# A NOTE ON THE ASYMPTOTIC EXPRESSIVENESS OF ZF AND ZFC 

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#### Abstract

We investigate the asymptotic densities of theorems provable in Zermelo-Fraenkel set theory ZF and its extension ZFC including the axiom of choice. Assuming a canonical De Bruijn representation of formulae, we construct asymptotically large sets of sentences unprovable within ZF, yet provable in ZFC. Furthermore, we link the asymptotic density of ZFC theorems with the provable consistency of ZFC itself. Consequently, if ZFC is consistent, it is not possible to refute the existence of the asymptotic density of ZFC theorems within zFC. Both these results address a recent question by Zaionc regarding the asymptotic equivalence of ZF and ZFC.


## 1. Introduction

In the current paper we are interested in the asymptotic expressiveness of first-order set theories ZF and ZFC. More specifically, we investigate the asymptotic density of sentences provable within these theories among all sentences expressible in the first-order language $\mathcal{L}$ consisting of a single binary membership predicate $(\in)$ and no function symbols.

We start with the following problem posted recently by Zaionc.
Problem. Consider the theories ZF and ZFC. What is the asymptotic density of theorems provable within ZFC? Is it true that ZFC is asymptotically more expressive than ZF?

To make the notion of asymptotic density of theorems sound, we have to assign to each formula $\varphi$ an integer size $|\varphi|$ in such a way that there exists a finite number of formulae of any given size. Having such a size notion, we then define the asymptotic expressiveness of a theory $\mathcal{T}$ as the asymptotic density $\mu(\mathcal{T})$ of its theorems among all possible sentences, i.e.

$$
\begin{equation*}
\mu(\mathcal{T})=\lim _{n \rightarrow \infty} \frac{|\{\varphi:|\varphi|=n \wedge \mathcal{T} \vdash \varphi\}|}{|\{\varphi:|\varphi|=n\}|} \tag{1}
\end{equation*}
$$

In order to start addressing the above problem, we have to establish a formal framework in which we fix certain technical, yet important details, such as the assumed size model of formulae, or their specific combinatorial representation. In this paper we choose to represent formulae using De Bruijn indices [4] instead of the usual notation involving named variables. Within this setup our contributions are twofold.

Firstly, we show that ZF and ZFC cannot share the same asymptotic expressiveness. Specifically, we construct an asymptotically large (i.e having positive asymptotic density) fraction of $\mathcal{L}$-sentences which, though provable in ZFC, cannot be proven in the weaker system ZF without the axiom of choice. Secondly, we show that it is not possible to refute the existence of $\mu(\mathrm{ZFC})$ within ZFC itself. For that purpose we link the provable existence of $\mu(\mathrm{ZFC})$ with the provable consistency of ZFC. In light of Gödel's second incompleteness theorem, the existence of $\mu(\mathrm{ZFC})$ becomes unprovable within ZFC.

We base our analysis on a mixture of methods from analytic combinatorics and, more specifically, recent advances in its application in the quantitative analysis of $\lambda$-terms $[1,3,5]$.

[^0]
## 2. Analysis

2.1. Formulae representation. Let $V$ be an infinite, denumerable set of De Bruijn indices $\underline{0}, \underline{1}, \underline{2}, \ldots$, and $\mathcal{F}$ be a finite, functionally complete set of proposition connectives, e.g. $\{\wedge, \vee, \neg\}$. Then, we define the set of $\mathcal{L}$-formulae $\Phi$ inductively as follows:

- If $\underline{n}, \underline{m} \in V$ are two indices, then $(\underline{n} \in \underline{m})$ is a formula in $\Phi$;
- If $\varphi_{1}, \ldots, \varphi_{n} \in \Phi$ and $\circ \in \mathcal{F}$ is an $n$-ary connective, then $\circ\left(\varphi_{1}, \ldots, \varphi_{n}\right) \in \Phi$;
- If $\varphi \in \Phi$, then both $(\forall \varphi)$ and $(\exists \varphi)$ are formulas in $\Phi$.

