# THE AXIOM OF CHOICE IN COMPUTABILITY THEORY AND REVERSE MATHEMATICS 

WITH A CAMEO FOR THE CONTINUUM HYPOTHESIS

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#### Abstract

The Axiom of Choice (AC for short) is the most (in)famous axiom of the usual foundations of mathematics, ZFC set theory. The (non-)essential use of AC in mathematics has been well-studied and thoroughly classified. Now, fragments of countable AC not provable in ZF have recently been used in Kohlenbach's higher-order Reverse Mathematics to obtain equivalences between closely related compactness and local-global principles. We continue this study and show that NCC, a weak choice principle provable in ZF and much weaker systems, suffices for many of these results. In light of the intimate connection between Reverse Mathematics and computability theory, we also study realisers for NCC, i.e. functionals that produce the choice functions claimed to exist by the latter from the other data. Our hubris of undertaking the hitherto underdeveloped study of the computational properties of (choice functions from) AC leads to interesting results. For instance, using Kleene's S1-S9 computation schemes, we show that various total realisers for NCC compute Kleene's $\exists^{3}$, a functional that gives rise to full second-order arithmetic, and vice versa. By contrast, partial realisers for NCC should be much weaker, but establishing this conjecture remains elusive. By way of catharsis, we show that the Continuum Hypothesis ( CH for short) is equivalent to the existence of a countably based partial realiser for NCC. The latter kind of realiser does not compute Kleene's $\exists^{3}$ and is therefore strictly weaker than a total one.


## 1. Introduction

Obviousness, much more than beauty, is in the eye of the beholder. For this reason, lest we be misunderstood, we formulate a blanket caveat: all notions (computation, continuity, function, open set, comprehension, et cetera) used in this paper are to be interpreted via their well-known definitions in higher-order arithmetic listed below, unless explicitly stated otherwise.
1.1. Short summary. The usual foundations of mathematics Zermelo-Fraenkel set theory with the Axiom of Choice and its acronym ZFC, explicitly reference a single axiom. The (in)essential use of the Axiom of Choice (AC for short) in mathematics, is well-studied and has been classified in detail ( $12,14,33,34$ ). In a nutshell, this paper deals with the (in)essential use of AC in Kohlenbach's higherorder Reverse Mathematics (RM for short; see Section (2.1), and the study of the

[^0]computational properties of the associated fragments of AC following Kleene's S1S9 computation schemes (see Section (2.2). Our hubris of undertaking the hitherto underdeveloped study of the computational properties of choice functions from AC leads to catharsis in that the latter properties turn out to be intimately connected to Cantor's Continuum Hypothesis, even in the most basic case.

In more detail, fragments of countable AC not provable in ZF, play a central role in the RM of local-global principles and compactness principles in [29, 30. The latter principles are generally believed to be intimately related (see e.g. Tao's description in [50, p. 168]), but they can have very different logical and computational properties, especially in the absence of countable AC, as shown in 29, 30 and discussed in detail below in Section 1.2.2,

In this paper, we show that countable AC can be replaced by the much weaker principle NCC (see Section 1.2.2) provable in higher-order arithmetic without choice, and hence ZF. Following the intimate connection between RM and computability theory, we also study the computational properties of NCC. A central role is played by the distinction between total and partial realisers of NCC. Intuitively, the former are strong as they compute Kleene's $\exists^{3}$ (yielding full second-order arithmetic; see Section(2.2), while there should be weak examples of the latter that in particular do not compute $\exists^{3}$. Establishing the latter fact, we run into the famous Continuum Hypothesis (CH for short). We explain the required background from [29, 30] in Sections 1.2 .1 and 1.2 .2 while the latter also sketches our main results.

Finally, ZF can prove certain choice principles and we refer to those as weak fragments of AC, whereas strong fragments are those not provable in ZF.
1.2. Overview. We discuss the starting point of this paper, namely Reverse Mathematics, in Section 1.2.1, while our main results are summarised in Section 1.2.2
1.2.1. A question with multiple answers. The starting point of our enterprise is the Main Question of the Reverse Mathematics program (RM hereafter; see Section 2.1 for an introduction), which is usually formulated as follows.

What are the minimal axioms needed to prove a given theorem of ordinary, i.e. non-set theoretic mathematics? (see [47, I.1])
Implicit in this question is the assumption that one can always find a unique and unambiguous set of such minimal axioms. As it turns out, there are basic theorems for which this question does not have an unique or unambiguous answer. The most basic example is Pincherle's theorem, published around 1882 in [32, p. 67] and studied in [29]. This third-order theorem expresses that a locally bounded function is bounded, say on Cantor space for simplicity.

As discussed in detail in Section 1.2.2, and with definitions in Section 2.1 Pincherle's theorem is equivalent to weak König's lemma from second-order RM, over Kohlenbach's base theory $\mathrm{RCA}_{0}^{\omega}$ plus QF-AC ${ }^{0,1}$; the latter is a strong fragment of countable choice. This equivalence is expected as compactness principles and local-global principles are intimately related in light of Tao's description in [50, p. 168]. By contrast, in the absence of countable choice, there are two conservative extensions of second-order arithmetic $Z_{2}$, called $Z_{2}^{\omega}$ and $Z_{2}^{\Omega}$, where the former cannot prove Pincherle's theorem and the latter can (and hence ZF can too). We note that $Z_{2}^{\omega}$ is based on third-order functionals $S_{k}^{2}$ deciding second-order $\Pi_{k}^{1}$-formulas, while $\mathbf{Z}_{2}^{\Omega}$ is based on Kleene's fourth-order quantifier $\exists^{3}$.

Similar results are available for the computability theory (in the sense of Kleene's S1-S9 from [17,21): weak König's lemma is equivalent to the Heine-Borel theorem for countable covers and the finite sub-cover claimed to exist by the latter is outright computable in terms of the data. Despite this equivalence and the similar syntactic form, no type two functional (which includes the aforementioned $S_{k}^{2}$ ) can compute the upper bound from Pincherle's theorem in terms of the data.

More results of the above nature can be found in [29, 30, as discussed in Section 1.2.2 Together, these results show that local-global principles (like Pincherle's theorem) are very similar to compactness (like weak König's lemma), yet can behave very differently, esp. in the absence of countable choice. In the spirit of RM, it is then a natural question whether countable choice is necessary in this context, or whether a weak(er) choice principle, say provable in ZF, suffices. A positive answer is provided in the next section, as well as the implications for (higher-order) computability theory. Indeed, the latter is intimately connected to RM, prompting the study of realisers for the aforementioned weak choice principles.

Finally, we note that, in the grander scheme of things, there is (was?) a movement to remove countable choice from Bishop's constructive analysis [2, 35, 36, 45] and constructive mathematics $([53, \S 3.9])$. While classical, our results do fit with the spirit of this constructive enterprise.
1.2.2. The Pincherle phenomenon. We formulate the results from 29, 30, the Pincherle phenomenon in particular, and sketch our results based on this phenomenon.

First of all, we have shown in [29] that Pincherle's theorem is closely related to (open-cover) compactness, but has fundamentally different logical and computational properties. Indeed, Pincherle's theorem, called PIT ${ }_{o}$ in [29], satisfies the following properties; definitions can be found in Section 2.2 and 3.2
(I) The systems $Z_{2}^{\omega}$ and $Z_{2}^{\Omega}$ are conservative extensions of $Z_{2}$ and $Z_{2}^{\omega}$ cannot prove $\mathrm{PIT}_{o}$ while $\mathrm{Z}_{2}^{\Omega}$ can; $\mathrm{RCA}_{0}^{\omega}+$ QF-AC ${ }^{0,1}$ proves $\mathrm{WKL} \leftrightarrow \mathrm{PIT}_{o}$.
(II) Even a weak realiser for $\mathrm{PIT}_{o}$ cannot be computed (Kleene S1-S9) in terms of any type two functional, including the comprehension functionals $\mathrm{S}_{k}^{2}$.
Secondly, we have established similar properties in 30 for many basic theorems pertaining to open sets given by (possibly discontinuous) characteristic functions. A number of results in 40 also make use of $\mathrm{QF}-\mathrm{AC}^{0,1}$ in (what seems like) an essential way. For instance, let HBC be the Heine-Borel theorem for countable covers of closed sets in $[0,1]$ which are complements of the aforementioned kind of open sets. Exactly the same properties as in items (I) and (II) hold for HBC, and a large number of similar theorems, by [30, §3].

We shall therefore say that HBC exhibits the Pincherle phenomemon, due to Pincherle's theorem $\mathrm{PIT}_{o}$ being the first theorem identified as exhibiting the behaviour as in (I) and (II), namely in [29. In other words, the aim of 30] was to establish the abundance of the Pincherele phenomenon in ordinary mathematics, beyond the few examples from [29].

[^1]Thirdly, since ZF cannot prove QF-AC ${ }^{0,1}$, it is a natural question, also implied by the Main Question of RM, whether a choice principle weaker than QF-AC ${ }^{0,1}$ also suffices to obtain equivalences like $\mathrm{HBC} \leftrightarrow \mathrm{WKL} \leftrightarrow \mathrm{PIT}_{o}$. In this paper, we show that a number of such results originally proved using QF-AC ${ }^{0,1}$, can be proved using the following weak choice principle.
Definition 1.1. [NCC] For $Y^{2}$ and $A(n, m) \equiv\left(\exists f \in 2^{\mathbb{N}}\right)(Y(f, m, n)=0)$ :

$$
\left(\forall n^{0}\right)\left(\exists m^{0}\right) A(n, m) \rightarrow\left(\exists g^{1}\right)\left(\forall n^{0}\right) A(n, g(n))
$$

Clearly, this principle is provable in ZF and even in $Z_{2}^{\Omega}$, a conservative extension of $Z_{2}$ introduced in Section 2.2. The replacement of QF-AC ${ }^{0,1}$ by NCC is for the most part non-trivial and introduces a lot more technical detail, as will become clear in Section3. An obvious RM-question is whether one can weaken NCC, e.g. by letting $g^{1}$ only provide an upper bound for the variable $m^{0}$ in NCC. The below proofs do not seem to go through with this modification. As discussed in Section 3.1, NCC is also connected to the uncountability of $\mathbb{R}$ in interesting ways.

Finally, since RM and computability theory are generally intimately connected (both in the second- and higher-order case), it is a natural next step to study the computational properties of NCC, even though choice functions provided by AC are often regarded as fundamentally non-constructive. We study realisers for NCC, which are functionals $\zeta$ that take as input $Y$ and output the choice function $\zeta(Y)=g$ from NCC. While NCC is quite weak, the associated realisers turn out to be rather strong, in that they compute the aforementioned $\exists^{3}$, a functional that yields full second-order arithmetic (and vice versa). We establish the same for weak realisers for NCC that only yield an upper bound for the variable $m^{0}$ from NCC.

Finally, the strength of the aforementioned realisers is due to their total nature, and it is therefore natural to study partial, i.e. not everywhere defined, realisers for NCC. In particular, we believe these realisers to be the key to answering the following question raised in [29]. Intuitively speaking, there should be a difference between the following two computational problems (A) and (B).
(A) For any $G: 2^{\mathbb{N}} \rightarrow \mathbb{N}$, compute a finite sub-cover of $\cup_{f \in 2^{\mathbb{N}}}[\bar{f} G(f)]$, i.e. compute $f_{1}, \ldots, f_{k} \in 2^{\mathbb{N}}$ such that $\cup_{i \leq k}\left[\overline{f_{i}} G\left(f_{i}\right)\right]$ covers $2^{\mathbb{N}}$.
(B) For any $G: 2^{\mathbb{N}} \rightarrow \mathbb{N}$, compute a number $k \in \mathbb{N}$ such that there exists a finite sub-cover $f_{1}, \ldots, f_{k} \in 2^{\mathbb{N}}$ of $\cup_{f \in 2^{\mathbb{N}}}[\bar{f} G(f)]$.
The problem (A) gives rise to $\Theta$-functionals, introduced in Section 2.2, while (B) gives rise to realisers for 'uniform' Pincherle's theorem, introduced in Section 4.3.3 Note that in item (A), one needs to provide elements in Cantor space (which can code infinitely much information), while item (B) only requires a natural number (which can only code finite information). In Section 4.3.1, we show that partial realisers for NCC can perform (B); we conjecture that they cannot perform (A).

Since we do not have any idea how to establish the aforementioned conjecture, we shall solve a weaker ${ }^{2}$ problem, namely finding a partial realiser of NCC that does not compute Kleene's $\exists^{3}$. In this context, the property countably based, a kind of higher-order continuity property as in Definition 2.6. is helpful. Indeed, countably based functionals cannot compute $\exists^{3}$, i.e. a countably based realiser for NCC is just what we want. Much to our surprise, this kind of construct does exist, but is rather elusive as the following is proved in Theorem 4.19.

[^2]The Continuum Hypothesis CH is equivalent to the existence of a countably based partial realiser for NCC.
This result perhaps constitutes a kind of catharsis following the hubris of studying the computational properties of choice functions from AC. Entertaining as this equivalence may be, it would be preferable to have a ZFC-proof of the existence of a partial realiser for NCC that does not compute $\exists^{3}$.

## 2. Preliminaries

We introduce Reverse Mathematics in Section 2.1 as well as its generalisation to higher-order arithmetic, and the associated base theory RCA ${ }_{0}^{\omega}$. We introduce some essential axioms in Section 2.2.
2.1. Reverse Mathematics. Reverse Mathematics is a program in the foundations of mathematics initiated around 1975 by Friedman ( $7,8,8$ ) and developed extensively by Simpson ([47]). The aim of RM is to identify the minimal axioms needed to prove theorems of ordinary, i.e. non-set theoretical, mathematics.

We refer to [48] for a basic introduction to RM and to [46, 47] for an overview of RM. We expect basic familiarity with RM, but do sketch some aspects of Kohlenbach's higher-order RM ([19) essential to this paper, including the base theory $\mathrm{RCA}_{0}^{\omega}$ (Definition 2.1). As will become clear, the latter is officially a type theory but can accommodate (enough) set theory.

