

On the Constructive Truth and Falsity in Peano Arithmetic

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Abstract. Recently, Artemov [4] offered the notion of constructive consistency for Peano Arithmetic and generalized it to constructive truth and falsity in the spirit of Brouwer-Heyting-Kolmogorov semantics and its formalization, the Logic of Proofs. In this paper, we provide a complete description of constructive truth and falsity for Friedman’s constant fragment of Peano Arithmetic. For this purpose, we generalize the constructive falsity to n -constructive falsity where n is any positive natural number. We also establish similar classification results for constructive truth and n -constructive falsity of Friedman’s formulas. Then, we discuss ‘extremely’ independent sentences in the sense that they are classically true but neither constructively true nor n -constructive false for any n .

§1 Introduction.

In the second incompleteness theorem, Gödel proved the impossibility to prove an arithmetical sentence, $Con(PA) = \forall x \neg Proof(x, 0 = 1)$, which is meant to be a formalization of consistency of Peano Arithmetic, **PA**: *For all x , x is not a code of a proof of $0 = 1$.* The formalization is concerned with arithmetization of the universal quantifier in the statement and the arithmetization cannot rule out the interpretability of the quantifier to range over both standard and nonstandard numbers. In a recent paper [4], Artemov pointed out that it is too strong to capture fairly Hilbert’s program on finitary consistency proof for arithmetic; it asked for a finitary proof that in a formal arithmetic *no finite sequence of formulas is a derivation of a contradiction*. Then, he proposed the notion of constructive consistency, $CCon(PA)$, and demonstrated that it is actually provable in **PA**.

Moreover, the generalization of constructive consistency was offered in [4] in the spirit of Brouwer-Heyting-Kolmogorov (BHK) semantics and its

formalization, the Logic of Proofs (LP): constructive falsity with its counterpart, the constructive truth. (On the family of systems called Justification Logics including the Logic of Proofs, we can refer to [2, 3, 6, 7].)

Definition 1.1 *An arithmetical sentence A is constructively false if PA proves: for any x , there is a proof that ‘ x is not a proof of A ’.*

This is also viewed as the result of a refinement of the interpretation of negation and implication in the BHK semantics by the framework of the Logic of Proofs, which is compliant with the Kreisel ‘second clause’ criticism. (Cf. [7])

On the other hand, the letterless fragment of the logic of provability GL has been an object of modal logical study of Peano Arithmetic, PA , since Friedman’s 35th problem in [12]. A letterless sentence is one built up from a constant for falsity \perp , boolean connectives, and the modality \Box . Boolos [9], J. van Benthem, C. Bernardi and F. Montagna showed that there is a specific normal form for these sentences and the fragment is decidable, which was an answer to the Friedman’s question.

Following Boolos [11], we call the counterpart of letterless sentences in PA *constant* sentences. Formally, they are built from the sentence $0 = 1$, a suitable provability predicate $\text{Prov}_{\text{PA}}(*)$ and boolean connectives. Any arithmetical interpretations convert a letterless sentence to the same constant sentence in PA . Here, for the sake of simplicity, we write \perp to mean $0 = 1$ and $\Box(*)$ to mean a fixed provability predicate of PA .

In this paper, we are primarily concerned with the constant fragment of PA ; in §2, we provide a complete delineation of the constant sentences in terms of the notions of constructive truth and falsity. Then, it turns out natural to generalize constructive falsity to n -constructive falsity, for each positive natural number n . Also, for each n , we provide classification results for constructive truth and n -constructive falsity for constant sentences.

The ‘constructive’ liar sentence was introduced and discussed in [4] along with the Rosser sentence. In §3, we generalize both of these two kinds of arithmetical sentences, and specify the logical status of them on the basis of generalized constructive falsity. Also, we clarify which constant sentences can be the generalized Rosser sentences.

In §4, we offer the notion of ‘extreme’ independence from PA for arithmetical sentences A : both they and their negation are neither provable in

PA nor belong to n -constructive falsity for any n . We show that there is an extremely independent arithmetical sentence but no constant sentence is extremely independent.

