf{Prob}_I(\beta) + \mathsf{Prob}_I(\alpha)$$

where $(*)$ holds by Proposition 5.1 and $(**)$ holds by the hypothesis $\vDash_{\mathsf{c}} \neg(\beta \wedge \alpha)$. $\qquad$ QED

Proposition 5.5 Given a quantum structure $I = (\mathcal{H}, \psi, B_j \mapsto \underline{B}_j)$, sets $A' \subseteq A \subseteq B$ such that $\{\underline{B}_j : B_j \in A\}$ is a compatible set, then

$$\mathsf{Prob}_I(\phi_{A'}^{U'}) = \sum_{\substack{U \subseteq A \\ U \cap A' = U'}} \mathsf{Prob}_I(\phi_A^U)$$

for every $U' \subseteq A'$.

Proof:
Assume, without loss of generality that, $A = \{B_1, \ldots, B_n\}$, $A' = \{B_1, \ldots, B_k\}$ and $U' = \{B_1, \ldots, B_\ell\}$ where $k \leq n$ and $\ell \leq k$. Observe that:

$$\mathsf{Prob}_I(\phi_{A'}^{U'}) = \sum_{\substack{v \,:\, A' \to \{0,1\} \\ v \,\Vdash_{\mathsf{c}}\, \phi_{A'}^{U'}}} \langle \psi, \mathrm{Proj}_{D_1^v}^1 \ldots \mathrm{Proj}_{D_k^v}^k \psi \rangle$$

$$= \langle \psi, \mathrm{Proj}_\uparrow^1 \ldots \mathrm{Proj}_\uparrow^\ell \mathrm{Proj}_\downarrow^{\ell+1} \ldots \mathrm{Proj}_\downarrow^k \psi \rangle$$

$$= \langle \psi, \mathrm{Proj}_\uparrow^1 \ldots \mathrm{Proj}_\uparrow^\ell \mathrm{Proj}_\downarrow^{\ell+1} \ldots \mathrm{Proj}_\downarrow^k$$
$$(\mathrm{Proj}_\uparrow^{k+1} + \mathrm{Proj}_\downarrow^{k+1}) \ldots (\mathrm{Proj}_\uparrow^n + \mathrm{Proj}_\downarrow^n) \psi \rangle$$

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$$= \sum_{(D_{k+1},\ldots,D_n)\in\{\uparrow,\downarrow\}^{n-k}} \langle\psi, \text{Proj}_\uparrow^1 \ldots \text{Proj}_\uparrow^\ell \text{Proj}_\downarrow^{\ell+1} \ldots \text{Proj}_\downarrow^k$$

$$\text{Proj}_{D_{k+1}}^{k+1} \ldots \text{Proj}_{D_n}^n \psi\rangle$$

$$= \sum_{(D_{k+1},\ldots,D_n)\in\{\uparrow,\downarrow\}^{n-k}} \text{Prob}_I(\phi_A^{U'\cup\{D_j:D_j=\uparrow \text{ and } j=k+1,\ldots,n\}})$$

$$= \sum_{\substack{U\subseteq A \\ U\cap A'=U'}} \text{Prob}_I(\phi_A^U)$$

QED

Now we are ready to show that the proposed axiomatization of PLQO is strongly sound.

Proposition 5.6 The logic PLQO is strongly sound.

Proof:
We only check that axiom RR is sound since the proof of the others is straightforward.

Rule (RR). Let I be a quantum structure and $\rho : X \to \mathbb{R}$ be an assignment. Assume that

$$I\rho \Vdash \beta_1 \wedge \cdots \wedge \beta_k$$

and that

$$\forall\left(\left(\bigwedge_{j=1}^k \beta_j^{\text{RCOF}}\right) \supset \beta^{\text{RCOF}}\right) \quad (*)$$

is in RCOF. Let $\rho' : X \to \mathbb{R}$ be an assignment such that $\rho'(x) = \rho(x)$ for $x \in X_\mathbb{N}$,

$$\rho'(x_\alpha) = \text{Prob}_I(\alpha)$$

for every $\alpha \in L_c$ such that $\underline{\alpha}$ is a compatible set of observables in I, and

$$\rho'(x_{\{B_i,B_j\}}) = \begin{cases} 0 & \text{if } \{\underline{B_i},\underline{B_j}\} \text{ is a compatible set of observables in } I \\ 1 & \text{otherwise} \end{cases}$$

for every $B_i, B_j \in B_{(\beta_1\wedge\cdots\wedge\beta_k)\supset\beta}$. We start by showing that

$$\mathbb{R}\rho' \Vdash_{\text{fo}} \beta_j^{\text{RCOF}},$$

for $j = 1,\ldots,k$. We consider three cases:
(1) β_j is $\odot\alpha_j$. Then, β_j^{RCOF} is

$$Q^{\odot\alpha_j} \wedge Q^{\int\alpha_j}.$$

Recall that $I\rho \Vdash \odot\alpha_j$. Hence, α_j is I-observable, that is $\underline{\alpha_j}$ is a compatible set of observables in I. Thus,

$$\rho'(x_{\{B_{i_1},B_{i_2}\}}) = 0$$

for every distinct $B_{i_1}, B_{i_2} \in B_{\alpha_j}$. Therefore,

$$\mathbb{R}\rho' \Vdash_{\text{fo}} Q^{\odot\alpha_j}.$$

Moreover, $\mathbb{R}\rho' \Vdash Q^{\int \alpha_j}$. In fact,

(a) $0 \leq \rho'(x_{\phi^U_{B_{\alpha_j}}}) \leq 1$. Let $U \subseteq B_{\alpha_j}$. Since

$$\rho'(x_{\phi^U_{B_{\alpha_j}}}) = \mathsf{Prob}_I(\phi^U_{B_{\alpha_j}})$$

then $0 \leq \rho'(x_{\phi^U_{B_{\alpha_j}}}) \leq 1$ by Proposition 5.4.

(b) $\sum_{U \subseteq B_{\alpha_j}} \rho'(x_{\phi^U_{B_{\alpha_j}}}) = 1$. Observe that

$$\bigvee_{U \subseteq B_{\alpha_j}} \phi^U_{B_{\alpha_j}}$$

is a tautology and so

$$\mathsf{Prob}_I\left(\bigvee_{U \subseteq B_{\alpha_j}} \phi^U_{B_{\alpha_j}}\right) = 1$$

by Proposition 5.4. Since, for distinct $U_1, U_2 \subseteq B_{\alpha_j}$,

$$\vDash_{\mathsf{c}} \neg(\phi^{U_1}_{B_{\alpha_j}} \wedge \phi^{U_2}_{B_{\alpha_j}}),$$

then, by Proposition 5.4,

$$\sum_{U \subseteq B_{\alpha_j}} \mathsf{Prob}_I(\phi^U_{B_{\alpha_j}}) = 1.$$

Thus,

$$\sum_{U \subseteq B_{\alpha_j}} \rho'(x_{\phi^U_{B_{\alpha_j}}}) = 1.$$

(c) $\rho'(x_{\phi^{U'}_{A'}}) = \displaystyle\sum_{\substack{U \subseteq B_{\alpha_j} \\ U \cap A' = U'}} \rho'(x_{\phi^U_{B_{\alpha_j}}})$ for every $A' \subseteq B_{\alpha_j}$ and $U' \subseteq A'$. Observe that

$$\rho'(x_{\phi^{U'}_{A'}}) = \mathsf{Prob}_I(\phi^{U'}_{A'})$$

$$= \sum_{\substack{U \subseteq B_{\alpha_j} \\ U \cap A' = U'}} \mathsf{Prob}_I(\phi^U_{B_{\alpha_j}}) \qquad (*)$$

$$= \sum_{\substack{U \subseteq B_{\alpha_j} \\ U \cap A' = U'}} \rho'(x_{\phi^U_{B_{\alpha_j}}})$$

where $(*)$ holds by Proposition 5.5.

