Relations between Propositional Normal Modal Logics: an Overview

RAJEEV GORÉ[1], *Automated Reasoning Project, Australian National University, Canberra, Australia.*
E-mail: rpg@arp.anu.edu.au

WOLFGANG HEINLE[2] and ALAIN HEUERDING[2], *IAM, University of Bern, Switzerland.*
E-mail: heinle,heuerd@iam.unibe.ch

Abstract

The modal logic literature is notorious for multiple axiomatizations of the same logic and for conflicting overloading of axiom names. Many of the interesting interderivability results are still scattered over the often hard to obtain classics. We catalogue the most interesting axioms, their numerous variants, and explore the relationships between them in terms of interderivability as both axiom (schema) and as simple formulae. In doing so we introduce the Logics Workbench (LWB, see `http://lwbwww.unibe.ch:8080/LWBinfo.html`), a versatile tool for proving theorems in numerous propositional (nonclassical) logics. As a side-effect we fulfill a call from the modal theorem proving community for a database of known theorems.

Keywords: Modal logic, automated theorem proving, sequent calculi, Logics Workbench.

1 Introduction

The modal logic literature is quite diverse but most beginners turn to the introductory works by Hughes and Cresswell [9, 10], Bull and Segerberg [1], Lemmon and Scott [11], or Chellas [2]. The classic by Segerberg [13] is indispensable but is probably the hardest to get hold of (legally). Unfortunately the nomenclature used in these works is not uniform, and the proof techniques range from semantic, to proof-theoretic, to algebraic. Consequently, it is difficult to form a clear picture of the many interesting results contained in these works, and even harder to keep track of the many different versions of the basic axioms, their different names, and even the overloading of names. In what follows we try to give a general picture of this nomenclature, the interesting interconnections between some of these logics, and the interderivability of different axiomatizations of the same logic.

Our basic tool is the Logics Workbench (LWB) [8], a program developed at the University of Bern that contains automated proof procedures based on modal Gentzen systems for numerous propositional (nonclassical) logics. Although the class of *modal* logics which have been automated in the LWB is relatively small, namely **K**, **KT**, **KT4**, **KT45** and **KW**, we can work with extensions of these logics by using a finite number of appropriate formula instances of the desired axioms, as explained below.

[1] Work supported by a visiting fellowship from the IAM.
[2] Work supported by the Swiss National Science Foundation, SPP 5003-34279.

2 Definitions and notational conventions

We assume that **formulae** are built as usual from the classical connectives $\wedge$, $\vee$, $\rightarrow$, $\neg$, $\top$, $\bot$ and the modal connectives $\Box$ and $\Diamond$. The binding strengths of the connectives from strongest to weakest is: $\neg$, $\Box$, $\Diamond$, $\wedge$, $\vee$, $\rightarrow$. We also assume that $\rightarrow$ associates so that $A \rightarrow B \rightarrow C$ is to be read as $A \rightarrow (B \rightarrow C)$. We use $\Rightarrow$ to separate the two halves of Gentzen's sequents, and use $\vdash$ to denote derivability in a Hilbert system for a particular modal logic. This notation has been chosen carefully to avoid the confusion that may arise from the fact that the 'deduction theorem' for modal logics is not the same as in classical logic.

We assume that our modal logics are formulated by taking all classical propositional tautologies as axioms, and then adding a finite set of formulae (called *the axioms*) together with the inference rules of modus ponens, universal substitution, and necessitation. The formula $\Box(p \rightarrow q) \rightarrow (\Box p \rightarrow \Box q)$ is always one of the axioms (hence we deal only with normal modal logics). The name of the logic is obtained by concatenating the names of the axioms to the prefix **K**. A formula A is **derivable** in a modal logic **L** axiomatized in this manner iff there is a finite sequence of formulae $A_1, A_2, \ldots, A_n$ such that $A_n = A$ and each member of the sequence is either a classical tautology, an axiom, or is obtained from previous members of the sequence by one of the inference rules. If the formula A is derivable in **L** we write **L** $\vdash A$ and call A a **theorem** of **L**. **Substitutions** σ are denoted as follows: $A\sigma := A\{p := B\}$ means that in A all occurrences of the propositional variable p are replaced simultaneously with B. $A\{p := B\}$ is called an **instance** of A. In general, just using *instances* of axioms is not sufficient for our purposes, so let $\Box^0 A = A$ and $\Box^{n+1} A = \Box(\Box^n A)$ for $n \geq 0$. For any given $m \geq 0$, the formula $\Box^m(A\sigma)$ is a **modalized instance** of A. We use $A^1, A^2, A^3, \ldots$ to denote modalized instances of A and use $\bigwedge_{i=1}^{k} A^i$ to denote the formula $A^1 \wedge A^2 \wedge \ldots \wedge A^k$ obtained by conjoining k modalized instances of A, for some finite $k \geq 1$.