§2 The Constant Fragment of Peano Arithmetic

In [4], Artemov clarified the status of some constant sentences on classical and constructive truth and falsity: $\text{Con}(\text{PA})$ is classically true and constructively false. $0 = 1$ is classically false and constructively false. $\neg\text{Con}(\text{PA})$ is classically false and neither constructively true nor constructively false. Then, it is natural to ask a general question: under which condition a constant sentence is said to be constructively true or constructively false.

First of all, we generalize the notion of constructive falsity to n -constructive falsity ($n \geq 1$). Put $cf^n(F) = \forall x \square^n \neg(x : F)$ for each $n \geq 1$, where $\square^n = \underbrace{\square \cdots \square}_n \perp$.

Definition 2.1 *An arithmetical sentence A is n -constructive false if and only if PA proves the sentence $cf^n(A)$.*

The original constructive falsehood is the special case with $n = 1$.

Theorem 2.2 (Normal Form Theorem) $\vdash_{\text{PA}} cf^n(F) \leftrightarrow \neg \square F \rightarrow \square^n \perp$.

Proof. Work in PA. Suppose $\square F$, that is, $\exists x(x : F)$ holds. Then, for some y , we have $y : F$. By applying Σ_1 -completeness n times, we obtain $\square^n(y : F)$. On the other hand, suppose $\forall x \square^n \neg(x : F)$. Then, $\square^n \neg(y : F)$ holds. Hence, we obtain $\square^n \perp$. Thus, $\forall x \square^n \neg(x : F) \rightarrow \neg \square F \rightarrow \square^n \perp$. For the other direction, obviously, $\square^n \perp \rightarrow \square^n \neg(x : F)$. By generalization, $\square^n \perp \rightarrow \forall x \square^n \neg(x : F)$. On the other hand, by applying Σ_1 -completeness n times, for any x , $\neg(x : F) \rightarrow \square^n \neg(x : F)$. By predicate calculus, $\neg \exists x(x : F) \rightarrow \forall x \square^n \neg(x : F)$, that is, $\neg \square F \rightarrow \forall x \square^n \neg(x : F)$. Therefore, $\neg \square F \vee \square^n \perp \rightarrow \forall x \square^n \neg(x : F)$. ■

Here we observe some simple facts.

(F1) If A is n -constructively false and PA proves $B \rightarrow A$, B is also n -constructively false.

(F2) If A is n -constructively false and $n \leq m$, A is m -constructively false.

(F3) If PA is n -consistent, that is, PA does not prove $\Box^n \perp$, then no n -constructively false sentence is constructively true.

We say that a sentence is n -constructively false *at the smallest* if and only if it is n -constructively false but not m -constructively false sentence for any $m < n$.

We introduce the following three types of arithmetical sentences.

(α) -sentences: of the form $\Box^n \perp \rightarrow \Box^m \perp$ ($0 \leq n \leq m$)

(β, n) -sentences: of the form $\Box^m \perp \rightarrow \Box^{n-1} \perp$ ($1 \leq n \leq m$)

(γ, n) -sentences: of the form: $\Box^{n-1} \perp$ ($1 \leq n$)

Lemma 2.3 (1) (β, n) - and (γ, n) -sentences are n -constructively false at the smallest.

(2) (α) -sentences are constructively true.

Proof. (2) is immediate. For (β, n) -sentences, consider the formula $\Box(\Box^m \perp \rightarrow \Box^{n-1} \perp) \rightarrow \Box^k \perp$ with $0 \leq n \leq m$. This is provably equivalent in PA to $\Box^n \perp \rightarrow \Box^k \perp$. Therefore, PA proves it if and only if $k \geq n$, in terms of Gödelean incompleteness theorems. The proof is similar for (γ, n) -sentences. ■

By $(\beta\gamma, n)$ -sentence we mean a conjunction of (β, a) - and (γ, b) -sentences such that n is the minimum of all such a 's and b 's. In particular, when it consists only of (β, a) -sentences, it is called a (β^+, n) sentence.

Lemma 2.4 $(\beta\gamma, n)$ -sentences are n -constructively false at the smallest.