(d) $\rho'(x_{\alpha_j}) = \sum_{\substack{v\,:\,B_{\alpha_j}\,\to\,\{0,1\} \\ v\,\Vdash_{\mathsf{c}}\,\alpha_j}} \rho'(x_{\phi^{\{B_j \in B_{\alpha_j}\,:\,v(B_j)=1\}}_{B_{\alpha_j}}})$. Observe that

$$\vDash_{\mathsf{c}} \alpha_j \equiv \left(\bigvee_{\substack{v\,:\,B_{\alpha_j}\,\to\,\{0,1\} \\ v\,\Vdash_{\mathsf{c}}\,\alpha_j}} \phi^{\{B_j \in B_{\alpha_j}\,:\,v(B_j)=1\}}_{B_{\alpha_j}} \right)$$

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and that

$$\models_{\mathsf{c}} \neg(\phi_{B_{\alpha_j}}^{\{B_j \in B_{\alpha_j} : v_1(B_j)=1\}} \wedge \phi_{B_{\alpha_j}}^{\{B_j \in B_{\alpha_j} : v_2(B_j)=1\}})$$

for every distinct pair $v_1, v_2 : B_{\alpha_j} \to \{0,1\}$. Thererefore, by Proposition 5.4, the thesis follows.

(2) β_j is $\int \alpha_j @ p_j$. Then, β_j^{RCOF} is

$$Q^{\odot \alpha_j} \wedge Q^{\int \alpha_j} \wedge (x_{\alpha_j} @ p).$$

Recall that $I\rho \Vdash \int \alpha_j @ p_j$. Then $I\rho \Vdash \odot \alpha_j$. Thus, by (1),

$$\mathbb{R}\rho' \Vdash_{\mathsf{fo}} Q^{\odot \alpha_j} \wedge Q^{\int \alpha_j}.$$

So it is enough to prove that

$$\mathbb{R}\rho' \Vdash_{\mathsf{fo}} x_{\alpha_j} @ p_j.$$

Since $I\rho \Vdash \int \alpha_j @ p_j$, then

$$\mathsf{Prob}_I(\alpha_j) @ p_j^{\mathbb{R}}.$$

Thus,

$$\rho'(x_{\alpha_j}) @ p_j^{\mathbb{R}}$$

and so, $\mathbb{R}\rho' \Vdash_{\mathsf{fo}} x_{\alpha_j} @ p_j$.

(3) β_j is $\neg \odot \alpha_j$. Then, β_j^{RCOF} is

$$Q^{\neg \odot \alpha_j}.$$

Recall that $I\rho \Vdash \neg \odot \alpha_j$. Hence, α_j is not an I-observable. Then, there are $B_i, B_j \in B_{\alpha_j}$ such that $\{\underline{B}_i, \underline{B}_j\}$ is an incompatible set of observables in I. So,

$$\rho'(x_{\{B_i, B_j\}}) = 1.$$

Therefore,

$$\mathbb{R}\rho' \Vdash_{\mathsf{fo}} \neg Q^{\odot \alpha_j}.$$

Moreover, by definition of ρ',

$$\mathbb{R}\rho' \Vdash_{\mathsf{fo}} \left(\bigwedge_{\substack{A' \subseteq B_{\alpha_j} \\ |A'| = 2}} x_{A'} \geq 0 \right).$$

In this way, we conclude the proof of

$$\mathbb{R}\rho' \Vdash_{\mathsf{fo}} \bigwedge_{j=1}^{k} \beta_j^{\mathsf{RCOF}}.$$

Returning to the main proof, observe that, by (*),

$$\mathbb{R}\rho' \Vdash_{\mathsf{fo}} \left(\bigwedge_{j=1}^{k} \beta_j^{\mathsf{RCOF}} \right) \supset \beta^{\mathsf{RCOF}},$$

so
$$\mathbb{R}\rho' \Vdash_{\mathsf{fo}} \beta^{\mathsf{RCOF}}.$$

It remains to prove that $I\rho \Vdash \beta$. There are three cases:

(1) β is $\odot \alpha$. Hence,
$$\mathbb{R}\rho' \Vdash_{\mathsf{fo}} Q^{\odot \alpha} \wedge Q^{\int \alpha}.$$

Therefore, $\rho'(x_{\{B_{i_1}, B_{i_2}\}}) = 0$ for every distinct $B_{i_1}, B_{i_2} \in B_\alpha$. Thus, by definition of ρ', $\{\underline{B_{i_1}}, \underline{B_{i_2}}\}$ is a compatible set of quantum variables. Therefore, $\underline{\alpha}$ is a compatible set of quantum variables, that is α is an I-observable formula and so $I\rho \Vdash \odot \alpha$.

(2) β is $\int \alpha \, @ \, p$. Thus,
$$\mathbb{R}\rho' \Vdash_{\mathsf{fo}} Q^{\odot \alpha} \wedge Q^{\int \alpha} \wedge (x_\alpha \, @ \, p).$$

Therefore, similarly to (1), $I\rho \Vdash \odot \alpha$. Moreover, $\rho'(x_\alpha) \, @ \, p^{\mathbb{R}\rho}$. Hence, by definition of ρ',
$$\mathsf{Prob}_I(\alpha) \, @ \, p^{\mathbb{R}\rho}$$

and so $I\rho \Vdash \int \alpha \, @ \, p$.

(3) β is $\neg \odot \alpha$. Thus,
$$\mathbb{R}\rho' \Vdash_{\mathsf{fo}} Q^{\neg \odot \alpha}.$$

Hence, $\rho'(x_{\{B_{i_1}, B_{i_2}\}}) = 1$ for some distinct $B_{i_1}, B_{i_2} \in B_\alpha$. Thus, by definition of ρ', $\{\underline{B_{i_1}}, \underline{B_{i_2}}\}$ is not a compatible set of quantum variables. Therefore, $\underline{\alpha}$ is not a compatible set of quantum variables, that is α is not an I-observable formula and so $I\rho \not\Vdash \odot \alpha$. Thus, $I\rho \Vdash \neg \odot \alpha$. QED

6 Weak completeness

The objective of this section is to show that PLQO is weakly complete. That is, for every formula $\varphi \in L_{\mathsf{PLQO}}$, if $\vDash \varphi$ then $\vdash \varphi$. There is no hope that there is a finitary calculus for which PLQO is strongly complete since, as we referred above, PLQO is not compact.