First of all, in contrast to 'classical' RM based on second-order arithmetic $\mathrm{Z}_{2}$, higher-order RM uses $\mathrm{L}_{\omega}$, the richer language of higher-order arithmetic. Indeed, while the former is restricted to natural numbers and sets of natural numbers, higher-order arithmetic can accommodate sets of sets of natural numbers, sets of sets of sets of natural numbers, et cetera. To formalise this idea, we introduce the collection of all finite types $\mathbf{T}$, defined by the two clauses:

$$
\text { (i) } 0 \in \mathbf{T} \text { and (ii) If } \sigma, \tau \in \mathbf{T} \text { then }(\sigma \rightarrow \tau) \in \mathbf{T}
$$

where 0 is the type of natural numbers, and $\sigma \rightarrow \tau$ is the type of mappings from objects of type $\sigma$ to objects of type $\tau$. In this way, $1 \equiv 0 \rightarrow 0$ is the type of functions from numbers to numbers, and $n+1 \equiv n \rightarrow 0$. Viewing sets as given by characteristic functions, we note that $Z_{2}$ only includes objects of type 0 and 1 .

Secondly, the language $\mathrm{L}_{\omega}$ includes variables $x^{\rho}, y^{\rho}, z^{\rho}, \ldots$ of any finite type $\rho \in \mathbf{T}$. Types may be omitted when they can be inferred from context. The constants of $\mathrm{L}_{\omega}$ include the type 0 objects 0,1 and $<_{0},+_{0}, \times_{0},={ }_{0}$ which are intended to have their usual meaning as operations on $\mathbb{N}$. Equality at higher types is defined in terms of ' $=0$ ' as follows: for any objects $x^{\tau}, y^{\tau}$, we have

$$
\begin{equation*}
\left[x={ }_{\tau} y\right] \equiv\left(\forall z_{1}^{\tau_{1}} \ldots z_{k}^{\tau_{k}}\right)\left[x z_{1} \ldots z_{k}={ }_{0} y z_{1} \ldots z_{k}\right] \tag{2.1}
\end{equation*}
$$

if the type $\tau$ is composed as $\tau \equiv\left(\tau_{1} \rightarrow \ldots \rightarrow \tau_{k} \rightarrow 0\right)$. Furthermore, $\mathrm{L}_{\omega}$ also includes the recursor constant $\mathbf{R}_{\sigma}$ for any $\sigma \in \mathbf{T}$, which allows for iteration on type $\sigma$-objects as in the special case (2.2). Formulas and terms are defined as usual. One obtains the sub-language $L_{n+2}$ by restricting the above type formation rule to produce only type $n+1$ objects (and related types of similar complexity).

Definition 2.1. The base theory $R C A_{0}^{\omega}$ consists of the following axioms.
(a) Basic axioms expressing that $0,1,<_{0},+_{0}, \times_{0}$ form an ordered semi-ring with equality $={ }_{0}$.
(b) Basic axioms defining the well-known $\Pi$ and $\Sigma$ combinators (aka $K$ and $S$ in [1]), which allow for the definition of $\lambda$-abstraction.
(c) The defining axiom of the recursor constant $\mathbf{R}_{0}$ : for $m^{0}$ and $f^{1}$ :

$$
\begin{equation*}
\mathbf{R}_{0}(f, m, 0):=m \text { and } \mathbf{R}_{0}(f, m, n+1):=f\left(n, \mathbf{R}_{0}(f, m, n)\right) \tag{2.2}
\end{equation*}
$$

(d) The axiom of extensionality: for all $\rho, \tau \in \mathbf{T}$, we have:

$$
\left(\forall x^{\rho}, y^{\rho}, \varphi^{\rho \rightarrow \tau}\right)\left[x={ }_{\rho} y \rightarrow \varphi(x)={ }_{\tau} \varphi(y)\right]
$$

(e) The induction axiom for quantifier-fred ${ }^{3}$ formulas of $\mathrm{L}_{\omega}$.
(f) QF-AC ${ }^{1,0}$ : the quantifier-free Axiom of Choice as in Definition 2.2

We let $\mathbf{I N D}^{\omega}$ be the induction axiom for all formulas in $\mathrm{L}_{\omega}$.
Definition 2.2. The axiom QF-AC consists of the following for all $\sigma, \tau \in \mathbf{T}$ :

$$
\left(\forall x^{\sigma}\right)\left(\exists y^{\tau}\right) A(x, y) \rightarrow\left(\exists Y^{\sigma \rightarrow \tau}\right)\left(\forall x^{\sigma}\right) A(x, Y(x)), \quad \quad\left(\mathrm{QF}^{-A C^{\sigma, \tau}}\right)
$$

for any quantifier-free formula $A$ in the language of $\mathrm{L}_{\omega}$.
As discussed in [19, §2], $\mathrm{RCA}_{0}^{\omega}$ and $\mathrm{RCA}_{0}$ prove the same sentences 'up to language' as the latter is set-based and the former function-based. Recursion as in (2.2) is called primitive recursion; the class of functionals obtained from $\mathbf{R}_{\rho}$ for all $\rho \in \mathbf{T}$ is called Gödel's system $T$ of all (higher-order) primitive recursive functionals.

We use the usual notations for natural, rational, and real numbers, and the associated functions, as introduced in [19, p. 288-289].
Definition 2.3 (Real numbers and related notions in $\mathrm{RCA}_{0}^{\omega}$ ).
(a) Natural numbers correspond to type zero objects, and we use ' $n$ ' and ' $n \in \mathbb{N}$ ' interchangeably. Rational numbers are defined as signed quotients of natural numbers, and ' $q \in \mathbb{Q}$ ' and ' $\angle_{\mathbb{Q}}$ ' have their usual meaning.
(b) Real numbers are coded by fast-converging Cauchy sequences $q_{(\cdot)}: \mathbb{N} \rightarrow$ $\mathbb{Q}$, i.e. such that $\left(\forall n^{0}, i^{0}\right)\left(\left|q_{n}-q_{n+i}\right|<_{\mathbb{Q}} \frac{1}{2^{n}}\right)$. We use Kohlenbach's 'hat function' from [19, p. 289] to guarantee that every $q^{1}$ defines a real number.
(c) We write ' $x \in \mathbb{R}$ ' to express that $x^{1}:=\left(q_{(\cdot)}^{1}\right)$ represents a real as in the previous item and write $[x](k):=q_{k}$ for the $k$-th approximation of $x$.
(d) Two reals $x, y$ represented by $q_{(\cdot)}$ and $r_{(\cdot)}$ are equal, denoted $x=_{\mathbb{R}} y$, if $\left(\forall n^{0}\right)\left(\left|q_{n}-r_{n}\right| \leq 2^{-n+1}\right)$. Inequality ' $<_{\mathbb{R}}$ ' is defined similarly. We sometimes omit the subscript ' $\mathbb{R}$ ' if it is clear from context.
(e) Functions $F: \mathbb{R} \rightarrow \mathbb{R}$ are represented by $\Phi^{1 \rightarrow 1}$ mapping equal reals to equal reals, i.e. extensionality as in $(\forall x, y \in \mathbb{R})\left(x=_{\mathbb{R}} y \rightarrow \Phi(x)=_{\mathbb{R}} \Phi(y)\right)$.
(f) The relation ' $x \leq_{\tau} y$ ' is defined as in (2.1) but with ' $\leq_{0}$ ' instead of ' $={ }_{0}$ '. Binary sequences are denoted ' $f^{1}, g^{1} \leq \leq_{1} 1^{\prime}$, but also ' $f, g \in C$ ' or ' $f, g \in 2^{\mathbb{N}}$ '. Elements of Baire space are given by $f^{1}, g^{1}$, but also denoted ' $f, g \in \mathbb{N}^{\mathbb{N}}$. .
(g) For a binary sequence $f^{1}$, the associated real in $[0,1]$ is $\llbracket(f):=\sum_{n=0}^{\infty} \frac{f(n)}{2^{n+1}}$.
(h) An object $\mathbf{Y}^{0 \rightarrow \rho}$ is called a sequence of type $\rho$ objects and also denoted $\mathbf{Y}=\left(Y_{n}\right)_{n \in \mathbb{N}}$ or $\mathbf{Y}=\lambda n . Y_{n}$ where $Y_{n}:=\mathbf{Y}(n)$ for all $n^{0}$.

Below, we shall discuss various different notions of open set, namely as in Definitions 3.4 and 3.12. Hence, we do not provide a general definition of set here. Next, we mention the highly useful ECF-interpretation.

[^3]Remark 2.4 (The ECF-interpretation). The (rather) technical definition of ECF may be found in [51, p. 138, §2.6]. Intuitively, the ECF-interpretation $[A]_{\mathrm{ECF}}$ of a formula $A \in \mathrm{~L}_{\omega}$ is just $A$ with all variables of type two and higher replaced by type one variables ranging over so-called 'associates' or 'RM-codes' (see [18, §4]); the latter are (countable) representations of continuous functionals. The ECFinterpretation connects $\mathrm{RCA}_{0}^{\omega}$ and $\mathrm{RCA}_{0}$ (see [19, Prop. 3.1]) in that if $\mathrm{RCA}_{0}^{\omega}$ proves $A$, then $\mathrm{RCA}_{0}$ proves $[A]_{\mathrm{ECF}}$, again 'up to language', as $\mathrm{RCA}_{0}$ is formulated using sets, and $[A]_{\text {ECF }}$ is formulated using types, i.e. using type zero and one objects.

In light of the widespread use of codes in RM and the common practise of identifying codes with the objects being coded, it is no exaggeration to refer to ECF as the canonical embedding of higher-order into second-order arithmetic. For completeness, we list the following notational convention for finite sequences.

Notation 2.5 (Finite sequences). We assume a dedicated type for 'finite sequences of objects of type $\rho^{\prime}$, namely $\rho^{*}$. Since the usual coding of pairs of numbers goes through in $\mathrm{RCA}_{0}^{\omega}$, we shall not always distinguish between 0 and $0^{*}$. Similarly, we do not always distinguish between ' $s^{\rho}$ ' and ' $\left\langle s^{\rho}\right\rangle^{\prime}$, where the former is 'the object $s$ of type $\rho$ ', and the latter is 'the sequence of type $\rho^{*}$ with only element $s^{\rho}$ '. The empty sequence for the type $\rho^{*}$ is denoted by ' $\left\rangle_{\rho}\right.$ ', usually with the typing omitted.

Furthermore, we denote by ' $|s|=n$ ' the length of the finite sequence $s^{\rho^{*}}=$ $\left\langle s_{0}^{\rho}, s_{1}^{\rho}, \ldots, s_{n-1}^{\rho}\right\rangle$, where $|\rangle|=0$, i.e. the empty sequence has length zero. For sequences $s^{\rho^{*}}, t^{\rho^{*}}$, we denote by ' $s * t$ ' the concatenation of $s$ and $t$, i.e. $(s * t)(i)=s(i)$ for $i<|s|$ and $(s * t)(j)=t(|s|-j)$ for $|s| \leq j<|s|+|t|$. For a sequence $s^{\rho^{*}}$, we define $\bar{s} N:=\langle s(0), s(1), \ldots, s(N-1)\rangle$ for $N^{0}<|s|$. For a sequence $\alpha^{0 \rightarrow \rho}$, we also write $\bar{\alpha} N=\langle\alpha(0), \alpha(1), \ldots, \alpha(N-1)\rangle$ for any $N^{0}$. By way of shorthand, $\left(\forall q^{\rho} \in Q^{\rho^{*}}\right) A(q)$ abbreviates $\left(\forall i^{0}<|Q|\right) A(Q(i))$, which is (equivalent to) quantifier-free if $A$ is.
2.2. Higher-order computability theory. As noted above, some of our main results are part of computability theory. Thus, we first make our notion of 'computability' precise as follows.

- We adopt ZFC, i.e. Zermelo-Fraenkel set theory with the Axiom of Choice, as the official metatheory for all results, unless explicitly stated otherwise.
- We adopt Kleene's notion of higher-order computation as given by his nine clauses S1-S9 (see [21, Ch. 5] or [17]) as our official notion of 'computable'.
We discuss our choice of framework, and a possible alternative, in Section 4.3.4
Secondly, similar to [25-29], one main aim of this paper is the study of functionals of type 3 that are natural from the perspective of mathematical practise. Our functionals are genuinely of type 3 in the sense that they are not computable from any functional of type 2 . The following definition is standard in this context.
Definition 2.6. A functional $\Phi^{3}$ is countably based if for every $F^{2}$ there is countable $X \subset \mathbb{N}^{\mathbb{N}}$ such that $\Phi(F)=\Phi(G)$ for every $G$ that agrees with $F$ on $X$.

Stanley Wainer (unpublished) has defined the countably based functionals of finite type as an analogue of the continuous functionals, while John Hartley has investigated the computability theory of this type structure in [11.

We only use countably based functionals of type at most 3 in this paper. Now, if $\Phi^{3}$ is computable in a functional of type 2 , then it is countably based, but the converse does not hold. However, Hartley proves in 11 that, assuming ZFC +CH
however, if $\Phi^{3}$ is not countably based, then there is some $F^{2}$ such that $\exists^{3}$ (see below) is computable in $\Phi$ and $F$. In other words, stating the existence of a noncountably based $\Phi$ brings us 'close to' $Z_{2}^{\Omega}$ (defined below). In the sequel, we shall explicitly point out where we use countably based functionals.

The importance of Definition [2.6 can be understood as follows: to answer whether a given functional $\Phi^{3}$ can compute another functional $\Psi^{3}$, the answer is automatically 'no' if $\Phi$ is countably based and $\Psi$ is not. A similar 'rule-of-thumb' is that if $\Phi$ does not compute $\exists^{2}$ (or a discontinuous functional on $\mathbb{R}$ or $\mathbb{N}^{\mathbb{N}}$; see below), while $\Psi$ does, the answer is similarly 'no'. We have used both rules-of-thumb throughout our project to provide a first 'rough' classification of new functionals.