LEMMA 2.1
Let **L** be a normal modal logic, and A, B formulae. Then there exist a finite number (say k) of modalized instances of A such that **L** $\vdash \bigwedge_{i=1}^{k} A^i \rightarrow B$ iff **LA** $\vdash B$.

Now suppose that we wish to use the LWB to prove that a particular formula P is a theorem of the logic **KT4A** obtained by adding A as an axiom to **KT4**. Since the LWB does not have a proof procedure for **KT4A** we must use the one for **KT4** by finding enough modalized instances of A such that **KT4** $\vdash \bigwedge_{i=1}^{k} A^i \rightarrow P$. Lemma 2.1 then allows us to conclude that **KT4A** $\vdash P$.

In this way, we can 'prove' theorems in the logic **KT4A** even though the LWB itself is capable of proving theorems only in the sublogic **KT4** (say). It is easy to generalize this process to use only the logic **K** by looking for modalized instances of the axioms *T* and *4* as well, and attempting to prove that:

$$\mathbf{K} \vdash A^1 \wedge A^2 \wedge \ldots \wedge A^i \wedge T^1 \wedge T^2 \wedge \ldots \wedge T^j \wedge 4^1 \wedge 4^2 \wedge \ldots \wedge 4^k \rightarrow P.$$

For many logics, the literature contains more than one axiomatization, say **KAB** and **KAC** where the axiom A is common but the axioms B and C differ. We can show the equivalence between these axiomatizations using the technique described above by proving that each logic contains the axioms of the other as theorems.

In the rest of this paper we do the following: in Section 3 we list the axiom names, associated formula and some variants that we use throughout our paper. In Section 4 we cover the relations between some of these axioms, namely the basic axioms, the axioms of convergence, axioms related to McKinsey's axiom, the axioms of provability logics, and finally some axioms from logics of time. The appendix contains a useful cross-reference to names used in the basic texts.

As a side-effect we provide a database of known theorems and their relationships for many propositional modal logics, thus answering a call from the modal theorem proving community (where testing an implementation calls for such a database).

3 Formulae

The tables below list all the formulae we consider and the names we use for them.

name	formula
D	$\Box p \to \Diamond p$
D_2	$\Diamond \top$
T	$\Box p \to p$
4	$\Box p \to \Box\Box p$
4_M	$\Box p \land \Diamond q \to \Diamond(\Box p \land q)$
5	$\Diamond p \to \Box\Diamond p$
5_M	$\Diamond p \land \Diamond q \to \Diamond(\Diamond p \land q)$
B	$p \to \Box\Diamond p$
B_M	$p \land \Diamond q \to \Diamond(\Diamond p \land q)$
G	$\Diamond\Box p \to \Box\Diamond p$
G_0	$\Diamond(p \land \Box q) \to \Box(p \lor \Diamond q)$
H	$\Box(p \lor q) \land \Box(\Box p \lor q) \land \Box(p \lor \Box q) \to \Box p \lor \Box q$
H^+	$\Box(\Box p \lor q) \land \Box(p \lor \Box q) \to \Box p \lor \Box q$
L	$\Box(p \land \Box p \to q) \lor \Box(q \land \Box q \to p)$
L^+	$\Box(\Box p \to q) \lor \Box(\Box q \to p)$
L^{++}	$\Box(\Box p \to \Box q) \lor \Box(\Box q \to \Box p)$
M	$\Box\Diamond p \to \Diamond\Box p$
M_2	$\Diamond\Box(p \to \Box p)$
M_3	$\Box\Diamond p \land \Box\Diamond q \to \Diamond(p \land q)$
Pt	$\Box(p \lor \Diamond p) \to \Diamond(p \land \Box p)$
W	$\Box(\Box p \to p) \to \Box p$
W_0	$\Box\Diamond\top \to \Box\bot$
Z	$\Box(\Box p \to p) \to (\Diamond\Box p \to \Box p)$