We prove the weak completeness by contrapositive, that is, if $\nvdash \varphi$ then $\nvDash \varphi$. Hence, assuming that φ is not derivable, we show that there is a quantum structure that falsifies φ. With this purpose in mind, we start by defining a generic quantum structure that will be relevant not only for weak completeness but also for conservativeness.

Given a finite set $B' \subseteq B$, $\mathsf{NC} \subseteq B' \times B'$ and a map $f : \{0, \ldots, 2^{|B'|} - 1\} \to \mathbb{C}$ such that NC is irreflexive and symmetric, and
$$\sqrt{\sum_{j=0}^{2^{|B'|}-1} f(j)\overline{f(j)}} = 1,$$

let
$$I^{B', \mathsf{NC}, f} = (\mathcal{H}, \psi, B_j \mapsto \underline{B_j})$$

be such that

- $\mathcal{H}$ is $\mathbb{C}^{2^{|B'|} + 2|\mathsf{NC}|}$ and fix
$$\{\zeta_{v_0}, \ldots, \zeta_{v_{2^{|B'|}-1}}\} \cup \bigcup_{A \in \mathsf{NC}} \{\zeta_{A^\bullet}, \zeta_{A_\bullet}\}$$

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as a basis for $\mathcal{H}$ where $v_k : B' \to \{0, 1\}$ is the valuation $B_j \mapsto b_j$ where b_j is the j-th bit in the binary decomposition of k (assuming left padding by zeros);

- ψ is

$$\sum_{j=0}^{2^{|B'|}-1} f(j) \zeta_{v_j};$$

- For each $B_j \in B'$, $\underline{B}_j = (O_j, \{\{0, 1\}, \{0, 1\}^{\mathsf{c}}\}, \{0, 1\})$ with

$$O_j = \left(\sum_{\{v_k : v_k(B_j)=1\}} \zeta_{v_k} \langle \zeta_{v_k}, \cdot \rangle \right) + \left(\sum_{\{v_k : v_k(B_j)=0\}} -\zeta_{v_k} \langle \zeta_{v_k}, \cdot \rangle \right) +$$

$$\left(\sum_{\{(B_i, B_j) \in \mathsf{NC} : j < i\}} \zeta_{(B_i, B_j)_{\bullet}^{\bullet}} \langle \zeta_{(B_i, B_j)_{\bullet}^{\bullet}}, \cdot \rangle \right) +$$

$$\left(\sum_{\{(B_i, B_j) \in \mathsf{NC} : j > i\}} \zeta_{(B_i, B_j)_{\bullet}} \langle \zeta_{(B_i, B_j)_{\bullet}}, \cdot \rangle \right) +$$

$$\left(\sum_{\{A \in \mathsf{NC} : B_j \notin A\}} \zeta_{A_{\bullet}} \langle \zeta_{A_{\bullet}}, \cdot \rangle + \zeta_{A^{\bullet}} \langle \zeta_{A^{\bullet}}, \cdot \rangle \right)$$

where

$$\zeta_{(B_i, B_j)_{\bullet}^{\bullet}} = \sqrt{\frac{1}{2}} \zeta_{(B_i, B_j)^{\bullet}} + \sqrt{\frac{1}{2}} \zeta_{(B_i, B_j)_{\bullet}};$$

- For $B_j \notin B'$, choose for O_j an arbitrary observable on $\mathcal{H}$, for $\mathcal{D}_j$ any partition of $\sigma_p(O_j)$ with two elements and for $\uparrow_j$ one of them.

Observe that the chosen basis for the Hilbert space $\mathcal{H}$ of $I^{B', \mathsf{NC}, f}$ is a finite set including vectors representing the valuations over the finite set B' of propositional symbols and vectors assigned to the non compatible pairs in NC. The map f corresponds to the "mass" of each valuation. So, the quantum state $|\psi\rangle$ is a superposition of the valuation vectors each weighted by its mass. The operator O_j associated with a propositional symbol $B_j \in B'$ is the sum of several operators:

- Projection $\sum_{\{v_k : v_k(B_j)=1\}} \zeta_{v_k} \langle \zeta_{v_k}, \cdot \rangle$ stating that the vectors associated with the valuations that satisfy B_j are eigenvectors with eigenvalue 1.

- Operator $\sum_{\{v_k : v_k(B_j)=0\}} -\zeta_{v_k} \langle \zeta_{v_k}, \cdot \rangle$ stating that the vectors associated with the valuations that do not satisfy B_j are eigenvectors with eigenvalue -1.

- Projections $\sum_{\{(B_i, B_j) \in \mathsf{NC} : j < i\}} \zeta_{(B_i, B_j)_{\bullet}^{\bullet}} \langle \zeta_{(B_i, B_j)_{\bullet}^{\bullet}}, \cdot \rangle$, $\sum_{\{(B_i, B_j) \in \mathsf{NC} : j > i\}} \zeta_{(B_i, B_j)_{\bullet}} \langle \zeta_{(B_i, B_j)_{\bullet}}, \cdot \rangle$ referring to the incompatible pairs where B_j appears which were defined so that O_i and O_j are not compatible (observe that that the two vectors $\zeta_{(B_i, B_j)^{\bullet}}$ and $\zeta_{(B_i, B_j)_{\bullet}}$ associated with the pair (B_i, B_j) are eigenvectors with eigenvalues 0 and 1, respectively).

- Projection $\sum_{\{A \in \mathsf{NC} : B_j \notin A\}} \zeta_{A_{\bullet}} \langle \zeta_{A_{\bullet}}, \cdot \rangle + \zeta_{A^{\bullet}} \langle \zeta_{A^{\bullet}}, \cdot \rangle$ stating that the vectors referring to the incompatible pairs where B_j does not appear are eigenvectors with eigenvalue 1.

So, O_j is an observable since it is a diagonizable, bounded and self-adjoint operator.

The next result states that the pairs of propositional symbols in NC correspond to incompatible quantum variables in $I^{B',\text{NC},f}$ and vice-versa.

Proposition 6.1 Let $B_\ell, B_m \in B'$ be distinct variables. Then, $\{\underline{B}_\ell, \underline{B}_m\}$ is not a compatible set of quantum variables in $I^{B',\text{NC},f}$ iff $(B_\ell, B_m) \in \text{NC}$.

Proof:
($\leftarrow$) Assume that $(B_\ell, B_m) \in \text{NC}$ and that $\ell < m$. Observe that

$$
O_\ell = \left(\sum_{\{v_k : v_k(B_\ell)=1\}} \zeta_{v_k} \langle \zeta_{v_k}, \cdot \rangle \right) + \left(\sum_{\{v_k : v_k(B_\ell)=0\}} -\zeta_{v_k} \langle \zeta_{v_k}, \cdot \rangle \right) +
$$

$$
\left(\sum_{\{(B_i,B_\ell)\in\text{NC}:\ell<i, i\neq m\}} \zeta_{(B_i,B_\ell)^\bullet_\bullet} \langle \zeta_{(B_i,B_\ell)^\bullet_\bullet}, \cdot \rangle \right) +
$$

$$
\zeta_{(B_m,B_\ell)^\bullet_\bullet} \langle \zeta_{(B_m,B_\ell)^\bullet_\bullet}, \cdot \rangle +
$$