For the rest of this section, we introduce some existing functionals which will be used below. In particular, we introduce some functionals which constitute the counterparts of second-order arithmetic $Z_{2}$, and some of the Big Five systems, in higher-order RM. We use the formulation from [19, 27].
First of all, $\mathrm{ACA}_{0}$ is readily derived from:

$$
\begin{aligned}
\left(\exists \mu^{2}\right)\left(\forall f^{1}\right)[(\exists n)(f(n)=0) \rightarrow[(f(\mu(f))=0) & \wedge(\forall i<\mu(f)) f(i) \neq 0] \\
& \wedge[(\forall n)(f(n) \neq 0) \rightarrow \mu(f)=0]],
\end{aligned}
$$

and $\mathrm{ACA}_{0}^{\omega} \equiv \mathrm{RCA}_{0}^{\omega}+\left(\mu^{2}\right)$ proves the same sentences as $\mathrm{ACA}_{0}$ by [15, Theorem 2.5]. The (unique) functional $\mu^{2}$ in $\left(\mu^{2}\right)$ is also called Feferman's $\mu$ ( 1 ), and is clearly discontinuous at $f={ }_{1} 11 \ldots$; in fact, $\left(\mu^{2}\right)$ is equivalent to the existence of $F: \mathbb{R} \rightarrow \mathbb{R}$ such that $F(x)=1$ if $x>_{\mathbb{R}} 0$, and 0 otherwise ( $[19, \S 3]$ ), and to

$$
\begin{equation*}
\left(\exists \varphi^{2} \leq_{2} 1\right)\left(\forall f^{1}\right)[(\exists n)(f(n)=0) \leftrightarrow \varphi(f)=0] . \tag{2}
\end{equation*}
$$

Secondly, $\Pi_{1}^{1}-\mathrm{CA}_{0}$ is readily derived from the following sentence:

$$
\begin{equation*}
\left(\exists \mathrm{S}^{2} \leq_{2} 1\right)\left(\forall f^{1}\right)\left[\left(\exists g^{1}\right)\left(\forall n^{0}\right)(f(\bar{g} n)=0) \leftrightarrow \mathrm{S}(f)=0\right] \tag{2}
\end{equation*}
$$

and $\Pi_{1}^{1}-\mathrm{CA}_{0}^{\omega} \equiv \mathrm{RCA}_{0}^{\omega}+\left(\mathrm{S}^{2}\right)$ proves the same $\Pi_{3}^{1}$-sentences as $\Pi_{1}^{1}-\mathrm{CA}_{0}$ by [37, Theorem 2.2]. The (unique) functional $\mathrm{S}^{2}$ in $\left(\mathrm{S}^{2}\right)$ is also called the Suslin functional ([19]). By definition, the Suslin functional $S^{2}$ can decide whether a $\Sigma_{1}^{1}$-formula as in the left-hand side of $\left(\mathrm{S}^{2}\right)$ is true or false. We similarly define the functional $S_{k}^{2}$ which decides the truth or falsity of $\Sigma_{k}^{1}$-formulas; we also define the system $\Pi_{k}^{1}$-CA $A_{0}^{\omega}$ as $\operatorname{RCA}_{0}^{\omega}+\left(\mathrm{S}_{k}^{2}\right)$, where $\left(\mathrm{S}_{k}^{2}\right)$ expresses that $\mathrm{S}_{k}^{2}$ exists. Note that we allow formulas with function parameters, but not functionals here. In fact, Gandy's Superjump ([9]) constitutes a way of extending $\Pi_{1}^{1}-\mathrm{CA}_{0}^{\omega}$ to parameters of type two. We identify the functionals $\exists^{2}$ and $\mathrm{S}_{0}^{2}$ and the systems $\mathrm{ACA}_{0}^{\omega}$ and $\Pi_{k}^{1}-\mathrm{CA}_{0}^{\omega}$ for $k=0$. We note that the operators $\nu_{n}$ from [4, p. 129] are essentially $S_{n}^{2}$ strengthened to return a witness (if existant) to the $\Sigma_{k}^{1}$-formula at hand.
Thirdly, full second-order arithmetic $Z_{2}$ is readily derived from $\cup_{k} \Pi_{k}^{1}-\mathrm{CA}_{0}^{\omega}$, or from:

$$
\begin{equation*}
\left(\exists E^{3} \leq_{3} 1\right)\left(\forall Y^{2}\right)\left[\left(\exists f^{1}\right)(Y(f)=0) \leftrightarrow E(Y)=0\right] \tag{3}
\end{equation*}
$$

and we therefore define $Z_{2}^{\Omega} \equiv \operatorname{RCA}_{0}^{\omega}+\left(\exists^{3}\right)$ and $Z_{2}^{\omega} \equiv \cup_{k} \Pi_{k}^{1}-C A_{0}^{\omega}$, which are conservative over $Z_{2}$ by [15, Cor. 2.6]. Despite this close connection, $Z_{2}^{\omega}$ and $Z_{2}^{\Omega}$ can behave quite differently, as discussed in e.g. [27, §2.2]. The functional from $\left(\exists^{3}\right)$ is also called ' $\exists$ '3', and we use the same convention for other functionals. Note that $\left(\exists^{3}\right) \leftrightarrow\left[\left(\exists^{2}\right)+\left(\kappa_{0}^{3}\right)\right]$ as shown in [25,43], where the latter is comprehension on $2^{\mathbb{N}}$ :

$$
\begin{equation*}
\left(\exists \kappa_{0}^{3} \leq_{3} 1\right)\left(\forall Y^{2}\right)\left[\kappa_{0}(Y)=0 \leftrightarrow(\exists f \in C)(Y(f)=0)\right] \tag{0}
\end{equation*}
$$

Other 'splittings' are studied in 43], including $\left(\kappa_{0}^{3}\right)$.
Fourth, the Heine-Borel theorem states the existence of a finite sub-covering for an open covering of certain spaces. Now, a functional $\Psi: \mathbb{R} \rightarrow \mathbb{R}^{+}$gives rise to the canonical covering $\cup_{x \in I} I_{x}^{\Psi}$ for $I \equiv[0,1]$, where $I_{x}^{\Psi}$ is the open interval $(x-\Psi(x), x+\Psi(x))$. Hence, the uncountable covering $\cup_{x \in I} I_{x}^{\Psi}$ has a finite subcovering by the Heine-Borel theorem; in symbols:

$$
\begin{equation*}
\left(\forall \Psi: \mathbb{R} \rightarrow \mathbb{R}^{+}\right)\left(\exists y_{1}, \ldots, y_{k} \in I\right)(\forall x \in I)\left(x \in \cup_{i \leq k} I_{y_{i}}^{\Psi}\right) \tag{HBU}
\end{equation*}
$$

Note that HBU is almost verbatim Cousin's lemma ([5, p. 22]), i.e. the Heine-Borel theorem restricted to canonical coverings. This restriction does not make a big difference, as shown in [44. By [27,29], $Z_{2}^{\Omega}$ proves HBU but $Z_{2}^{\omega}+\mathrm{QF}-\mathrm{AC}^{0,1}$ cannot, and basic properties of the gauge integral ([22,49]) are equivalent to HBU.

Fifth, since Cantor space (denoted $C$ or $2^{\mathbb{N}}$ ) is homeomorphic to a closed subset of $[0,1]$, the former inherits the same property. In particular, for any $G^{2}$, the corresponding 'canonical covering' of $2^{\mathbb{N}}$ is $\cup_{f \in 2^{\mathbb{N}}}[\bar{f} G(f)]$ where $\left[\sigma^{0^{*}}\right]$ is the set of all binary extensions of $\sigma$. By compactness, there are $f_{0}, \ldots, f_{n} \in 2^{\mathbb{N}}$ such that the set of $\cup_{i \leq n}\left[\bar{f}_{i} G\left(f_{i}\right)\right]$ still covers $2^{\mathbb{N}}$. By [27, Theorem 3.3], HBU is equivalent to the same compactness property for $C$, as follows:

$$
\left(\forall G^{2}\right)\left(\exists f_{1}, \ldots, f_{k} \in C\right)(\forall f \in C)\left(f \in \cup_{i \leq k}\left[\overline{f_{i}} G\left(f_{i}\right)\right]\right) . \quad\left(\mathrm{HBU}_{\mathrm{c}}\right)
$$

We now introduce the specification $\operatorname{SFF}(\Theta)$ for a (non-unique) functional $\Theta$ which computes a finite sequence as in $\mathrm{HBU}_{\mathrm{c}}$. We refer to such a functional $\Theta$ as a realiser for the compactness of Cantor space, and simplify its type to ' 3 '.

$$
\begin{equation*}
\left(\forall G^{2}\right)\left(\forall f^{1} \leq_{1} 1\right)(\exists g \in \Theta(G))(f \in[\bar{g} G(g)]) \tag{SFF}
\end{equation*}
$$

Clearly, there is no unique such $\Theta$ (just add more binary sequences to $\Theta(G)$ ) and any functional satisfying the previous specification is referred to as a ' $\Theta$-functional' or a 'special fan functional' or a 'realiser for HBU'. As to their provenance, $\Theta$ functionals were introduced as part of the study of the Gandy-Hyland functional in [38, §2] via a slightly different definition. These definitions are identical up to a term of Gödel's $T$ of low complexity by [26, Theorem 2.6].

Sixth, a number of higher-order axioms are studied in 40 including:

$$
\begin{equation*}
\left(\forall Y^{2}\right)(\exists X \subset \mathbb{N})(\forall n \in \mathbb{N})\left(n \in X \leftrightarrow\left(\exists f \in \mathbb{N}^{\mathbb{N}}\right)(Y(f, n)=0)\right) \tag{BOOT}
\end{equation*}
$$

We only mention that this axiom is equivalent to e.g. the monotone convergence theorem for nets indexed by Baire space (see [40, §3]). As it turns out, the coding principle open ${ }^{+}$from Section 3.4 is closely related to BOOT and fragments, as shown in 40. Historical remarks related to BOOT are as follows.

Remark 2.7 (Historical notes). First of all, BOOT is definable in Hilbert-Bernays' system $H$ from the Grundlagen der Mathematik ([13, Supplement IV]). In particular, one uses the functional $\nu$ from [13, p. 479] to define the set $X$ from BOOT. In this way, BOOT and subsystems of second-order arithmetic can be said to 'go back' to the Grundlagen in equal measure, although such claims may be controversial.

Secondly, after the completion of [40, it was observed by the second author that Feferman's 'projection' axiom (Proj1) from [6] is similar to BOOT. The former is however formulated using sets, which makes it more 'explosive' than BOOT in that full $Z_{2}$ follows when combined with $\left(\mu^{2}\right)$, as noted in [6, I-12]. Note that [6] is Paper 154 in Feferman's publication list from [16], going back to about 1980.

## 3. Reverse Mathematics and the Axiom of Choice

3.1. Introduction and basic results. A number of results in 29, 30, 40, exhibit the Pincherle phenomenon from Section 1.2.2. In particular, certain equivalences are established using QF-AC ${ }^{0,1}$, while they (often) cannot be established without QF-AC ${ }^{0,1}$. At the same time, a much stronger system not involving QF-AC ${ }^{0,1}$ proves both members of these equivalences. In this section, we show that countable choice can be avoided in favour of NCC from Section 1.2.2. Unsurprisingly, the proofs become more complex and require greater attention to detail. Here is a list of theorems from [29, 30,40 , to be treated in the aforementioned way.

- Pincherle's original theorem for Cantor Space (Section 3.2).
- The Heine-Borel theorem for countable coverings (Section 3.3.2).
- The Urysohn lemma and Tietze extension theorem (Section 3.3.3).
- The bootstrap axiom BOOT and the coding of open sets (Section 3.4).

We only establish the sufficiency of NCC for these results, while similar results can be treated in the same way.

The above results are established based on [19, §3] as follows. As noted in Section [2.1] $\left(\exists^{2}\right)$ is equivalent to the existence of a discontinuous function on $\mathbb{R}$. Hence, $\neg\left(\exists^{2}\right)$ is equivalent to the statement all functions on $\mathbb{R}$ are continuous. In the latter case, higher-order statements, like e.g. $\mathrm{PIT}_{o}$, often reduce to well-known second-order results. Since all systems here are classical, we can therefore invoke the law of excluded middle $\left(\exists^{2}\right) \vee \neg\left(\exists^{2}\right)$ and split a given proof in e.g. RCA ${ }_{0}^{\omega}$ or RCA $_{0}^{\omega}+$ WKL into two parts: one assuming $\neg\left(\exists^{2}\right)$ which often reduces to secondorder results, and a second part assuming $\left(\exists^{2}\right)$, where the latter is much stronger than the base theory and WKL. This 'excluded middle trick' was pioneered in [29].

For the rest of this section, we discuss some basic results and observations regarding NCC. First of all, consider the following axiom, called $\Delta$-comprehension, essential for many 'lifted' proofs from 40 42.

$$
\begin{align*}
&\left(\forall Y^{2}, Z^{2}\right)\left[\left(\forall n^{0}\right)\left(\left(\exists f^{1}\right)(Y(f, n)=0) \leftrightarrow\left(\forall g^{1}\right)(Z(g, n)=0)\right)\right. \\
& \rightarrow\left(\exists X^{1}\right)\left(\forall n^{0}\right)\left(n \in X \leftrightarrow\left(\exists f^{1}\right)(Y(f, n)=0)\right]
\end{align*}
$$

Now, $\Delta$-CA is mapped to recursive comprehension from $\mathrm{RCA}_{0}$ by ECF, i.e. the former axiom is needed to do higher-order RM in a fashion similar to second-order RM. We have the following theorem, establishing the basic properties of NCC.

## Theorem 3.1.

- The system $\mathrm{RCA}_{0}^{\omega}+$ BOOT proves NCC.
- The system $\mathrm{RCA}_{0}^{\omega}$ proves $\mathrm{QF}-\mathrm{AC}^{0,1} \rightarrow \mathrm{NCC} \rightarrow \Delta$-CA.