Dum	$\Box(\Box(p \to \Box p) \to p) \to (\Diamond\Box p \to p)$
Dum_1	$\Box(\Box(p \to \Box p) \to p) \to (\Diamond\Box p \to \Box p)$
Dum_2	$\Box(\Box(p \to \Box p) \to \Box p) \to (\Diamond\Box p \to p)$
Dum_3	$\Box(\Box(p \to \Box p) \to \Box p) \to (\Diamond\Box p \to \Box p)$
Dum_4	$\Box(\Box(p \to \Box p) \to p) \to (\Diamond\Box p \to p \vee \Box p)$
Grz	$\Box(\Box(p \to \Box p) \to p) \to p$
Grz_1	$\Box(\Box(p \to \Box p) \to p) \to \Box p$
Grz_2	$\Box(\Box(p \to \Box p) \to \Box p) \to p$
Grz_3	$\Box(\Box(p \to \Box p) \to \Box p) \to \Box p$
Grz_4	$\Box(\Box(p \to \Box p) \to p) \to p \vee \Box p$
Grz_5	$\Box(\Box(p \to \Box q) \to \Box q) \wedge \Box(\Box(\neg p \to \Box q) \to \Box q) \to \Box q$
F	$(\Diamond\Box p \to q) \vee \Box(\Box q \to p)$
H_s	$p \to \Box(\Diamond p \to p)$
P	$\Diamond\Box\Diamond p \to (p \to \Box p)$
R	$\Diamond\Box p \to (p \to \Box p)$
X	$\Box\Box p \to \Box p$
Zem	$\Box\Diamond\Box p \to (p \to \Box p)$

4 Tables and diagrams

In the diagrams, $\mathbf{L_1} \to \mathbf{L_2}$ means every theorem of $\mathbf{L_1}$ is also a theorem of $\mathbf{L_2}$.

4.1 Relations concerning D, T, 4, 5, B

The most basic normal modal logics are obtained by adding combinations of the formulae *T*, *D*, *4*, *5* and *B* as axioms to the basic normal modal logic **K**. There are fifteen distinct logics obtained in this way and the relationships between them are quite well known (see [11] or [2] for a complete diagram). In the table, we simply confirm some of these relationships, and show the equivalence between different versions of these axioms.

$\mathbf{KA_1 \ldots A_k} \vdash C$		
$A_1, \ldots, A_k$	C	modalized instances of $A_1, \ldots, A_k$
4	4_M	4
4_M	4	$4_M\{q := \neg\Box p\}$
5	5_M	5
5_M	5	$5_M\{q := \neg\Diamond p\}$
B	B_M	B
B_M	B	$B_M\{q := \neg\Diamond p\}$

$\mathbf{KA_1 \ldots A_k} \vdash C$		
$A_1, \ldots, A_k$	C	modalized instances of $A_1, \ldots, A_k$
D	D_2	D
D_2	D	D_2
T	D	$T\{p := \bot\}$
$T, 5$	4	$T\{p := \Diamond\neg p\}, 5\{p := \Box p\}, \Box 5\{p := \neg p\}$
$T, 5$	B	$T\{p := \neg p\}, 5$
$4, B$	5	$\Box 4\{p := \neg p\}, B\{p := \Diamond p\}$
$5, B$	4	$\Box 5\{p := \neg p\}, B\{p := \Box p\}$
$D, 4, B$	T	$D, 4, B\{p := \neg p\}$

4.2 Relations concerning $G, G_0, H, H^{+}, L, L^{+}, L^{++}$

$$
\begin{tikzcd}
\mathbf{K5} & & & & \\
\mathbf{KL^{++}} \arrow[u] \arrow[r] \arrow[ddd] & \mathbf{KDL^{++}} \arrow[r] & \mathbf{KTL^{++}} & & \\
 & \mathbf{KG} = \mathbf{KG_0} \arrow[u] \arrow[r] & \mathbf{KL^{+}} \arrow[u] \arrow[r] & \mathbf{KDL^{+}} \arrow[dr] & \\
 & & & & \mathbf{KTL} = \mathbf{KTL^{+}} \\
\mathbf{K4L} & & \mathbf{KL} = \mathbf{KH} \arrow[ll] \arrow[uu] \arrow[r] & \mathbf{KDL} \arrow[uu] \arrow[ur] &
\end{tikzcd}
$$