$$
\left(\sum_{\{(B_i,B_\ell)\in\text{NC}:\ell>i\}} \zeta_{(B_i,B_\ell)_\bullet} \langle \zeta_{(B_i,B_\ell)_\bullet}, \cdot \rangle \right) +
$$

$$
\left(\sum_{\{A\in\text{NC}:B_\ell\notin A\}} \zeta_{A_\bullet} \langle \zeta_{A_\bullet}, \cdot \rangle + \zeta_{A^\bullet} \langle \zeta_{A^\bullet}, \cdot \rangle \right)
$$

and

$$
O_m = \left(\sum_{\{v_k : v_k(B_m)=1\}} \zeta_{v_k} \langle \zeta_{v_k}, \cdot \rangle \right) + \left(\sum_{\{v_k : v_k(B_m)=0\}} -\zeta_{v_k} \langle \zeta_{v_k}, \cdot \rangle \right) +
$$

$$
\left(\sum_{\{(B_i,B_m)\in\text{NC}:m<i\}} \zeta_{(B_i,B_m)^\bullet_\bullet} \langle \zeta_{(B_i,B_m)^\bullet_\bullet}, \cdot \rangle \right) +
$$

$$
\zeta_{(B_m,B_\ell)_\bullet} \langle \zeta_{(B_m,B_\ell)_\bullet}, \cdot \rangle +
$$

$$
\left(\sum_{\{(B_i,B_m)\in\text{NC}:m>i, i\neq\ell\}} \zeta_{(B_i,B_m)_\bullet} \langle \zeta_{(B_i,B_m)_\bullet}, \cdot \rangle \right) +
$$

$$
\left(\sum_{\{A\in\text{NC}:B_m\notin A\}} \zeta_{A_\bullet} \langle \zeta_{A_\bullet}, \cdot \rangle + \zeta_{A^\bullet} \langle \zeta_{A^\bullet}, \cdot \rangle \right).
$$

We now show that

$$
O_m O_\ell \neq O_\ell O_m.
$$

Indeed,

$$
O_m O_\ell = \left(\sum_{\{v_k : v_k(B_\ell) + v_k(B_m) \neq 1\}} \zeta_{v_k} \langle \zeta_{v_k}, \cdot \rangle \right) +
$$

$$
\left(\sum_{\{v_k : v_k(B_\ell) + v_k(B_m) = 1\}} -\zeta_{v_k} \langle \zeta_{v_k}, \cdot \rangle \right) +
$$

$$
\left(\sum_{\{(B_i, B_\ell) \in \mathsf{NC} : \ell < i, i \neq m\}} \zeta_{(B_i, B_\ell)_{\substack{\bullet \\ \bullet}}} \langle \zeta_{(B_i, B_\ell)_{\substack{\bullet \\ \bullet}}}, \cdot \rangle \right) +
$$

$$
\sqrt{\frac{1}{2}} \, \zeta_{(B_m, B_\ell)_\bullet} \langle \zeta_{(B_m, B_\ell)_{\substack{\bullet \\ \bullet}}}, \cdot \rangle +
$$

$$
\left(\sum_{\{(B_i, B_\ell) \in \mathsf{NC} : \ell > i\}} \zeta_{(B_i, B_\ell)_\bullet} \langle \zeta_{(B_i, B_\ell)_\bullet}, \cdot \rangle \right) +
$$

$$
\left(\sum_{\{(B_i, B_m) \in \mathsf{NC} : m < i\}} \zeta_{(B_i, B_m)_{\substack{\bullet \\ \bullet}}} \langle \zeta_{(B_i, B_m)_{\substack{\bullet \\ \bullet}}}, \cdot \rangle \right) +
$$

$$
\left(\sum_{\{(B_i, B_m) \in \mathsf{NC} : m > i, i \neq \ell\}} \zeta_{(B_i, B_m)_\bullet} \langle \zeta_{(B_i, B_m)_\bullet}, \cdot \rangle \right) +
$$

$$
\left(\sum_{\{A \in \mathsf{NC} : B_\ell, B_m \notin A\}} \zeta_{A_\bullet} \langle \zeta_{A_\bullet}, \cdot \rangle + \zeta_{A^\bullet} \langle \zeta_{A^\bullet}, \cdot \rangle \right).
$$

On the other hand,

$$
O_\ell O_m = \left(\sum_{\{v_k : v_k(B_\ell) + v_k(B_m) \neq 1\}} \zeta_{v_k} \langle \zeta_{v_k}, \cdot \rangle \right) +
$$

$$
\left(\sum_{\{v_k : v_k(B_\ell) + v_k(B_m) = 1\}} -\zeta_{v_k} \langle \zeta_{v_k}, \cdot \rangle \right) +
$$

$$
\left(\sum_{\{\{B_i, B_m\} \in \mathsf{NC} : m < i\}} \zeta_{(B_i, B_m)_{\substack{\bullet \\ \bullet}}} \langle \zeta_{(B_i, B_m)_{\substack{\bullet \\ \bullet}}}, \cdot \rangle \right) +
$$

$$
\sqrt{\frac{1}{2}} \, \zeta_{(B_m, B_\ell)_{\substack{\bullet \\ \bullet}}} \langle \zeta_{(B_m, B_\ell)_\bullet}, \cdot \rangle +
$$

$$
\left(\sum_{\{(B_i, B_m) \in \mathsf{NC} : m > i, i \neq \ell\}} \zeta_{(B_i, B_m)_\bullet} \langle \zeta_{(B_i, B_m)_\bullet}, \cdot \rangle \right) +
$$

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$$\left(\sum_{\{(B_i,B_\ell)\in\mathsf{NC}:\ell<i,i\neq m\}} \zeta_{(B_i,B_\ell)^\bullet_\bullet} \big\langle \zeta_{(B_i,B_\ell)^\bullet_\bullet}, \cdot \big\rangle \right) +$$

$$\left(\sum_{\{(B_i,B_\ell)\in\mathsf{NC}:\ell>i\}} \zeta_{(B_i,B_\ell)_\bullet} \big\langle \zeta_{(B_i,B_\ell)_\bullet}, \cdot \big\rangle \right) +$$

$$\left(\sum_{\{A\in\mathsf{NC}:B_\ell,B_m\notin A\}} \zeta_{A_\bullet} \big\langle \zeta_{A_\bullet}, \cdot \big\rangle + \zeta_{A^\bullet} \big\langle \zeta_{A^\bullet}, \cdot \big\rangle \right).$$

Hence

$$O_m O_\ell - O_\ell O_m = \frac{1}{2}\left(\zeta_{(B_m,B_\ell)_\bullet} \big\langle \zeta_{(B_m,B_\ell)^\bullet}, \cdot \big\rangle - \zeta_{(B_m,B_\ell)^\bullet} \big\langle \zeta_{(B_m,B_\ell)_\bullet}, \cdot \big\rangle \right) \neq 0.$$

Therefore, $\{\underline{B}_m, \underline{B}_\ell\}$ is not a compatible set of quantum variables.

