# Topological semantics of conservativity and interpretability logics 

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#### Abstract

We introduce and develop a topological semantics of conservativity logics and interpretability logics. We prove the topological compactness theorem of consistent normal extensions of the conservativity logic CL by extending Shehtman's ultrabouquet construction method to our framework. As a consequence, we prove that several extensions of CL such as IL, ILM, ILP and ILW are strongly complete with respect to our topological semantics.


## 1 Introduction

The present paper is devoted to solving a natural problem of whether the topological semantics of the propositional modal logic GL can be extended to that of conservativity logics and interpretability logics, which are extensions of GL. We newly introduce a topological semantics of these logics, and investigate several basic properties of our semantics such as the topological strong completeness of them.

The logic GL is known as the logic of provability (cf. Boolos [2]). Let $\operatorname{Pr}_{\mathbf{P A}}(x)$ be a natural provability predicate of Peano Arithmetic PA. Then, the logic $\mathbf{G L}$ is precisely the set of all $\mathbf{P A}$-verifiable modal formulas under all arithmetical interpretations where the modal operator $\square$ is interpreted by $\operatorname{Pr}_{\mathbf{P A}}(x)$. This is called Solovay's arithmetical completeness theorem [18. In his proof, the completeness theorem of $\mathbf{G L}$ with respect to Kripke semantics plays an essential role. Actually, it is well-known that $\mathbf{G L}$ is complete with respect to the class of all transitive and conversely well-founded finite Kripke frames. On the other hand, it is also known that $\mathbf{G L}$ is not strongly complete with respect to Kripke semantics, that is, there exists a set $\Gamma$ of modal formulas such that $\Gamma$ is finitely satisfiable in a transitive and conversely well-founded Kripke model, but $\Gamma$ itself is not satisfiable (See also Boolos [2]).

[^0]This obstacle can be avoided by dealing with topological semantics of modal logics. Topological semantics of modal logic based on derived sets were initiated by McKinsey and Tarski [13. Also topological semantics of GL was founded by Simmons [17] and Esakia [5, and has been developed by many authors (See Beklemishev and Gabelaia [1]). One of important results in this research is the fact that GL is determined by the class of all scattered topological spaces. Moreover, as opposed to Kripke semantics, Shehtman [15] proved that GL is strongly complete with respect to scattered spaces by using so-called the method of ultrabouquet construction.

The language of interpretability logics has the additional binary modal operator $\triangleright$. The modal formula $\varphi \triangleright \psi$ is intended to be read as " $T+\psi$ is interpretable in $T+\varphi^{\prime \prime}$, where $T$ is a suitable theory of arithmetic, such as PA. The logic IL is a basis for the modal logical investigations of the notion of interpretability between theories, and it has been proved that the extensions ILM and ILP of IL are arithmetically complete. Also it is known that the notion of interpretability is closely related to that of partial conservativity. Actually, the logic ILM is exactly the logic of $\Pi_{1}$-conservativity of theories of arithmetic (See Japaridze and de Jongh [10] for a detailed extensive survey of these results). From this point of view, Ignatiev [8] introduced the sublogic CL of IL as a basis for modal logical study of capturing properties of the notion of partial conservativity.

A relational semantics of interpretability logics was introduced by de Jongh and Veltman [3] that is called Veltman semantics. A Veltman frame is a Kripke frame equipped with a family of binary relations. Then, de Jongh and Veltman [3] proved that the logics IL, ILM and ILP are complete with respect to Veltman semantics. Several alternative relational semantics of interpretability logics are also known, and one of important semantics was introduced by Visser [20] that is called simplified Veltman semantics or Visser semantics. By constructing bisimulations between corresponding Visser and Veltman frames, Visser proved that IL, ILM and ILP are also complete with respect to Visser semantics. Moreover, Ignatiev [8] proved that the logic CL is complete with respect to both Veltman and Visser semantics. However, it can be shown that $\mathbf{C L}$ and IL lack strong completeness in both Veltman and Visser semantics, as in GL.

On the other hand, there is a possibility of finding out the strong completeness of these logics with respect to another semantics. Particularly, one with respect to topological semantics is strongly suggested by Shehtman's strong completeness theorem of GL. From this perspective, in the present paper, we propose a topological semantics of $\mathbf{C L}$ and its extensions, and prove the strong completeness theorem of some of these logics by extending Shehtman's method of ultrabouquet construction.

This paper is organized as follows. We briefly summarize Kripke and topological semantics of GL and Visser semantics of $\mathbf{C L}$ and its extensions in the next section. In Section 3, we introduce a new topological semantics of normal extensions of $\mathbf{C L}$, and investigate some basic properties of our semantics. Our topological semantics is based on bitopological spaces with Visser semantics in mind. In Section 4, we extend Shehtman's ultrabouquet construction
to our framework, and then we prove the topological compactness theorem of consistent normal extensions of CL. As a consequence, the topological strong completeness theorem of the logics CL, CLM, IL, ILM, ILP and ILW are obtained. Finally, in Section 5, we discuss topological aspects of the logic IL.

## 2 Preliminaries

The language $\mathcal{L}(\square)$ of propositional modal logic consists of countably many propositional variables $p_{0}, p_{1}, p_{2}, \ldots$, logical constants $\top, \perp$, logical connectives $\neg, \wedge, \vee, \rightarrow$ and unary modal operators $\square, \diamond$. A set $L$ of $\mathcal{L}(\square)$-formulas is said to be a normal modal logic if $L$ contains all tautologies in the language $\mathcal{L}(\square)$ and the formula $\square(p \rightarrow q) \rightarrow(\square p \rightarrow \square q)$, and is closed under Modus Ponens $\frac{\varphi \rightarrow \psi \quad \varphi}{\psi}$, Necessitation $\frac{\varphi}{\square \varphi}$ and Substitution $\frac{\varphi\left(p_{0}, \ldots, p_{n}\right)}{\varphi\left(\psi_{0}, \ldots, \psi_{n}\right)}$. For any normal modal logic $L$, any set $\Gamma$ of $\mathcal{L}(\square)$-formulas and any $\mathcal{L}(\square)$-formula $\varphi$, we write $\Gamma \vdash_{L} \varphi$ to indicate that there exists a finite subset $\Gamma_{0}$ of $\Gamma$ such that $\bigwedge \Gamma_{0} \rightarrow \varphi \in L$.

The logic GL is defined as the smallest normal modal logic containing the additional axiom $\square(\square p \rightarrow p) \rightarrow \square p$.

This section consists of three subsections. In the first subsection, we introduce Kripke semantics of GL. The second subsection is devoted to introducing topological semantics of $\mathbf{G L}$, and reviewing some basic results relating to our study. In the last subsection, we introduce the conservativity logic CL and its extensions, and also introduce their relational semantics, namely, Visser semantics.

### 2.1 Kripke semantics of GL

Definition 2.1 (Kripke frames and models).

- A pair $\langle W, R\rangle$ is said to be a Kripke frame if $W$ is a non-empty set and $R$ is a binary relation on $W$.
- A triple $\langle W, R, \Vdash\rangle$ is said to be a Kripke model if $\langle W, R\rangle$ is a Kripke frame and $\Vdash$ is a binary relation between $W$ and the set of all $\mathcal{L}(\square)$-formulas satisfying the following conditions:

1. $x \nVdash \perp$ and $x \Vdash \top$;
2. $x \Vdash \neg \varphi \Longleftrightarrow x \nVdash \varphi$;
3. $x \Vdash \varphi \wedge \psi \Longleftrightarrow x \Vdash \varphi$ and $x \Vdash \psi$;
4. $x \Vdash \varphi \vee \psi \Longleftrightarrow x \Vdash \varphi$ or $x \Vdash \psi$;
5. $x \Vdash \varphi \rightarrow \psi \Longleftrightarrow x \nVdash \varphi$ or $x \Vdash \psi$;
6. $x \Vdash \square \varphi \Longleftrightarrow \forall y \in W[x R y \Rightarrow y \Vdash \varphi]$;
7. $x \Vdash \diamond \varphi \Longleftrightarrow \exists y \in W[x R y \& y \Vdash \varphi]$.

- An $\mathcal{L}(\square)$-formula $\varphi$ is said to be valid in $\langle W, R\rangle$ if for any Kripke model $\langle W, R, \Vdash\rangle$ and any $x \in W, x \Vdash \varphi$.
- Let $\log (W, R)$ denote the set of all $\mathcal{L}(\square)$-formulas valid in $\langle W, R\rangle$, and this set is called the logic of $\langle W, R\rangle$.

Notice that every $\log (W, R)$ is a normal modal logic. We say that a binary relation $R$ on a set $W$ is conversely well-founded if there is no infinite $R$-increasing sequence of elements of $W$. Then, the validity of the logic GL in a Kripke frame is characterized by a property of the relation $R$.

Fact 2.2 (See Boolos [2, Theorem 10 in Chapter 4]). For any Kripke frame $\langle W, R\rangle$, GL $\subseteq \log (W, R)$ if and only if $R$ is transitive and conversely wellfounded.

We introduce the consequence relation $\models_{L}^{K}$ with respect to Kripke semantics where $K$ stands for "Kripke".

Definition 2.3. Let $L$ be a normal modal logic, $\Gamma$ be a set of $\mathcal{L}(\square)$-formulas and $\varphi$ be an $\mathcal{L}(\square)$-formula.

- $\Gamma \models_{L}^{K} \varphi: \Longleftrightarrow$ for any Kripke model $\langle W, R, \Vdash\rangle$ satisfying $L \subseteq \log (W, R)$ and any $x \in W$, if $x \Vdash \psi$ for all $\psi \in \Gamma$, then $x \Vdash \varphi$.

Clearly, $\Gamma \vdash_{L} \varphi$ implies $\Gamma \models_{L}^{K} \varphi$. For GL, the converse implication also holds in the case of $\Gamma=\varnothing$. This is the Kripke completeness theorem of GL.

Fact 2.4 (Kripke completeness of GL (Segerberg [14])). For any $\mathcal{L}(\square)$-formula $\varphi, \varnothing \vdash_{\mathbf{G L}} \varphi$ if and only if $\varnothing \models_{\text {GL }}^{K} \varphi$.

On the other hand, GL is not strongly complete with respect to Kripke semantics, that is, the equivalence of $\Gamma \vdash_{\mathbf{G L}} \varphi$ and $\Gamma \models_{\mathbf{G L}}^{K} \varphi$ does not hold in general.

Fact 2.5 (Fine and Rautenberg (see Boolos [2, pp. 102-103])). Let

$$
\Delta:=\left\{\diamond p_{0}\right\} \cup\left\{\square\left(p_{n} \rightarrow \diamond p_{n+1}\right) \mid n \in \mathbb{N}\right\},
$$

then $\Delta \models_{\mathrm{GL}}^{K} \perp$ but $\Delta \nVdash_{\mathrm{GL}} \perp$.

### 2.2 Topological semantics of GL

For a non-empty set $X$ and a family $\tau$ of its subsets, we say that $\tau$ is a topology on $X$ if they enjoy the following conditions:

1. $X, \varnothing \in \tau$;
2. If $U_{0}, U_{1} \in \tau$, then $U_{0} \cap U_{1} \in \tau$;
3. For any family $\left\{U_{i}\right\}_{i \in I}$ of sets of $\tau, \bigcup_{i \in I} U_{i} \in \tau$.

Then, the pair $\langle X, \tau\rangle$ is called a topological space. Every $U \in \tau$ containing $x \in X$ is called a $\tau$-neighborhood of $x$.

Definition 2.6 (Derived sets and co-derived sets). Let $\langle X, \tau\rangle$ be a topological space and $Y \subseteq X$.

- The derived set $d_{\tau}(Y)$ of $Y$ (with respect to $\tau$ ) is the subset of $X$ defined as follows:

$$
d_{\tau}(Y):=\{x \in X \mid \forall U \in \tau[x \in U \Rightarrow \exists y \neq x(y \in U \cap Y)]\}
$$

- The co-derived set $c d_{\tau}(Y)$ of $Y$ (with respect to $\tau$ ) is the set $\overline{d_{\tau}(\bar{Y})}$, where $\bar{Y}$ is the complement of $Y$.

In topological semantics of modal logic, every topological space plays a role of a frame, and $\mathcal{L}(\square)$-formulas are interpreted as subsets of the topological space by valuations.

Definition 2.7 (Valuations on topological spaces). Let $\langle X, \tau\rangle$ be a topological space.

- A valuation on $\langle X, \tau\rangle$ is a mapping $v: \mathcal{L}(\square) \rightarrow \mathcal{P}(X)$ satisfying the following conditions:

1. $v(\perp)=\varnothing$ and $v(\top)=X$;
2. $v(\neg \varphi)=\overline{v(\varphi)}$;
3. $v(\varphi \wedge \psi)=v(\varphi) \cap v(\psi)$;
4. $v(\varphi \vee \psi)=v(\varphi) \cup v(\psi)$;
5. $v(\varphi \rightarrow \psi)=\overline{v(\varphi)} \cup v(\psi)$;
6. $v(\square \varphi)=c d_{\tau}(v(\varphi))$;
7. $v(\diamond \varphi)=d_{\tau}(v(\varphi))$.

- We say that an $\mathcal{L}(\square)$-formula $\varphi$ is valid in $\langle X, \tau\rangle$ if $v(\varphi)=X$ for all valuations $v$ on $\langle X, \tau\rangle$.
- Let $\log (X, \tau)$ be the set of all $\mathcal{L}(\square)$-formulas valid in $\langle X, \tau\rangle$, and we call this set the logic of $\langle X, \tau\rangle$.