Proof. The first item is trivial as $\mathrm{RCA}_{0}^{\omega}$ includes QF-AC ${ }^{0,0}$. The first implication in the second item is immediate. For the second implication in the second item, consider $Y^{2}, Z^{2}$ that satisfy the antecedent of $\Delta$-CA, i.e.

$$
\left(\forall n^{0}\right)\left(\left(\exists f^{1}\right)(Y(f, n)=0) \leftrightarrow\left(\forall g^{1}\right)(Z(g, n)>0)\right) .
$$

Now apply NCC to the following (trivial) formula

$$
\left(\forall n^{0}\right)\left(\exists m^{0}\right)\left[m=0 \rightarrow\left(\exists f^{1}\right)(Y(f, n)=0) \wedge\left(\forall g^{1}\right)(Z(g, n)>0) \rightarrow m=0\right]
$$

to obtain the set required for $\Delta-C A$.

We also note that the axiom $\mathrm{A}_{0}$ from [40, §5] trivially implies NCC. The former axiom is used in 40] to calibrate theorems based on fragments of the neighbourhood function principle NFP ([52]), a scale finer than (higher-order) comprehension.

Finally, NCC is also interesting for conceptual reasons: as shown in [31, §3.2], NCC implies the principle NBI, that there is no bijection from $[0,1]$ to $\mathbb{N}$, but NCC cannot prove NIN, that there is no injection from $[0,1]$ to $\mathbb{N}$, even when combined with $Z_{2}^{\omega}$. Thus, NCC is intimately connected to the uncountability of $\mathbb{R}$.
3.2. Pincherle's theorem. In this section, we show that NCC suffices to obtain the equivalence $\mathrm{WKL} \leftrightarrow \mathrm{PIT}_{o}$, where the latter is Pincherle's 'original' theorem, which is mentioned in Section 1.2 .2 and defined as in $\mathrm{PIT}_{o}$ below:

$$
\begin{array}{cr}
(\forall f, g \in C)[g \in[\bar{f} G(f)] \rightarrow F(g) \leq G(f)], & (\mathrm{LOC}(F, G)) \\
(\forall F, G: C \rightarrow \mathbb{N})[\operatorname{LOC}(F, G) \rightarrow(\exists N \in \mathbb{N})(\forall g \in C)(F(g) \leq N)] .
\end{array}
$$

Note that $\operatorname{LOC}(F, G)$ expresses that $F$ is locally bounded on $2^{\mathbb{N}}$ and $G$ realises this fact. As discussed in [29], Pincherle explicitly assumes such realisers in [32]. Corollary 3.3 deals with PIT $_{o}$ without such realisers.

Theorem 3.2. The system $\mathrm{RCA}_{0}^{\omega}+$ NCC proves $\mathrm{WKL} \leftrightarrow \mathrm{PIT}_{o}$.
Proof. The reverse implication is proved in [29, Cor. 4.8] over $\mathrm{RCA}_{0}^{\omega}$. For the forward direction, let $F: 2^{\mathbb{N}} \rightarrow \mathbb{N}$ be a totally bounded function with realiser $G: 2^{\mathbb{N}} \rightarrow \mathbb{N}$, i.e. we have $\left(\forall f, g \in 2^{\mathbb{N}}\right)(g \in[\bar{f} G(f)] \rightarrow F(g) \leq G(f))$. In case $\neg\left(\exists^{2}\right), F$ is continuous by [19, §3] and it is well-known that WKL suffices to prove that $F$ has an upper bound in this case (see [18, §4]). In case $\left(\exists^{2}\right)$, suppose $F$ is unbounded on $2^{\mathbb{N}}$, i.e. $\left(\forall n^{0}\right)\left(\exists f \in 2^{\mathbb{N}}\right)(F(f) \geq n)$. The following is immediate:

$$
\begin{equation*}
\left(\forall n^{0}\right)\left(\exists \sigma^{0^{*}} \leq_{0^{*}} 1\right)\left[|\sigma|=n \wedge\left(\exists g \in 2^{\mathbb{N}}\right)(F(\sigma * g) \geq n)\right] \tag{3.1}
\end{equation*}
$$

The formula in big square brackets has the right form (modulo coding) to apply NCC. Let $H^{0 \rightarrow 0^{*}}$ be the sequence thus obtained and define $f_{n}:=H(n) * 00 \ldots$ Since $\left(\exists^{2}\right)$ is given, the sequence $f_{n}$ has a convergent subsequence $f_{h(n)}$ with limit $g_{0}$ (see [47, III.2]), i.e. we have

$$
\begin{equation*}
\left(\forall k^{0}\right)\left(\exists n^{0}\right)\left(\forall m^{0} \geq n\right)\left(\overline{g_{0}} k=_{0^{*}} \overline{f_{h(m)}} k\right) \tag{3.2}
\end{equation*}
$$

Now, apply (3.2) for $k_{0}=G\left(g_{0}\right)+1$ and obtain the associated $n_{0}$. For $m_{0}=$ $\max \left(n_{0}, G\left(g_{0}\right)+1\right)$, we then have that $\overline{f_{h\left(m_{0}\right)}} h\left(m_{0}\right) * g \in\left[\overline{g_{0}} G\left(g_{0}\right)\right]$ for any $g \in 2^{\mathbb{N}}$ as $h\left(m_{0}\right) \geq m_{0} \geq G\left(g_{0}\right)+1$, and hence $F\left(\overline{f_{h\left(m_{0}\right)}} h\left(m_{0}\right) * g\right) \leq G\left(g_{0}\right)$ for any $g \in 2^{\mathbb{N}}$ by local boundedness. However, the definition of $f_{h\left(m_{0}\right)}$ implies that there is $g_{1} \in 2^{\mathbb{N}}$ such that $F\left(\overline{f_{h\left(m_{0}\right)}} h\left(m_{0}\right) * g_{1}\right)=F\left(H\left(h\left(m_{0}\right)\right) * g_{1}\right) \geq h\left(m_{0}\right)$. The assumption $h\left(m_{0}\right) \geq m_{0} \geq G\left(g_{0}\right)+1$ thus yields a contradiction.

Finally, let $\mathrm{PIT}_{o}^{\prime}$ be $\mathrm{PIT}_{o}$ with the antecedent weakened as follows:

$$
\begin{equation*}
(\forall f \in C)\left(\exists n^{0}\right)(\forall g \in C)[g \in[\bar{f} n] \rightarrow F(g) \leq n] \tag{3.3}
\end{equation*}
$$

As expected, (3.3) gives rise to the following corollary.
Corollary 3.3. The system $\mathrm{RCA}_{0}^{\omega}+\mathrm{NCC}$ proves $\mathrm{WKL} \leftrightarrow \mathrm{PIT}_{o}^{\prime}$.
Proof. Replace $G\left(g_{0}\right)$ with the number $n_{1}^{0}$ obtained for $f=g_{0}$ in (3.3).
The previous results should be contrasted with the fact that $Z_{2}^{\omega}$ cannot prove $\mathrm{PIT}_{o}$, while $\mathrm{PIT}_{o}$ is provable in $\mathrm{Z}_{2}^{\Omega}$ (and hence ZF ).
3.3. Closed and open sets. We study theorems named after Tietze and Urysohn (Section 3.3.3) and Heine and Borel (Section 3.3.2), formulated using higher-order open and closed sets. The latter notion is introduced in Section 3.3.1, along with more details. In each case, we show that NCC can replace the use of QF-AC ${ }^{0,1}$.
3.3.1. Introduction. In this section, we study theorems from 30 that exhibit the Pincherle phenomenon. In particular, we show that QF-AC ${ }^{0,1}$ is not necessary, but that NCC suffices in these results. These theorems pertain to open and closed sets given by characteristic functions, defined as follows.

Definition 3.4. [Open sets in $\mathrm{RCA}_{0}^{\omega}$ from [30]] We let $Y: \mathbb{R} \rightarrow \mathbb{R}$ represent open subsets of $\mathbb{R}$ as follows: we write ' $x \in Y^{\prime}$ for ' $|Y(x)|>_{\mathbb{R}} 0^{\prime}$ ' and call a set $Y \subseteq \mathbb{R}$ 'open' if for every $x \in Y$, there is an open ball $B(x, r) \subset Y$ with $r^{0}>0$. A set $Y$ is called 'closed' if the complement, denoted $Y^{c}=\{x \in \mathbb{R}: x \notin Y\}$, is open.

We have argued in [30] that this definition remains close to the ' $\Sigma_{1}^{0}$-definition' of open set used in RM. In the case of sequential compactness, Definition 3.4 yields the known results involving $\mathrm{ACA}_{0}$, while countable open-cover compactness already gives rise to the Pincherle phenomenon, as sketched in Section 3.3.2.

For the rest of this section, 'open' and 'closed' refer to Definition 3.4, while 'RM-open' and 'RM-closed' refer to the usual RM-definition from [47, II.4].
3.3.2. Heine-Borel theorem. We now study the Heine-Borel theorem for countable covers of closed sets as in Definition [3.4. Note that the associated theorem for RM-codes is equivalent to WKL by 3, Lemma 3.13].
Definition 3.5. $[\mathrm{HBC}]$ Let $C \subseteq[0,1]$ be a closed set and let $a_{n}, b_{n}$ be sequences of reals such that $C \subseteq \cup_{n \in \mathbb{N}}\left(a_{n}, b_{n}\right)$. Then there is $n_{0}$ such that $C \subseteq \cup_{n \leq n_{0}}\left(a_{n}, b_{n}\right)$.
It is shown in 30 that HBC has the following properties.

- The system $\mathrm{RCA}_{0}^{\omega}+$ QF-AC ${ }^{0,1}$ proves $W K L \leftrightarrow H B C$.
- The system $Z_{2}^{\omega}$ cannot prove HBC, while $Z_{2}^{\Omega}$ (and $\left.R C A_{0}^{\omega}+H B U\right)$ can.

By the these items, HBC clearly exhibits the Pincherle phenomenon. Note that by the second item, HBC is provable without countable choice and has weak first-order strength. We let $\mathrm{HBC}_{\mathrm{rm}}$ be HBC with $C \subseteq[0,1]$ represented by RM-codes.

We now prove the following theorem.
Theorem 3.6. The system $\mathrm{RCA}_{0}^{\omega}+$ NCC proves $W K L \leftrightarrow H B C$.
Proof. The reversal can be found in [30, Cor. 3.4] over $\mathrm{RCA}_{0}^{\omega}$. It also follows from taking $C=[0,1]$ in HBC and applying [47, IV.1.2]. For the forward direction, in case $\neg\left(\exists^{2}\right)$, all functions on $\mathbb{R}$ are continuous by [19, §3]. Following the results in [18, §4], continuous functions have an RM-code on $[0,1]$ given WKL, i.e. our definition of open set reduces to an $L_{2}$-formula in $\Sigma_{1}^{0}$, which (equivalently) defines a code for an open set by [47, II.5.7]. In this way, HBC is merely $\mathrm{HBC}_{r m}$, which follows from WKL by [3, Lemma 3.13]. In case $\left(\exists^{2}\right)$, let $C \subseteq[0,1]$ be a closed set and let $a_{n}, b_{n}$ be as in HBC . If there is no finite sub-cover, then we also have that

$$
\begin{equation*}
\left(\forall m^{0}\right)(\exists q \in \mathbb{Q})(\exists x \in C)\left[[x](m)=q \wedge x \notin \cup_{n \leq m}\left(a_{n}, b_{n}\right)\right] . \tag{3.4}
\end{equation*}
$$

Apply NCC and $\left(\exists^{2}\right)$ to (3.4), yielding a sequence $\left(q_{n}\right)_{n \in \mathbb{N}}$ of rationals in $C$ with this property. Since $\left(\exists^{2}\right) \rightarrow \mathrm{ACA}_{0}$, any sequence in $[0,1]$ has a convergent sub-sequence [47, III.2]. Let $h: \mathbb{N} \rightarrow \mathbb{N}$ be such that $y_{n}:=q_{h(n)}$ converges to $y \in[0,1]$.

If $y \notin C$, then there is $N^{0}$ such that $B\left(y, \frac{1}{2^{N}}\right) \subset C^{c}$, as the complement of $C$ is open by definition. However, $y_{n}$ is eventually in $B\left(y, \frac{1}{2^{N}}\right)$ by definition, a contradiction. Note that $y_{n}$ may not be in $C$, but elements of $C$ are arbitrarily close to $y_{n}$ for large enough $n$ by (3.4).

Hence, we may assume $\lim _{n \rightarrow \infty} y_{n}=y \in C$. However, then $y \in\left(a_{k}, b_{k}\right)$ for some $k$, and $y_{n}$ is eventually in this interval. In the same way as in the previous case, this yields a contradiction. The law of excluded middle now finishes the proof.

As shown in [30, §3], the following theorems imply HBC over $\mathrm{RCA}_{0}^{\omega}$ :
(a) Pincherle's theorem for $[0,1]$ : a locally bounded function on $[0,1]$ is bounded.
(b) If $F^{2}$ is continuous on a closed set $D \subset 2^{\mathbb{N}}$, it is bounded on $D$.
(c) If $F^{2}$ is continuous on a closed set $D \subset 2^{\mathbb{N}}$, it is uniformly cont. on $D$.
(d) If $F$ is continuous on a closed set $D \subset[0,1]$, it is bounded on $D$.
(e) If $F$ is continuous on a closed set $D \subset[0,1]$, it is uniformly cont. on $D$.
(f) If $F$ is continuous on a closed set $D \subset[0,1]$, then for every $\varepsilon>0$ there is a polynomial $p(x)$ such that $|p(x)-F(x)|<\varepsilon$ for all $x \in D$.
In the same way as above, one obtains an equivalence between these theorems and $W_{K L}$, using NCC instead of QF-AC ${ }^{0,1}$.

We finish this section with a remark on the Baire category theorem.
Remark 3.7. The Baire category theorem for open sets as in Definition 3.4 is studied in [30, §6]. Similar to e.g. HBC, the Baire category theorem exhibits (part of) the Pincherle phenomenon. The associated proofs for the latter theorem are however very different from all other proofs. Similarly, NCC does not seem to suffice to prove the Baire category theorem and the following one does.