($\rightarrow$) Assume that $(B_\ell, B_m) \notin \mathsf{NC}$. Then,

$$O_\ell = \left(\sum_{\{v_k:v_k(B_\ell)=1\}} \zeta_{v_k}\big\langle \zeta_{v_k}, \cdot \big\rangle \right) + \left(\sum_{\{v_k:v_k(B_\ell)=0\}} -\zeta_{v_k}\big\langle \zeta_{v_k}, \cdot \big\rangle \right) +$$

$$\left(\sum_{\{(B_i,B_\ell)\in\mathsf{NC}:\ell<i\}} \zeta_{(B_i,B_\ell)^\bullet_\bullet} \big\langle \zeta_{(B_i,B_\ell)^\bullet_\bullet}, \cdot \big\rangle \right) +$$

$$\left(\sum_{\{(B_i,B_\ell)\in\mathsf{NC}:\ell>i\}} \zeta_{(B_i,B_\ell)_\bullet} \big\langle \zeta_{(B_i,B_\ell)_\bullet}, \cdot \big\rangle \right) +$$

$$\left(\sum_{\{A\in\mathsf{NC}:B_\ell\notin A\}} \zeta_{A_\bullet} \big\langle \zeta_{A_\bullet}, \cdot \big\rangle + \zeta_{A^\bullet} \big\langle \zeta_{A^\bullet}, \cdot \big\rangle \right)$$

and

$$O_m = \left(\sum_{\{v_k:v_k(B_m)=1\}} \zeta_{v_k}\big\langle \zeta_{v_k}, \cdot \big\rangle \right) + \left(\sum_{\{v_k:v_k(B_m)=0\}} -\zeta_{v_k}\big\langle \zeta_{v_k}, \cdot \big\rangle \right) +$$

$$\left(\sum_{\{(B_i,B_m)\in\mathsf{NC}:m<i\}} \zeta_{(B_i,B_m)^\bullet_\bullet} \big\langle \zeta_{(B_i,B_m)^\bullet_\bullet}, \cdot \big\rangle \right) +$$

$$\left(\sum_{\{(B_i,B_m)\in\mathsf{NC}:m>i\}} \zeta_{(B_i,B_m)_\bullet} \big\langle \zeta_{(B_i,B_m)_\bullet}, \cdot \big\rangle \right) +$$

$$\left(\sum_{\{A\in\mathsf{NC}:B_m\notin A\}} \zeta_{A_\bullet} \big\langle \zeta_{A_\bullet}, \cdot \big\rangle + \zeta_{A^\bullet} \big\langle \zeta_{A^\bullet}, \cdot \big\rangle \right).$$

Hence,

$$
\begin{aligned}
O_m O_\ell = {}& \left(\sum_{\{v_k : v_k(B_\ell) + v_k(B_m) \neq 1\}} \zeta_{v_k} \langle \zeta_{v_k}, \cdot \rangle \right) + \\[2mm]
& \left(\sum_{\{v_k : v_k(B_\ell) + v_k(B_m) = 1\}} -\zeta_{v_k} \langle \zeta_{v_k}, \cdot \rangle \right) + \\[2mm]
& \left(\sum_{\{(B_i, B_m) \in \mathsf{NC} : m < i\}} \zeta_{(B_i, B_m)_\bullet^\bullet} \langle \zeta_{(B_i, B_m)_\bullet^\bullet}, \cdot \rangle \right) + \\[2mm]
& \left(\sum_{\{(B_i, B_m) \in \mathsf{NC} : m > i\}} \zeta_{(B_i, B_m)_\bullet} \langle \zeta_{(B_i, B_m)_\bullet}, \cdot \rangle \right) + \\[2mm]
& \left(\sum_{\{(B_i, B_\ell) \in \mathsf{NC} : \ell < i\}} \zeta_{(B_i, B_\ell)_\bullet^\bullet} \langle \zeta_{(B_i, B_\ell)_\bullet^\bullet}, \cdot \rangle \right) + \\[2mm]
& \left(\sum_{\{(B_i, B_\ell) \in \mathsf{NC} : \ell > i\}} \zeta_{(B_i, B_\ell)_\bullet} \langle \zeta_{(B_i, B_\ell)_\bullet}, \cdot \rangle \right) + \\[2mm]
& \left(\sum_{\{A \in \mathsf{NC} : B_\ell, B_m \notin A\}} \zeta_{A_\bullet} \langle \zeta_{A_\bullet}, \cdot \rangle + \zeta_{A^\bullet} \langle \zeta_{A^\bullet}, \cdot \rangle \right).
\end{aligned}
$$

On the other hand,

$$
\begin{aligned}
O_\ell O_m = {}& \left(\sum_{\{v_k : v_k(B_\ell) + v_k(B_m) \neq 1\}} \zeta_{v_k} \langle \zeta_{v_k}, \cdot \rangle \right) + \\[2mm]
& \left(\sum_{\{v_k : v_k(B_\ell) + v_k(B_m) = 1\}} -\zeta_{v_k} \langle \zeta_{v_k}, \cdot \rangle \right) + \\[2mm]
& \left(\sum_{\{(B_i, B_m) \in \mathsf{NC} : m < i\}} \zeta_{(B_i, B_m)_\bullet^\bullet} \langle \zeta_{(B_i, B_m)_\bullet^\bullet}, \cdot \rangle \right) + \\[2mm]
& \left(\sum_{\{(B_i, B_m) \in \mathsf{NC} : m > i\}} \zeta_{(B_i, B_m)_\bullet} \langle \zeta_{(B_i, B_m)_\bullet}, \cdot \rangle \right) + \\[2mm]
& \left(\sum_{\{(B_i, B_\ell) \in \mathsf{NC} : \ell < i\}} \zeta_{(B_i, B_\ell)_\bullet^\bullet} \langle \zeta_{(B_i, B_\ell)_\bullet^\bullet}, \cdot \rangle \right) + \\[2mm]
\end{aligned}
$$

$$\left(\sum_{\{(B_i,B_\ell)\in\mathsf{NC}:\ell>i\}} \zeta_{(B_i,B_\ell)_\bullet} \left\langle \zeta_{(B_i,B_\ell)_\bullet}, \cdot \right\rangle \right) +$$

$$\left(\sum_{\{A\in\mathsf{NC}:B_\ell,B_m\notin A\}} \zeta_{A_\bullet} \left\langle \zeta_{A_\bullet}, \cdot \right\rangle + \zeta_{A^\bullet} \left\langle \zeta_{A^\bullet}, \cdot \right\rangle \right).$$

Therefore,

$$O_m O_\ell - O_\ell O_m = 0.$$

Therefore, $\{\underline{B}_m, \underline{B}_\ell\}$ is a compatible set of quantum variables. QED

We now proceed towards the weak completeness of the calculus. We start by proving a finite dimensional model existence lemma showing that we can move back and forth between satisfaction of certain RCOF formulas and satisfaction of PLQO formulas.