Proof. Temporarily, let (β, n_i) and (γ, m_i) denote a (β, n_i) - and a (γ, m_i) -sentence, respectively. Consider the following sentence.

$$(*) \quad \Box(\bigwedge_i (\beta, n_i) \wedge \bigwedge_j (\gamma, m_j)) \rightarrow \Box^k \perp$$

where $n = \min_{i,j}(n_i, m_j)$. By using derivability conditions on the provability predicate \Box , this is provably equivalent in PA to the following.

$$\bigwedge_i \Box(\beta, n_i) \wedge \bigwedge_j \Box(\gamma, m_j) \rightarrow \Box^k \perp.$$

Furthermore, we can execute the following transformations, keeping equivalence in **PA**.

$$\bigwedge_i \Box^{n_i} \perp \wedge \bigwedge_j \Box^{m_j} \perp \rightarrow \Box^k \perp;$$

$$\Box^n \perp \rightarrow \Box^k \perp.$$

Thus, in terms of Gödelean incompleteness theorems, $(*)$ is provable in **PA** if and only if $k \geq n$. ■

Lemma 2.5 *Any constant sentence is provably in **PA** equivalent to an (α) -sentence or a $(\beta\gamma, n)$ -sentence for some $n \geq 1$.*

Proof. Boolos' normal form theorem for constant sentences in [11] states that any constant sentence is equivalent in **PA** to a boolean combination of $\Box^n \perp$. By propositional transformation, it is further equivalent to a conjunction of sentences of the form of (α) , (β, n) and (γ, m) . If it contains only conjuncts which are (α) -sentences, it is equivalent to an (α) -sentence. Suppose that it is of the form $X \wedge Y$ where X contains no (α) -sentence and Y contains only (α) -sentences. As $X \wedge Y$ is equivalent in **PA** to X , it is a $(\beta\gamma, n)$ sentence with some n . ■

Theorem 2.6 *Any constant sentence is provably in **PA** equivalent to a constructively true sentence or an n -consistently false sentence for some n .*

Proof. Derived by Lemmas 2.4, 2.5. ■

Theorem 2.7 *Let A be any constant sentence and n be any positive natural number. Suppose that **PA** is n -consistent. Then, we have the following.*

- (1) *A is n -constructively false and classically true, if and only if, A is provably in **PA** equivalent to a (β^+, m) -sentence for some $m \leq n$.*
- (2) *A is n -constructively false and classically false, if and only if, A is provably in **PA** equivalent to a (γ, m) -sentence for some $m \leq n$.*
- (3) *A is constructively true, if and only if, A is provably in **PA** equivalent to an (α) -sentence.*

Proof. The 'if' directions in (1-3) are immediate by Lemma 2.4. For the 'only if' direction. (3) is obvious. We prove (1, 2). Suppose that A is m -constructively false at the smallest for some $m \leq n$. By (F3), A is not

constructively true and so, is not an (α) -sentence. Since A is constant, by Lemma 2. 5, A is equivalent to a $(\beta\gamma, a)$ -sentence for some $a \geq 1$. By Lemma 2. 4, $a = m$.

Now, if it is classically true, A is equivalent to (β^+, m) -sentence; if it is classically false, A is equivalent to a conjunction of (γ, m_i) -sentences where $\min_i(m_i) = m$, which is equivalent to a (γ, m) -sentence, that is, $\Box^{m-1}\perp$. ■

§3 Generalized ‘Constructive’ Liar Sentences and Rosser Sentences

In [4], Artemov offered a constructive version, L , of ‘Liar Sentence’ by applying the diagonal lemma:

$$\begin{aligned} \vdash_{\text{PA}} L &\leftrightarrow \forall x \Box \neg(x : L) \\ &\leftrightarrow (\Box L \rightarrow \Box \perp) \end{aligned}$$

And he pointed out that L is classically true but neither constructively true nor constructively false. We show that L is 2-constructively false and $\neg L$ is (1-)constructively false.

We shall introduce a general version of ‘Constructive Liar Sentence’. For each $n \geq 1$, L_n is provided by the following.

$$\begin{aligned} \vdash_{\text{PA}} L_n &\leftrightarrow \forall x \Box^n \neg(x : L_n) \\ &\leftrightarrow (\Box L_n \rightarrow \Box^n \perp) \end{aligned}$$

The existence of L_n , we call *n-constructive liar*, is guaranteed by the diagonal lemma.