Definition 3.8. [MCC] For $Y^{2}$ and $A(n, m) \equiv\left(\forall g \in 2^{\mathbb{N}}\right)(Y(g, m, n)=0)$ :

$$
\left(\forall n^{0}\right)\left(\exists m^{0}\right) A(n, m) \rightarrow\left(\exists h^{1}\right)\left(\forall n^{0}\right) A(n, h(n)) .
$$

We have not found any use for MCC besides, but it shall be seen to yield the same class as realisers as NCC in Section 4.
3.3.3. Urysohn's lemma and Tietze's theorem. We study the equivalence between the Urysohn lemma (URY) and the Tietze extension theorem (TIE), formulated using open sets as in Definition 3.4. In particular, this equivalence is proved in [30, §5] using QF-AC ${ }^{0,1}$ and we now show that NCC suffices.

We first consider the following necessary definitions.
Definition 3.9. [URY] For closed disjoint sets $C_{0}, C_{1} \subseteq \mathbb{R}$, there is a continuous function $g: \mathbb{R} \rightarrow[0,1]$ such that $x \in C_{i} \leftrightarrow g(x)=i$ for any $x \in \mathbb{R}$ and $i \in\{0,1\}$.

Definition 3.10. [TIE] For $f: \mathbb{R} \rightarrow \mathbb{R}$ continuous on the closed $D \subset[0,1]$, there is $g: \mathbb{R} \rightarrow \mathbb{R}$, continuous on $[0,1]$ such that $f(x)=_{\mathbb{R}} g(x)$ for $x \in D$.

Secondly, URY $\leftrightarrow$ TIE is proved in [30, §5] using QF-AC ${ }^{0,1}$ and coco, where the latter is the statement that every continuous $Y: \mathbb{R} \rightarrow \mathbb{R}$ has an RM-code, as studied in [18, §4] for Baire space. Note that the ECF-interpretation of coco is a tautology. We have the following nice equivalence.

Theorem 3.11. The system $\mathrm{RCA}_{0}^{\omega}+\mathrm{NCC}+$ coco proves TIE $\leftrightarrow$ URY.

Proof. The implication URY $\rightarrow$ TIE is proved in [30, §5] over RCA ${ }_{0}^{\omega}$.
For TIE $\rightarrow$ URY, in case $\neg\left(\exists^{2}\right)$, all functions on $\mathbb{R}$ are continuous by [19, §3] and open sets reduce to RM-codes via coco; the usual proof of URY from [47, II.7] then goes through. In case $\left(\exists^{2}\right)$, let $C_{i}$ be as in URY for $i=0,1$ and define $f$ on $C_{2}:=C_{0} \cup C_{1}$ as follows: $f(x)=0$ if $x \in C_{0}$ and 1 otherwise. If $f$ is continuous on $C_{2}$, then its extension $g$ provided by TIE is as required for URY. To show that $f$ is continuous on $C_{2}$, we prove that

$$
\begin{equation*}
\left(\forall N^{0}\right)\left(\exists n^{0}\right)\left(\forall x \in C_{0}, y \in C_{1}\right)\left(x, y \in[-N, N] \rightarrow|x-y| \geq \frac{1}{2^{n}}\right) \tag{3.5}
\end{equation*}
$$

If (3.5) is false, there is $N \in \mathbb{N}$ such that for $n \in \mathbb{N}$, there are $q, r \in \mathbb{Q}$ such that:
$\left(\exists x \in C_{0}, y \in C_{1}\right)\left([x](n+1)=q \wedge[y](n+1)=r \wedge x, y \in[-N, N] \wedge|x-y|<\frac{1}{2^{n+1}}\right)$.
Applying NCC yields sequences $\left(q_{n}\right)_{n \in \mathbb{N}},\left(r_{n}\right)_{n \in \mathbb{N}}$ in $[-N-1, N+1]$ such that for all $n^{0}$, there are $x \in C_{0}, y \in C_{1}$ such that

$$
\begin{equation*}
[x](n+1)=q_{n} \wedge[y](n+1)=r_{n} \wedge x, y \in[-N, N] \wedge|x-y|<\frac{1}{2^{n+1}} \tag{3.6}
\end{equation*}
$$

As these sequences are bounded, there are $x_{0}, y_{0} \in[-N, N]$ such that $q_{h_{0}(n)} \rightarrow x_{0}$ and $r_{h_{1}(n)} \rightarrow y_{0}$ for subsequences provided by $h_{0}, h_{1}: \mathbb{N} \rightarrow \mathbb{N}$. Since $C_{0}$ is closed, we have the following: if $x_{0} \notin C_{0}$, then there is $r>0$ such that $B\left(x_{0}, r\right) \cap C_{0}=\emptyset$. This however contradicts the convergence $q_{h_{0}(n)} \rightarrow x_{0}$ and (3.6). Hence $x_{0} \in C_{0}$ and $y_{0} \in C_{1}$ in the same way. Now note that $\left(\forall n^{0}\right)\left(\left|r_{n}-q_{n}\right|<\frac{1}{2^{n}}\right)$ by (3.6), which implies that $x_{0}=_{\mathbb{R}} y_{0}$, a contradiction since $C_{0} \cap C_{1}=\emptyset$. Finally, since (3.5) provides a positive 'distance' between $C_{0}$ and $C_{1}$ in every interval [ $-N, N$ ], we can always chose a small enough neighbourhood to exclude points from one of the parts of $C_{2}$, thus guaranteeing continuity for $f$ everywhere on $C_{2}$.

Finally, we point out that while Definition 3.4 gives rise to interesting results in [30], we could not obtain (all) the expected RM-equivalences try as we might. A better definition of open set, namely Definition 3.12, that does yield the expected RM-equivalences was introduced in 40. We now study this 'better' definition.
3.4. Bootstrap axioms. We study equivalences from 40 involving the 'bootstrap' axiom BOOT and show that the use of QF-AC ${ }^{0,1}$ can be replaced with NCC.

First of all, $\left[\mathrm{BOOT}+\mathrm{ACA}_{0}\right] \leftrightarrow$ open $^{+}$was proved using QF-AC ${ }^{0,1}$ in 40, §4.2]. The 'coding principle' open ${ }^{+}$connects open sets as in RM, given by countable unions, and open sets given by uncountable unions. In this section, 'open' and 'closed' and refers to the below definition, while 'RM-open' refers to the well-known RM-definition from [47, II.5] involving countable unions of basic open balls.

Definition 3.12. [Open sets in $\mathrm{RCA}_{0}^{\omega}$ from [40]] An open set $O$ in $\mathbb{R}$ is represented by a functional $\psi: \mathbb{R} \rightarrow \mathbb{R}^{2}$. We write ' $x \in O^{\prime}$ for $(\exists y \in \mathbb{R})\left(x \in I_{y}^{\psi}\right)$, where $I_{y}^{\psi}$ is the open interval $(\psi(y)(1), \psi(y)(1)+|\psi(y)(2)|)$ in case the end-points are different, and $\emptyset$ otherwise. We write $O=\cup_{y \in \mathbb{R}} I_{y}^{\psi}$ to emphasise the connection to uncountable unions. A closed set is represented by the complement of an open set.

Intuitively, open sets are given by uncountable unions $\cup_{y \in \mathbb{R}} I_{y}^{\psi}$, just like RM-open sets are given by countable such unions. Hence, our notion of open set reduces to the notion RM-open set when applying ECF or when all functions on $\mathbb{R}$ are continuous. Moreover, writing down the definition of elementhood in an RM-open set, one observes that such sets are also open (in our sense). Finally, closed sets are readily seen to be sequentially closed, and the same for nets instead of sequences.

The following 'coding principle' turns out to have nice properties. Note that open, a weaker version of open ${ }^{+}$, was introduced and studied in 30. We fix an enumeration of all basic open balls $B\left(q_{n}, r_{n}\right) \subset \mathbb{R}$ for rational $q_{n}, r_{n}$ with $r_{n}>_{\mathbb{Q}} 0$.
Definition 3.13. [open ${ }^{+}$] For every open set $Z \subseteq \mathbb{R}$, there is $X \subseteq \mathbb{N}$ such that $(\forall n \in \mathbb{N})\left(n \in X \leftrightarrow B\left(q_{n}, r_{n}\right) \subseteq Z\right)$.

Note that given the set $X$ from open ${ }^{+}$, we can write $Z=\cup_{n \in X} B\left(q_{n}, r_{n}\right)$ as expected. We now have the following equivalence.

Theorem 3.14. The system $\mathrm{RCA}_{0}^{\omega}+\mathrm{NCC}_{\text {proves }} \mathrm{BOOT} \leftrightarrow\left[\right.$ open $\left.{ }^{+}+\mathrm{ACA}_{0}\right]$.
Proof. The implication $\left[\mathrm{ACA}_{0}+\right.$ open $\left.{ }^{+}\right] \rightarrow$ BOOT over $\mathrm{RCA}_{0}^{\omega}$ is immediate from [40, Theorem 4.4]. We now prove the 'crux' implication BOOT $\rightarrow$ open ${ }^{+}$using NCC. In case $\neg\left(\exists^{2}\right)$, all functionals on $\mathbb{R}$ or $\mathbb{N}^{\mathbb{N}}$ are continuous by [19, §3]. Thus, an open set $\cup_{y \in \mathbb{R}} I_{y}^{\psi}$ reduces to the countable union $\cup_{q \in \mathbb{Q}} I_{q}^{\psi}$, yielding open ${ }^{+}$in this case. In case $\left(\exists^{2}\right)$, let $O$ be an open set given by $\psi: \mathbb{R} \rightarrow \mathbb{R}^{2}$ as in Definition 3.12 Now use BOOT and $\left(\exists^{2}\right)$ to define the following set $X \subset \mathbb{N} \times \mathbb{Q}$ :

$$
\begin{equation*}
(\forall n \in \mathbb{N}, q \in \mathbb{Q})\left((n, q) \in X \leftrightarrow(\exists y \in \mathbb{R})\left(B\left(q, \frac{1}{2^{n}}\right) \subset I_{y}^{\psi}\right)\right) \tag{3.7}
\end{equation*}
$$

Trivially, for the set $X$ from (3.7), we have for all $n \in \mathbb{N}, q \in \mathbb{Q}$ that:

$$
\begin{align*}
(n, q) \in X \rightarrow(\exists m \in \mathbb{N}, r \in \mathbb{Q}) & \left(B\left(q, \frac{1}{2^{n}}\right) \subseteq B\left(r, \frac{1}{2^{m}}\right)\right. \\
& \left.\wedge(\exists y \in \mathbb{R})\left(B\left(r, \frac{1}{2^{m}}\right) \subseteq I_{y}^{\psi}\right)\right) \tag{3.8}
\end{align*}
$$

Apply NCC to the implication in (3.8) to obtain $\Phi$ such that for all $n \in \mathbb{N}, q \in \mathbb{Q}$ :

$$
\begin{align*}
(n, q) \in X \rightarrow\left(B\left(q, \frac{1}{2^{n}}\right)\right. & \subseteq B\left(\Phi(n, q)(1), \frac{1}{2^{\Phi(n, q)(2)}}\right) \\
& \left.\wedge(\exists y \in \mathbb{R})\left(B\left(\Phi(n, q)(1), \frac{1}{2^{\Phi(n, q)(2)}}\right) \subseteq I_{y}^{\psi}\right)\right) \tag{3.9}
\end{align*}
$$

Now consider the following formula defined in terms of the above $X$ and $\Phi$.

$$
\begin{gather*}
x \in O \leftrightarrow(\exists n \in \mathbb{N}, q \in \mathbb{Q})\left((n, q) \in X \wedge x \in B\left(\Phi(n, q)(1), \frac{1}{2^{\Phi(n, q)(2)}}\right)\right. \\
\left.\wedge(\exists y \in \mathbb{R})\left(B\left(\Phi(n, q)(1), \frac{1}{2^{\Phi(n, q)(2)}}\right) \subseteq I_{y}^{\psi}\right)\right) . \tag{3.10}
\end{gather*}
$$

Note that BOOT provides a set $Y$ such that $(q, n) \in Y$ if and only $q, n$ satisfy the underlined formula in (3.10). Thus, the right-hand side of (3.10) is decidable given $\left(\exists^{2}\right)$. The formula (3.10) provides a representation of $O$ as a countable union of open balls, and of course gives rise to open ${ }^{+}$. What is left is to prove (3.10).

For the reverse implication in (3.10), $x \in O$ follows by definition from the righthand side of (3.10). For the forward implication, $x_{0} \in O$ implies $B\left(x_{0}, \frac{1}{2^{n_{0}}}\right) \subset I_{y_{0}}^{\psi}$ for some $y_{0} \in \mathbb{R}$ and $n_{0} \in \mathbb{N}$ by definition. For $n_{1}$ large enough, the rational $q_{0}:=\left[x_{0}\right]\left(n_{1}\right)$ is inside $B\left(x_{0}, \frac{1}{2^{n_{0}+1}}\right)$. Hence, $\left(q_{0}, n_{0}+1\right) \in X$ by (3.7) for $y=y_{0}$. Applying (3.9) then yields

$$
\begin{align*}
B\left(q_{0}, \frac{1}{2^{n_{0}+1}}\right) \subseteq & B\left(\Phi\left(n_{0}+1, q_{0}\right)(1), \frac{1}{2^{\Phi\left(n_{0}+1, q_{0}\right)(2)}}\right)  \tag{3.11}\\
& \left.\wedge(\exists y \in \mathbb{R})\left(B\left(\Phi\left(n_{0}+1, q_{0}\right)(1), \frac{1}{2^{\Phi\left(n_{0}+1, q_{0}\right)(2)}}\right) \subseteq I_{y}^{\psi}\right)\right)
\end{align*}
$$

By assumption, we also have $x_{0} \in B\left(q_{0}, \frac{1}{2^{n_{0}+1}}\right)$, and the right-hand side of (3.10) thus follows from (3.11), and we are done.