Recall that in the De Bruijn notation, we replace named variables with indices denoting the relative distance between the represented variable occurrence and its binding quantifier. For instance, in the De Bruijn notation the empty set axiom $\exists x \forall y(y \notin x)$ becomes $\exists \forall(\underline{0} \notin \underline{1})$. The index $\underline{0}$ denotes a variable bound by the nearest quantifier, i.e. $\forall$. Likewise, $\underline{1}$ denotes a variable bound by the second nearest quantifier, i.e. $\exists$. In general, the index $\underline{n}$ represents a variable occurrence which is bound by the $(n+1)$ st quantifier on the path between the index and the top node of the corresponding expression tree. If no such quantifier exists, the index occurs free.

Remark 1. De Bruijn introduced integer indices to facilitate the automatic manipulation of $\lambda$-terms [4]. From our point of view, adopting his notation to first-order formulae presents a few important advantages. Most notably, each sentence admits one, canonical representative. Consequently, we do not have to concern ourselves with counting formulae up to $\alpha$-equivalence, i.e. up to bound variable names. For instance, formulae $\exists x \forall y(y \notin x)$ and $\exists y \forall z(z \notin y)$ admit the same representation $\exists \forall(\underline{0} \notin \underline{1})$. Further advantages of the De Bruijn notation will become clear once we start a more detailed quantitative analysis of formulae.
2.2. Size model. The set $\Phi$ of formulae we consider can be neatly encapsulated in the following, more symbolic specification:

$$
\begin{equation*}
\Phi_{\infty}:=V \in V\left|\forall \Phi_{\infty}\right| \exists \Phi_{\infty} \mid \bigcup_{(0) \in \mathcal{F}} \circ\left(\boldsymbol{\Phi}_{\infty}\right) \tag{2}
\end{equation*}
$$

Here $V$ denotes the class of De Bruijn indices whereas the boldface notation $\boldsymbol{\Phi}_{\infty}$ denotes vectors of lengths matching the arities of respective connectives. So, for instance, if $\mathcal{F}=\{$ NAND $\}$ then the final alternative in (2) becomes NAND $\left(\Phi_{\infty}, \Phi_{\infty}\right)$.

Given such a general combinatorial specification of formulae, we assume a unary representation of indices. In other words, we represent the index $\underline{\mathrm{n}}$ as an $n$-fold successor of zero, i.e. $S^{(n)} 0$. In doing so, the class $V$ admits a simple, recursive definition:

$$
\begin{equation*}
V:=0 \mid S V \tag{3}
\end{equation*}
$$

Note that combined, (2) and (3) constitute a simple algebraic specification of formulae. To complete the above size model, we assume a so-called natural size model, $c f$. [2], which adheres with the previously mentioned finiteness condition. In this model, each constructor, i.e. successor $S$, zero 0 , membership predicate $(\in)$, quantifiers $\forall, \exists$, and connectives ( $\circ$ ), is assigned weight one. The size of a formula becomes then the total weight of all constructors it is built from. So, for instance, $\exists \forall(\underline{0} \notin \underline{1})$ has size 7 as it consists of seven constructors. Note that $\notin$ is a shorthand for two constructors $(\in)$ and $(\neg)$, and that $\underline{1} \equiv S 0$ consists of two constructors.

Remark 2. We choose the natural size notion for technical simplicity. Let us mention that our analysis extends onto more sophisticated weighing systems, though it is specific to the unary representation of indices and does not immediately apply to other representations, e.g. involving a more compact binary encoding of indices, or counting formulae with named variables up to $\alpha$-equivalence.
2.3. Counting formulae. Given the algebraic specification (2) of formulae, we can easily lift it onto the level of generating functions using so-called symbolic methods, see [5, Part A. Symbolic Methods]. Let $\Phi_{\infty}(z)=\sum_{n \geq 0} a_{n} z^{n}$ be the generating function in which the $n$th coefficient $a_{n}=\left[z^{n}\right] \Phi_{\infty}(z)$ denotes the number of formulae of size $n$.