Proposition 6.2 Let $\beta_1, \ldots, \beta_k$ be formulas of PLQO of the form $\odot\alpha$, $\neg\odot\alpha$, $\int\alpha @ p$ and ρ be an assignment over $\mathbb{R}$ such that

$$\mathbb{R}\rho \Vdash_{\mathsf{fo}} Q^{\int\alpha_j},$$

for each j such that β_j is either of the form $\odot\alpha_j$ or $\int\alpha_j@p_j$. Then, there is a finite dimensional quantum structure I such that

$$I\rho \Vdash \bigwedge_{j=1}^{k} \beta_j \qquad \text{iff} \qquad \mathbb{R}\rho \Vdash_{\mathsf{fo}} \bigwedge_{j=1}^{k} \beta_j^{\mathsf{RCOF}}.$$

Proof: Consider two cases:
(1) There is $j = 1, \ldots, k$ such β_j is either of the form $\odot\alpha_j$ or $\int\alpha_j @ p_j$. Let B' is $\cup_{j=1}^{k} B_{\beta_j}$. Assume without loss of generality that B' is the set $\{B_1, \ldots, B_{|B'|}\}$. Consider the quantum structure

$$I = I^{B',\mathsf{NC},f}$$

where

$$\mathsf{NC} = \{(B_i, B_j) : \rho(x_{\{B_i,B_j\}}) \neq 0, B_i, B_j \in B', i \neq j\}$$

and

$$f(j) = \sqrt{\rho(x_{\phi_{B'}^{U_j}})}$$

where $U_j = \{B_i \in B' : v_j(B_i) = 1\}$ for each $j = 0, \ldots, 2^{|B'|} - 1$ and each valuation v_j is as defined in $I^{B',\mathsf{NC},f}$.

We now prove that

$$I\rho \Vdash \beta_j \quad \text{iff} \quad \mathbb{R}\rho \Vdash_{\mathsf{fo}} \beta_j^{\mathsf{RCOF}}$$

for each $j = 1, \ldots, k$.

There are three cases to consider:
(a) β_j is $\odot\alpha$. Recall that $(\odot\alpha)^{\mathsf{RCOF}}$ is $Q^{\odot\alpha} \wedge Q^{\int\alpha}$.

($\leftarrow$) Assume that $\mathbb{R}\rho \Vdash_{\mathsf{fo}} Q^{\odot\alpha} \wedge Q^{\int\alpha}$. Hence, $\mathbb{R}\rho \Vdash_{\mathsf{fo}} x_{A'} = 0$ for every $A' \subseteq B_\alpha$ with $|A'| = 2$

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since $\mathbb{R}\rho \Vdash_{\mathsf{fo}} Q^{\odot\alpha}$. We must show that $\underline{\alpha} = \{\underline{B}_j : B_j \in B_\alpha\}$ is a compatible set of quantum variables. Let $\{\underline{B}_i, \underline{B}_j\}$ be a set of quantum variables in $\underline{\alpha}$. Observe that $(B_i, B_j) \notin \mathsf{NC}$. Thus, by Proposition 6.1, $\{\underline{B}_i, \underline{B}_j\}$ is a compatible set of quantum variables. Thus, $I\rho \Vdash \odot\alpha$.

($\rightarrow$) Assume that $\mathbb{R}\rho \nVdash_{\mathsf{fo}} (\odot\alpha)^{\mathsf{RCOF}}$. Then, $\mathbb{R}\rho \nVdash_{\mathsf{fo}} Q^{\odot\alpha}$ since, by hypothesis,

$$\mathbb{R}\rho \Vdash_{\mathsf{fo}} Q^{\int\alpha}.$$

Then, there is $\{B_\ell, B_m\} \subseteq B_\alpha$ such that

$$\rho(x_{\{B_\ell, B_m\}}) \neq 0.$$

Hence, $(B_\ell, B_m) \in \mathsf{NC}$. Thus, by Proposition 6.1, $\{\underline{B}_m, \underline{B}_\ell\}$ is not a compatible set of quantum variables. Therefore, also $\underline{\alpha}$ is not a compatible set of quantum variables and so α is not an I-observable formula. Thus, $I\rho \nVdash \odot\alpha$.

(b) β_j is $\int\alpha @ p$. Recall that $(\int\alpha @ p)^{\mathsf{RCOF}}$ is $Q^{\odot\alpha} \wedge Q^{\int\alpha} \wedge (x_\alpha @ p)$. Observe that

$$\mathsf{Prob}_I(\alpha) = \sum_{\substack{v\,:\,B_\alpha \to \{0,1\} \\ v \Vdash_{\mathsf{c}} \alpha}} \left(\sum_{u \in U_v} \langle \psi, u \rangle^2 \right) \qquad (*)$$

$$= \sum_{\substack{v\,:\,B_\alpha \to \{0,1\} \\ v \Vdash_{\mathsf{c}} \alpha}} \left(\sum_{\substack{v'\,:\,B' \to \{0,1\} \\ v'|_{B_\alpha} = v}} \langle \psi, \zeta_{v'} \rangle^2 \right) \qquad (**)$$

$$= \sum_{\substack{v\,:\,B_\alpha \to \{0,1\} \\ v \Vdash_{\mathsf{c}} \alpha}} \left(\sum_{\substack{v'\,:\,B' \to \{0,1\} \\ v'|_{B_\alpha} = v}} \rho(x_{\phi_{B'}^{\{B_k \in B' : v'(B_k)=1\}}}) \right)$$

$$= \sum_{\substack{v\,:\,B_\alpha \to \{0,1\} \\ v \Vdash_{\mathsf{c}} \alpha}} \rho(x_{\phi_{B_\alpha}^{\{B_j \in B_\alpha : v(B_j)=1\}}})$$

$$= \rho(x_\alpha)$$

since $\mathbb{R}\rho \Vdash_{\mathsf{fo}} Q^{\int\alpha}$, (*) holds by Proposition 5.2 and (**) holds since

$$U_v \cap \{\zeta_{v_0}, \ldots, \zeta_{v_{2|B'|-1}}\} = \{v' : v' : B' \to \{0,1\} \text{ and } v'|_{B_\alpha} = v\}$$

and

$$\langle \psi, u \rangle = 0$$

for each $u \in \bigcup_{A \in \mathsf{NC}} \{\zeta_{A^\bullet}, \zeta_{A_\bullet}\}$.

($\leftarrow$) Assume that $\mathbb{R}\rho \Vdash_{\mathsf{fo}} \beta_j^{\mathsf{RCOF}}$. Therefore,

$$\mathbb{R}\rho \Vdash_{\mathsf{fo}} Q^{\odot\alpha} \wedge Q^{\int\alpha} \wedge (x_\alpha @ p)$$

and following a similar reasoning to the one in (a), we can conclude that $I\rho \Vdash \odot\,\alpha$. Moreover, $\rho(x_\alpha)\,@\,p^{\mathbb{R}\rho}$. Then,

$$\mathsf{Prob}_I(\alpha) = \rho(x_\alpha)\,@\,p^{\mathbb{R}\rho}.$$

So $I\rho \Vdash \beta_j$.

($\rightarrow$) Assume that $\mathbb{R}\rho \not\Vdash_{\mathsf{fo}} \beta_j^{\mathsf{RCOF}}$. Since, by hypothesis, $\mathbb{R}\rho \Vdash_{\mathsf{fo}} Q^{\int\alpha}$, there are only two cases to consider.

(i) $\mathbb{R}\rho \not\Vdash_{\mathsf{fo}} Q^{\odot\,\alpha}$. Then, by (a), $I\rho \not\Vdash \odot\,\alpha$ and so $I\rho \not\Vdash \beta_j$.