The previous theorem has numerous implications. For instance, it is proved in [40, §4] that $\left[A C A_{0}+C B T\right] \leftrightarrow\left[\Pi_{1}^{1}-C A_{0}+B O O T\right]$ over $\mathrm{RCA}_{0}^{\omega}+\mathrm{QF}-\mathrm{AC}^{0,1}$, where CBT is the Cantor-Bendixson theorem, defined as follows.

Principle 3.15 (CBT). For any closed set $C \subseteq[0,1]$, there exist $P, S \subset C$ such that $C=P \cup S, P$ is perfect and closed, and $S^{0 \rightarrow 1}$ is a sequence of reals.

It goes without saying that the above equivalence involving CBT can be obtained using only NCC instead. The same holds for the perfect set theorem and theorems pertaining to separably closed sets from [40, §4].

## 4. Computability theory and the Axiom of Choice

4.1. Introduction. We study the computational properties of NCC and related principles. To this end, we first introduce the concept of 'realiser for NCC'.

Definition 4.1. $[\mathrm{NCC}(\zeta)]$ For $Y^{2}$ and $A(n, m) \equiv\left(\exists f \in 2^{\mathbb{N}}\right)(Y(f, m, n)=0)$ :

$$
\left(\forall n^{0}\right)\left(\exists m^{0}\right) A(n, m) \rightarrow\left(\forall n^{0}\right) A(n, \zeta(Y)(n)) .
$$

We refer to $\zeta^{2 \rightarrow 1}$ satisfying $\operatorname{NCC}(\zeta)$ as a 'realiser for NCC' or ' $\zeta$-functional'.
Note that $\zeta$-functionals as in the previous definition are trivially computable in $\exists^{3}$ via a term of Gödel's $T$ of very low complexity. We are also interested in weak realisers for NCC, which are $\zeta_{\mathrm{w}}^{2 \rightarrow 1}$ such that $\left(\forall n^{0}\right)\left(\exists m^{0} \leq \zeta_{\mathrm{w}}(Y)(n)\right) A(n, m)$ in the above specification. Thus, $\zeta_{\mathrm{w}}$-functionals only provide an upper bound for the choice function in NCC, while $\zeta$-functionals provide such a function, as is clear from Definition 4.8. This modification has been discussed in Section 1.2.2.

We are also interested in the following related specification for $\vartheta$-functionals, which are realisers for MCC as in Remark 3.7.

Definition 4.2. $[\mathrm{MCC}(\vartheta)]$ For $Y^{2}$ and $A(n, m) \equiv\left(\forall g \in 2^{\mathbb{N}}\right)(Y(g, m, n)=0)$ :

$$
\left(\forall n^{0}\right)\left(\exists m^{0}\right) A(n, m) \rightarrow\left(\forall n^{0}\right) A(n, \vartheta(Y)(n)) .
$$

As noted in Remark 3.7, the Pincherle phenomenon also pops up when studying the Baire category theorem for open sets given by characteristic functions. However, the associated proofs are completely different from those for the Heine-Borel or Pincherle theorem. Similarly, $\vartheta$-functionals give rise to a realiser for the Baire category theorem, while the former seem fundamentally different from $\zeta$-functionals.

In Section 4.2, we show that while NCC and MCC are rather weak (from a first-order strength perspective), its total realisers are quite strong (from a computational perspective) in that they are exactly $\exists^{3}$. Interestingly, this result makes use of a relatively strong fragment of the axiom of extensionality; the latter is included in $\mathrm{RCA}_{0}^{\omega}$ as $\mathrm{E}_{\rho, \tau}$ for all finite types.

In Section 4.3, we study partial realisers of NCC; one expects those to be weaker than their total counterparts. We show that such partial realisers can perform the computational task (B) from Section 1.2 .2 we also conjecture that such partial realisers cannot perform the seemingly stronger task (A). Since we do not have a proof of this conjecture, we will tackle a weaker problem, namely to find a partial realiser of NCC that does not compute $\exists^{3}$. As noted above, a useful concept is that of a countably based functional as in Definition 2.6. Indeed, since $\exists^{3}$ is not countably based, a countably based partial realiser for NCC cannot compute $\exists^{3}$. In other words, such a partial realiser would be exactly what we want. This construct does exist, but is rather elusive: by Theorem 4.19 the existence of countably based partial realiser for NCC is equivalent to the Continuum Hypothesis ( CH for short).
4.2. Total realisers. We show that Kleene's $\exists^{3}$ and various total realisers for NCC are one and the same thing, even in weak systems.
4.2.1. The power of total realisers for NCC. In this section, we show that $\zeta$-functionals compute $\exists^{3}$ and vice versa. We also obtain associated equivalences over the base theory $\mathrm{RCA}_{0}^{\omega}$. To this end, we first establish the following two lemmas.

Lemma 4.3. Any $\zeta$-functional computes $\kappa_{0}$ via a term of Gödel's T. The system $\mathrm{RCA}_{0}^{\omega}$ proves $(\exists \zeta) \mathrm{NCC}(\zeta) \rightarrow\left(\kappa_{0}^{3}\right)$.
Proof. Fix some functional $Y^{2}$ and define the following sequence:

$$
Y_{k}(n, m, f):= \begin{cases}0 & Y(f)=0 \wedge m=k  \tag{4.1}\\ 1 & \text { otherwise }\end{cases}
$$

Let $\zeta$ be as in $\operatorname{NCC}(\zeta)$ and consider the following formula.

$$
\begin{equation*}
\left(\exists f \in 2^{\mathbb{N}}\right)(Y(f)=0) \leftrightarrow \zeta\left(Y_{0}\right)(0) \neq 0 \zeta\left(Y_{1}\right)(0) \tag{4.2}
\end{equation*}
$$

Since the right-hand side of (4.2) is decidable, this formula gives rise to $\kappa_{0}^{3}$, as required by the lemma. To prove (4.2), note that $\left(\exists f \in 2^{\mathbb{N}}\right)(Y(f)=0)$ implies $\zeta\left(Y_{0}\right)(0)=0$ and $\zeta\left(Y_{1}\right)(0)=1$ by the definition in (4.1). For the remaining implication, $\left(\forall f \in 2^{\mathbb{N}}\right)(Y(f)>0)$ implies $Y_{0}={ }_{2} Y_{1}={ }_{2}$ 1, i.e. the latter functionals are constant 1. The axiom of extensionality $(\mathrm{E})_{2,0}$ then yields $\zeta\left(Y_{0}\right)(0)={ }_{0} \zeta\left(Y_{1}\right)(0)$, as required. The equivalence (4.2) now finishes the proof.

We note that the above proof fails if the $\zeta$-functional at hand is not total, while we can prove that there is a partial $\zeta$-functional computable in $\kappa_{0}^{3}$. We also point out that the axiom of extensionality (for a relatively high type) is used in an essential way in the reverse implication in (4.2).
Lemma 4.4. Any $\zeta$-functional computes $\exists^{2}$ via a term of Gödel's T. The system $\mathrm{RCA}_{0}^{\omega}$ proves $(\exists \zeta) \mathrm{NCC}(\zeta) \rightarrow\left(\exists^{2}\right)$.
Proof. Fix $Y^{2}$ and $\zeta$ as in $\operatorname{NCC}(\zeta)$. Using dummy variables and $\zeta(Y)(0)$, we can define $\zeta_{0}^{2 \rightarrow 0}$ such that whenever $\left(\exists m^{0}, \exists f \in 2^{\mathbb{N}}\right)(Y(f, m)=0)$ then $\zeta_{0}(Y)=m_{0}$ such that $\left(\exists f \in 2^{\mathbb{N}}\right)\left(Y\left(f, m_{0}\right)=0\right)$. Now fix $g^{1}$ and define $Y$ as follows

$$
Y(n, f)= \begin{cases}0 & \text { if } f(n)=0 \\ 1 & \text { otherwise }\end{cases}
$$

Then $\left(\exists n^{0}\right)(g(n)=0) \leftrightarrow g\left(\zeta_{0}(Y)\right)=0$, and we are done.
We again point out that the axiom of extensionality (for a relatively high type) is used in an essential way in the final equivalence in the proof. To the best of our knowledge, the axiom of extensionality has not been used in higher-order RM beyond $(\mathrm{E})_{1,0}$ in formalising Grilliot's trick in $\mathrm{RCA}_{0}^{\omega}$ (see [19, 20]).

We now have the following main theorem of this section.
Theorem 4.5. The functional $\exists^{3}$ computes a $\zeta$-functional via a term of Gödel's $T$, and vice versa. The system $\mathrm{RCA}_{0}^{\omega}$ proves $(\exists \zeta) \mathrm{NCC}(\zeta) \leftrightarrow\left(\exists \exists^{3}\right)$.
Proof. That $\exists^{3}$ computes a $\zeta$-functional is immediate from the fact that the former computes Feferman's $\mu^{2}$. The reverse computational direction is similarly immediate in light of the above lemmas. For the forward implication, the splitting
$\left(\exists^{3}\right) \leftrightarrow\left[\left(\kappa_{0}^{3}\right) \leftrightarrow\left(\exists^{2}\right)\right]$ can be found in [25, §6], going back to Kohlenbach. Combining the two above lemmas yields the forward implication, while the reverse one is immediate in light of $\left(\exists^{2}\right) \leftrightarrow\left(\mu^{2}\right)$ over $\operatorname{RCA}_{0}^{\omega}$ (see [20]).

The following corollary is immediate by the theorem, while the second corollary follows mutatis mutandis.

Corollary 4.6. In the specification $\operatorname{NCC}(\zeta)$, we may assume that $\zeta(Y)(n)$ provides the least witness to $m$.

Corollary 4.7. The functional $\exists^{3}$ is computable from a $\vartheta$-functional via a term of Gödel's $T$, and vice versa. The system $\mathrm{RCA}_{0}^{\omega}$ proves $(\exists \vartheta) \mathrm{MCC}(\vartheta) \leftrightarrow\left(\exists^{3}\right)$.

In light of the above, realisers for NCC and MCC are (too) strong and we shall study weaker objects in the next section. Nonetheless, it is interesting that we have obtained a very different equivalent formulation for $\left(\exists^{3}\right)$ based on a fragment AC, namely NCC. It is also interesting that we seem to need a relatively strong fragment of the axiom of extensionality. Similar to [20], it is a natural question whether the above equivalences go through without the latter axiom.
4.2.2. The power of weak total realisers for NCC. Similar to the previous section, we study weak realisers for NCC as in Defintion 4.8 below. As noted in Section 1.2 .2 weakening NCC as in the latter definition means that the proofs in Section 3 do not (seem to) go through. Nonetheless, we show that these weak realisers for NCC still compute $\exists^{3}$, and vice versa.

Definition 4.8. $\left[\mathrm{NCC}_{\mathrm{w}}\left(\zeta_{\mathrm{w}}\right)\right]$ For $Y^{2}$ and $A(n, m) \equiv\left(\exists f \in 2^{\mathbb{N}}\right)(Y(f, m, n)=0)$ :

$$
\left(\forall n^{0}\right)\left(\exists m^{0}\right) A(n, m) \rightarrow\left(\forall n^{0}\right)\left(\exists m \leq \zeta_{\mathrm{w}}(n)\right) A(n, m)
$$

The following lemma is proved in the same way as for Lemma 4.4.
Lemma 4.9. Any $\zeta_{w}$-functional computes $\exists^{2}$ via a term of Gödel's T. The system $\operatorname{RCA}_{0}^{\omega}$ proves $\left(\exists \zeta_{w}\right) \operatorname{NCC}_{w}\left(\zeta_{w}\right) \rightarrow\left(\exists^{2}\right)$.
Proof. Use the same functional $Y$ as in the proof of Lemma 4.4 observing that $\left(\exists n^{0}\right)(g(n)=0) \leftrightarrow\left(\exists n \leq \zeta_{\mathrm{w}}(Y)\right)(g(n)=0)$.

We also have the following (surprising) result showing that even weak realisers for NCC are in fact strong.

Lemma 4.10. Any $\zeta_{\mathrm{w}}$-functional computes $\kappa_{0}$ via a term of Gödel's T. The system $\mathrm{RCA}_{0}^{\omega}$ proves $\left(\exists \zeta_{\mathrm{w}}\right) \mathrm{NCC}_{\mathrm{w}}\left(\zeta_{\mathrm{w}}\right) \rightarrow\left(\kappa_{0}^{3}\right)$.

Proof. In the same way as in the proof of Lemma 4.4, use $\zeta_{\mathrm{w}}$ to define $\zeta_{0}^{2 \rightarrow 1}$ such that if $\left(\exists n^{0}\right)\left(\exists f \in 2^{\mathbb{N}}\right)(Y(n, f)=0)$ then $\left(\exists n \leq \zeta_{0}(Y)\right)\left(\exists f \in 2^{\mathbb{N}}\right)(Y(n, f)=0)$. Fix some $Z^{2}$ and define two functionals $Y_{i}$ for $i=0,1$ as follows:

$$
Y_{0}(n, f):=\left\{\begin{array}{lc}
0 & Z(f)=0 \\
1 & \text { otherwise }
\end{array} \quad Y_{1}(n, f):= \begin{cases}0 & Z(f)=0 \wedge n>\zeta_{0}\left(Y_{0}\right) \\
1 & \text { otherwise }\end{cases}\right.
$$

Then $\kappa_{0}^{3}$ is obtained by the previous lemma and the following equivalence:

$$
\begin{equation*}
\left(\exists f \in 2^{\mathbb{N}}\right)(Z(f)=0) \leftrightarrow \zeta_{0}\left(Y_{0}\right) \neq 1 \zeta_{0}\left(Y_{1}\right) \tag{4.3}
\end{equation*}
$$

For the forward direction in (4.3), note that $\left(\exists f \in 2^{\mathbb{N}}\right)(Z(f)=0)$ implies that $\left.\zeta_{0}\left(Y_{1}\right)\right)(0)>_{0} \zeta_{0}\left(Y_{0}\right)(0)$ by the definition of $Y_{1}$. For the reverse direction in (4.3),
assuming $\left(\forall f \in 2^{\mathbb{N}}\right)(Z(f)>0)$ yields $Y_{0}={ }_{2} Y_{1}={ }_{2} 1$, and the axiom of extensionality $(\mathrm{E})_{2,1}$ yields $\zeta_{0}\left(Y_{0}\right)={ }_{1} \zeta_{0}\left(Y_{1}\right)$, as required.