Then, based on (2) $\Phi_{\infty}(z)$ satisfies the relation

$$
\begin{equation*}
\Phi_{\infty}(z)=z\left(\frac{z}{1-z}\right)^{2}+2 z \Phi_{\infty}(z)+\sum_{(0) \in \mathcal{F}} z \Phi_{\infty}(z)^{\operatorname{ar}(0)} \tag{4}
\end{equation*}
$$

Here $\operatorname{ar}(\cdot)$ denotes the the arity of the respective symbol.
Proposition 3. The generating function $\Phi_{\infty}(z)$ admits a Puiseux expansion in form of

$$
\begin{equation*}
\Phi_{\infty}(z)=a-b \sqrt{1-\frac{z}{\rho}}+O\left(\left|1-\frac{z}{\rho}\right|\right) \tag{5}
\end{equation*}
$$

where $0<\rho<1$, and $a, b>0$.
Proof. We apply the Drmota-Lalley-Woods theorem, see e.g. [5, Theorem VII.6].
Recall that the set $\mathcal{F}$ of connectives is functionally complete and so by Post's theorem it must contain a connective (o) of arity $n \geq 2, c f$. [7]. It means that (4) is non-linear in $\Phi_{\infty}(z)$. By construction, it is also algebraic positive and algebraic irreducible. Algebraic properness, sometimes referred to as well-foundedness, follows for instance from Joyal and Labelle's implicit species theorem, cf. [8]. Indeed, consider the Jacobian $\mathcal{H}\left(z, \Phi_{\infty}\right)$ associated with $\Phi_{\infty}$ :

$$
\begin{equation*}
\mathcal{H}\left(z, \Phi_{\infty}\right)=2 z+\sum_{(\circ) \in \mathcal{F}} z \operatorname{ar}(\circ) \Phi_{\infty}(z)^{\operatorname{ar}(\circ)-1} \tag{6}
\end{equation*}
$$

Note that $\mathcal{H}(0,0)=0$ and so, as a $1 \times 1$ matrix, it is trivially nilpotent. Algebraic aperiodicity is a consequence of the fact that for all sufficiently large $n$ there exists a formula of size $n$ (for instance the sole index $\underline{\mathrm{n}-1})$. Hence $\left[z^{n}\right] \Phi_{\infty}(z) \neq 0$.

The generating function $\Phi_{\infty}(z)$ meets therefore all necessary requirements. We can apply the Drmota-Lalley-Woods theorem and conclude that (4) has a unique dominant singularity $\rho$ and a suitable Puiseux expansion of declared form.

Remark 4. Given the Puiseux expansion of $\Phi_{\infty}(z)$ we can readily use transfer theorems [5, Section VI.3] to obtain an asymptotic estimate for $\left[z^{n}\right] \Phi_{\infty}(z)$ standing for the number of formulae of size $n$ :

$$
\begin{equation*}
\left[z^{n}\right] \Phi_{\infty}(z) \sim C \cdot \rho^{n} n^{-3 / 2} \tag{7}
\end{equation*}
$$

2.4. Counting sentences. Since we are interested in the asymptotic density of theorems, we need to establish asymptotic estimates for the number of sentences, i.e. formulae without free indices. We follow a method similar to $[1,3]$ developed for the purpose of counting closed $\lambda$-terms in the De Bruijn representation.

Let us start with introducing $m$-open formulae. Analogously to $m$-open $\lambda$-terms, we call a formula $\varphi m$-open if prepending $\varphi$ with $m$ head quantifiers, be it universal or existential, turns $\varphi$ into a sentence. For instance, the formula $(\forall \underline{0} \rightarrow(\exists \underline{0} \wedge \underline{2}))$ is 1-open as $\underline{2}$ occurs free, however becomes bound once we introduce a single head quantifier. Note that by such a definition, $m$-open formulae are at the same time $(m+1)$-open. In particular, sentences are just 0 -open (closed) formulae.