It is known that every $\log (X, \tau)$ is a normal modal logic validating $p \wedge \square p \rightarrow$ $\square \square p$ (See Esakia [6] and van Benthem and Bezhanishvili [19). As well as Fact 2.2 the validity of the logic $\mathbf{G L}$ in a topological space $\langle X, \tau\rangle$ is characterized by a property of $\tau$.

Definition 2.8 (Scattered spaces). A topological space $\langle X, \tau\rangle$ is said to be scattered if for any $Y \subseteq X, Y \neq \varnothing$ implies $Y \backslash d_{\tau}(Y) \neq \varnothing$.

Fact 2.9 (Simmons [17; Esakia [5]). For any topological space $\langle X, \tau\rangle$, GL $\subseteq$ $\log (X, \tau)$ if and only if $\langle X, \tau\rangle$ is scattered.

The following fact is a summary of basic properties of derived sets.
Fact 2.10. Let $\langle X, \tau\rangle$ be a topological space and let $Y, Z \subseteq X$.

1. $d_{\tau}(\varnothing)=\varnothing$;
2. If $Y \subseteq Z$, then $d_{\tau}(Y) \subseteq d_{\tau}(Z)$;
3. $d_{\tau}(Y \cup Z)=d_{\tau}(Y) \cup d_{\tau}(Z)$;
4. $Y \in \tau \Longleftrightarrow d_{\tau}(\bar{Y}) \cap Y=\varnothing$;
5. If $\langle X, \tau\rangle$ is scattered, then $d_{\tau}\left(d_{\tau}(Y)\right) \subseteq d_{\tau}(Y)$ (cf. [1, Corollary 2.3]).

Each transitive and irreflexive Kripke frame can be considered as a topological space having the same logic via the topology of $R$-upward closed subsets.

Definition 2.11. Let $\langle W, R\rangle$ be a Kripke frame.

- For each $x \in W, R(x):=\{y \in W \mid x R y\} ;$
- A subset $Y \subseteq W$ is said to be $R$-upward closed if for any $x \in Y, R(x) \subseteq Y$;
- Define $\tau_{R}:=\{Y \subseteq W \mid Y$ is $R$-upward closed $\}$.

Definition 2.12 (Alexandroff spaces). A topological space $\langle X, \tau\rangle$ is said to be Alexandroff if for any family $\left\{U_{i}\right\}_{i \in I}$ of members of $\tau, \bigcap_{i \in I} U_{i} \in \tau$.

Fact 2.13 (cf. van Benthem and Bezhanishvili [19). Let $\langle W, R\rangle$ be a Kripke frame. Then,

1. $\left\langle W, \tau_{R}\right\rangle$ is an Alexandroff topological space;
2. If $R$ is transitive and irreflexive, then for any $Y \subseteq W, d_{\tau_{R}}(Y)=\{x \in$ $W \mid R(x) \cap Y \neq \varnothing\}$;
3. If $R$ is transitive and irreflexive, then $\log (W, R)=\log \left(W, \tau_{R}\right)$.

Alexandroff spaces will be studied precisely in Sections 3 and 5
As in the case of Kripke semantics, we introduce the consequence relation $\models_{L}^{T}$ with respect to topological semantics where $T$ stands for "Topology".

Definition 2.14. Let $L$ be a normal modal logic, $\Gamma$ be a set of $\mathcal{L}(\square)$-formulas and $\varphi$ be an $\mathcal{L}(\square)$-formula.

- $\Gamma \neq_{L}^{T} \varphi: \Longleftrightarrow$ for any topological space $\langle X, \tau\rangle$ satisfying $L \subseteq \log (X, \tau)$, any valuation $v$ on $X$ and any $x \in X$, if $x \in v(\psi)$ for all $\psi \in \Gamma$, then $x \in v(\varphi)$.

From Facts 2.4 and 2.13, we obtain the topological completeness of GL.

Fact 2.15 (Topological completeness of GL). For any $\mathcal{L}(\square)$-formula $\varphi, \varnothing \vdash_{\mathbf{G L}}$ $\varphi$ if and only if $\varnothing \models_{\mathbf{G L}}^{T} \varphi$.

Moreover, as opposed to Fact 2.5. Shehtman proved that GL is strongly complete with respect to topological semantics ${ }^{1}$

Fact 2.16 (Topological strong completeness of GL (Shehtman [15, Theorem 3.3])). Let $\Gamma$ be any set of $\mathcal{L}(\square)$-formulas and $\varphi$ be any $\mathcal{L}(\square)$-formula. Then, $\Gamma \vdash_{\mathbf{G L}} \varphi$ if and only if $\Gamma \models_{\mathbf{G L}}^{T} \varphi$.

### 2.3 Conservativity and interpretability logics and their Visser semantics

In this section, we introduce the conservativity logic $\mathbf{C L}$ and its extensions. Also we introduce their relational semantics. The language $\mathcal{L}(\square, \triangleright)$ is obtained from $\mathcal{L}(\square)$ by adding the binary modal operator $\triangleright$.

Definition 2.17 (The conservativity logic CL). The conservativity logic CL is a logic in the language $\mathcal{L}(\square, \triangleright)$ obtained from $\mathbf{G L}$ by adding the following axioms:

J1 $\square(p \rightarrow q) \rightarrow(p \triangleright q) ;$
J2 $(p \triangleright q) \wedge(q \triangleright r) \rightarrow(p \triangleright r) ;$
J3 $(p \triangleright r) \wedge(q \triangleright r) \rightarrow((p \vee q) \triangleright r) ;$
$\mathbf{J 4}(p \triangleright q) \rightarrow(\diamond p \rightarrow \diamond q)$.
We say that a set $L$ of $\mathcal{L}(\square, \triangleright)$-formulas is a normal extension of $\mathbf{C L}$ if $\mathbf{C L} \subseteq L$ and $L$ is closed under Modus Ponens, Necessitation and Substitution. There are well-known normal extensions of CL having some of the following additional axioms:

J5 $\diamond p \triangleright p$;
$\mathbf{M}(p \triangleright q) \rightarrow((p \wedge \square r) \triangleright(q \wedge \square r)) ;$
$\mathbf{P}(p \triangleright q) \rightarrow \square(p \triangleright q) ;$
$\mathbf{W}(p \triangleright q) \rightarrow(p \triangleright(q \wedge \square \neg p))$.
The smallest normal extension containing $\mathbf{M}$ is called CLM. In this case, we write $\mathbf{C L M}=\mathbf{C L}+\mathbf{M}$. The logics $\mathbf{C L}$ and $\mathbf{C L M}$ were introduced by Ignatiev [8]. Also let $\mathbf{I L}=\mathbf{C L}+\mathbf{J 5}, \mathbf{I L M}=\mathbf{I L}+\mathbf{M}, \mathbf{I L P}=\mathbf{I L}+\mathbf{P}$ and $\mathbf{I L W}=\mathbf{I L}+\mathbf{W}$. The logic $\mathbf{I L}$ is called the basic interpretability logic.

[^1]One of well-known relational semantics of $\mathbf{C L}$ and its extensions is Veltman semantics which was introduced by de Jongh and Veltman [3. A triple $\left\langle W, R,\left\{S_{w}\right\}_{w \in W}\right\rangle$ is called a Veltman frame if $\langle W, R\rangle$ is a transitive and conversely well-founded Kripke frame and for each $w \in W, S_{w}$ is a binary relation on $R(w)$ satisfying some additional conditions. One of the purposes of the present paper is to find an appropriate topological semantics of extensions of CL. From the point of view of Fact 2.13, every binary relation $P$ on a set $W$ is associated to the topology $\tau_{P}$ on $W$ consisting of $P$-upward closed subsets. However, each binary relation $S_{w}$ of Veltman frames is not a binary relation on full $W$, and so Veltman frames are not directly recognized as topological frames.

For this reason, we adopt the alternative relational semantics of extensions of CL introduced by Visser [20.

Definition 2.18 (Visser frames and models).

- A triple $\langle W, R, S\rangle$ is said to be a Visser frame if $\langle W, R\rangle$ is a transitive and conversely well-founded Kripke frame and $S$ is a binary transitive and reflexive relation on $W$;
- A quadruple $\langle W, R, S, \Vdash\rangle$ is said to be a Visser model if $\langle W, R, S\rangle$ is a Visser frame and $\Vdash$ is a binary relation as in Definition 2.1 with the following additional clause:

$$
\begin{aligned}
& -x \Vdash \varphi \triangleright \psi \Longleftrightarrow \forall y \in W[x R y \& y \Vdash \varphi \Rightarrow \exists z \in W(x R z \& y S z \& z \Vdash \\
& \psi)] .
\end{aligned}
$$

- The validity of an $\mathcal{L}(\square, \triangleright)$-formula in Visser frames and models, and the $\operatorname{logic} \log (W, R, S)$ of $\langle W, R, S\rangle$ are defined as in Definition 2.1.

Visser actually introduced the notion of Visser frames as a relational semantics for extensions of $\mathbf{I L}$, and Definition 2.18 is an adaptation of Visser's definition to our framework obtained by removing the condition $R \subseteq S$ from his original definition. Visser frames are also known as simplified Veltman frames. Then, the following fact holds.

Fact 2.19 (See Ignatiev [8] and Visser [20]). Let $\langle W, R, S\rangle$ be any Visser frame. Then,

1. $\log (W, R, S)$ is a normal extension of $\mathbf{C L}$;
2. If $\forall x, y, z \in W[x S y R z \Rightarrow x R z]$, then $\mathbf{C L M} \subseteq \log (W, R, S)$;
3. If $R \subseteq S$, then $\mathbf{I L} \subseteq \log (W, R, S)$;
4. If $R \subseteq S$ and $\forall x, y, z \in W[x R y S z \Rightarrow x R z]$, then $\mathbf{I L P} \subseteq \log (W, R, S)$;
5. If $R \subseteq S$ and the composition $R \circ S$ is conversely well-founded, then $\mathbf{I L W} \subseteq \log (W, R, S)$.

In Section 5, we will investigate the condition $R \subseteq S$ of Visser frames from a topological viewpoint.

We also define the consequence relation $\models_{L}^{V}$ with respect to Visser semantics.
Definition 2.20. Let $L$ be a normal extension of $\mathbf{C L}, \Gamma$ be a set of $\mathcal{L}(\square, \triangleright)$ formulas and $\varphi$ be an $\mathcal{L}(\square, \triangleright)$-formula.

- $\Gamma \quad=_{L}^{V} \varphi: \Longleftrightarrow$ for any Visser model $\langle W, R, S, \Vdash\rangle$ satisfying $L \subseteq \log (W, R, S)$ and any $x \in W$, if $x \Vdash \psi$ for all $\psi \in \Gamma$, then $x \Vdash \varphi$.

Clearly, $\Gamma \vdash_{L} \varphi$ implies $\Gamma \models{ }_{L}^{V} \varphi$. The completeness theorems of CL, CLM, IL, ILP, ILM and ILW with respect to Visser semantics are proved by Ignatiev, de Jongh and Veltman and Visser.

Fact 2.21 (Visser completeness of CL and CLM (Ignatiev [8])). Let $L \in$ $\{\mathbf{C L}, \mathbf{C L M}\}$. For any $\mathcal{L}(\square, \triangleright)$-formula $\varphi, \varnothing \vdash_{L} \varphi$ if and only if $\varnothing \models_{L}^{V} \varphi$.

Fact 2.22 (Visser completeness of IL, ILM, ILP and ILW (de Jongh and Veltman [3, 4] and Visser [20])). Let $L \in\{\mathbf{I L}, \mathbf{I L M}, \mathbf{I L P}, \mathbf{I L W}\}$. For any $\mathcal{L}(\square, \triangleright)$ formula $\varphi, \varnothing \vdash_{L} \varphi$ if and only if $\varnothing \vDash{ }_{L}^{V} \varphi$.

However, every logic $L \in\{\mathbf{C L}, \mathbf{C L M}, \mathbf{I L}, \mathbf{I L M}, \mathbf{I L P}, \mathbf{I L W}\}$ lacks strong completeness with respect to Visser semantics as in the case of GL. That is, $\Delta \models_{L}^{V} \perp$ but $\Delta \vdash_{L} \perp$ where $\Delta$ is the set of formulas defined in Fact 2.5

## 3 Topological semantics of normal extensions of CL

In this section, we newly introduce a topological semantics of normal extensions of CL. Our topological semantics is based on bitopological spaces.
Definition 3.1 (Bitopological spaces). Let $X$ be a non-empty set and $\tau^{0}, \tau^{1}$ be families of subsets of $X$. A triple $\left\langle X, \tau^{0}, \tau^{1}\right\rangle$ is called a bitopological space if both $\tau^{0}$ and $\tau^{1}$ are topologies on $X$.

The following definition is an essential part of our work.
Definition 3.2. Let $\left\langle X, \tau^{0}, \tau^{1}\right\rangle$ be a bitopological space. For subsets $Y$ and $Z$ of $X$, we define a subset $e_{\tau^{0}, \tau^{1}}(Y, Z)$ of $X$ as follows:

$$
e_{\tau^{0}, \tau^{1}}(Y, Z):=\left\{x \in X \mid \forall U \in \tau^{1}\left[x \in d_{\tau^{0}}(Y \cap U) \Rightarrow x \in d_{\tau^{0}}(Z \cap U)\right]\right\}
$$

If there is no room for confusion, we simply write $e(Y, Z)$ instead of $e_{\tau^{0}, \tau^{1}}(Y, Z)$. Using our sets $e_{\tau^{0}, \tau^{1}}(Y, Z)$, we define valuations on bitopological spaces.