$\mathbf{KA_1 \ldots A_k} \vdash C$		
$A_1, \ldots, A_k$	C	modalized instances of $A_1, \ldots, A_k$
G	G_0	$G\{p := q\}$
D, G_0	G	$\Box D, G_0\{p := \Box p, q := p\}$
L	H	L
H	L	$H\{p := p \wedge \Box p \to q, q := q \wedge \Box q \to p\}$
L^{+}	H^{+}	L^{+}
H^{+}	L^{+}	$H^{+}\{p := \Box p \to q, q := \Box q \to p\}$

$\mathbf{KA_1 \ldots A_k} \vdash C$		
$A_1, \ldots, A_k$	C	modalized instances of $A_1, \ldots, A_k$
L^+	L	L^+
T, L	L^+	$\Box T, \Box T\{p := q\}, L$
T, L^{++}	L^+	$\Box T, \Box T\{p := q\}, L^{++}$
$4, L$	L^{++}	$4\{p := p \wedge \Box p \rightarrow q\}, 4\{p := q \wedge \Box q \rightarrow p\}, \Box 4, \Box 4\{p := q\}, L$
5	L^{++}	$5\{p := \Box p\}, \Box 5\{p := \neg p\}$
D, L^{++}	G	$\Box D\{p := \neg p\}, L^{++}\{q := \neg p\}$
5	G	$5, 5\{p := \neg p\}$

4.3 *Relations concerning M, Pt*

The axioms discussed in this section are loosely grouped around the famed McKinsey-axiom M. Variants of M are discussed in [14]. The formula M appears to be one of the simplest axioms defining a pure second-order property (M is not canonical), thus falling out of the class of Sahlqvist axioms [12]. In the presence of 4, however, M defines a first-order property: every point in the frame has a successor which is a (reflexive) dead end point.

$$\mathbf{KD} \longrightarrow \mathbf{KM} \longrightarrow \mathbf{KPt} \longrightarrow \mathbf{K4M} = \mathbf{K4Pt}$$

$\mathbf{KA_1 \ldots A_k} \vdash C$		
$A_1, \ldots, A_k$	C	modalized instances of $A_1, \ldots, A_k$
$T, 4, M$	M_2	$T\{p := M\}, \Box 4, \Box M$
$T, 4, M_2$	M	$T\{p := M_2\}, 4\{p := \Diamond\neg p\}, M_2$
$4, M$	M_3	$4\{p := \Diamond q \wedge (\neg p \vee \neg q)\}, M$
M_3	M	$M_3\{q := \neg p\}$
M	D	M
$4, M$	Pt	$4\{p := (p \vee \Diamond p) \wedge (\neg p \vee \Diamond\neg p)\}$, $\Box 4\{p := p \wedge \Diamond\neg p\}$, $\Box 4\{p := \neg p \wedge \Diamond p\}$, M
Pt	M	Pt

4.4 *Relations concerning W, W_0, Z*

The formula we call W (and sometimes called G) has only one variant but it is related to the formula Z, used by Goldblatt [6] to describe the (temporal) logic of the frame $\langle \omega, \leq \rangle$ where ω is the set of natural numbers and $\leq$ is the usual ordering on the natural numbers. (See also the figure in Section 4.5.)

$\mathbf{KA_1 \ldots A_k} \vdash C$		
$A_1, \ldots, A_k$	C	modalized instances of $A_1, \ldots, A_k$
W	4	$W\{p := p \wedge \Box p\}$
W	W_0	$W\{p := \bot\}$
W	Z	W
W_0, Z	W	W_0, Z
W	Grz_1	$W\{p := p \wedge \Box(\Box p \to p)\}, \Box\Box W$
$4, Z$	Dum_1	$\Box 4, Z$