By symbolic methods, the definition of $m$-open formulae gives rise to an infinite specification involving all the classes of $m$-open formulae $\left(\Phi_{m}\right)_{m \geq 0}$ :

$$
\begin{align*}
\Phi_{0} & :=\forall \Phi_{1}\left|\exists \Phi_{1}\right| \bigcup_{(0) \in \mathcal{F}} \circ\left(\boldsymbol{\Phi}_{0}\right) \\
\Phi_{1} & :=\forall \Phi_{2}\left|\exists \Phi_{2}\right| \bigcup_{(0) \in \mathcal{F}} \circ\left(\boldsymbol{\Phi}_{1}\right) \mid V_{<1} \in V_{<1} \\
\Phi_{2} & :=\forall \Phi_{3}\left|\exists \Phi_{3}\right| \bigcup_{(\circ) \in \mathcal{F}} \circ\left(\boldsymbol{\Phi}_{2}\right) \mid V_{<2} \in V_{<2}  \tag{8}\\
& \ldots \\
\Phi_{m} & :=\forall \Phi_{m+1}\left|\exists \Phi_{m+1}\right| \bigcup_{(\circ) \in \mathcal{F}} \circ\left(\boldsymbol{\Phi}_{m}\right) \mid V_{<m} \in V_{<m}
\end{align*}
$$

In words, a formula $\varphi$ is $m$-open if it is in form of $(\forall \tau)$ or $(\exists \tau)$ where $\tau$ is $(m+1)$-open, or in form of $\varphi=\circ\left(\tau_{1}, \ldots, \tau_{n}\right)$ where $\tau_{1}, \ldots, \tau_{n}$ are again $m$-open, or finally, if $\varphi$ is in form of ( $\underline{\mathfrak{n}} \in \underline{\mathbf{k}}$ ) where both $\underline{\mathrm{n}}$ and $\underline{\mathrm{k}}$ are two of the $m$ initial indices $V_{<m}=\{\underline{0}, \ldots, \underline{\mathrm{~m}}-\mathbf{1}\}$.

Note that $\Phi_{0} \subsetneq \Phi_{1} \subsetneq \cdots \subsetneq \Phi_{m} \subsetneq \cdots \subsetneq \Phi_{\infty}$. Consecutive classes subsume and extend all the previous ones so $\Phi_{m}$ resembles more and more $\Phi_{\infty}$ as $m \rightarrow \infty$. Let us formalise this intuition and show that the infinite $\left(\Phi_{m}\right)_{m \geq 0}$ is a forward recursive system in the sense of the following definition, cf. [1, Definition 5.5] ${ }^{1}$.
Definition 5 (Forward recursive systems). Let $z$ be a formal variable. Consider the infinite sequences $\left(L^{\langle m\rangle}\right)_{m \geq 0}$ and $\left(\mathcal{K}^{\langle m\rangle}\right)_{m \geq 0}$ of formal power series $L^{\langle m\rangle}(z)$ and $\mathcal{K}^{\langle m\rangle}\left(\ell_{1}, \ell_{2}, z\right)$. Assume that $\left(L^{\langle m\rangle}\right)_{m \geq 0}$ and $\left(\mathcal{K}^{\langle m\rangle}\right)_{m \geq 0}$ satisfy

$$
\begin{equation*}
L^{\langle m\rangle}=\mathcal{K}^{\langle m\rangle}\left(L^{\langle m\rangle}, L^{\langle m+1\rangle}, z\right) . \tag{9}
\end{equation*}
$$

Then, we say that the system (9) is forward recursive.
Furthermore, consider a limiting system in form of

$$
\begin{equation*}
L^{\langle\infty\rangle}=\mathcal{K}^{\langle\infty\rangle}\left(L^{\langle\infty\rangle}, L^{\langle\infty\rangle}, z\right) \tag{10}
\end{equation*}
$$

where $L^{\langle\infty\rangle}(z)$ and $\mathcal{K}^{\langle\infty\rangle}\left(\ell_{1}, \ell_{2}, z\right)$ are formal power series, and moreover $K^{\langle\infty\rangle}$ is analytic at $\left(\ell_{1}, \ell_{2}, z\right)=(0,0,0)$. In this setting, we say that the system (9):
a) is infinitely nested if $\mathcal{K}^{\langle m\rangle}\left(\ell_{1}, \ell_{2}, z\right) \preceq \mathcal{K}^{\langle\infty\rangle}\left(\ell_{1}, \ell_{2}, z\right)$ meaning that for all $n \geq 0$