Definition 3.3. Let $\left\langle X, \tau^{0}, \tau^{1}\right\rangle$ be a bitopological space. A valuation on $\left\langle X, \tau^{0}, \tau^{1}\right\rangle$ is a mapping $v: \mathcal{L}(\square, \triangleright) \rightarrow \mathcal{P}(X)$ defined as in Definition 2.7 with the following clauses:

- $v(\square \varphi)=c d_{\tau^{0}}(v(\varphi))$;
- $v(\diamond \varphi)=d_{\tau^{0}}(v(\varphi))$;
- $v(\varphi \triangleright \psi)=e_{\tau^{0}, \tau^{1}}(v(\varphi), v(\psi))$.

The validity of an $\mathcal{L}(\square, \triangleright)$-formula in a bitopological space and the logic $\log \left(X, \tau^{0}, \tau^{1}\right)$ of $\left\langle X, \tau^{0}, \tau^{1}\right\rangle$ are also defined as in Definition 2.7

For a normal extension $L$ of $\mathbf{C L}$, we say that a bitopological space $\left\langle X, \tau^{0}, \tau^{1}\right\rangle$ is an $L$-space if $L \subseteq \log \left(X, \tau^{0}, \tau^{1}\right)$. We prove that every $\tau^{0}$-scattered bitopological space is a CL-space.

Proposition 3.4. All axioms J1, J2, J3 and J4 in Definition 2.17 are valid in any bitopological space $\left\langle X, \tau^{0}, \tau^{1}\right\rangle$.

Proof. (J1): It suffices to show that for any $Y, Z \subseteq X, c d_{\tau^{0}}(\bar{Y} \cup Z) \subseteq e(Y, Z)$. Suppose $x \in c d_{\tau^{0}}(\bar{Y} \cup Z)$, that is, $x \notin d_{\tau^{0}}(Y \cap \bar{Z})$. Then there exists a $\tau^{0}$ neighborhood $W$ of $x$ such that $Y \cap \bar{Z} \cap W \subseteq\{x\}$.

Take $U \in \tau^{1}$ arbitrarily, and suppose $x \in d_{\tau^{0}}(Y \cap U)$. We would like to show $x \in d_{\tau^{0}}(Z \cap U)$. Let $V$ be any $\tau^{0}$-neighborhood of $x$. Then $V \cap W$ is also a $\tau^{0}$-neighborhood of $x$. Since $x \in d_{\tau^{0}}(Y \cap U)$, there exists $y \neq x$ such that $y \in Y \cap U \cap V \cap W$, and hence $y \in Y \cap W$. On the other hand, since $Y \cap \bar{Z} \cap W \subseteq\{x\}$, we have $y \notin Y \cap \bar{Z} \cap W$. Therefore $y \in Z$, and hence $y \in Z \cap U \cap V$. This implies $x \in d_{\tau^{0}}(Z \cap U)$. We have shown $x \in e(Y, Z)$.
$(\mathbf{J} 2):$ We show $e(Y, Z) \cap e(Z, W) \subseteq e(Y, W)$. Suppose $x \in e(Y, Z) \cap e(Z, W)$. Take $U \in \tau^{1}$ arbitrarily. If $x \in d_{\tau^{0}}(Y \cap U)$, then $x \in d_{\tau^{0}}(Z \cap U)$ by $x \in e(Y, Z)$. Moreover, $x \in d_{\tau^{0}}(W \cap U)$ by $x \in e(Z, W)$. Thus $x \in e(Y, W)$.
(J3): We show $e(Y, W) \cap e(Z, W) \subseteq e(Y \cup Z, W)$. Suppose $x \in e(Y, W) \cap$ $e(Z, W)$. Take $U \in \tau^{1}$ arbitrarily, and assume $x \in d_{\tau^{0}}((Y \cup Z) \cap U)$. By Fact 2.10, we have

$$
d_{\tau^{0}}((Y \cup Z) \cap U)=d_{\tau^{0}}((Y \cap U) \cup(Z \cap U))=d_{\tau^{0}}(Y \cap U) \cup d_{\tau^{0}}(Z \cap U)
$$

Then $x \in d_{\tau^{0}}(Y \cap U)$ or $x \in d_{\tau^{0}}(Z \cap U)$. In either case, we obtain $x \in d_{\tau^{0}}(W \cap U)$ by $x \in e(Y, W) \cap e(Z, W)$. Thus $x \in e(Y \cup Z, W)$.
(J4): We show $e(Y, Z) \cap d_{\tau^{0}}(Y) \subseteq d_{\tau^{0}}(Z)$. Suppose $x \in e(Y, Z) \cap d_{\tau^{0}}(Y)$. Then $x \in d_{\tau^{0}}(Y \cap X)$. Since $X \in \tau^{1}$, it follows from $x \in e(Y, Z)$ that $x \in$ $d_{\tau^{0}}(Z \cap X)$. Equivalently, $x \in d_{\tau^{0}}(Z)$.

Since each inference rule of CL preserves validity in bitopological spaces, we obtain the following corollary from Fact 2.9 and Proposition 3.4

Corollary 3.5. For any bitopological space $\left\langle X, \tau^{0}, \tau^{1}\right\rangle$, it is a CL-space if and only if $\left\langle X, \tau^{0}\right\rangle$ is scattered.

As well as Kripke frames, Visser fames $\langle W, R, S\rangle$ can be considered as bitopological spaces by considering topologies $\tau_{R}$ and $\tau_{S}$ (see Definition 2.11). In truth, our new operation $e_{\tau^{0}, \tau^{1}}$ is defined with the intention of satisfying the following proposition.

Proposition 3.6. Let $\langle W, R, S, \Vdash\rangle\rangle$ be a Visser model. Let $v$ be a valuation on $\left\langle W, \tau_{R}, \tau_{S}\right\rangle$ satisfying $v(p)=\{x \in W \mid x \Vdash p\}$ for any propositional variable $p$, then $v(\varphi)=\{x \in W \mid x \Vdash \varphi\}$ for any $\mathcal{L}(\square, \triangleright)$-formula $\varphi$.

Proof. We prove by induction on the construction of $\varphi$. We provide proofs of only two cases that $\varphi$ is $\diamond \psi$ and $\varphi$ is $\psi \triangleright \chi$.

Case of $\varphi \equiv \diamond \psi$ :

$$
\begin{array}{rlr}
x \Vdash \diamond \psi & \Longleftrightarrow \exists y \in W(x R y \& y \Vdash \psi), \\
& \Longleftrightarrow R(x) \cap v(\psi) \neq \varnothing, & \text { (by induction hypothesis) } \\
& \Longleftrightarrow x \in d_{\tau_{R}}(v(\psi)), \\
& \Longleftrightarrow x \in v(\diamond \psi) .
\end{array}
$$

Case of $\varphi \equiv \psi \triangleright \chi$ :

$$
\begin{aligned}
x \Vdash \psi \triangleright \chi & \Longleftrightarrow \forall y[x R y \& y \Vdash \psi \Rightarrow \exists z(x R z \& y S z \& z \Vdash \chi)], \\
& \Longleftrightarrow \forall y[y \in R(x) \cap v(\psi) \Rightarrow R(x) \cap S(y) \cap v(\chi) \neq \varnothing],
\end{aligned}
$$

(by induction hypothesis)
$\stackrel{(*)}{\Longleftrightarrow} \forall U \in \tau_{S}[R(x) \cap v(\psi) \cap U \neq \varnothing \Rightarrow R(x) \cap U \cap v(\chi) \neq \varnothing]$, $\Longleftrightarrow \forall U \in \tau_{S}\left[x \in d_{\tau_{R}}(v(\psi) \cap U) \Rightarrow x \in d_{\tau_{R}}(U \cap v(\chi))\right]$,
(by Fact 2.13 2)
$\Longleftrightarrow x \in e_{\tau_{R}, \tau_{S}}(v(\psi), v(\chi))$,
$\Longleftrightarrow x \in v(\psi \triangleright \chi)$.
Here we give a proof of the equivalence marked by $(*)$.
$(\Rightarrow)$ : Let $U$ be any element of $\tau_{S}$ with $R(x) \cap v(\psi) \cap U \neq \varnothing$. Let $y \in$ $R(x) \cap v(\psi) \cap U$. Then, $R(x) \cap S(y) \cap v(\chi)$ is non-empty. Since $U$ is $S$-upward closed, $S(y) \subseteq U$. Thus $R(x) \cap U \cap v(\chi)$ is also non-empty.
$(\Leftarrow)$ : Let $y$ be any element of $R(x) \cap v(\psi)$. Since $S$ is reflexive, $y \in S(y)$, and hence $y \in R(x) \cap v(\psi) \cap S(y)$. It follows from the transitivity of $S$ that $S(y)$ is $S$-upward closed. Hence $S(y) \in \tau_{S}$. Then, we obtain that $R(x) \cap S(y) \cap v(\chi)$ is non-empty.

From Proposition 3.6. we obtain the following corollary.
Corollary 3.7. For any Visser frame $\langle W, R, S\rangle, \log (W, R, S)=\log \left(W, \tau_{R}, \tau_{S}\right)$.

Since every transitive and conversely well-founded Kripke frame can be extended to a Visser frame, Corollary 3.7 is an extension of Fact 2.13 .3. Conversely, we show that $\tau^{0}$-scattered Alexandroff bitopological spaces can be considered as Visser frames.

Theorem 3.8. Let $\left\langle X, \tau^{0}, \tau^{1}\right\rangle$ be any bitopological space. Then, the following are equivalent:

1. $\tau^{0}$ is scattered and both $\tau^{0}$ and $\tau^{1}$ are Alexandroff.
2. There exists a Visser frame $\langle X, R, S\rangle$ such that $\tau^{0}=\tau_{R}$ and $\tau^{1}=\tau_{S}$.

Proof. $(\Rightarrow)$ : We define binary relations $R$ and $S$ on $X$ as follows:

- $x R y: \Longleftrightarrow x \neq y \& \forall U \in \tau^{0}(x \in U \Rightarrow y \in U)$
$\left(\Longleftrightarrow x \in d_{\tau^{0}}(\{y\})\right)$;
- $x S y: \Longleftrightarrow \forall U \in \tau^{1}(x \in U \Rightarrow y \in U)$.

Clearly, $R$ is irreflexive and $S$ is transitive and reflexive. We show that $R$ is transitive. Let $x R y$ and $y R z$. Then $x \in d_{\tau^{0}}(\{y\})$ and $y \in d_{\tau^{0}}(\{z\})$. By Fact 2.10. $2, d_{\tau^{0}}(\{y\}) \subseteq d_{\tau^{0}}\left(d_{\tau^{0}}(\{z\})\right)$. Since $\tau^{0}$ is scattered, $d_{\tau^{0}}\left(d_{\tau^{0}}(\{z\})\right) \subseteq$ $d_{\tau^{0}}(\{z\})$ by Fact 2.10.5. Thus $d_{\tau^{0}}(\{y\}) \subseteq d_{\tau^{0}}(\{z\})$. Then, $x \in d_{\tau^{0}}(\{z\})$ and hence $x R z$.

We prove $\tau^{0}=\tau_{R}$, and the proof of $\tau^{1}=\tau_{S}$ is similar.
$(\subseteq)$ : Let $U \in \tau^{0}$. If $x \in U$ and $x R y$, then $y \in U$ by the definition of $R$. This means that $U$ is $R$-upward closed. Thus $U \in \tau_{R}$.
$(\supseteq):$ Let $U \in \tau_{R}$ and $x$ be an arbitrary element of $U$. Define $V^{\prime}:=\bigcap\{V \in$ $\left.\tau^{0} \mid x \in V\right\}$. Since $\tau^{0}$ is Alexandroff, $V^{\prime}$ is a $\tau^{0}$-neighborhood of $x$. Since $V^{\prime}$ is a subset of every $\tau^{0}$-neighborhood of $x$, for any $y \in V^{\prime}$, either $x=y$ or $x R y$. Since $U$ is $R$-upward closed, $U$ contains such $y$. Thus $V^{\prime} \subseteq U$. We have shown that an arbitrary element of $U$ has a $\tau^{0}$-neighborhood inside of $U$. Thus $U \in \tau^{0}$.

Since $\left\langle X, \tau^{0}\right\rangle$ is scattered, by Fact $2.9, \mathbf{G L} \subseteq \log \left(X, \tau^{0}\right)$. By Fact 2.133 , $\log (X, R)=\log \left(X, \tau_{R}\right)=\log \left(X, \tau^{0}\right)$. Then $\mathbf{G L} \subseteq \log (X, R)$, and thus $R$ is conversely well-founded by Fact 2.2 . Therefore $\langle W, R, S\rangle$ is a Visser frame.
$(\Leftarrow)$ : By Fact 2.131 , both $\tau^{\sigma}=\tau_{R}$ and $\tau^{1}=\tau_{S}$ are Alexandroff. Since $R$ is transitive and conversely well-founded, $\mathbf{G L} \subseteq \log (W, R)=\log \left(W, \tau_{R}\right)$ by Facts 2.2 and 2.13 3. Then it follows from Fact 2.9 that $\tau^{0}=\tau_{R}$ is scattered.

To summarize the previous investigations, Visser semantics is exactly a topological semantics restricted to $\tau^{0}$-scattered Alexandroff bitopological spaces. Some extensions of $\mathbf{C L}$ such as $\mathbf{I L}$ are complete but not strongly complete with respect to this restricted version of topological semantics.

As in the previous section, we introduce the consequence relation $\models_{L}^{T}$ with respect to our topological semantics.

Definition 3.9. Let $L$ be a normal extension of $\mathbf{C L}, \Gamma$ be a set of $\mathcal{L}(\square, \triangleright)$ formulas, and $\varphi$ be an $\mathcal{L}(\square, \triangleright)$-formula.