We now easily obtain the other main result of this section.
Theorem 4.11. The functional $\exists^{3}$ computes a $\zeta_{w}$-functional via a term of Gödel's $T$, and vice versa. The system $\mathrm{RCA}_{0}^{\omega}$ proves $\left(\exists \zeta_{w}\right) \mathrm{NCC}\left(\zeta_{w}\right) \leftrightarrow\left(\exists^{3}\right)$.

What makes the results in this section interesting is that realisers for NCC grew out of a principle that seemed natural and weak, and these functionals then turned out to be strong. This illustrates the power of assuming that realisers are total, and supports our view that the partial $\zeta$-functionals reflect in a more natural way the principle NCC that is meant to replace countable choice. Hence, we shall study partial realisers for NCC in Section 4.3 .
4.3. Partial realisers. In this section, we study partial realisers for NCC and show that they are weaker and have more interesting computational properties than total realisers for NCC.

In Section 4.3.1 we connect these realisers to the computational study of compactness as in items (A) and (B) from Section 1.2.2. In Section 4.3.2 we show that the existence of countably based partial realisers for NCC is equivalent to the Continuum Hypothesis. We further provide a foundational discussion of partial versus total functionals in Section 4.3.3. Finally, in Section 4.3 .4 we discuss the role of Kleene computability in our endeavour and a possible weaker alternative.
4.3.1. The power of partial realisers for NCC. We introduce the notion of 'partial realiser for NCC' and prove some basic properties.

First of all, the following definition is as expected.
Definition 4.12. [Partial realisers for NCC]
(a) A partial NCC-realiser is a partial functional $\zeta_{p}$ taking objects $Y$ of type $\left(\mathbb{N}^{2} \times 2^{\mathbb{N}}\right) \rightarrow \mathbb{N}$ as arguments such that if

$$
\left(\forall n^{0}\right)\left(\exists m^{0}\right)\left(\exists f \in 2^{\mathbb{N}}\right)(Y(n, m, f)=0)
$$

then $\zeta_{\mathrm{p}}(Y)=g$ is a choice function satisfying

$$
\left.\left(\forall n^{0}\right)\left(\exists f \in 2^{\mathbb{N}}\right)(Y(n, g(n), f))=0\right)
$$

(b) A weak partial NCC-realiser is a partial functional $\zeta_{\mathrm{p}_{0}}$ taking objects $Y$ of type $\left(\mathbb{N} \times 2^{\mathbb{N}}\right) \rightarrow \mathbb{N}$ as arguments, such that if $\left(\exists m^{0}\right)\left(\exists f \in 2^{\mathbb{N}}\right)(Y(m, f)=0)$ then $\zeta_{\mathbf{p}_{0}}(Y)$ terminates and yields an $m$ such that $\left(\exists f \in 2^{\mathbb{N}}\right)(Y(m, f)=0)$.
While seemingly different, items (a) and (b) yield the same computational class.
Lemma 4.13. The classes of partial NCC-realisers and weak partial NCC-realisers are computationally equivalent.

Proof. Clearly a partial NCC-realiser computes a weak one: to compute $\zeta_{\mathrm{p}_{0}}(Y)$, one computes $\zeta_{\mathrm{p}}(\lambda(n, m, f) . Y(m, f))(0)$. Given $\zeta_{\mathrm{p}_{0}}$ we can compute $\zeta_{\mathrm{p}}(Y)(n)=$ $\zeta_{\mathfrak{p}_{0}}(\lambda(m, f) . Y(n, m, f))$.

In the sequel, we sometimes identify a function $Y$ as above with its set of zeros. The functional $\nu$ in the following theorem is called a selector, for obvious reasons.

Theorem 4.14. Let $\zeta_{p_{0}}$ be a weak partial NCC-realiser. Then there is a partial functional $\nu$ taking subsets $X$ of $2^{\mathbb{N}}$ as arguments and with values in $2^{\mathbb{N}}$ such that if $X$ is closed and nonempty, then $\nu(X) \in X$.

Proof. By recursion on $n$, we use $\zeta_{\mathrm{p}_{0}}$ and primitive recursion to find (compute) a binary function $f$ such that $X \cap[\bar{f} n] \neq \emptyset$ for each $n$. Now note that $f \in X$.

While seemingly basic, selectors are hard to compute as follows.
Lemma 4.15. There is no selector $\nu$ computable in any functional of type 2.
Proof. Assume that the selector $\nu$ is computable in $F$ and let $f, g \in 2^{\mathbb{N}}$ be distinct and not computable in $F$. Let $X_{f}=\{f\}$ and $X_{g}=\{g\}$. When we compute $\nu\left(X_{f}\right)$ and $\nu\left(X_{g}\right)$ using the algorithm for $\nu$ from $F$, we will only use oracle calls for $h \in X$ for $h$ computable in $F$, and will get the same negative answer for both inputs. Thus $\nu\left(X_{f}\right)=\nu\left(X_{g}\right)$, contradicting what $\nu$ should do.

The background for this argument is treated in the proof of [31, Lem. 2.14].
Corollary 4.16. There is no partial NCC-realiser computable in any type 2 functional.

Proof. Follows directly from Lemma 4.13. Theorem4.14 and Lemma 4.15.
Corollary 4.16 can also be seen a consequence of the fact that partial NCCrealisers can deal with the computational problem (B) from the introduction.

Theorem 4.17. Any weak partial NCC-realiser $\zeta_{\mathrm{p}_{0}}$ can perform the following task: for $G: 2^{\mathbb{N}} \rightarrow \mathbb{N}$, compute $k \in \mathbb{N}$ such that there exists a finite sub-covering of size $k$ of the covering $\cup_{f \in 2^{\mathbb{N}}}[\bar{f} G(f)]$.
Proof. Given $G$, let $Y(k, f)=0$ if $f=\left\langle f_{1}, \ldots, f_{k}\right\rangle$ and the set of neighbourhoods $\left[\overline{f_{i}} G\left(f_{i}\right)\right]$ for $i=1, \ldots, k$ form a sub-covering of $\cup_{f \in 2^{\mathrm{N}}}[\bar{f} G(f)]$. Clearly, $Y$ is uniformly computable in $G$, only requiring explicit elementary constructions. Then $\zeta_{\mathrm{p}_{0}}(Y)$ answers the computational task.

On a related note, consider the following computational task (C), intermediate between (A) and (B) from Section 1.2.2, A Lebesgue number for $\cup_{f \in 2^{\mathbb{N}}}[\bar{f} G(f)]$ is $k \in \mathbb{N}$ such that $\left(\forall f \in 2^{\mathbb{N}}\right)\left(\exists g \in 2^{\mathbb{N}}\right)\left(G(g) \leq 2^{k} \wedge f \in[\bar{g} G(g)]\right)$. This notion has been studied in RM in e.g. [10, 27].
(C) For any $G: 2^{\mathbb{N}} \rightarrow \mathbb{N}$, compute a Lebesgue number for $\cup_{f \in 2^{\mathbb{N}}}[\bar{f} G(f)]$.

Clearly, the proof of Theorem 4.17 yields that partial NCC-realisers can perform the task (C). It can be shown that (B) and (C) are equivalent, but we do not have a proof of this equivalence for $2^{\mathbb{N}}$ replaced by $[0,1]$.

Finally, we conjecture that partial NCC-realisers cannot perform the computation task (A) from the introduction, i.e. compute the sub-covering itself, rather than just a bound on its size. We do not know how to establish this conjecture at the moment, and we therefore consider an 'easier' problem: to show the existence of partial NCCrealisers that do not compute $\exists^{3}$. As discussed below Definition [2.6, this easier problem can be solved by exhibiting a countably based partial NCC-realiser. This is the topic of Section 4.3.2, where we encounter CH.
4.3.2. Partial realisers and the Continuum Hypothesis. In this section, we show that the existence of a countably based partial NCC realiser is equivalent to CH .

First, let us observe that the computational power of partial NCC-realisers depends on a symbiosis with discontinuity in the form of $\exists^{2}$.
Lemma 4.18. Assuming $\neg\left(\exists^{2}\right)$ there is a computable partial NCC-realiser $\zeta_{\mathrm{p}}$
Proof. Given $Y(n, m, f)$ and $n$, we search for a pair $(m, s)$ where $s$ is a binary sequence, and where $Y(n, m, s * 00 \ldots)=0$. If there is an $m$ and an $f$ such that $Y(n, m, f)=0$, the continuity of $Y$ will ensure that we find $(m, s)$ as above. We then let $\zeta_{\mathbf{p}}(Y)(n)=m$.

There is noting dramatic about the previous lemma: the class of realisers for HBU has the same property. Hence, if we are interested in the relative computational power of partial NCC-realisers, we may as well assume that $\exists^{2}$ is given.

Our next result is not within the scope of usual RM, but we include it in order to illustrate the special character of partial NCC-realisers.

Theorem 4.19. Assuming ZFC, the following are equivalent:
(1) There is a countably based partial NCC-realiser $\zeta_{\mathrm{p}}$.
(2) The continuum hypothesis CH .

Proof. First assume CH. Define the set $\left\{\left(m_{\alpha, n}, f_{\alpha, n}\right): n \in \mathbb{N} \wedge \alpha<\aleph_{1}\right\}$ where $m_{n, \alpha} \in \mathbb{N}$ and $f_{n, \alpha} \in 2^{\mathbb{N}}$, and such that whenever $\left\{\left(m_{n}, f_{n}\right)\right\}_{n \in \mathbb{N}}$ is a sequence from $\mathbb{N} \times 2^{\mathbb{N}}$ there is an $\alpha<\aleph_{1}$ such that $m_{n}=m_{n, \alpha}$ and $f_{n}=f_{n, \alpha}$ for all $n$. We can then define $\zeta_{\mathrm{p}}$ by $\zeta_{\mathrm{p}}(Y)(n)=m_{n, \alpha}$ for the least $\alpha$ such that $\left(\forall n^{0}\right)\left(Y\left(n, m_{n, \alpha}, f_{n, \alpha}\right)=0\right)$. This $\zeta_{\mathrm{p}}$ will be countably based, since when terminating we only have to evaluate $Y\left(n, m_{n, \beta}, f_{n, \beta}\right)$ for countably many $\beta$ in order to find a suitable $\alpha$.

Now assume that $\zeta_{p}$ is a countably based partial NCC-realiser. For each $f \in 2^{\mathbb{N}}$ let $Y_{f} \leq 1$ be defined by $Y_{f}(n, m, g)=0$ if and only if $f={ }_{1} g$ and $m=f(n)$. Then $\zeta_{\mathrm{p}}\left(Y_{f}\right)=f$. Let $Z_{f} \subseteq Y_{f}$ be a countable basis for $\zeta_{\mathrm{p}}\left(Y_{f}\right)$, i.e. for all $Y$ such that $Z_{f} \subseteq Y$ we have that $\zeta_{\mathrm{p}}(Y)=f$. Let $A_{f}$ be the set of $g \in 2^{\mathbb{N}}$ such that $Z_{f}(n, m, g)$ is defined for some $n$ and $m$. Then $A_{f}$ is countable and satisfies $f \in A_{f}$. Indeed, otherwise $Z_{f}$ is a sub-function of the constant 1 , and actually a sub-function of all but countably many $Y_{g}$. This is impossible and CH follows from:
Claim Let $X \subseteq 2^{\mathbb{N}}$ have cardinality $\aleph_{1}$. Then $2^{\mathbb{N}}=\bigcup_{f \in X} A_{f}$.
Proof of Claim Assume not, and let $g \notin \bigcup_{f \in X} A_{f}$. Let $f \in X$. Since $\zeta_{\mathrm{p}}\left(Y_{f}\right) \neq$ $\zeta_{\mathrm{p}}\left(Y_{g}\right)$ we must have that $Z_{f}$ and $Z_{g}$ are incompatible, which again means that there is a triple $(n, m, h)$ such that both $Z_{f}(n, m, h)$ and $Z_{g}(n, m, h)$ are defined, but different. Since $h \in A_{f}$ and $g \notin A_{f}$ by the choice of $g$ we must have that $h \neq g$, so $Y_{g}(n, m, h)=1$, and consequently $Y_{f}(n, m, h)=0$ (since the values differ), with the further consequence that $h=f$. Since $f \in X$ was arbitrary, this shows that $X \subseteq A_{g}$, which is impossible since $X$ is uncountable, while $A_{g}$ is countable. So, the assumption leads to a contradiction, and our claim follows.

There are two observations to be made from this theorem. One is that in the case of $\mathrm{CH}, \exists^{3}$ cannot be computable in all partial NCC-realisers, since $\exists^{3}$ is not countably based. The argument readily generalises to the case when the cardinality of the continuum is a successor cardinal, but we have no fully general proof. The argument in case of successor cardinal is outside the scope of this paper.

We conjecture that it is provable in ZFC that there is a partial NCC-realiser that does not compute $\exists^{3}$ relative to any functional of type 2 . On the other hand, if CH fails, there is no partial NCC-realiser that is computable in any of the countably based functionals we have considered, like $\Theta$-functionals from Section 2.2 and the functional for non-monotone inductive definitions from [23], studied in more detail in [24]. We again conjecture that CH is not needed, i.e. the existence of such a realiser is provable in ZFC.

Finally, from the point of view of higher-order computability, partial NCCrealisers are of interest because they are natural enough and represent a hitherto unobserved level of complexity in light of Theorem 4.19. This is clearly related to the fact that they are partial, and in the next section we discuss the general problem of how concepts of higher order computability extends to cases like this.
4.3.3. Total versus partial functionals. We discuss the foundational role of partial versus total functionals via some interesting examples based on Pincherle's theorem (Example 4.20), transfinite recursion (Example 4.21), and representations of open sets (Example 4.22).