$$
\left[z^{n}\right] \mathcal{K}^{\langle m\rangle}\left(\ell_{1}, \ell_{2}, z\right) \leq\left[z^{n}\right] \mathcal{K}^{\langle\infty\rangle}\left(\ell_{1}, \ell_{2}, z\right) ;
$$

b) tends to an irreducible context-free schema if it is infinitely nested and its corresponding limiting system (10) satisfies the premises of the Drmota-Lalley-Woods theorem [5, Theorem VII.6], i.e. is a polynomial, non-linear functional equation which is algebraic positive, proper, irreducible and aperiodic;
c) is exponentially converging if there exists a formal power series $A(z)$ and $B(z)$ such that

$$
\begin{equation*}
\mathcal{K}^{\langle\infty\rangle}\left(L^{\langle\infty\rangle}, L^{\langle\infty\rangle}, z\right)-\mathcal{K}^{\langle m\rangle}\left(L^{\langle\infty\rangle}, L^{\langle\infty\rangle}, z\right) \preceq A(z) \cdot B(z)^{m} \tag{11}
\end{equation*}
$$

and both $A(z)$ and $B(z)$ are analytic in the disk $|z|<\rho+\varepsilon$ for some $\varepsilon>0$ where $\rho$ is the dominant singularity of the limit system (10). Moreover, we have $|B(\rho)|<1$.
Proposition 6. The infinite system $\left(\Phi_{m}\right)_{m \geq 0}$ is an infinitely nested, forward recursive system which tends to the irreducible context-free schema $\Phi_{\infty}$ at an exponential convergence rate.

[^1]Proof. Let us unpack the all the definitions one at a time. First, since $\Phi_{m}$ involves $\Phi_{m+1}$ in its specification (8), note that the infinite system $\left(\Phi_{m}\right)_{m \geq 0}$ fits the definition of a forward recursive system assuming $\Phi_{\infty}$ as its limiting system.

Next, let us consider $\Phi_{m}$ and the limiting system $\Phi_{\infty}$. Note that each $m$-open formula is accounted for in $\left[z^{n}\right] \Phi_{m}(z)$ and in $\left[z^{n}\right] \Phi_{\infty}(z)$. Hence $\Phi_{m}(z) \preceq \Phi_{\infty}(z)$ and $\left(\Phi_{m}\right)_{m \geq 0}$ is indeed infinitely nested. By the arguments presented in the proof of Proposition 3 it is at the same time tending to the irreducible context-free schema $\Phi_{\infty}$.

Finally, note that by (8) $\Phi_{m}(z)$ satisfies the equation

$$
\begin{equation*}
\Phi_{m}(z)=z\left(\frac{z\left(1-z^{m}\right)}{1-z}\right)^{2}+2 z \Phi_{m+1}(z)+\sum_{(\circ) \in \mathcal{F}} z \Phi_{m}(z)^{\operatorname{ar}(\circ)} \tag{12}
\end{equation*}
$$

Therefore

$$
\begin{align*}
\Phi_{\infty}\left(\Phi_{\infty}, \Phi_{\infty}, z\right)-\Phi_{m}\left(\Phi_{\infty}, \Phi_{\infty}, z\right) & =z\left(\frac{z}{1-z}\right)^{2}-z\left(\frac{z\left(1-z^{m}\right)}{1-z}\right)^{2} \\
& =\frac{z^{3}}{(1-z)^{2}}\left(2 z^{m}-z^{2 m}\right) \preceq \frac{2 z^{3}}{(1-z)^{2}} z^{m} . \tag{13}
\end{align*}
$$