Theorem 3. 1 (1) L_n is classically true and $(n + 1)$ -constructively false at the smallest.

(2) $\neg L_k$ is classically false and 1-constructively false. ($k \geq 1$)

Proof. For (1). Suppose that L_n is not true. Then, $\Box L_n$ is not true and $\Box L_n \rightarrow \Box^n \perp$ is true. This means L_n is true by definition of L_n . Hence, a contradiction.

Next, again by definition, PA proves $\Box[L_n \rightarrow (\Box L_n \rightarrow \Box^n \perp)]$ and so $\Box L_n \rightarrow \Box \Box^n \perp$. This means L_n is $(n + 1)$ -constructively false. To show

that $(n + 1)$ is the smallest, suppose that **PA** proves $\Box L_n \rightarrow \Box^n \perp$, that is, $\Box(\Box L_n \rightarrow \Box^n \perp) \rightarrow \Box^n \perp$. Then, **PA** also proves $\Box \Box^n \perp \rightarrow \Box^n \perp$, which is impossible in terms of Gödelean incompleteness theorems.

The proof for (2) is similar. ■

How about Gödelean Liar Sentence? It is considered to be $\text{Con}(\text{PA})$, that is, $\neg \Box \perp$. We can generalize this as follows: *n-Gödelean Liar Sentence*, or *n-Liar Sentence* is defined to be $\text{Con}(\text{PA}^n)$, which is well known to be equivalent to $\neg \Box^n \perp$.¹ About this, we already know its status from the result of the previous section. $\text{Con}(\text{PA}^n)$ is a $(\beta, 1)$ -sentence and, by Lemma 2.3, it is 1-constructively false at the smallest. As to $\neg \text{Con}(\text{PA}^n)$, it is equivalent to $\Box^n \perp$, which is a $(\gamma, n + 1)$ -sentence and, by Lemma 2.3, $(n + 1)$ -constructively false at the smallest.

In [4], Artemov pointed out that the Rosser sentence, R , is classically true and constructively false; $\neg R$ is classically false and constructively false. Therefore, the result of Rosser's incompleteness theorem is said to have been the discovery of such a sentence which is 1-constructively false and the negation of which is also 1-constructively false.

Here again, we can make a generalization: an arithmetical sentence R_n is an *n-Rosser sentence* if both R_n and $\neg R_n$ are *n*-constructively false at the smallest ($n \geq 1$). This condition is equivalent to the following: **PA** proves

$$\neg \Box^k \perp \rightarrow (\neg \Box R_n \wedge \neg \Box \neg R_n)$$

for any $k \geq n$ and does not for any $k < n$. The original Rosser sentence R is an instance of 1-Rosser sentence R_1 . It is well-known that such an R_n can be constructed in **PA**.

Now, we can naturally ask: is it possible to construct constant *n*-Rosser sentences?

Lemma 3.2 *Let A be any constant sentence containing the provability predicate \Box . If A is *n*-constructively false, $\neg A$ is 1-constructively false.*

Proof. If A is classically true, by Theorem 2.7, $\neg A$ is equivalent to the form: $\bigvee_i (\Box^{k_i} \perp \wedge \neg \Box^{a_i} \perp)$ where for each i , $a_i < n$ and $a_i < k_i$. Note that in **PA**, $\bigvee_i (\Box^{k_i} \perp \wedge \neg \Box^{a_i} \perp)$ implies $\neg \Box^{\min_i(a_i)} \perp$. We have a derivation in **PA**:

¹ PA^n is usually defined: $\text{PA}^0 = \text{PA}$; $\text{PA}^{n+1} = \text{PA}^n + \text{Con}(\text{PA}^n)$

$$\begin{aligned}
\Box(\bigvee_i(\Box^{k_i}\perp \wedge \neg\Box^{a_i}\perp)) &\rightarrow \Box(\neg\Box^{\min_i(a_i)}\perp) \\
&\rightarrow \Box(\Box^{\min_i(a_i)}\perp \rightarrow \perp) \\
&\rightarrow \Box(\Box\perp \rightarrow \perp) \\
&\rightarrow \Box\perp
\end{aligned}$$