Most abstractly, given a statement of the form $(\forall x)(\exists y)(\Phi(x) \rightarrow \Psi(x, y))$, say provable in ZFC, there are two main questions of interest in computability theory.
(1) How hard is it to compute a realiser $\zeta$ such that $(\forall x)(\Phi(x) \rightarrow \Psi(x, \zeta(x)))$ ?
(2) What can we compute from such a realiser $\zeta$ ?

For item (1), the existence of a computable realiser implies that the theorem is constructively true (for some notion of 'constructive'). In the non-computable case, the complexity of a realiser indicates to what extent contra-positive arguments or AC are needed. Of course, we obtain more information from a total realiser than just a partial one. For item (2), we get more information about an implication $A \rightarrow B$ if we can compute realisers for $B$ just from partial realisers for $A$.

As to naming, we have taken the liberty to talk about 'realisers', without introducing a specific realisability semantics or a precise definition of what we mean by a realiser. This is deliberate, as we want to use the expression in any situation where we have some functional that transforms information about an assumption to information about a conclusion. The main point of this section is now that:

> it generally makes a huge difference whether we require our realisers to be total objects or not.

We will consider a couple of examples backing the above claim, but let us first make one point clear: combining Kleene's S1-S9 and partial functionals the way we do is not problematic or strange in the least. Indeed, it is part of the nature of computability theory that one computes partial objects, directly or relative to other objects. In our context, when we discuss computability relative to a partial object, this partial object will only take total objects as arguments, so the scheme S8 of functional composition needs no adjustment.

As a preliminary example, in the case of total NCC-realisers, it is clear from the proofs in Section 4.2.1 that it does not matter what the output is when the input does not satisfy the assumption of $\left(\forall n^{0}\right)\left(\exists m^{0}\right)\left(\exists f \in 2^{\mathbb{N}}\right)(Y(n, m, f)=0)$, the computational strength, namely $\exists^{3}$, stems from the assumption that there will always be a value. The following three examples are more conceptual in nature.

Example 4.20 (Pincherle's theorem). We discuss how the realisers for the original and uniform versions of Pincherle's theorem are related. The original version $\mathrm{PIT}_{o}$ was introduced in Section 3.2, while the 'uniform' version $\mathrm{PIT}_{\mathrm{u}}$ is as follows:

$$
(\forall G: C \rightarrow \mathbb{N})(\exists N \in \mathbb{N})(\forall F: C \rightarrow \mathbb{N})[\operatorname{LOC}(F, G) \rightarrow(\forall g \in C)(F(g) \leq N)]
$$

where $\operatorname{LOC}(F, G)$ from Section 3.2 expresses that $F$ is locally bounded with $G$ realising this fact. These theorems were studied in detail in [29], including a reasonable definition of realiser, inspired by the work of Pincherle (32), as follows.

For the uniform version $\mathrm{PIT}_{\mathrm{u}}$, we considered Pincherle realisers $M_{\mathrm{u}}^{3}$ in [29] such that whenever $G: 2^{\mathbb{N}} \rightarrow \mathbb{N}$ then $M_{\mathrm{u}}(G)$ is an upper bound for all functions $F$ satisfying $\operatorname{LOC}(F, G)$ from Section 3.2. It is shown in [29] that computing an upper bound in this way amounts to the task (B) from Section 1.2.2. What is interesting is that Pincherle realisers are naturally total: $M_{\mathrm{u}}(F)$ must be defined for all $F$. By contrast, for the original version $\mathrm{PIT}_{o}$, a weak Pincherle realiser $M_{o}$ has two variables: the number $M_{o}(F, G)=m$ is such that if $\operatorname{LOC}(F, G)$ then $F$ is bounded by $m$ on $2^{\mathbb{N}}$. Even though we considered total functionals $M_{o}$ in [29], we do not need $M_{o}(F, G)$ to be defined unless $\operatorname{LOC}(F, G)$ is satisfied, so here it is equally natural to consider a partial realiser. In fact, the proof of [29, Cor. 3.8] mentioned in Example 4.21 can be adjusted to show that any partial weak Pincherle realiser $M_{o}$, together with $\exists^{2}$, computes a total realiser for transfinite recursion. This proof is however beyond the scope of this paper.

Moreover, computing a Pincherle realiser from a partial NCC-realiser via Theorem 4.17 demonstrates how uniform Pincherle's theorem $\mathrm{PIT}_{u}$ can be proved from NCC (assuming HBU), while computing it from a total NCC-realiser just witnesses that the theorem is provable in $Z_{2}^{\Omega}$. A similar observation holds for $\mathrm{PIT}_{o}$ and WKL .

The next example deals with transfinite recursion and Pincherle realisers.
Example 4.21 (Transfinite recursion). We assume there is a partial functional $\Gamma$ such that if $(X, \prec)$ is a well-ordering of a subset of $\mathbb{N}$ and $F: \mathbb{N}^{\mathbb{N}} \rightarrow \mathbb{N}^{\mathbb{N}}$, then $\Gamma(X, \prec, F)$ is a sequence of functions $f_{x}$ such that if $x \in X$ then $f_{x}=F\left(f_{\prec x}\right)$, where $f_{\prec x}(\langle y, z\rangle)=f_{y}(z)$ if $y \prec x$ and 0 otherwise. This gives rise to three different functionals of increasing power, as follows.

- If we are satisfied with $\Gamma$ being partial, it is outright computable by using the recursion theorem for S1-S9.
- If we want a total extension of $\Gamma$, but it does not matter what the value is when $(X, \prec)$ is not a well-ordering, we can use $\exists^{2}$ and a Pincherle realiser $M_{\mathrm{u}}$ to compute such a total extension by [29, Cor. 3.8].
- Given a $\Theta$-functional and $\exists^{2}$, we may expand $\Gamma$ so that it extracts an infinite descending sequence in $(X, \prec)$ when $\Gamma$ does not provide a fixed point to the recursion equation for iterating $F$ along $(X, \prec)$ (see [26, Cor. 3.16]).

An important aspect of these three results is the difference in what we mean by 'computability'. In the first case, we use the full power of Kleene-computability, and the full use of S1-S9 will only make sense assuming principles of transfinite recursion anyhow, so there is not much insight to be gained from this. For the other two cases, we only use a fragment of Gödel's $T$, and thereby illustrate the computational power of compactness in various guises.

Another example is provided by the $\Delta$-functional introduced in [30, §7]. Note that modulo $\exists^{2}$, (R.3) below is exactly the usual 'countable union of open balls' representation of open sets from RM, called (R.4) in [30] and introduced in 47, II].
Example 4.22 (Representations of open sets). The $\Delta$-functional outputs a 'highlevel' representation (R.3) of an open set $O$ from a 'low-level' representation (R.2) of $O$, as defined below the following two clauses.
(R.2) If $O \subseteq[0,1]$ is open, an (R.2)-representation is a function $Y:[0,1] \rightarrow \mathbb{R}$ such that $x \in O \leftrightarrow Y(x)>0$ and moreover such that if $Y(x)>0$ then $(x-Y(x), x+Y(x)) \cap[0,1] \subseteq O$.
(R.3) If $O \subseteq[0,1]$ is open, the (R.3)-representation is the continuous function $Y^{\prime}$ where $Y^{\prime}(x)$ is the distance from $x$ to $[0,1] \backslash O$, where the distance to the empty set is defied as 1 .
With $Y$ and $Y^{\prime}$ as in (R.2) and (R.3), we have that $\Delta(Y)=Y^{\prime}$. The functional $\Delta$ is of low complexity among the genuine type 3 functionals; it is however unknown what happens with the complexity if we extend $\Delta$ to a total object. Indeed, the point is that if $\Delta$ can be partial, we never (have to) specify what to do if the input does not represent an open set at all. Hence, when we say that $\Delta$ is computable from a Pincherle realiser $M_{u}$ (see [28, Theorem 7.5]), the algorithm works under the assumption that the input is an (R.2)-representation of an open set. In this case, $\Delta$ is also computable from a partial NCC-realiser $\zeta_{p}$ and $\exists^{2}$ as well.
4.3.4. Alternatives to Kleene computability. We briefly discuss the possibility of using computational frameworks other than Kleene's S1-S9.

On one hand, we have seen that the partial functional for transfinite recursion is outright S1-S9 computable. On the other hand, the step from $A C A_{0}$ to $A T R_{0}$ is a significant step in logical strength. The explanation is of course that the assumption the definition of computability via S1-S9 is sound
is itself quite strong. In fact, this soundness goes beyond the strength of transfinite recursion, as it involves the termination of monotone inductions (see [23]). We will not pursue this discussion here, or make any precise mathematical claims related to it, but let us emphasise the following observation.

On one hand, positive computability results are more interesting when the concept of higher-order computability at hand is (far) simpler than full S1-S9. A natural such simple framework is finite type theory with constants for the arithmetical operations and the partial $\mu$-operator. Note that in the proof of Theorem 4.14, we go slightly beyond this, but generally our positive results are witnessed by terms in Gödel's $T$ of low complexity.

On the other hand, non-computability results are better the stronger the concept of relative computability involved is. In this case, $\mathrm{S} 1-\mathrm{S} 9$ is of great interest. In fact, our non-computability results generally make use of S1-S9, while infinite time Turing machines would be too strong (see [24]). Finally, as explored systematically in 31, it should be noted that computability theory based on S1-S9 is a crucial tool in constructing models for fragments of $Z_{2}^{\Omega}$, as in e.g. [25-31.
4.4. Turing machines and higher types. We finish this paper with a section on accommodating higher types in Turing's framework.

Now, Turing's famous 'machine' framework (54) introduces an intuitively convincing concept of 'computing with real numbers'. Certain higher type objects, like
continuous functions on $\mathbb{R}$, can be represented as real numbers, but this 'coding' is not without its problems (see [28,39]) By contrast, Kleene's S1-S9 has the advantage of providing a notion of 'computing with objects of finite type', at the cost of the simplicity of Turing's framework, like e.g. the lack of a counterpart of Kleene's $T$-predicate or the axiomatic encoding of the recursion theorem in S9.

It is then a natural question whether we can discuss certain higher-order results in terms of Turing computability. An example from 40, §3.2.1] is as follows: let ' $\leq_{T}$ ' be the usual Turing reducibility relation and let $J(Y)$ be the set $\{n \in \mathbb{N}$ : $\left.\left(\exists f \in \mathbb{N}^{\mathbb{N}}\right)(Y(f, n)=0)\right\}$, i.e. the set $X$ claimed to exist by BOOT. Now, BOOT follows from the monotone convergence theorem for nets indexed by Baire space in $[0,1]$ by [40, Theorem 3.7]. This implication yields the following:
for any $Y^{2}$, there is a net $x_{d}: D \rightarrow[0,1]$ such that $x=\lim _{d} x_{d}$ implies $J(Y) \leq_{T} x$.
Note that the net $x_{d}: D \rightarrow[0,1]$ can be defined in terms of $Y^{2}$ via a term of Gödel's T. A similar result for the Baire category theorem can be found in [28, §6.2.2].

We now discuss a similar result based on $[\mathrm{WKL}+\mathrm{NCC}] \rightarrow \mathrm{PIT}_{o}$ as in Theorem 3.2. The computational properties of WKL in Turing's framework are (very) well-studied, and the aforementioned implication suggests the possibility of studying Pincherle's theorem in the same way, namely as follows.

Given $Y$ as in NCC, define $C(Y)$ as the function $g$ therein, i.e. $m=C(Y)(n)$ yields $\left(\exists f \in 2^{\mathbb{N}}\right)(Y(f, n, m)=0)$. Then clearly we have $C(Y) \leq_{T} J(Y)$, where we assume the two number variables are coded into one. Now consider the contraposition of Pincherle's theorem (without realisers):
if a functional $F$ is unbounded on $2^{\mathbb{N}}$, there is a point $x_{0} \in 2^{\mathbb{N}}$ such that $F$ is unbounded on all its neighbourhoods.
Similar to the above, $\exists^{3}$ can (S1-S9) compute $x_{0}$ in terms of $F$, but no type two functional can. However, we can state the following:
if a functional $F$ is unbounded on $2^{\mathbb{N}}$, there is a point $x_{0} \leq_{T}\left(C\left(F_{0}\right)\right)^{\prime}$ in $2^{\mathbb{N}}$ such that $F$ is unbounded on all its neighbourhoods.
Note that $\left(C\left(F_{0}\right)\right)^{\prime}$ is the Turing jump of $C\left(F_{0}\right)$, which is well-defined. The exact definition of $F_{0}$ is of course based on the formula in square brackets in (3.1), with obvious/minimal coding. Clearly, we could apply QF-AC ${ }^{0,1}$ to ' $F$ is unbounded on $2^{\mathbb{N}}$, and the jump of the resulting sequence would also yield a point like $x_{0}$. However, the point thus obtained is not Turing computable from e.g. oracles provided by $J$.

Finally, the same can be established for the contraposition of HBC mutatis mutandis and many similar theorems about open sets as in Definition 3.4

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[^1]:    ${ }^{1}$ Two kinds of realisers for Pincherle's theorem were introduced in 29: a weak Pincherle realiser $M_{o}$ takes as input $F^{2}$ that is locally bounded on $2^{\mathbb{N}}$ together with $G^{2}$ such that $G(f)$ is an upper bound for $F$ in $[\bar{f} G(f)]$ for any $f \in 2^{\mathbb{N}}$, and outputs an upper bound $M_{o}(F, G)$ for $F$ on $2^{\mathbb{N}}$. A (normal) Pincherle realiser $M_{\mathrm{u}}$ outputs an upper bound $M_{\mathrm{u}}(G)$ without access to $F$. We discuss these functionals in some detail in Section 4.3.3

[^2]:    ${ }^{2}$ It is shown in [25] that $\exists^{3}$ can perform the computational task (A).

[^3]:    ${ }^{3}$ To be absolutely clear, variables (of any finite type) are allowed in quantifier-free formulas of the language $L_{\omega}$ : only quantifiers are banned.