4.5 Relations concerning Dum, Grz

The formulae known as *Dum* and *Grz* are two of the most bizarre formulae that occur in the literature. They each have numerous variants and the relationships between these variants has attracted considerable attention in the literature. For example, Segerberg [13] proves (using semantic means) that these variants are 'deductively equivalent' [13, p. 108] in **KT4**. We not only show that **KT4Dum** = $\mathbf{KT4Dum_1}$ but we give the exact substitutions necessary to prove (using LWB) that 'Dum_1 is deductively equivalent to Dum in **KT4**'. A detailed syntactical analysis of these axioms based on K4 (explicitly benefitting from the usage of modal rules), has been undertaken in [7].

$$\mathbf{KT4Dum} = \mathbf{KTDum_1} = \mathbf{KTDum_2} = \mathbf{KTDum_3} = \mathbf{KTDum_4}$$

$$\mathbf{KT4Grz} = \mathbf{KTGrz_1} = \mathbf{KTGrz_2} = \mathbf{KTGrz_3} = \mathbf{KTGrz_4} = \mathbf{KTGrz_5}$$

$$\mathbf{KGrz} = \mathbf{KT4MDum}$$

$$\begin{tikzcd}
\mathbf{KT4} \arrow[r] & \mathbf{KGrz} & \mathbf{KT4Dum} \arrow[l] & \mathbf{K4Dum} \arrow[l] \\
\mathbf{K4} \arrow[u] \arrow[r] & \mathbf{K4Grz_1} \arrow[u] \arrow[d] & & \mathbf{K4Dum_1} \arrow[u] \arrow[ll] \arrow[d] \\
& \mathbf{KW} & & \mathbf{K4Z} \arrow[ll]
\end{tikzcd}$$

$\mathbf{KA_1 \ldots A_k} \vdash C$		
$A_1, \ldots, A_k$	C	modalized instances of $A_1, \ldots, A_k$
$4, Dum$	Dum_1	$4\{p := \Box(p \to \Box p) \to p\}$, $\Box 4$, Dum, $Dum\{p := p \to \Box p\}$
T, Dum_1	Dum	T, Dum_1
T, Dum	Dum_2	$\Box T$, Dum
T, Dum_2	Dum	T, $\Box T\{p := p \to \Box p\}$, Dum_2
$T, 4, Dum$	Dum_3	$\Box T$, $4\{p := \Box(p \to \Box p) \to p\}$, $\Box 4$, Dum, $Dum\{p := p \to \Box p\}$
T, Dum_3	Dum	T, $\Box T\{p := p \to \Box p\}$, Dum_3
Dum	Dum_4	Dum
T, Dum_4	Dum	T, Dum_4
Grz	Grz_1	$Grz\{p := (\Box(p \to \Box p) \to p) \wedge 4\{p := \Box(p \to \Box p) \to p\}\}$, $\Box Grz$
T, Grz_1	Grz	T, $T\{p := Grz_1\}$, $\Box Grz_1$
Grz	Grz_2	Grz, $\Box Grz$
T, Grz_2	Grz	$\Box T\{p := p \to \Box p\}$, $T\{p := \neg\Box(\Box(p \to \Box p) \to p)\}$, Grz_2
Grz	Grz_3	$Grz\{p := (\Box(p \to \Box p) \to \Box p) \wedge Grz$ $\wedge 4\{p := (\Box(p \to \Box p) \to \Box p) \wedge Grz\}\}$, $\Box Grz$
T, Grz_3	Grz	T, $\Box T\{p := p \to \Box p\}$, Grz_3
Grz	Grz_4	$Grz\{p := (\Box(p \to \Box p) \to p) \wedge 4\{p := \Box(p \to \Box p) \to p\}\}$, Grz
T, Grz_4	Grz	T, Grz_4
Grz	Grz_5	$Grz\{p := (\Box(p \to \Box q) \to \Box q) \wedge 4\{p := \Box(p \to \Box q) \to \Box q\}\}$, $Grz\{p := (\Box((p \to \Box q) \to \Box(p \to \Box q)) \to (p \to \Box q))$ $\wedge 4\{p := \Box((p \to \Box q) \to \Box(p \to \Box q)) \to (p \to \Box q)\}\}$, $\Box Grz\{p := p \to \Box q\}$
$T, 4, Grz_5$	Grz	$T\{p := \Box(p \to \Box p) \to p\}$, $\Box\Box T\{p := p \to \Box p\}$, $4\{p := \Box(p \to \Box p) \to p\}$, $\Box 4$, $Grz_5\{q := p \to \Box p\}$
Grz	T	Grz
Grz	4	$Grz\{p := p \wedge 4\}$
Grz	M	$\Box Grz$, $Grz\{p := \Diamond\neg p\}$, $Grz\{p := \Diamond\neg p \wedge 4\{p := \Diamond\neg p\}\}$, $Grz\{p := \neg\Box(p \to \Box p) \wedge 4\{p := \neg\Box(p \to \Box p)\}\}$
Grz	Dum	Grz
Grz_1	Dum_1	Grz_1
T, M, Dum	Grz	$\Box T\{p := \neg p\}$, M, Dum