If A is classically false, by Theorem 2. 7, $\neg A$ is equivalent to the form: $\neg\Box^a\perp$ with $a < n$. By the hypothesis, $a \neq 0$. We have a derivation in PA:

$$\begin{aligned}
\Box\neg\Box^a\perp &\rightarrow \Box(\Box^a\perp \rightarrow \perp) \\
&\rightarrow \Box(\Box\perp \rightarrow \perp) \\
&\rightarrow \Box\perp
\end{aligned}$$

Thus, in any case, $\neg A$ is 1-constructively false. ■

Theorem 3. 3 *Let A be any constant sentence containing the provability predicate \Box . Then, the following are equivalent.*

- (1) A is an n -Rosser sentence for some n ;
- (2) A is a 1-Rosser sentence;
- (3) A is 1-constructively false.

Proof. Proofs from (2) to (1) and from (2) to (3) are immediate.

From (1) to (2): If (1) holds, both A and $\neg A$ are both n -constructively false and, by Lemma 3. 2, $n = 1$.

From (3) to (2): If (3) holds, by Lemma 3. 2, $\neg A$ is 1-constructively false. Then, (2) holds. ■

By Theorem 3. 3, constant sentences can be n -Rosser sentences only when $n = 1$. Of course, we can weaken the definition of n -Rosser sentences: R_n is a *weak n -Rosser sentence* if and only if both R_n and $\neg R_n$ are n -constructively false (not necessarily at the smallest).

Corollary 3. 4 *Any constant sentence containing the provability predicate \Box is a weak n -Rosser sentence for some n , unless it is constructively true.*

Proof. For any constant sentence A containing \Box , if A is not constructively true, by Theorem 2. 6, A is n -constructively false for some $n \geq 1$. By Lemma 3. 2, $\neg A$ is 1-constructively, therefore, n -constructively false. ■

Also, we obtain a relationship between n -constructive liar sentences and n -Rosser sentences.

Corollary 3.5 (1) No one of constructive liar sentences and the negation of them is an n -Rosser sentence for any n .
(2) For any $n \geq 1$, any n -constructive liar sentence L_n is a weak $(n+1)$ -Rosser sentence.

Proof. Derived by Theorem 3.1. ■

Here is a table to sum up some of the results from §§2, 3.

	Classically True	Classically False
\vdots	\vdots	\vdots
n -const. false	L_{n-1}, R_n $\Box^m \perp \rightarrow \Box^{n-1} \perp \ (m \geq n)$	$\Box^{n-1} \perp$
\vdots	\vdots	\vdots
3-const. false	L_2, R_3 $\Box^m \perp \rightarrow \Box^2 \perp \ (m \geq 3)$	$\Box^2 \perp$
2-const. false	L_1, R_2 $\Box^m \perp \rightarrow \Box^1 \perp \ (m \geq 2)$	$\Box \perp$
1-const. false	R_1 $\neg \Box^m \perp \ (m \geq 1)$	$\neg L_i, \neg R_i \ (1 \leq i)$ $\Box^m \perp \wedge \neg \Box^{n-1} \perp \ (m \geq n \geq 2)$ \perp

§4 ‘Extremely’ Independent Sentences

We showed that any constant sentence is n -constructively false for some n , unless it is constructively true (Theorem 2.6). This implies that well-known constant Gödelean sentences such as $\text{Con}(PA^n)$ and $\neg \text{Con}(PA^n)$ are m -constructively false for some m .

How about the ‘Reflection Principles’? For any sentence A , let $\text{Ref}(A)$ denote $\Box A \rightarrow A$ (what we call the local Reflection Principle for A). We claim the following.

Theorem 4.1 For any sentence A , $\neg \text{Ref}(A)$ is 2-constructively false.

Proof. In PA , we have the following derivation.

$$\begin{aligned}
\Box(\Box A \wedge \neg A) &\rightarrow \Box\Box A \wedge \Box\neg A \\
&\rightarrow \Box\Box A \wedge \Box\Box\neg A \\
&\rightarrow \Box\Box\perp
\end{aligned}$$

This finishes the proof. ■

We note that the above argument does not generally hold for what we call the uniform Reflection Principle.