- $\Gamma \models_{L}^{T} \varphi: \Longleftrightarrow$ for any $L$-space $\left\langle X, \tau^{0}, \tau^{1}\right\rangle$, any valuation $v$ on $\left\langle X, \tau^{0}, \tau^{1}\right\rangle$ and any $x \in X$, if $x \in v(\psi)$ for all $\psi \in \Gamma$, then $x \in v(\varphi)$;
- We say that $L$ is topologically complete if for any $\mathcal{L}(\square, \triangleright)$-formula $\varphi, \varnothing \models_{L}^{T}$ $\varphi$ implies $\varnothing \vdash_{L} \varphi$;
- We say that $L$ is topologically strongly complete if for any $\mathcal{L}(\square, \triangleright)$-formula $\varphi$ and set $\Gamma$ of $\mathcal{L}(\square, \triangleright)$-formulas, $\Gamma \models{ }_{L}^{T} \varphi$ implies $\Gamma \vdash_{L} \varphi$.
From Facts 2.21 and 2.22, and the above discussions, we obtain the following topological completeness of $\mathbf{C L}$ and its some extensions.

Theorem 3.10 (Topological completeness of some extensions of CL). The logics CL, CLM, IL, ILM, ILP and ILW are topologically complete.

The main purpose of the present paper is to strengthen Theorem 3.10, that is, we prove that these logics are topologically strongly complete.

## 4 Topological compactness and topological strong completeness

In this section, we prove the topological strong completeness theorem of some extensions of CL. This directly follows from the the topological compactness theorem (Theorem 4.13) and the topological completeness theorem (Theorem 3.10). Thus the main purpose of this section is to prove the topological compactness theorem. We prove this theorem by extending the method of Shehtman's ultrabouquet construction for topological spaces (cf. Shehtman [15, 16]) to our framework.

### 4.1 The ultrabouquet construction for bitopological spaces

We introduce the notion of the ultrabouquet of a countable family $\left\{\left\langle X_{n}, \tau_{n}^{0}, \tau_{n}^{1}\right\rangle\right\}_{n \in \mathbb{N}}$ of bitopological spaces, and investigate properties of ultrabouquets used in our proof of the topological compactness theorem. Before introducing it, we recall the following fact.

Fact 4.1 (cf. Shehtman [16, Lemma 61]). Let $\langle X, \tau\rangle$ be a scattered space. Then for any $x \in X$, there exists $Y \subseteq X$ such that $Y$ is a $\tau$-neighborhood of $x$ and $Y \backslash\{x\} \in \tau$.

In this subsection, we fix a countable family $\left\{\left\langle X_{n}, \tau_{n}^{0}, \tau_{n}^{1}\right\rangle\right\}_{n \in \mathbb{N}}$ of bitopological spaces satisfying the following conditions:

- All topological spaces $\left\langle X_{n}, \tau_{n}^{0}\right\rangle$ are scattered;
- The family $\left\{X_{n}\right\}_{n \in \mathbb{N}}$ is pairwise disjoint.

We also fix a family $\left\{x_{n}\right\}_{n \in \mathbb{N}}$ of elements such that $x_{n} \in X_{n}$ for every $n \in \mathbb{N}$. Then by Fact 4.1, for each $n \in \mathbb{N}$, there exists $Y_{n} \subseteq X_{n}$ such that $Y_{n}$ is $\tau_{n}^{0}$ neighborhood of $x_{n}$ and $Y_{n} \backslash\left\{x_{n}\right\} \in \tau_{n}^{0}$. Let $\mathcal{U}$ be a non-principal ultrafilter on $\mathbb{N}$. Let $x_{*}$ be a new element not contained in $\bigcup_{n \in \mathbb{N}} X_{n}$.

Definition 4.2. We define an ultrabouquet $\mathfrak{X}:=\left\langle X, \tau^{0}, \tau^{1}\right\rangle$ based on the families $\left\{\left\langle X_{n}, \tau_{n}^{0}, \tau_{n}^{1}\right\rangle\right\}_{n \in \mathbb{N}}$ and $\left\{x_{n}\right\}_{n \in \mathbb{N}}$ as follows:

- $X:=\bigcup_{n \in \mathbb{N}}\left(X_{n} \backslash\left\{x_{n}\right\}\right) \cup\left\{x_{*}\right\}$.

For each $V \subseteq X$ and $n \in \mathbb{N}$, we sometimes restrict $V$ to $X_{n}$ or $Y_{n}$. In these situations, we would like to identify $x_{*}$ with $x_{n}$. From this perspective,
we let:

$$
V \upharpoonright X_{n}:= \begin{cases}V \cap X_{n} & \text { if } x_{*} \notin V \\ \left(\left(V \backslash\left\{x_{*}\right\}\right) \cup\left\{x_{n}\right\}\right) \cap X_{n} & \text { if } x_{*} \in V\end{cases}
$$

Also $V \upharpoonright Y_{n}$ is defined in a similar way.

- $U \in \tau^{0}: \Longleftrightarrow$
(i) For each $n \in \mathbb{N}, U \cap\left(Y_{n} \backslash\left\{x_{n}\right\}\right) \in \tau_{n}^{0}$; and
(ii) If $x_{*} \in U$, then $\left\{n \in \mathbb{N} \mid U \upharpoonright Y_{n} \in \tau_{n}^{0}\right\} \in \mathcal{U}$.
- $U \in \tau^{1}: \Longleftrightarrow$ for each $n \in \mathbb{N}, U \upharpoonright X_{n} \in \tau_{n}^{1}$.

Lemma 4.3. The ultrabouquet $\mathfrak{X}$ is a bitopological space.
Proof. We only prove that $\tau^{0}$ is a topology on $X$. A proof for $\tau^{1}$ is similar.

- $\varnothing \in \tau^{0}$ : (i) $\varnothing \cap\left(Y_{n} \backslash\left\{x_{n}\right\}\right)=\varnothing \in \tau_{n}^{0}$; and (ii) $x_{*} \notin \varnothing$.
- $X \in \tau^{0}:$ (i) $X \cap\left(Y_{n} \backslash\left\{x_{n}\right\}\right)=Y_{n} \backslash\left\{x_{n}\right\} \in \tau_{n}^{0}$; and (ii) Since $X \upharpoonright Y_{n}=$ $Y_{n} \in \tau_{n}^{0},\left\{n \in \mathbb{N}|X| Y_{n} \in \tau_{n}^{0}\right\}=\mathbb{N} \in \mathcal{U}$ because $\mathcal{U}$ is a non-trivial filter.
- Let $U_{0}, U_{1} \in \tau^{0}$. We show $U_{0} \cap U_{1} \in \tau^{0}$. (i): By condition (i) for $U_{0}$ and $U_{1}$, the sets $U_{0} \cap\left(Y_{n} \backslash\left\{x_{n}\right\}\right)$ and $U_{1} \cap\left(Y_{n} \backslash\left\{x_{n}\right\}\right)$ are elements of $\tau_{n}^{0}$. Then

$$
\left(U_{0} \cap U_{1}\right) \cap\left(Y_{n} \backslash\left\{x_{n}\right\}\right)=\left(U_{0} \cap\left(Y_{n} \backslash\left\{x_{n}\right\}\right)\right) \cap\left(U_{1} \cap\left(Y_{n} \backslash\left\{x_{n}\right\}\right)\right) \in \tau_{n}^{0}
$$

(ii): If $x_{*} \in U_{0} \cap U_{1}$, then $x_{*}$ is in both $U_{0}$ and $U_{1}$. By condition (ii) for $U_{0}$ and $U_{1}$, the sets $Z_{0}=\left\{n \in \mathbb{N} \mid U_{0} \upharpoonright Y_{n} \in \tau_{n}^{0}\right\}$ and $Z_{1}=\left\{n \in \mathbb{N} \mid U_{1} \upharpoonright\right.$ $\left.Y_{n} \in \tau_{n}^{0}\right\}$ are in $\mathcal{U}$. Then,

$$
Z_{0} \cap Z_{1} \subseteq\left\{n \in \mathbb{N} \mid\left(U_{0} \cap U_{1}\right) \upharpoonright Y_{n} \in \tau_{n}^{0}\right\} \in \mathcal{U}
$$

because $\mathcal{U}$ is a filter.

- $\left\{U_{i}\right\}_{i \in I}$ be any family of elements of $\tau^{0}$. We show $\bigcup_{i \in I} U_{i} \in \tau^{0}$. (i): Since $U_{i} \cap\left(Y_{n} \backslash\left\{x_{n}\right\}\right) \in \tau_{n}^{0}$ for all $i \in I$,

$$
\left(\bigcup_{i \in U} U_{i}\right) \cap\left(Y_{n} \backslash\left\{x_{n}\right\}\right)=\bigcup_{i \in U}\left(U_{i} \cap\left(Y_{n} \backslash\left\{x_{n}\right\}\right)\right) \in \tau_{n}^{0}
$$

(ii): If $x_{*} \in \bigcup_{i \in I} U_{i}$, then $x_{*} \in U_{j}$ for some $j \in I$. By condition (ii) for $U_{j},\left\{n \in \mathbb{N} \mid U_{j} \upharpoonright Y_{n} \in \tau_{n}^{0}\right\} \in \mathcal{U}$.

Claim 1. For $n \in \mathbb{N}$, if $U_{j} \upharpoonright Y_{n} \in \tau_{n}^{0}$, then $\left(\bigcup_{i \in I} U_{i}\right) \upharpoonright Y_{n} \in \tau_{n}^{0}$.

Proof of Claim 1. Let $x$ be an arbitrary element of $\left(\bigcup_{i \in I} U_{i}\right) \upharpoonright Y_{n}$. We show that there exists a $\tau_{n}^{0}$-neighborhood $V$ of $x$ satisfying $V \subseteq\left(\bigcup_{i \in I} U_{i}\right) \upharpoonright$ $Y_{n}$. We distinguish the following two cases:
If $x=x_{n}$, then $U_{j} \upharpoonright Y_{n}$ is a required $\tau_{n}^{0}$-neighborhood of $x$.
If $x \neq x_{n}$, then $x \in U_{k} \cap\left(Y_{n} \backslash\left\{x_{n}\right\}\right)$ for some $k \in I$. By condition (i) for $U_{k}$, this set is a required $\tau_{n}^{0}$-neighborhood of $x$.
Therefore $\left(\bigcup_{i \in I} U_{i}\right) \upharpoonright Y_{n} \in \tau_{n}^{0}$.
From Claim 1, we have

$$
\left\{n \in \mathbb{N} \mid U_{j} \upharpoonright Y_{n} \in \tau_{n}^{0}\right\} \subseteq\left\{n \in \mathbb{N} \mid\left(\bigcup_{i \in I} U_{i}\right) \upharpoonright Y_{n} \in \tau_{n}^{0}\right\} \in \mathcal{U}
$$

For each $n \in \mathbb{N}$, let $v_{n}$ be a valuation on $\left\langle X_{n}, \tau_{n}^{0}, \tau_{n}^{1}\right\rangle$. We define a valuation $v$ on $\mathfrak{X}$ as follows:

## Definition 4.4.

- For $x \in X_{n} \backslash\left\{x_{n}\right\}, x \in v(p): \Longleftrightarrow x \in v_{n}(p)$;
- $x_{*} \in v(p): \Longleftrightarrow\left\{n \in \mathbb{N} \mid x_{n} \in v_{n}(p)\right\} \in \mathcal{U}$.

Let $Y$ denote the set $\bigcup_{n \in \mathbb{N}}\left(Y_{n} \backslash\left\{x_{n}\right\}\right) \cup\left\{x_{*}\right\}$. We investigate the images of the valuation $v$ by dividing $X$ into three parts, namely, $Y \backslash\left\{x_{*}\right\}, X \backslash Y$ and $\left\{x_{*}\right\}$.

First, we investigate in $Y \backslash\left\{x_{*}\right\}$. If $x \in Y \backslash\left\{x_{*}\right\}$, then $x$ is in $Y_{n} \backslash\left\{x_{n}\right\}$ for some $n \in \mathbb{N}$. In the set $Y_{n} \backslash\left\{x_{n}\right\}$, the first clause of Definition 4.4 is extended to all $\mathcal{L}(\square, \triangleright)$-formulas as follows.

Lemma 4.5. For any $\mathcal{L}(\square, \triangleright)$-formula $\varphi, n \in \mathbb{N}$ and $x \in Y_{n} \backslash\left\{x_{n}\right\}$,

$$
x \in v(\varphi) \Longleftrightarrow x \in v_{n}(\varphi)
$$

Proof. We prove by induction on the construction of $\varphi$. We only give a proof of the case $\varphi \equiv \psi \triangleright \chi$.
$(\Rightarrow)$ : Suppose $x \in v(\psi \triangleright \chi)$. Then

$$
\begin{equation*}
\forall U \in \tau^{1}\left[x \in d_{\tau^{0}}(v(\psi) \cap U) \Rightarrow x \in d_{\tau^{0}}(v(\chi) \cap U)\right] \tag{1}
\end{equation*}
$$

In order to prove $x \in v_{n}(\psi \triangleright \chi)$, let $U$ be an arbitrary element of $\tau_{n}^{1}$ and assume $x \in d_{\tau_{n}^{0}}\left(v_{n}(\psi) \cap U\right)$. We would like to show $x \in d_{\tau_{n}^{0}}\left(v_{n}(\chi) \cap U\right)$. Let

$$
U^{\prime}:= \begin{cases}U & \text { if } x_{n} \notin U \\ \left(\left(U \backslash\left\{x_{n}\right\}\right) \cup \bigcup_{m \neq n} X_{m} \backslash\left\{x_{m}\right\}\right) \cup\left\{x_{*}\right\} & \text { if } x_{n} \in U\end{cases}
$$

Then, it is easily shown that $U^{\prime} \in \tau^{1}$ and $U^{\prime} \upharpoonright X_{n}=U$.