4.6 Miscellaneous relations

$\mathbf{KA_1 \ldots A_k} \vdash C$		
$A_1, \ldots, A_k$	C	modalized instances of $A_1, \ldots, A_k$
R	L^+	$R\{p := \Box p \rightarrow q\}$
T, R	Zem	$T\{p := \Diamond\Box p\}, R$
$T, 4, L^+, Zem$	R	$\Box\Box T\{p := 4\}, \Box\Box T\{p := \neg(\Box\Diamond\Box p \land \Box p)\}, \Box 4,$ $L^+\{p := \Diamond\Box p, q := \neg\Box p\}, Zem$
$T, 4, P$	M	$T\{p := \Diamond p\}, \Box T\{p := \neg\Box\Diamond p\}, 4\{p := \Diamond p\}, \Box P$
T, P	R	$\Box\Box T\{p := \neg p\}, P$
$4, M, R$	P	$4\{p := \neg\Box p\}, \Box M, R$
T, P	H_s	$\Box\Box\Box T, P, \Box P\{p := \neg p\}$
$T, 4, R$	Dum	$T\{p := \Box(p \rightarrow \Box p) \rightarrow p\}, \Box 4\{p := p \land (\Box(p \rightarrow \Box p) \rightarrow p)\},$ $R\{p := p \rightarrow \Box p\}$

5 Checking the results and obtaining purely syntactic proofs

Since we give the instances required for the proofs, it is easy to check all the results using a theorem prover for **K**.

One possibility is the Logics Workbench (see [8]). You can use it via the Internet. Open `http://lwbwww.unibe.ch:8080/LWBinfo.html` in your WWW browser, choose the item `run a session` and type in your request.

It is also possible to check the results by hand (although it is a bit tiring in some cases). One can for example make a backward search in the usual sequent calculus. (See [3] or [4] for more information on such calculi.) Such proofs can then be converted into Hilbert-style proofs (see [5] for details).

Acknowledgement

We would like to thank Zhang Jian for pointing out a simplification concerning *Dum*.

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Appendix

The following table shows the names used in the literature. Note that the formulae are only equal modulo substitutions of the form $\{p := \neg p\}$.

here	[1]	[13]	[11]	[9]	[10]	[2]	[6]
D	D		D (p. 50)		D	D	D
D_2		D (p. 47)					
T	T	T (p. 47)	T (p. 50)		T	T	T
4	4	4 (p. 47)	4 (p. 50)		4	4	4
5	E	E (p. 47)	E (p. 50)		5	5	5
B	B	B (p. 47)	B (p. 50)	B	B	B	B
G	G	G (p. 47)	G (p. 50)	$G1$	$G1$	G	
G_0		G_0 (p. 47)					
H	H		H (p. 69)			H	
H^+			H^+ (p. 69)			H^+	
L		Lem_0 (p. 47)	H_0^+ (p. 80)		DI_0		L
L^+		Lem (p. 47)	H_0 (p. 80)		DI	L^+	L_1
L^{++}				$D2$		L^{++}	
M	M	M (p. 107)	M (p. 47)	Kb	M	G_c	
M_3				Ka			
W	W	W (p. 84)			W	Gr	W
W_0		W_0 (p. 93)			W_0		
Z		Z (p. 84)					Z
Dum	Dum			$N1$	$N1$		
Dum_1		Dum_1 (p. 107)					Dum
Grz	Grz		Grz (p. 81)	$J1$	$J1$		
Grz_1		Grz_1 (p. 107)					
Grz_5		(p. 108)					
F		F (p. 161)					
H_s		H (p. 148)		$H1$			
P		P (p. 152)					
R		R (p. 160)		$R1$			
X							X
Zem		Zem (p. 152)					

Received 12 October 1995