(ii) $\mathbb{R}\rho \not\Vdash_{\mathsf{fo}} x_\alpha\,@\,p$. Hence,

$$\mathsf{Prob}_I(\alpha) = \rho(x_\alpha)\,\cancel{@}\,p^{\mathbb{R}\rho}.$$

Therefore, $I\rho \not\Vdash \beta_j$.

(c) β_j is $\neg\odot\,\alpha$. Recall that $(\neg\odot\,\alpha)^{\mathsf{RCOF}}$ is $\neg Q^{\odot\,\alpha}$. Hence,

$$\mathbb{R}\rho \Vdash_{\mathsf{fo}} (\neg\odot\,\alpha)^{\mathsf{RCOF}}$$
$$\text{iff}$$
$$\mathbb{R}\rho \Vdash_{\mathsf{fo}} \neg Q^{\odot\,\alpha}$$
$$\text{iff}$$
$$\mathbb{R}\rho \Vdash_{\mathsf{fo}} \neg(Q^{\odot\,\alpha}\wedge Q^{\int\alpha})$$
$$\text{iff}$$
$$\mathbb{R}\rho \Vdash_{\mathsf{fo}} \neg(\odot\,\alpha)^{\mathsf{RCOF}}$$
$$\text{iff}$$
$$\mathbb{R}\rho \not\Vdash_{\mathsf{fo}} (\odot\,\alpha)^{\mathsf{RCOF}}$$
$$\text{iff}$$
$$I\rho \not\Vdash \odot\,\alpha$$
$$\text{iff}$$
$$I\rho \Vdash \neg\odot\,\alpha.$$

Therefore,

$$\mathbb{R}\rho \Vdash_{\mathsf{fo}} \beta_1 \wedge \cdots \wedge \beta_k$$
$$\text{iff}$$
$$\mathbb{R}\rho \Vdash_{\mathsf{fo}} \beta_1 \quad \ldots \quad \mathbb{R}\rho \Vdash_{\mathsf{fo}} \beta_k$$
$$\text{iff}$$
$$I\rho \Vdash \beta_1 \quad \ldots \quad I\rho \Vdash_{\mathsf{fo}} \beta_k$$
$$\text{iff}$$
$$I\rho \Vdash \beta_1 \wedge \cdots \wedge \beta_k$$

as we wanted to show. $\hfill$ QED

Theorem 6.3 The logic PLQO is weakly complete.

Proof:
Let $\varphi \in L_{\mathsf{PLQO}}$. Assume that $\not\vdash \varphi$. We proceed to show that $\not\models \varphi$. First observe that $\neg\varphi$ must be consistent because otherwise from $\neg\varphi$ one would be able to derive every formula,

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including in particular, φ, and so $\vdash (\neg \varphi) \supset \varphi$, and in that case one would have $\vdash \varphi$. Observe that there are conjunctions $\eta_1, \ldots, \eta_\ell$ of literals of the form $\odot \alpha$, $\neg \odot \alpha$ and $\int \alpha \, @ \, p$ such that

$$\vdash (\neg \varphi) \equiv \bigvee_{i=1}^{\ell} \eta_i.$$

Since $\neg \varphi$ is consistent, at least one of the disjuncts must also be consistent. Let η_m be one such consistent disjunct. Assume that η_m is

$$\beta_1 \wedge \cdots \wedge \beta_k$$

where each β_j is a literal. For proving that $\nvdash \varphi$ it is enough to show that $\neg \varphi$ is satisfiable. Hence, it is enough to show that there is one satisfiable disjunct. Indeed, η_m is satisfiable as we proceed to prove. Towards a contradiction, assume that there are no I and ρ such that $I\rho \Vdash \eta_m$ holds. Then, by Proposition 6.2, there would not exist ρ such that

$$\mathbb{R}\rho \Vdash_{\mathsf{fo}} \bigwedge_{j \, : \, \beta_j \text{ is either } \odot \alpha_j \text{ or } \int \alpha_j @ p_j} Q^{\int \alpha_j} \wedge \bigwedge_{j=1}^{k} \beta_j^{\mathsf{RCOF}}.$$

So, there would not exist ρ such that

$$\mathbb{R}\rho \Vdash_{\mathsf{fo}} \bigwedge_{j=1}^{k} \beta_j^{\mathsf{RCOF}}.$$

Hence, we would have

$$\forall \left(\left(\bigwedge_{j=1}^{k} \beta_j^{\mathsf{RCOF}} \right) \supset \mathsf{ff}^{\mathsf{RCOF}} \right) \in \mathsf{RCOF}.$$

Then, by RR, we would establish

$$\eta_m \vdash \mathsf{ff}$$

in contradiction with the consistency of η_m. \hfill QED

7 Conservativeness

In this section, we show that PLQO is a conservative extension of classical propositional logic modulo a translation of classical propositional formulas into the language of PLQO.

Proposition 7.1 Let $\alpha \in L_{\mathsf{c}}$. Then

$$\odot \alpha \vDash \int \alpha = 1 \qquad \text{iff} \qquad \vDash_{\mathsf{c}} \alpha.$$

Proof:
($\rightarrow$) Assume that $\odot \alpha \vDash \int \alpha = 1$. Consider the quantum structure

$$I = I^{B_\alpha, \emptyset, f}$$

where $f(j) = \sqrt{\frac{1}{2^{|B_\alpha|}}}$ for every $j = 0, \ldots, 2^{|B_\alpha|} - 1$, and an arbitrary assignment ρ over $\mathbb{R}$. Thus, by Proposition 6.1, $\{\underline{B}_m, \underline{B}_\ell\}$ is a compatible set of quantum variables for I for every $\underline{B}_m, \underline{B}_\ell \in \underline{\alpha}$. Therefore,

$$I\rho \Vdash \odot \alpha.$$

Hence, by the hypothesis, $I\rho \Vdash \int \alpha = 1$. Then, $\mathsf{Prob}_I(\alpha) = 1$. Observe that, by Proposition 5.2,

$$\mathsf{Prob}_I(\alpha) = \sum_{\substack{v\,:\,B_\alpha \to \{0,1\} \\ v \Vdash_c \alpha}} \left(\sum_{u \in U_v} \langle \psi, u \rangle^2 \right)$$

and that

$$\sum_{u \in U_v} \langle \psi, u \rangle^2 = \frac{1}{2^{|B_\alpha|}},$$

for each $v : B_\alpha \to \{0, 1\}$, because $U_v = \{\zeta_v\}$. Thus, since

$$\sum_{\substack{v\,:\,B_\alpha \to \{0,1\} \\ v \Vdash_c \alpha}} \left(\sum_{u \in U_v} \langle \psi, u \rangle^2 \right) = 1$$

we have that $\{v : v : B_\alpha \to \{0,1\}\} = \{v : v : B_\alpha \to \{0,1\}, v \Vdash_c \alpha\}$. Therefore, α is a tautology.

($\leftarrow$) Assume that $\vDash_c \alpha$. Let I be a quantum structure and ρ an arbitrary assignment over $\mathbb{R}$. Assume $I\rho \Vdash \odot \alpha$. Since Prob_I is a probability assignment by Proposition 5.4, $\mathsf{Prob}_I(\alpha) = 1$ and so $I\rho \Vdash \int \alpha = 1$. $\hfill$ QED

Then, writing

$$\alpha^* \quad \text{for} \quad (\odot \alpha) \supset (\textstyle\int \alpha = 1),$$

for the translation of classical formula α into the language of PLQO, the following corollary holds.