Claim 2. $x \in d_{\tau^{0}}\left(v(\psi) \cap U^{\prime}\right)$.
Proof of Claim 2. Let $V$ be any $\tau^{0}$-neighborhood of $x$. By Definition 4.2, $V \cap$ $\left(Y_{n} \backslash\left\{x_{n}\right\}\right) \in \tau_{n}^{0}$, and hence the set $V \cap\left(Y_{n} \backslash\left\{x_{n}\right\}\right)$ is a $\tau_{n}^{0}$-neighborhood of $x$. Since $x \in d_{\tau_{n}^{0}}\left(v_{n}(\psi) \cap U\right)$, there exists $y \neq x$ such that $y \in v_{n}(\psi) \cap U \cap V \cap$ $\left(Y_{n} \backslash\left\{x_{n}\right\}\right)$. By the induction hypothesis, $y \in v(\psi) \cap U \cap V \cap\left(Y_{n} \backslash\left\{x_{n}\right\}\right)$. Hence $y \in v(\psi) \cap U^{\prime} \cap V$. This implies $x \in d_{\tau^{0}}\left(v(\psi) \cap U^{\prime}\right)$.

From (1) and Claim 2, we have $x \in d_{\tau^{0}}\left(v(\chi) \cap U^{\prime}\right)$.
Claim 3. $x \in d_{\tau_{n}^{0}}\left(v_{n}(\chi) \cap U\right)$.
Proof of Claim [3. Let $V$ be any $\tau_{n}^{0}$-neighborhood of $x$. Then, $V \cap\left(Y_{n} \backslash\left\{x_{n}\right\}\right) \in$ $\tau_{n}^{0}$ and $x \in V \cap\left(Y_{n} \backslash\left\{x_{n}\right\}\right)$. Together with $x_{*} \notin V \cap\left(Y_{n} \backslash\left\{x_{n}\right\}\right)$, it is shown that the set $V \cap\left(Y_{n} \backslash\left\{x_{n}\right\}\right)$ is a $\tau^{0}$-neighborhood of $x$. Since $x \in d_{\tau^{0}}\left(v(\chi) \cap U^{\prime}\right)$, there exists $y \neq x$ such that $y \in v(\chi) \cap U^{\prime} \cap V \cap\left(Y_{n} \backslash\left\{x_{n}\right\}\right)$. By the induction hypothesis, $y \in v_{n}(\chi) \cap U^{\prime} \cap V \cap\left(Y_{n} \backslash\left\{x_{n}\right\}\right)$. Since $U^{\prime} \upharpoonright X_{n}=U$, we conclude $y \in v_{n}(\chi) \cap U \cap V$.

We have shown $x \in e_{\tau_{n}^{0}, \tau_{n}^{1}}\left(v_{n}(\psi), v_{n}(\chi)\right)=v_{n}(\psi \triangleright \chi)$.
$(\Leftarrow)$ : Suppose $x \in v_{n}(\psi \triangleright \chi)$. Then

$$
\begin{equation*}
\forall U \in \tau_{n}^{1}\left[x \in d_{\tau_{n}^{0}}\left(v_{n}(\psi) \cap U\right) \Rightarrow x \in d_{\tau_{n}^{0}}\left(v_{n}(\chi) \cap U\right)\right] \tag{2}
\end{equation*}
$$

Let $U$ be an arbitrary element of $\tau^{1}$ and assume $x \in d_{\tau^{0}}(v(\psi) \cap U)$. We would like to show $x \in d_{\tau^{0}}(v(\chi) \cap U)$. Let $U^{\prime}:=U \upharpoonright X_{n}$, then $U^{\prime} \in \tau_{n}^{1}$.
Claim 4. $x \in d_{\tau_{n}^{0}}\left(v_{n}(\psi) \cap U^{\prime}\right)$.
Proof of Claim 4. Let $V$ be any $\tau_{n}^{0}$-neighborhood of $x$. Then $V \cap\left(Y_{n} \backslash\left\{x_{n}\right\}\right) \in \tau_{n}^{0}$ and $x \in V \cap\left(Y_{n} \backslash\left\{x_{n}\right\}\right)$. Together with $x_{*} \notin V \cap\left(Y_{n} \backslash\left\{x_{n}\right\}\right)$, it is shown that the set $V \cap\left(Y_{n} \backslash\left\{x_{n}\right\}\right)$ is a $\tau^{0}$-neighborhood of $x$. Since $x \in d_{\tau^{0}}(v(\psi) \cap U)$, there exists $y \neq x$ such that $y \in v(\psi) \cap U \cap V \cap\left(Y_{n} \backslash\left\{x_{n}\right\}\right)$. By the induction hypothesis, $y \in v_{n}(\psi) \cap U \cap V \cap\left(Y_{n} \backslash\left\{x_{n}\right\}\right)$, and hence $y \in v_{n}(\psi) \cap U^{\prime} \cap V$. Thus we conclude $x \in d_{\tau_{n}^{0}}\left(v_{n}(\psi) \cap U^{\prime}\right)$.

From (2) and Claim 4 , $x \in d_{\tau_{n}^{0}}\left(v_{n}(\chi) \cap U^{\prime}\right)$.
Claim 5. $x \in d_{\tau^{0}}(v(\chi) \cap U)$.
Proof of Claim 5. Let $V$ be any $\tau^{0}$-neighborhood of $x$. By Definition 4.2, $V \cap$ $\left(Y_{n} \backslash\left\{x_{n}\right\}\right) \in \tau_{n}^{0}$ and hence $V \cap\left(Y_{n} \backslash\left\{x_{n}\right\}\right)$ is a $\tau_{n}^{0}$-neighborhood of $x$. Since $x \in d_{\tau_{n}^{0}}\left(v_{n}(\chi) \cap U^{\prime}\right)$, there exists $y \neq x$ such that $y \in v_{n}(\chi) \cap U^{\prime} \cap V \cap\left(Y_{n} \backslash\left\{x_{n}\right\}\right)$. By the induction hypothesis, $y \in v(\chi) \cap U^{\prime} \cap V \cap\left(Y_{n} \backslash\left\{x_{n}\right\}\right)$, and hence $y \in$ $v(\chi) \cap U \cap V$. Thus we conclude $x \in d_{\tau^{0}}(v(\chi) \cap U)$.

We have proved $x \in e_{\tau^{0}, \tau^{1}}(v(\psi), v(\chi))=v(\psi \triangleright \chi)$. This completes our proof of Lemma 4.5

Secondly, we investigate the behavior of valuations on $\mathfrak{X}$ in $X \backslash Y$.

Lemma 4.6. For any subset $U$ of $X \backslash Y, U \in \tau^{0}$.
Proof. We show that each $U \subseteq X \backslash Y$ satisfies conditions (i) and (ii) in Definition 4.2. Clearly $U \cap\left(Y_{n} \backslash\left\{x_{n}\right\}\right)=\varnothing$ for any $n \in \mathbb{N}$, and hence (i) holds. Moreover, (ii) vacuously holds since $U$ does not contain $x_{*}$.

The following lemma shows that every element of $X \backslash Y$ behaves as a dead end of Kripke frames.

Lemma 4.7. For any $x \in X \backslash Y$ and any $Z \subseteq X, x \in c d_{\tau^{0}}(Z)$.
Proof. Let $x \in X \backslash Y$. Then, by Lemma 4.6. $\{x\} \in \tau^{0}$. Since $\bar{Z} \cap\{x\} \subseteq\{x\}$, we have $x \notin d_{\tau^{0}}(\bar{Z})$. That is, $x \in c d_{\tau^{0}}(Z)$.

For $x \in X_{n} \backslash Y_{n}$, even if $x \in v_{n}(\diamond \varphi)$, by Lemma 4.7, $x \notin v(\diamond \varphi)$. So the equivalence of Lemma 4.5 cannot be extended to elements of $X_{n} \backslash\left\{x_{n}\right\}$.

Thirdly, the following lemma is a generalization of the second clause of Definition 4.4. In particular, it plays a key role in our proof of the topological compactness theorem.

Lemma 4.8. For any $\mathcal{L}(\square, \triangleright)$-formula $\varphi$,

$$
x_{*} \in v(\varphi) \Longleftrightarrow\left\{n \in \mathbb{N} \mid x_{n} \in v_{n}(\varphi)\right\} \in \mathcal{U} .
$$

Proof. We prove by induction on the construction of $\varphi$. We only give a proof of the case $\varphi \equiv \psi \triangleright \chi$.
$(\Rightarrow)$ : We prove the contrapositive. Assume $\left\{n \in \mathbb{N} \mid x_{n} \in v_{n}(\psi \triangleright \chi)\right\} \notin \mathcal{U}$. Since $\mathcal{U}$ is an ultrafilter on $\mathbb{N}, Z_{0}:=\left\{n \in \mathbb{N} \mid x_{n} \notin v_{n}(\psi \triangleright \chi)\right\} \in \mathcal{U}$. For each $n \in Z_{0}$, there exists $U_{n} \in \tau_{n}^{1}$ such that

$$
\begin{equation*}
x_{n} \in d_{\tau_{n}^{0}}\left(v_{n}(\psi) \cap U_{n}\right) \& x_{n} \notin d_{\tau_{n}^{0}}\left(v_{n}(\chi) \cap U_{n}\right) . \tag{3}
\end{equation*}
$$

Let $Z_{00}:=\left\{n \in Z_{0} \mid x_{n} \notin U_{n}\right\}$ and $Z_{01}:=\left\{n \in Z_{0} \mid x_{n} \in U_{n}\right\}$. Then, $Z_{0}=Z_{00} \cup Z_{01}$. Since $\mathcal{U}$ is an ultrafilter, we get an $i \in\{0,1\}$ such that $Z_{0 i} \in \mathcal{U}$. Let

$$
U:= \begin{cases}\bigcup_{n \in Z_{0 i}} U_{n} & \text { if } i=0 \\ \left(\bigcup_{n \in Z_{0 i}} U_{n} \backslash\left\{x_{n}\right\}\right) \cup\left(\bigcup_{n \notin Z_{0 i}} X_{n} \backslash\left\{x_{n}\right\}\right) \cup\left\{x_{*}\right\} & \text { if } i=1\end{cases}
$$

Then, it is shown that $U$ is an element of $\tau^{1}$ satisfying $U \upharpoonright X_{n}=U_{n}$ for all $n \in Z_{0 i}$.

First, we prove $x_{*} \in d_{\tau^{0}}(v(\psi) \cap U)$. Let $V$ be any $\tau^{0}$-neighborhood of $x_{*}$. By Definition 4.2, $Z_{1}:=\left\{n \in \mathbb{N}|V| Y_{n} \in \tau_{n}^{0}\right\} \in \mathcal{U}$. Since $Z_{0 i} \cap Z_{1} \in \mathcal{U}, Z_{0 i} \cap Z_{1}$ is non-empty, and fix some $n \in Z_{0 i} \cap Z_{1}$. Since the set $V \upharpoonright Y_{n}$ is a $\tau_{n}^{0}$-neighborhood of $x_{n}$, by (3), there exists $y \in X_{n} \backslash\left\{x_{n}\right\}$ such that $y \in v_{n}(\psi) \cap U_{n} \cap V \cap\left(Y_{n} \backslash\left\{x_{n}\right\}\right)$. Applying Lemma 4.5, $y \in v(\psi) \cap U_{n} \cap V \cap\left(Y_{n} \backslash\left\{x_{n}\right\}\right)$. Since $U_{n}=U \upharpoonright X_{n}$, we obtain $y \in v(\psi) \cap \overline{U \cap V}$ and $y \neq x_{*}$. Thus $x_{*} \in d_{\tau^{0}}(v(\psi) \cap U)$.

Secondly, we prove $x_{*} \notin d_{\tau^{0}}(v(\chi) \cap U)$. By (3), for each $n \in Z_{0 i}$, there exists a $\tau_{n}^{0}$-neighborhood $W_{n}$ of $x_{n}$ such that $v_{n}(\chi) \cap U_{n} \cap W_{n} \subseteq\left\{x_{n}\right\}$. Let $W:=\bigcup_{n \in Z_{0 i}}\left(W_{n} \cap\left(Y_{n} \backslash\left\{x_{n}\right\}\right) \cup\left\{x_{*}\right\}\right.$. We show $W \in \tau^{0}$. (i) For each $n \in \mathbb{N}$,

$$
W \cap\left(Y_{n} \backslash\left\{x_{n}\right\}\right)= \begin{cases}W_{n} \cap\left(Y_{n} \backslash\left\{x_{n}\right\}\right) & \text { if } n \in Z_{0 i} \\ \varnothing & \text { otherwise }\end{cases}
$$

Then, $W \cap\left(Y_{n} \backslash\left\{x_{n}\right\}\right) \in \tau_{n}^{0}$. (ii) If $n \in Z_{0 i}$, then $W \upharpoonright Y_{n}=W_{n} \cap Y_{n} \in \tau_{n}^{0}$. Hence $Z_{0 i} \subseteq\left\{n \in \mathbb{N} \mid W \upharpoonright Y_{n} \in \tau_{n}^{0}\right\} \in \mathcal{U}$ because $\mathcal{U}$ is a filter. Thus $W$ is a $\tau^{0}$-neighborhood of $x_{*}$.

Suppose, towards a contradiction, that $x_{*} \in d_{\tau^{0}}(v(\chi) \cap U)$. Then there exists $y \neq x_{*}$ such that $y \in v(\chi) \cap U \cap W$. Since $y \in W$, for some $n \in Z_{0 i}$, $y \in v(\chi) \cap U \cap W_{n} \cap\left(Y_{n} \backslash\left\{x_{n}\right\}\right)$. Applying Lemma 4.5. $y \in v_{n}(\chi) \cap U \cap W_{n} \cap$ $\left(Y_{n} \backslash\left\{x_{n}\right\}\right)$. Since $U \upharpoonright X_{n}=U_{n}, y \in v_{n}(\chi) \cap U_{n} \cap W_{n}$. This contradicts $v_{n}(\chi) \cap U_{n} \cap W_{n} \subseteq\left\{x_{n}\right\}$. Therefore $x_{*} \notin d_{\tau^{0}}(v(\chi) \cap U)$.