In addition, it is known that $Ref(A^{\Pi_1})$ for Π_1 -sentences A^{Π_1} is provably equivalent to $Con(PA)$ in PA . Hence, $Ref(A^{\Pi_1})$ is 1-constructively false and $\neg Ref(A^{\Pi_1})$ is 2-constructive false.

These results raise the question of the status of independence of a kind of Gödelean sentences (such as $Con(PA)$, $Ref(A)$, and other constant ones) from PA . So, we can naturally ask if there is a ‘truly’ independent arithmetical sentence from PA or not. We consider stronger notions of independence.

Definition 4.2 1. *An arithmetical sentence A is strongly independent from PA if and only if A is neither constructively true nor n -constructively false for any n .*

2. *An arithmetical sentence A is extremely independent from PA if and only if both A and $\neg A$ are strongly independent from PA .*

Note that if a sentence A is extremely independent, so is $\neg A$.

Theorem 4.3 *No instance of the local Reflection Principle is extremely independent from PA .*

Proof. Derived by Theorem 4.1. ■

Theorem 4.4 *No arithmetical constant sentence is strongly nor extremely independent from PA .*

Proof. Derived by Theorem 2.6. ■

In [4], Artemov showed that there is an arithmetical sentence A such that both A and $\neg A$ are not 1-constructively false by using the uniform arithmetical completeness for the modal logic GL. We extend this result to our general setting.

Proposition 4.5 (*Uniform Arithmetical Completeness for GL*) *There is an arithmetical interpretation $*$ such that for any formula A of modal logic, $\vdash_{\text{GL}} A$ iff $\vdash_{\text{PA}} A^*$.*

This was established independently in [1, 8, 10, 13, 14].

Theorem 4.6 *There is an extremely independent sentence.*

Proof. Fix a propositional variable p . It is easily seen that for any positive natural number n , $\not\vdash_{\text{GL}} \Box p \rightarrow \Box^n \perp$ and $\not\vdash_{\text{GL}} \Box \neg p \rightarrow \Box^n \perp$. (This can be proved by an argument of Kripke completeness or the arithmetical completeness for GL.) Therefore, by the above proposition, there is an arithmetical sentence F such that for any positive natural number n , $\not\vdash_{\text{PA}} \Box F \rightarrow \Box^n \perp$ and $\not\vdash_{\text{PA}} \Box \neg F \rightarrow \Box^n \perp$. This sentence F is extremely independent from PA. ■

Corollary 4.7 *There is an instance of the local Reflection Principle which is strongly independent from PA.*

Proof. In the proof of Theorem 4.6, we obtain the sentence F such that for any positive natural number n , $\not\vdash_{\text{PA}} \Box F \rightarrow \Box^n \perp$. This sentence is equivalent to $\Box(\Box F \rightarrow F) \rightarrow \Box^n \perp$. Therefore, $\text{Ref}(F) = \Box F \rightarrow F$ is the desired instance. ■

Theorem 4.4 could signify a limitation of the expressibility of arithmetical constant sentences, as contrasted with Theorem 4.6.

§5 Concluding Remark

In this paper, we reported some results on the notion of constructive truth and falsity in PA, which was just invented and has been reported to offer a ‘real’ solution to the Hilbert program in Artemov [4]. In particular, we showed some theorems on the relationship of those notions and the ‘constant’ fragment of PA, which has been actively studied a lot since Friedman [12].

As is easily observed, an arithmetical sentence is n -constructively false if and only if its unprovability in PA is provable in PA plus $\text{Con}(\text{PA}^n)$. As an extension of the work of this paper, a natural research problem would

be to examine whether or not things change in an essential way, if we are permitted to talk about extensions of **PA** in the well-known ‘transfinite progression’ since Turing. Then, we have the notion of α -constructive falsity, where α is an ordinal in an ordinal system. As the research subject of the transfinite progression is known to form a vast area of mathematical logic, we report this further study in a separate paper.

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