In this form, it is clear that $\left(\Phi_{m}(z)\right)_{m \geq 0}$ is exponentially converging.
Having established the behaviour of $\left(\Phi_{m}\right)_{m \geq 0}$ we are ready to apply the following general result [1, Theorem 5.9] tailored here so to fit our specific application.
Theorem 7. Let $\mathcal{S}$ be an infinitely nested, forward recursive system which tends to an irreducible context-free schema at an exponential convergence rate. Then, the respective solutions $L^{\langle m\rangle}(z)$ of $\mathcal{S}$ admit for each $m \geq 0$ an asymptotic expansion of their coefficients as $n \rightarrow \infty$ in form of

$$
\begin{equation*}
\left[z^{n}\right] L^{\langle m\rangle}(z) \sim\left[z^{n}\right] \sum_{k \geq 0} c_{k}^{\langle m\rangle}\left(1-\frac{z}{\rho}\right)^{k / 2} \tag{14}
\end{equation*}
$$

where $\rho$ is the dominant singularity of the corresponding limiting system (10).
As for general formulae, a direct application of transfer theorems gives us the following asymptotic estimate for the number of $m$-open formulae.
Proposition 8. The number $\left[z^{n}\right] \Phi_{m}(z)$ of $m$-open formulae of size $n$ satisfies the following asymptotic estimate:

$$
\begin{equation*}
\left[z^{n}\right] \Phi_{m}(z) \sim C_{m} \cdot \rho^{n} n^{-3 / 2} . \tag{15}
\end{equation*}
$$

Note that estimates (15) share the same exponential $\rho^{n}$ and sub-exponential $n^{-3 / 2}$ factors. These are also the same for the number of all formulae, cf. (7). What differentiates them are the respective multiplicative constants $C_{m}$ and $C$.

## 3. Applications

Given the asymptotic estimate for the number of $m$-open formulae, we continue our investigations into the asymptotic expressiveness of ZF and zFC . Let us introduce the following useful concept of formulae templates.

Definition 9 (Formulae templates). A template $\mathcal{C}$ is a formula with a single hole [.] instead of some sub-formula in form of $(\underline{n} \in \underline{m})$. To denote the result of substituting a formula $\varphi$ for [.] in $\mathcal{C}$ we write $\mathcal{C}[\varphi]$. Note that the outcome of such a substitution is always a valid formula.

We call a template $m$-permissive if for each $m$-open formula $\varphi$ the resulting $\mathcal{C}[\varphi]$ is a sentence, i.e. is 0 -open. By analogy to formulae, the size of a template is the total weight of its building constructors, assuming that [.] weights zero.

For instance, consider the template $\mathcal{C}=\exists \forall[\cdot]$ of size two. The result of substituting ( $\underline{0} \notin \underline{1}$ ) into $\mathcal{C}$ is the formula $\mathcal{C}[(\underline{0} \notin \underline{1})]=\exists \forall(\underline{0} \notin \underline{1})$. The hole $[\cdot]$ is proceeded by two quantifiers in
$\mathcal{C}$ so $\mathcal{C}$ is 2-permissive. Note that, in general, if $\mathcal{C}$ is $m$-permissive, then it is also 0-permissive, 1-permissive, etc.
Lemma 10. Let $\mathcal{C}$ be an $m$-permissive template and $\mathcal{L}(\mathcal{C})=\{\mathcal{C}[\varphi]: \varphi$ is $m$-open $\}$. Then, the set $\mathcal{L}(\mathcal{C})$ has positive asymptotic density in the set of all sentences.
Proof. Assume that the template $\mathcal{C}$ has size $d$. It means that the sentence $\mathcal{C}[\varphi]$ is of size $d+|\varphi|$. Let us estimate the number of formulae of this form. By Proposition 8 the number of sentences $\phi \in \mathcal{L}(\mathcal{C})$ of size $n$ satisfies the asymptotic estimate $C_{m} \cdot \rho^{n-d}(n-d)^{-3 / 2}$. Likewise, the number of all sentences of size $n$ is estimated by $C_{0} \cdot \rho^{n} n^{-3 / 2}$. Hence, the asymptotic density of $\mathcal{L}(\mathcal{C})$ in the set of all sentences admits the following estimate:

$$
\begin{equation*}
\mu(\mathcal{L}(\mathcal{C})) \sim \frac{C_{m} \cdot \rho^{n-d}(n-d)^{-3 / 2}}{C_{0} \cdot \rho^{n} n^{-3 / 2}} \sim \frac{C_{m}}{C_{0} \cdot \rho^{d}}>0 . \tag{16}
\end{equation*}
$$