We conclude $x_{*} \notin e_{\tau^{0}, \tau^{1}}(v(\psi), v(\chi))$, and hence $x_{*} \notin v(\psi \triangleright \chi)$.
$(\Leftarrow)$ : Suppose $Z_{0}:=\left\{n \in \mathbb{N} \mid x_{n} \in v_{n}(\psi \triangleright \chi)\right\} \in \mathcal{U}$. In order to prove $x_{*} \in v(\psi \triangleright \chi)$, suppose that $U \in \tau^{1}$ and $x_{*} \in d_{\tau^{0}}(v(\psi) \cap U)$. We would like to show $x_{*} \in d_{\tau^{0}}(v(\chi) \cap U)$. Let $V$ be any $\tau^{0}$-neighborhood of $x_{*}$. By Definition 4.2. $Z_{1}:=\left\{n \in \mathbb{N} \mid V \upharpoonright Y_{n} \in \tau_{n}^{0}\right\} \in \mathcal{U}$. For each $n \in \mathbb{N}$, let $U_{n}:=U \upharpoonright X_{n}$. Then $U_{n} \in \tau_{n}^{1}$.
Claim 6. There exists $n \in Z_{0} \cap Z_{1}$ such that $x_{n} \in d_{\tau_{n}^{0}}\left(v_{n}(\psi) \cap U_{n}\right)$.
Proof of Claim 6. Suppose, towards a contradiction, that for all $n \in Z_{0} \cap Z_{1}$, $x_{n} \notin d_{\tau_{n}^{0}}\left(v_{n}(\psi) \cap U_{n}\right)$. Then for each $n \in Z_{0} \cap Z_{1}$, there exists $W_{n} \in \tau_{n}^{0}$ such that $x_{n} \in W_{n}$ and $v_{n}(\psi) \cap U_{n} \cap W_{n} \subseteq\left\{x_{n}\right\}$. Let $W:=\bigcup_{n \in Z_{0} \cap Z_{1}}\left(W_{n} \cap\left(Y_{n} \backslash\right.\right.$ $\left.\left\{x_{n}\right\}\right) \cup\left\{x_{*}\right\}$.

We show $W \in \tau^{0}$. (i) For each $n \in \mathbb{N}$,

$$
W \cap\left(Y_{n} \backslash\left\{x_{n}\right\}\right)= \begin{cases}W_{n} \cap\left(Y_{n} \backslash\left\{x_{n}\right\}\right) & \text { if } n \in Z_{0} \cap Z_{1} \\ \varnothing & \text { otherwise }\end{cases}
$$

and this set is in $\tau_{n}^{0}$. (ii) If $n \in Z_{0} \cap Z_{1}$, then $W \upharpoonright Y_{n}=W_{n} \cap Y_{n} \in \tau_{n}^{0}$. Thus $Z_{0} \cap Z_{1} \subseteq\left\{n \in \mathbb{N} \mid W \upharpoonright Y_{n} \in \tau_{n}^{0}\right\} \in \mathcal{U}$ because $\mathcal{U}$ is a filter. Therefore $W \in \tau^{0}$.

Since $x_{*} \in d_{\tau^{0}}(v(\psi) \cap U)$, there exists $y \neq x_{*}$ such that $y \in v(\psi) \cap U \cap W$. Since $y \in W$, there exists $m \in Z_{0} \cap Z_{1}$ such that $y \in v(\psi) \cap U_{m} \cap W_{m} \cap\left(Y_{m} \backslash\right.$ $\left.\left\{x_{m}\right\}\right)$. Applying Lemma 4.5, $y \in v_{m}(\psi) \cap U_{m} \cap W_{m} \cap\left(Y_{m} \backslash\left\{x_{m}\right\}\right)$. Then $y \neq x_{m}$ and $y \in v_{m}(\psi) \cap U_{m} \cap W_{m}$. This contradicts $v_{m}(\psi) \cap U_{m} \cap W_{m} \subseteq\left\{x_{m}\right\}$. Our proof of Claim 6 is completed.

We continue the proof of $x_{*} \in d_{\tau^{0}}(v(\chi) \cap U)$. From Claim6 there exists $n \in$ $Z_{0} \cap Z_{1}$ such that $x_{n} \in d_{\tau_{n}^{0}}\left(v_{n}(\psi) \cap U_{n}\right)$. Since $n \in Z_{0}$, we have $x_{n} \in v_{n}(\psi \triangleright \chi)$. Therefore $x_{n} \in d_{\tau_{n}^{0}}\left(v_{n}(\chi) \cap U_{n}\right)$. Moreover, since $n \in Z_{1}$, we have $V \upharpoonright Y_{n} \in \tau_{n}^{0}$. This set is a $\tau_{n}^{0}$-neighborhood of $x_{n}$, and thus there exists $y \neq x_{n}$ such that $y \in v_{n}(\chi) \cap U_{n} \cap\left(V \upharpoonright Y_{n}\right)$. Since $y \neq x_{n}$, we obtain $y \in v(\chi) \cap U_{n} \cap V \cap\left(Y_{n} \backslash\left\{x_{n}\right\}\right)$
by Lemma 4.5. In particular, $y \neq x_{*}$ and $y \in v(\chi) \cap U \cap V$. This implies $x_{*} \in d_{\tau^{0}}(v(\chi) \cap U)$. We conclude $x_{*} \in v(\psi \triangleright \chi)$.

The following lemma is an adaptation of Shehtman's result on the preservation of validity in ultrabouquets to our framework (See Shehtman [15, Lemma 5.6]).

Lemma 4.9. If an $\mathcal{L}(\square, \triangleright)$-formula $\varphi$ is valid in all $\left\langle X_{n}, \tau_{n}^{0}, \tau_{n}^{1}\right\rangle$, then for all valuations $v^{\prime}$ on $\mathfrak{X}$ and all $x \in Y, x \in v^{\prime}(\varphi)$.

Proof. We prove the contrapositive. Suppose that there exist a valuation $v^{\prime}$ on $\mathfrak{X}$ and $x \in Y$ such that $x \notin v^{\prime}(\varphi)$. For each $n \in \mathbb{N}$, we define a valuation $v_{n}^{\prime}$ on $\left\langle X_{n}, \tau_{n}^{0}, \tau_{n}^{1}\right\rangle$ as follows:

- For $x \in X_{n} \backslash\left\{x_{n}\right\}, x \in v_{n}^{\prime}(p): \Longleftrightarrow x \in v^{\prime}(p)$;
- $x_{n} \in v_{n}^{\prime}(p): \Longleftrightarrow x_{*} \in v^{\prime}(p)$.

Then the valuation on $\mathfrak{X}$ defined from $\left\{v_{n}^{\prime}\right\}_{n \in \mathbb{N}}$ in Definition 4.4 coincides with $v^{\prime}$ because $\varnothing \notin \mathcal{U}$ and $\mathbb{N} \in \mathcal{U}$. We distinguish the following two cases.

If $x \in Y_{n} \backslash\left\{x_{n}\right\}$, then by Lemma 4.5. we obtain $x \notin v_{n}^{\prime}(\varphi)$.
If $x=x_{*}$, then by Lemma 4.8, $\left\{n \in \mathbb{N} \mid x_{n} \in v_{n}^{\prime}(\varphi)\right\} \notin \mathcal{U}$. Since $\mathbb{N} \in \mathcal{U}$, for some $n \in \mathbb{N}, x_{n} \notin v_{n}^{\prime}(\varphi)$.

Thus in either case, $\varphi$ is not valid in $\left\langle X_{n}, \tau_{n}^{0}, \tau_{n}^{1}\right\rangle$ for some $n \in \mathbb{N}$.
From the viewpoint of Lemma 4.7, the set $Y$ in the statement of Lemma 4.9 does not seem to be replaceable by $X$ in general. However, we prove that this is actually the case. First, we prove that the validity of the axiom $\square(\square p \rightarrow p) \rightarrow$ $\square p$ of $\mathbf{G L}$ is preserved.

Lemma 4.10. The topological space $\left\langle X, \tau^{0}\right\rangle$ is scattered. That is, the ultrabouquet $\mathfrak{X}$ is a $\mathbf{C L}$-space.
Proof. Since each space $\left\langle X_{n}, \tau_{n}^{0}, \tau_{n}^{1}\right\rangle$ is scattered, $\varphi: \equiv \square(\square p \rightarrow p) \rightarrow \square p$ is valid in $\left\langle X_{n}, \tau_{n}^{0}, \tau_{n}^{1}\right\rangle$ by Fact 2.9. Let $v^{\prime}$ be any valuation on $\mathfrak{X}$. By Lemma 4.9 for all $y \in Y, y \in v^{\prime}(\varphi)$. Moreover, by Lemma 4.7, for all $x \in X \backslash Y$, $x \in c d_{\tau^{0}}\left(v^{\prime}(p)\right)$, that is, $x \in v^{\prime}(\square p)$. Hence $x \in v^{\prime}(\varphi)$. Thus $\varphi$ is valid in $\mathfrak{X}$, and hence $\mathbf{G L} \subseteq \log (\mathfrak{X})$. We conclude that $\left\langle X, \tau^{0}\right\rangle$ is scattered.

The following lemma is a version of a part of Makinson's theorem (See Makinson [12]). Our proof is a modification of that in Hughes and Cresswell [7, Lemma 3.2]).

Lemma 4.11. Let $L$ be any consistent normal extension of $\mathbf{C L}$ and $\varphi$ be any $\mathcal{L}(\square, \triangleright)$-formula. If $\varphi \in L$, then $\square \perp \rightarrow \varphi \in \mathbf{C L}$.

Proof. Let $L$ be a normal extension of $\mathbf{C L}$ and suppose that there exists an $\mathcal{L}(\square, \triangleright)$-formula $\varphi$ such that $\varphi \in L$ but $\square \perp \rightarrow \varphi \notin \mathbf{C L}$. We would like to show that $L$ is inconsistent. From axioms $\mathbf{J} \mathbf{1}$ and $\mathbf{J} 4$, we have that for any $\mathcal{L}(\square, \triangleright)$ formula $\psi, \square \psi$ is equivalent to $(\neg \psi) \triangleright \perp$ in CL. So we may assume that neither
$\square$ nor $\diamond$ occurs in $\varphi$. Also we assume that $\varphi$ is in a conjunctive normal form $\varphi_{0} \wedge \varphi_{1} \wedge \cdots \wedge \varphi_{k}$ where each $\varphi_{i}$ is a disjunction of formulas, and each disjunct of $\varphi_{i}$ is either a formula without $\triangleright$, or a formula of the form $\psi \triangleright \chi$, or a formula of the form $\neg(\psi \triangleright \chi)$.

By the choice of $\varphi$, for some $i \leq k, \varphi_{i} \in L$ and $\square \perp \rightarrow \varphi_{i} \notin \mathbf{C L}$. From J1, we have that $\square \perp \rightarrow \psi \triangleright \chi \in \mathbf{C L}$. Then $\varphi_{i}$ does not contain a formula of the form $\psi \triangleright \chi$ as a disjunct because $\square \perp \rightarrow \varphi_{i} \notin \mathbf{C L}$. Thus, we may assume that $\varphi_{i}$ is of the form

$$
\gamma \vee \bigvee_{j=0}^{m} \neg\left(\psi_{j} \triangleright \chi_{j}\right)
$$

where $\gamma$ is a classical propositional formula. Since $\square \perp \rightarrow \varphi_{i} \notin \mathbf{C L}, \gamma$ is not a tautology of the classical propositional logic. Then, there exists a substitution instance $\gamma^{\prime}$ of $\gamma$ such that $\neg \gamma^{\prime}$ is a tautology (cf. [7] p. 47]). So $\neg \gamma^{\prime} \in L$.

Suppose $m=0$. Then $L$ contains both $\gamma^{\prime}$ and $\neg \gamma^{\prime}$, and hence is inconsistent. Suppose $m>0$. Since each $\neg\left(\psi_{j} \triangleright \chi_{j}\right)$ implies $\diamond T$ in $\mathbf{C L}, L$ contains $\gamma \vee \diamond T$. Then $\gamma^{\prime} \vee \diamond \top \in L$, and thus $\diamond T \in L$. Since $L$ is normal, $\square \diamond \top \in L$. Therefore $\square \perp \in L$ because $L$ is an extension of $\mathbf{C L}$. We conclude that $L$ is inconsistent.

Theorem 4.12. If an $\mathcal{L}(\square, \triangleright)$-formula $\varphi$ is valid in all $\left\langle X_{n}, \tau_{n}^{0}, \tau_{n}^{1}\right\rangle$, then $\varphi$ is also valid in $\mathfrak{X}$.

Proof. Since $\left\langle X_{0}, \tau_{0}^{0}\right\rangle$ is scattered, $\log \left(X_{0}, \tau_{0}^{0}, \tau_{0}^{1}\right)$ is a consistent normal extension of CL by Corollary 3.5. Since $\varphi \in \log \left(X_{0}, \tau_{0}^{0}, \tau_{0}^{1}\right)$, we obtain $\square \perp \rightarrow \varphi \in$ CL by Lemma 4.11 .