Proposition 7.2 Let $\alpha \in L_c$. Then

$$\vDash \alpha^* \quad \text{iff} \quad \vDash_c \alpha.$$

8 Epistemic nature of observability

We now analyse the epistemic aspects of quantum reasoning of observability in PLQO following the steps in [HS15]. Recall that $\odot \alpha$ intuitively means that it is possible to do a simultaneous measurement over the essential propositional symbols of α in order to know its truth value. With this in mind, we discuss the behaviour of $\odot$ with respect to the propositional connectives.

We start by considering the law

$$((\odot \alpha_1) \wedge (\odot \alpha_2)) \equiv (\odot(\alpha_1 \wedge \alpha_2))$$

which is characteristic of an epistemic operator (see for instance [MvdH95]). Observe that there are classical formulas α_1 and α_2 such that

$$\nvDash ((\odot \alpha_1) \wedge (\odot \alpha_2)) \equiv (\odot(\alpha_1 \wedge \alpha_2)).$$

Indeed, consider the quantum structure I_1 defined in Example 8.1 and let α_1 be B_1 and α_2 be B_2.

Example 8.1 Recall Example 2.1. Let

$$I = (\mathbb{C}^2, \psi, B_j \mapsto \underline{B}_j)$$

be the quantum structure where

- $\underline{B}_1$ is a propositional quantum variable such that O_1 is the observable O_x for the x component of a spin-$\frac{1}{2}$ particle;

- $\underline{B}_2$ is a propositional quantum variable such that O_2 is the observable O_y for the y component of a spin-$\frac{1}{2}$ particle;

- $\underline{B}_3$ is a propositional quantum variable such that O_3 is the observable O^2 for the total angular momentum of a spin-$\frac{1}{2}$ particle.

Hence, $\underline{B}_1$ and $\underline{B}_2$ are not compatible. Therefore,

$$I \not\Vdash \odot(B_1 \wedge B_2).$$

On the other hand, $I \Vdash \odot(B_1)$ and $I_1 \Vdash \odot(B_2)$ and so

$$I \not\Vdash ((\odot B_1) \wedge (\odot B_2)) \equiv (\odot(B_1 \wedge B_2)).$$

However,

$$I \Vdash ((\odot B_1) \wedge (\odot B_3)) \equiv (\odot(B_1 \wedge B_3))$$

since $\underline{B}_1$ and $\underline{B}_3$ are compatible.

Finally, there is no surprise at all due to the intended meaning of $\odot$ that the following distributive law

$$(\odot(\alpha_1 \wedge (\alpha_2 \vee \alpha_3))) \equiv ((\odot(\alpha_1 \wedge \alpha_2) \vee (\odot(\alpha_1 \wedge \alpha_3))))$$

does not hold. In fact,

$$I \not\Vdash (\odot(B_3 \wedge (B_1 \vee B_2))) \equiv ((\odot(B_3 \wedge B_1) \vee (\odot(B_3 \wedge B_2))))$$

holds. ∇

In the same vein, we can show that

$$((\odot \alpha_1) \vee (\odot \alpha_2)) \equiv (\odot(\alpha_1 \vee \alpha_2))$$

is satisfiable for some classical formulas α_1 and α_2 but does not hold in general, and the same for the law

$$((\odot \alpha_1) \supset (\odot \alpha_2)) \supset (\odot(\alpha_1 \supset \alpha_2)).$$

Concerning other laws of epistemic logic, note that the introspection axioms cannot be considered in our setting since the language of PLQO does not allow formulas with nested $\odot$ operators. On the other hand, the knowledge axiom

$$(\odot \alpha) \supset \alpha^*$$

holds in general for any tautology α.

Finally, observe that the formula

$$(\odot\, \alpha) \equiv (\odot(\neg\, \alpha))$$

holds in general since $eB_\alpha = eB_{\neg\,\alpha}$. Indeed, assume that $B_j \in eB_\alpha$. Then, for every valuation v on B_α,

$$v^{B_j \leftarrow 0} \Vdash_{\mathsf{c}} \alpha \quad \text{iff} \quad v^{B_j \leftarrow 1} \nVdash_{\mathsf{c}} \alpha.$$

Hence,

$$v^{B_j \leftarrow 0} \Vdash_{\mathsf{c}} \neg\,\alpha \quad \text{iff} \quad v^{B_j \leftarrow 1} \nVdash_{\mathsf{c}} \neg\,\alpha.$$

taking into account v is also a valuation on $B_{\neg\,\alpha}$. So, $B_j \in eB_{\neg\,\alpha}$. Similarly, for the other inclusion.

A similar epistemic behaviour occurs for example for the operator M defined as an abbreviation of $\int \alpha > 0$. The intuitive meaning of $\mathsf{M}\alpha$ is that α is observable and possibly true. Since, $\mathsf{M}\alpha$ can be seen as a conjunction involving $\odot\,\alpha$, then, as a consequence of the previous discussion, it is immediate to extrapolate its behaviour with respect to the propositional connectives. On the other hand,

$$\nvDash (\mathsf{M}\alpha) \equiv (\mathsf{M}(\neg\,\alpha))$$

when α is a tautology. In fact, when α is a tautology, $\mathsf{M}\alpha$ holds but $\mathsf{M}(\neg\,\alpha)$ does not hold since although $\odot(\neg\,\alpha)$ holds we have $\int \neg\,\alpha = 0$. However, $\vDash (\mathsf{M}\alpha) \equiv (\mathsf{M}(\neg\,\alpha))$ when both α and $\neg\,\alpha$ are not tautologies.

9   Concluding remarks

Probabilistic reasoning about quantum observability was introduced into classical propositional logic by developing the logic PLQO. The semantics of PLQO was given by quantum structures each one composed by a Hilbert space representing the quantum system together with a unit vector representing the current state plus the interpretation of each propositional symbol as a propositional quantum variable over a diagonizable and bounded observable. The logic was axiomatized by relying on the decidable theory of real closed ordered fields and shown to be strongly sound and weakly complete. Furthermore, PLQO was proved to be a conservative extension of classical propositional logic (modulo a suitable translation). The epistemic nature of the observability operator $\odot$ was discussed. Although $\odot$ satisfies at least in part some epistemic properties, it is of a different nature of other quantum-like epistemic operators, like for instance those in [BS10], defined for different purposes.

Concerning future work, we intend to extend the logic to cope with unbounded operators. Moreover, we intend to investigate the consequence of discarding the assumption that simultaneous measurements imply compatibility of observables.

Furthermore, it seems worthwhile to study other meta-properties of PLQO starting with decidability and complexity of the decision problem. We expect this complexity to be much lower than the complexity of RCOF theoremhood since we only need to recognize RCOF theorems of a very simple clausal form.

Strong completeness of the PLQO axiomatization is not possible because $\mathbb{R}$ is an Archimedean real closed ordered field which led to a non compact entailment. Relaxing the semantics may open the door to establishing strong completeness.

Finally, it seems worthwhile to analyze the observability operator $\odot$ as an analog of the consistency operator in some paraconsistent and paracomplete logics [CCM07].

Acknowledgments

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