The above lemma allows us to construct asymptotically large (i.e. having positive asymptotic density) sets of sentences whose structure fits the imposed template pattern. We are going to exploit this construction in the following sections.
3.1. Consistency, extensions, and asymptotic expressiveness. To investigate the asymptotic expressiveness of both ZF and ZFC we start with several propositions regarding general, abstract axiomatic set theories and their properties. For convenience, we use $\mu^{-}(\mathcal{T})$ and $\mu^{+}(\mathcal{T})$ to denote $\liminf _{n \rightarrow \infty}$ and $\limsup _{n \rightarrow \infty}$ of $\frac{|\{\varphi:|\varphi|=n \wedge \mathcal{T} \vdash \varphi\}|}{|\{\varphi:|\varphi|=n\}|}$, respectively.
Proposition 11. Let $\mathcal{T}$ be a consistent axiomatic system. Then, the set of $\mathcal{T}$-theorems cannot have a trivial asymptotic density, i.e. $0<\mu^{-}(\mathcal{T})$ and $\mu^{+}(\mathcal{T})<1$.
Proof. Let $\tau$ be an arbitrary tautology. Consider the following templates:

$$
\begin{equation*}
\mathcal{C}=([\cdot] \vee \tau) \quad \text { and } \quad \overline{\mathcal{C}}=([\cdot] \wedge \neg \tau) . \tag{17}
\end{equation*}
$$

Note that $\mathcal{L}(\mathcal{C})=\left\{\mathcal{C}[\varphi]: \varphi \in \Phi_{0}\right\}$ consists of tautologies (in particular, $\mathcal{T}$-theorems). Likewise, $\overline{\mathcal{L}}(\mathcal{C})=\left\{\overline{\mathcal{C}}[\varphi]: \varphi \in \Phi_{0}\right\}$ consists of anti-tautologies. By Lemma 10 both have positive asymptotic density in the set of all sentences. Therefore, $0<\mu^{-}(\mathcal{T})$ and $\mu^{+}(\mathcal{T})<1$.

In other words, consistent theories cannot have a trivial asymptotic expressiveness. In particular, this remark applies to ZF and ZFC (assuming, of course, their consistency). The following slight refinement gives an if and only if condition linking inconsistent theories and trivial asymptotic expressiveness.
Proposition 12. An axiomatic system $\mathcal{T}$ is inconsistent if and only if $\mu(\mathcal{T})=1$.
Proof. Assume that $\mathcal{T}$ is inconsistent. It means that we can derive a contradiction within $\mathcal{T}$. Since ex falso quodilibet, any sentence $\varphi$ is a theorem of $\mathcal{T}$. Trivially, it means that $\mu(\mathcal{T})$ exists and is equal to one. Now, let us assume that $\mathcal{T}$ is consistent. Note that $\mathcal{T}$ cannot have an asymptotic expressiveness one as for consistent theories we can construct asymptotically large sets of anti-tautologies witnessing $\mu^{+}(\mathcal{T})<1, c f$. Proposition 11.

Template formulae allow us to derive also the following result.
Proposition 13. Let $\mathcal{T}$ be a consistent axiomatic system and $\phi$ be an sentence independent of $\mathcal{T}$, i.e. $\mathcal{T} \nvdash \phi$ nor $\mathcal{T} \nvdash \neg \phi$. Then, there exists a set of theorems of the extended $\mathcal{T}+\phi$ which has positive asymptotic density.
Proof. Let $\tau$ be an arbitrary tautology. Consider the context $\mathcal{C}=(\tau \vee[\cdot]) \rightarrow \phi$ and the set $\mathcal{L}(\mathcal{C})=\left\{\mathcal{C}[\varphi]: \varphi \in \Phi_{0}\right\}$ it generates. Note that by Lemma 10 the set $\mathcal{L}(\mathcal{C})$ is asymptotically large. Let us choose an arbitrary sentence $\xi \in \mathcal{L}(\mathcal{C})$. Note that $\mathcal{T}+\phi \vdash \xi$ as we can simply assume $\xi$ 's premise and discharge the axiom $\phi$. Moreover, note that $\mathcal{T} \nvdash \xi$. Indeed, suppose to the contrary that $\mathcal{T} \vdash \xi$. Since $\xi$ 's premise is a tautology it has a proof in $\mathcal{T}$. By modus ponens we conclude that $\mathcal{T} \vdash \phi$ which contradicts the fact that $\phi$ is independent of $\mathcal{T}$.