Let $v^{\prime}$ be any valuation on $\mathfrak{X}$, then for all $y \in Y, y \in v^{\prime}(\varphi)$ by Lemma 4.9 . Also, for all $x \in X \backslash Y, x \in v^{\prime}(\square \perp)$ by Lemma 4.7. Since $\mathfrak{X}$ is a CL-space by Lemma 4.10, it follows from $\square \perp \rightarrow \varphi \in \mathbf{C L}$ that $x \in v^{\prime}(\varphi)$. Therefore $\varphi$ is valid in $\mathfrak{X}$.

### 4.2 Proofs of the theorems

We are ready to prove the topological compactness theorem.
Theorem 4.13 (Topological compactness theorem). Let $L$ be a consistent normal extension of $\mathbf{C L}, \Gamma$ be a set of $\mathcal{L}(\square, \triangleright)$-formulas and $\varphi$ be an $\mathcal{L}(\square, \triangleright)$ formula. If $\Gamma \not \models_{L}^{T} \varphi$, then $\Gamma_{0} \models_{L}^{T} \varphi$ for some finite subset $\Gamma_{0}$ of $\Gamma$.

Proof. Suppose that for all finite subsets $\Gamma_{0}$ of $\Gamma, \Gamma_{0} \not \vDash_{L}^{T} \varphi$. Let $\left\{\psi_{n}\right\}_{n \in \mathbb{N}}$ be an enumeration of elements of $\Gamma$, and let $\chi_{n}:=\bigwedge_{i=0}^{n} \psi_{n}$. Then, for each $n \in \mathbb{N}$, $\left\{\chi_{n}\right\} \not \vDash_{L}^{T} \varphi$. Hence there exist an $L$-space $\left\langle X_{n}, \tau_{n}^{0}, \tau_{n}^{1}\right\rangle$, a valuation $v_{n}$ on the space and $x_{n} \in X_{n}$ such that $x_{n} \in v_{n}\left(\chi_{n}\right)$ and $x_{n} \notin v_{n}(\varphi)$. By Corollary 3.5. $\left\langle X_{n}, \tau_{n}^{0}\right\rangle$ is scattered. Also we may assume that the family $\left\{X_{n}\right\}_{n \in \mathbb{N}}$ is pairwise disjoint. Let $\mathfrak{X}$ be an ultrabouquet based on the families $\left\{\left\langle X_{n}, \tau_{n}^{0}, \tau_{n}^{1}\right\rangle\right\}_{n \in \mathbb{N}}$ and $\left\{x_{n}\right\}_{n \in \mathbb{N}}$. Since every $\varphi \in L$ is valid in all $\left\langle X_{n}, \tau_{n}^{0}, \tau_{n}^{1}\right\rangle$, by Lemma 4.12, $\varphi$ is also valid in $\mathfrak{X}$. Therefore $\mathfrak{X}$ is also an $L$-space.

Let $v$ be the valuation on $\mathfrak{X}$ defined from $\left\{v_{n}\right\}_{n \in \mathbb{N}}$ in Definition 4.4. We claim that for every $\psi_{i} \in \Gamma, x_{*} \in v\left(\psi_{i}\right)$. Indeed, for any $n \geq i, x_{n} \in v_{n}\left(\psi_{i}\right)$. Then the set $\left\{n \in \mathbb{N} \mid x_{n} \in v_{n}\left(\psi_{i}\right)\right\}$ is cofinite, and hence in $\mathcal{U}$ because $\mathcal{U}$ is a non-principal ultrafilter. By Lemma 4.8, $x_{*} \in v\left(\psi_{i}\right)$.

On the other hand, $\left\{n \in \mathbb{N} \mid x_{n} \in v_{n}(\varphi)\right\}=\varnothing \notin \mathcal{U}$. Again by Lemma 4.8. $x_{*} \notin v(\varphi)$. Thus we conclude $\Gamma \not \vDash_{L}^{T} \varphi$.

Theorem 4.14. For any normal extension $L$ of $\mathbf{C L}, L$ is topologically complete if and only if $L$ is topologically strongly complete.

Proof. It suffices to prove the implication $(\Rightarrow)$. Suppose $\Gamma \models_{L}^{T} \varphi$. By the topological compactness theorem, $\Gamma_{0} \models_{L}^{T} \varphi$ for some finite subset $\Gamma_{0}$ of $\Gamma$, and we have $\varnothing \models_{L}^{T} \bigwedge \Gamma_{0} \rightarrow \varphi$. By the topological completeness of $L, \varnothing \vdash_{L} \bigwedge \Gamma_{0} \rightarrow \varphi$. Thus $\Gamma \vdash_{L} \varphi$.

From Theorems 3.10 and 4.14 , we obtain the following topological strong completeness theorem.

Theorem 4.15 (Topological strong completeness theorem of some extensions of CL). The logics CL, CLM, IL, ILM, ILP and ILW are topologically strongly complete.

## 5 Topological investigations of IL

In this section, we investigate topological aspects of IL. First, we investigate necessary and sufficient conditions for a CL-space to be an IL-space. Secondly, we explore Alexandroff IL-spaces.

Theorem 5.1. Let $\left\langle X, \tau^{0}, \tau^{1}\right\rangle$ be a CL-space. Then the following are equivalent:

1. $\left\langle X, \tau^{0}, \tau^{1}\right\rangle$ is an IL-space.
2. For any $U \in \tau^{1}$ and $Y \subseteq X, d_{\tau^{0}}\left(d_{\tau^{0}}(Y) \cap U\right) \subseteq d_{\tau^{0}}(Y \cap U)$.
3. For any $U \in \tau^{1}, d_{\tau^{0}}\left(d_{\tau^{0}}(\bar{U}) \cap U\right)=\varnothing$.
4. For any $U \in \tau^{1}$, there exists $V \in \tau^{0}$ such that $V \subseteq U$ and $d_{\tau^{0}}(U \backslash V)=\varnothing$.

Proof. ( $1 \Leftrightarrow 2$ ): Notice that a CL-space $\left\langle X, \tau^{0}, \tau^{1}\right\rangle$ is an IL-space if and only if $\diamond p \triangleright p$ is valid in $\left\langle X, \tau^{0}, \tau^{1}\right\rangle$. The latter condition is equivalent to the condition that for all $Y \subseteq X, e_{\tau^{0}, \tau^{1}}\left(d_{\tau^{0}}(Y), Y\right)=X$. Then it follows from Definition 3.2 that this is equivalent to Clause 2.
$(2 \Rightarrow 3)$ : Let $U \in \tau^{1}$. From Clause 2 for $Y=\bar{U}$, we have $d_{\tau^{0}}\left(d_{\tau^{0}}(\bar{U}) \cap\right.$ $U) \subseteq d_{\tau^{0}}(\bar{U} \cap U)=d_{\tau^{0}}(\varnothing)$. Since $d_{\tau^{0}}(\varnothing)=\varnothing$ by Fact 2.10 , we obtain $d_{\tau^{0}}\left(d_{\tau^{0}}(\bar{U}) \cap U\right)=\varnothing$.
(3 $\Rightarrow 2)$ : Let $U \in \tau^{1}$ and $Y \subseteq X$. Since $Y \backslash U \subseteq \bar{U}$, by Fact 2.10 2, $d_{\tau^{0}}(Y \backslash U) \cap U \subseteq d_{\tau^{0}}(\bar{U}) \cap U$. Then $d_{\tau^{0}}\left(d_{\tau^{0}}(Y \backslash U) \cap U\right) \subseteq d_{\tau^{0}}\left(d_{\tau^{0}}(\bar{U}) \cap U\right)=\varnothing$. We get $d_{\tau^{0}}\left(d_{\tau^{0}}(Y \backslash U) \cap U\right)=\varnothing$.

Since $Y=(Y \cap U) \cup(Y \backslash U)$, by Fact 2.10 ,

$$
\begin{aligned}
d_{\tau^{0}}\left(d_{\tau^{0}}(Y) \cap U\right) & =d_{\tau^{0}}\left(d_{\tau^{0}}(Y \cap U) \cap U\right) \cup d_{\tau^{0}}\left(d_{\tau^{0}}(Y \backslash U) \cap U\right) \\
& =d_{\tau^{0}}\left(d_{\tau^{0}}(Y \cap U) \cap U\right) \\
& \subseteq d_{\tau^{0}}\left(d_{\tau^{0}}(Y \cap U)\right) \\
& \subseteq d_{\tau^{0}}(Y \cap U)
\end{aligned}
$$

$(3 \Rightarrow 4)$ : Let $U \in \tau^{1}$. Let $V$ denote the set $U \backslash d_{\tau^{0}}(\bar{U})$. Then $V \subseteq U$ and $d_{\tau^{0}}(U \backslash V)=d_{\tau^{0}}\left(d_{\tau^{0}}(\bar{U}) \cap U\right)=\varnothing$. So it suffices to show that $V$ is an element of $\tau^{0}$. Let $x$ be an arbitrary element of $V$. Since $x \notin d_{\tau^{0}}\left(d_{\tau^{0}}(\bar{U}) \cap U\right)$, there exists a $\tau^{0}$-neighborhood $W_{0}$ of $x$ such that $W_{0} \cap d_{\tau^{0}}(\bar{U}) \cap U \subseteq\{x\}$. Since $x \notin d_{\tau^{0}}(\bar{U}), W_{0} \cap d_{\tau^{0}}(\bar{U}) \cap U=\varnothing$. Furthermore, from $x \notin d_{\tau^{0}}(\bar{U})$, there exists a $\tau^{0}$-neighborhood $W_{1}$ of $x$ such that $W_{1} \cap \bar{U} \subseteq\{x\}$. Since $x \notin \bar{U}$, we also have $W_{1} \cap \bar{U}=\varnothing$. Equivalently, $W_{1} \subseteq U$. Then, we have $W_{0} \cap W_{1} \in \tau^{0}$, $x \in W_{0} \cap W_{1}$ and $W_{0} \cap W_{1} \subseteq V$. We have shown that an arbitrary element of $V$ has a $\tau^{0}$-neighborhood which is included in $V$. Therefore $V \in \tau^{0}$.
$(4 \Rightarrow 3)$ : Let $U \in \tau^{1}$, then for some $V \in \tau^{0}, V \subseteq U$ and $d_{\tau^{0}}(U \backslash V)=\varnothing$. Since $\bar{U} \subseteq \bar{V}$ and $V \in \tau^{0}$, by Fact 2.10, $d_{\tau^{0}}(\bar{U}) \cap V \subseteq d_{\tau^{0}}(\bar{V}) \cap V=\varnothing$. Then $d_{\tau^{0}}(\bar{U}) \cap V=\varnothing$ and so $d_{\tau^{0}}\left(d_{\tau^{0}}(\bar{U}) \cap V\right)=\varnothing$.

Since $U=V \cup(U \backslash V)$, we obtain

$$
\begin{aligned}
d_{\tau^{0}}\left(d_{\tau^{0}}(\bar{U}) \cap U\right) & =d_{\tau^{0}}\left(d_{\tau^{0}}(\bar{U}) \cap V\right) \cup d_{\tau^{0}}\left(d_{\tau^{0}}(\bar{U}) \cap(U \backslash V)\right) \\
& =d_{\tau^{0}}\left(d_{\tau^{0}}(\bar{U}) \cap(U \backslash V)\right) \\
& \subseteq d_{\tau^{0}}(U \backslash V)
\end{aligned}
$$

Therefore we conclude $d_{\tau^{0}}\left(d_{\tau^{0}}(\bar{U}) \cap U\right)=\varnothing$.
Corollary 5.2. For any CL-space $\left\langle X, \tau^{0}, \tau^{1}\right\rangle$, if $\tau^{1} \subseteq \tau^{0}$, then $\left\langle X, \tau^{0}, \tau^{1}\right\rangle$ is an IL-space.

Proof. Let $U \in \tau^{1}$, then $U \in \tau^{0}$. By Fact 2.10 $d_{\tau^{0}}(\bar{U}) \cap U=\varnothing$, and hence $d_{\tau^{0}}\left(d_{\tau^{0}}(\bar{U}) \cap U\right)=\varnothing$. By Theorem 5.1. $\left\langle X, \tau^{0}, \tau^{1}\right\rangle$ is an IL-space.

We have already stated that IL is complete with respect to Visser semantics (Fact 2.22). Actually, Visser proved the following stronger result saying that IL is sound and complete with respect to a smaller class of Visser frames than the class of all Visser frames validating IL (See also Fact 2.19.3).

Fact 5.3 (Visser [20]). For any $\mathcal{L}(\square, \triangleright)$-formula $\varphi$, the following are equivalent:

1. $\varnothing \vdash_{\mathrm{IL}} \varphi$.
2. $\varphi$ is valid in all Visser frames $\langle W, R, S\rangle$ with $R \subseteq S$.

We explain how Fact 5.3 follows from Fact 2.22 in our framework. For this purpose, we prepare the following lemmas.

Lemma 5.4. For any topological space $\langle X, \tau\rangle$, the following are equivalent:

1. $\langle X, \tau\rangle$ is Alexandroff.
2. For any family $\left\{Y_{i}\right\}_{i \in I}$ of subsets of $X, d_{\tau}\left(\bigcup_{i \in I} Y_{i}\right) \subseteq \bigcup_{i \in I} d_{\tau}\left(Y_{i}\right)$.