Remark 14. The axiom of choice is a prominent example of a sentence independent of ZermeloFraenkel's set theory ZF, cf. [6]. Consequently, it is possible to construct asymptotically large sets of ZFC-theorems which cannot be proven in the weaker system ZF.

Note that our argument holds for any consistent axiomatic set theory and independent sentences. Moreover, for sufficiently rich theories Gödel's first incompleteness theorem, see [9], guarantees that such an argumentation can be carried out ad infinitum.
3.2. Unprovable existence of asymptotic expressiveness. Having discussed the properties of general axiomatic theories, let us now concentrate on ZFC. The following result is a simple consequence of Gödel's second incompleteness theorem, cf. [9].

Proposition 15. Let $\mu$ be a predicate definable in ZFC such that $\mathrm{ZFC} \vdash \mu(g)$ if and only if $\mu(\mathrm{ZFC})$ exists and is equal to $g$, and CONSistent be the canonical predicate encoding the consistency of ZFC. Assume that ZFC is consistent. If

$$
\begin{equation*}
\text { ZFC } \vdash \text { CONSISTENT } \longleftrightarrow \neg \mu(1), \tag{18}
\end{equation*}
$$

then ZFC $\forall \neg \exists g: \mu(g)$.
Proof. Suppose that $\neg \exists g: \mu(g)$ can be proven in ZFC , i.e. $\mathrm{ZFC} \vdash \neg \exists g: \mu(g)$. Equivalently, zFC $\vdash \forall g: \neg \mu(g)$. In particular ZFC $\vdash \neg \mu(1)$. By (18) it holds ZFC $\vdash$ COnsistent. Gödel's second incompleteness theorem, as instantiated for ZFC , provides the required contradiction.

Remark 16. Note that the assumption (18) states that Proposition 12 can be formalised in zFC. We can safely assume that this laborious task can be accomplished. It is unclear, however, if the same results hold for the weaker theory ZF. In the proof of Proposition 12 we use quite deep results from analytic combinatorics whose use of the axiom of choice is, as far as we know, undetermined.

## 4. Conclusions

We investigated the asymptotic expressiveness of ZF and ZFC developing a general argument that within ZFC it is possible to prove an asymptotically large set of theorems unprovable in ZF alone. By linking provable consistency of ZFC and its asymptotic expressiveness, we argued that within ZFC it is not possible to disprove the existence of ZFC's asymptotic expressiveness. Moreover, by the same argument, it is not possible to prove the existence of a non-trivial asymptotic expressiveness of ZFC within itself.

Let us note that ZFC $\forall$ CONSISTENT along with the assumption that ZFC is consistent imply that ZFC has a model $\mathcal{M}$ such that $\mathcal{M} \vDash \mu(1)$. This model witnesses the fact that zFC $\forall$ $\neg \exists g: \mu(g)$. Nevertheless, it is not clear whether zFC $\forall \exists g: \mu(g)$ holds, i.e. if there exists a model of ZFC in which theorems of ZFC do not have an asymptotic density. We speculate that establishing a model $\mathcal{M}$ such that $\mathcal{M} \not \forall \exists \mu(g)$ would require a clever mixture of forcing, and analytic methods.

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[^1]:    ${ }^{1}$ The current definition is a simplified version of [1, Definition 5.5]. The original one involves systems of functional equations and permits additional vectors of so-called marking variables which can be used, for instance, to track the behaviour of certain interesting sub-patterns of random structures.