Proof. $(1 \Rightarrow 2)$ : Let $\left\{Y_{i}\right\}_{i \in I}$ be any family of subsets of $X$. Let $x \notin \bigcup_{i \in I} d_{\tau}\left(Y_{i}\right)$. Then, for all $i \in I$, there exists a $\tau$-neighborhood $U_{i}$ of $x$ such that $Y_{i} \cap U_{i} \subseteq\{x\}$. Let $U:=\bigcap_{i \in I} U_{i}$, then $U$ is also a $\tau$-neighborhood of $x$ because $\tau$ is Alexandroff. Suppose, towards a contradiction, $x \in d_{\tau}\left(\bigcup_{i \in I} Y_{i}\right)$. Then there exists $y \neq x$ such that $y \in\left(\bigcup_{i \in I} Y_{i}\right) \cap U$. For some $j \in I, y \in Y_{j} \cap U \subseteq Y_{j} \cap U_{j}$, and this is a contradiction. Therefore $x \notin d_{\tau}\left(\bigcup_{i \in I} Y_{i}\right)$.
$(2 \Rightarrow 1)$ : Let $\left\{U_{i}\right\}_{i \in I}$ be any family of sets of $\tau$. Then for each $i \in I$, $d_{\tau}\left(\overline{U_{i}}\right) \cap U_{i}=\varnothing$ by Fact 2.10 .4 .

$$
\begin{align*}
d_{\tau}\left(\overline{\bigcap_{i \in I} U_{i}}\right) \cap \bigcap_{i \in I} U_{i} & =d_{\tau}\left(\bigcup_{i \in I} \overline{U_{i}}\right) \cap \bigcap_{i \in I} U_{i} \\
& \subseteq \bigcup_{i \in I} d_{\tau}\left(\overline{U_{i}}\right) \cap \bigcap_{i \in I} U_{i}  \tag{byClause1}\\
& \subseteq \bigcup_{i \in I}\left(d_{\tau}\left(\overline{U_{i}}\right) \cap U_{i}\right)=\varnothing
\end{align*}
$$

Therefore $\bigcap_{i \in I} U_{i}$ is a member of $\tau$.
Notice that the converse inclusion $\bigcup_{i \in I} d_{\tau}\left(Y_{i}\right) \subseteq d_{\tau}\left(\bigcup_{i \in I} Y_{i}\right)$ in Lemma 5.42 is easily obtained from Fact 2.102.

Lemma 5.5. Let $\langle X, \tau\rangle$ be a topological space and $V, U \subseteq X$. If $V \subseteq U$ and $d_{\tau}(U \backslash V)=\varnothing$, then $d_{\tau}(Y \cap U)=d_{\tau}(Y \cap V)$ for all subsets $Y$ of $X$.

Proof. Notice that $d_{\tau^{0}}(Y \cap(U \backslash V))$ is also empty because $Y \cap(U \backslash V) \subseteq U \backslash V$. Since $U=(U \backslash V) \cup V$, by Fact 2.10. 3 ,

$$
d_{\tau}(Y \cap U)=d_{\tau}(Y \cap(U \backslash V)) \cup d_{\tau}(Y \cap V)=d_{\tau}(Y \cap V)
$$

Theorem 5.6. Let $\left\langle X, \tau^{0}, \tau^{1}\right\rangle$ be a bitopological space with both $\tau^{0}$ and $\tau^{1}$ are Alexandroff. Then, the following are equivalent:

1. $\left\langle X, \tau^{0}, \tau^{1}\right\rangle$ is an IL-space.
2. $\tau^{0}$ is scattered and there exists an Alexandroff topology $\tau^{2}$ on $X$ such that $\tau^{0} \cap \tau^{1} \subseteq \tau^{2} \subseteq \tau^{0}$ and $\log \left(X, \tau^{0}, \tau^{1}\right)=\log \left(X, \tau^{0}, \tau^{2}\right)$.
3. There exists a Visser frame $\langle X, R, S\rangle$ such that $R \subseteq S$ and $\log \left(X, \tau^{0}, \tau^{1}\right)=$ $\log (X, R, S)$.

Proof. $(1 \Rightarrow 2)$ : Since $\left\langle X, \tau^{0}, \tau^{1}\right\rangle$ is a CL-space, $\tau^{0}$ is scattered by Corollary 3.5. Define

$$
\tau^{2}:=\left\{V \in \tau^{0} \mid \exists U \in \tau^{1}\left[V \subseteq U \& d_{\tau^{0}}(U \backslash V)=\varnothing\right]\right\}
$$

Then, obviously $\tau^{2} \subseteq \tau^{0}$. Let $V \in \tau^{0} \cap \tau^{1}$. Since $V \subseteq V$ and $d_{\tau^{0}}(V \backslash V)=$ $d_{\tau^{0}}(\varnothing)=\varnothing$ by Fact 2.10. 1, we have $V \in \tau^{2}$. Thus $\tau^{0} \cap \tau^{1} \subseteq \tau^{2}$.

First, we prove that $\tau^{2}$ is a topology on $X$.

- Since $X$ and $\varnothing$ are in $\tau^{0} \cap \tau^{1}$, they are also in $\tau^{2}$.
- Let $V_{0}, V_{1} \in \tau^{2}$. Then there exist elements $U_{0}$ and $U_{1}$ of $\tau^{1}$ such that $V_{i} \subseteq U_{i}$ for $i \in\{0,1\}$ and $d_{\tau^{0}}\left(U_{0} \backslash V_{0}\right)=d_{\tau^{0}}\left(U_{1} \backslash V_{1}\right)=\varnothing$. We have $V_{0} \cap V_{1} \subseteq U_{0} \cap U_{1} \in \tau^{1}$ and

$$
\begin{aligned}
& d_{\tau^{0}}\left(\left(U_{0} \cap U_{1}\right) \backslash\left(V_{0} \cap V_{1}\right)\right)=d_{\tau^{0}}\left(\left(\left(U_{0} \cap U_{1}\right) \backslash V_{0}\right) \cup\left(\left(U_{0} \cap U_{1}\right) \backslash V_{1}\right)\right), \\
& \subseteq d_{\tau^{0}}\left(\left(U_{0} \backslash V_{0}\right) \cup\left(U_{1} \backslash V_{1}\right)\right), \\
& \quad(\text { by Fact } 2.10,2) \\
&=d_{\tau^{0}}\left(U_{0} \backslash V_{0}\right) \cup d_{\tau^{0}}\left(U_{1} \backslash V_{1}\right)=\varnothing \\
&\quad \text { (by Fact 2.10. } 3)
\end{aligned}
$$

Hence $V_{0} \cap V_{1} \in \tau^{2}$.

- Let $\left\{V_{i}\right\}_{i \in I}$ be any family of elements of $\tau^{2}$. Then for each $i \in I$, there exists $U_{i} \in \tau^{1}$ such that $V_{i} \subseteq U_{i}$ and $d_{\tau^{0}}\left(U_{i} \backslash V_{i}\right)=\varnothing$. We get $\bigcup_{i \in I} V_{i} \subseteq$ $\bigcup_{i \in I} U_{i} \in \tau^{1}$ and

$$
\begin{aligned}
d_{\tau^{0}}\left(\left(\bigcup_{i \in I} U_{i}\right) \backslash\left(\bigcup_{i \in I} V_{i}\right)\right) & \subseteq d_{\tau^{0}}\left(\bigcup_{i \in I}\left(U_{i} \backslash V_{i}\right)\right), & & \text { (by Fact 2.10. } 2) \\
& \subseteq \bigcup_{i \in I} d_{\tau^{0}}\left(U_{i} \backslash V_{i}\right)=\varnothing & & \text { (by Lemma 5.4) }
\end{aligned}
$$

Therefore $\bigcup_{i \in I} V_{i}$ is an element of $\tau^{2}$.
Secondly, we prove $\tau^{2}$ is Alexandroff. Let $\left\{V_{i}\right\}_{i \in I}$ be any family of elements of $\tau^{2}$. Then for each $i \in I$, there exists $U_{i} \in \tau^{1}$ such that $d_{\tau^{0}}\left(U_{i} \backslash V_{i}\right)=\varnothing$. Since $\tau^{1}$ is Alexandroff, $\bigcap_{i \in I} V_{i} \subseteq \bigcap_{i \in I} U_{i} \in \tau^{1}$. Since $\tau^{0}$ is also Alexandroff,

$$
\begin{array}{rlrl}
d_{\tau^{0}}\left(\left(\bigcap_{i \in I} U_{i}\right) \backslash\left(\bigcap_{i \in I} V_{i}\right)\right) & \subseteq d_{\tau^{0}}\left(\bigcup_{i \in I}\left(U_{i} \backslash V_{i}\right)\right), & & (\text { by Fact 2.10, } 2) \\
& =\bigcup_{i \in I} d_{\tau^{0}}\left(U_{i} \backslash V_{i}\right)=\varnothing
\end{array}
$$

Therefore $\bigcap_{i \in I} V_{i} \in \tau^{2}$.
Finally, we prove $\log \left(X, \tau^{0}, \tau^{1}\right)=\log \left(X, \tau^{0}, \tau^{2}\right)$. It suffices to prove that for all subsets $Y, Z$ of $X, e_{\tau^{0}, \tau^{1}}(Y, Z)=e_{\tau^{0}, \tau^{2}}(Y, Z)$.
$(\subseteq):$ Let $x \in e_{\tau^{0}, \tau^{1}}(Y, Z), V \in \tau^{2}$ and $x \in d_{\tau^{0}}(Y \cap V)$. We would like to show $x \in d_{\tau^{0}}(Z \cap V)$. Then, there exists $U \in \tau^{1}$ such that $V \subseteq U$ and $d_{\tau^{0}}(U \backslash V)=\varnothing$.

By Lemma 5.5, $d_{\tau^{0}}(Y \cap U)=d_{\tau^{0}}(Y \cap V)$ and so $x \in d_{\tau^{0}}(Y \cap U)$. Since $x \in e_{\tau^{0}, \tau^{1}}(Y, Z), x \in d_{\tau^{0}}(Z \cap U)$. By Lemma 5.5again, $d_{\tau^{0}}(Z \cap U)=d_{\tau^{0}}(Z \cap V)$ and thus $x \in d_{\tau^{0}}(Z \cap V)$.
$(\supseteq):$ Let $x \in e_{\tau^{0}, \tau^{2}}(Y, Z), U \in \tau^{1}$ and $x \in d_{\tau^{0}}(Y \cap U)$. We would like to show $x \in d_{\tau^{0}}(Z \cap U)$. Since $\left\langle X, \tau^{0}, \tau^{1}\right\rangle$ is an IL-space, by Theorem 5.1, there exists $V \in \tau^{0}$ such that $V \subseteq U$ and $d_{\tau^{0}}(U \backslash V)=\varnothing$. Then, $V \in \tau^{2}$. As above, by Lemma 5.5, $x \in d_{\tau^{0}}(Y \cap U)=d_{\tau^{0}}(Y \cap V)$. Since $x \in e_{\tau^{0}, \tau^{2}}(Y, Z)$, $x \in d_{\tau^{0}}(Z \cap V)$. Also by Lemma 5.5 again, $x \in d_{\tau^{0}}(Z \cap V)=d_{\tau^{0}}(Z \cap U)$.
$(2 \Rightarrow 3)$ : Let $R$ and $S$ be binary relations on $X$ defined as follows:

- $x R y: \Longleftrightarrow x \neq y \& \forall U \in \tau^{0}(x \in U \Rightarrow y \in U)$;
- $x S y: \Longleftrightarrow \forall U \in \tau^{2}(x \in U \Rightarrow y \in U)$.

As proved in the proof of Theorem $3.8,\langle X, R, S\rangle$ is a Visser frame, $\tau^{0}=\tau_{R}$ and $\tau^{2}=\tau_{S}$. By Corollary 3.7, $\log (X, R, S)=\log \left(X, \tau_{R}, \tau_{S}\right)=\log \left(X, \tau^{0}, \tau^{2}\right)=$ $\log \left(X, \tau^{0}, \tau^{1}\right)$. Also $R \subseteq S$ follows from the definitions of $R$ and $S$ and $\tau^{2} \subseteq \tau^{0}$. $(3 \Rightarrow 1)$ : This is a direct consequence of Fact 2.19. 3 .

Corollary 5.7. For any Visser frame $\langle W, R, S\rangle$, the following are equivalent:

1. $\mathbf{I L} \subseteq \log (W, R, S)$.
2. There exists a Visser frame $\left\langle W, R, S^{\prime}\right\rangle$ such that $R \subseteq S^{\prime}$ and $\log (W, R, S)=$ $\log \left(W, R, S^{\prime}\right)$.

Proof. $(1 \Rightarrow 2)$ : By Fact 2.13 , both $\tau_{R}$ and $\tau_{S}$ are Alexandroff. By Corollary 3.7. $\log (W, R, S)=\log \left(W, \tau_{R}, \tau_{S}\right)$, and hence $\left\langle W, \tau_{R}, \tau_{S}\right\rangle$ is an IL-space. By Theorem 5.6, there exists a Visser frame $\left\langle W, R^{\prime}, S^{\prime}\right\rangle$ such that $R^{\prime} \subseteq S^{\prime}$ and $\log \left(W, R^{\prime}, S^{\prime}\right)=\log \left(W, \tau_{R}, \tau_{S}\right)$. Then $\log (W, R, S)=\log \left(W, R^{\prime}, S^{\prime}\right)$. Furthermore, since $R$ is irreflexive and transitive, it is easily shown that for any $x, y \in W$,

$$
x R y \Longleftrightarrow x \neq y \& \forall U \in \tau_{R}(x \in U \Rightarrow y \in U)
$$

Notice that the right-to-left direction of this equivalence is proved by letting $U=\{x\} \cup R(x)$. From our proof of Theorem 5.6. $R^{\prime}=R$.
( $2 \Rightarrow 1$ ): Immediate from Fact 2.19.3.

## 6 Concluding remarks

In this paper, we newly introduced a topological semantics of $\mathbf{C L}$ and its extensions, and proved the topological compactness theorem. As a consequence, we proved that the logics CL, CLM, IL, ILM, ILP and ILW are strongly complete with respect to our topological semantics. These results are just the starting point for research in this direction. Obviously, investigating the topological completeness of other logics which are not listed above is an important further task.

As we have described in Section 3, we introduced our new topological semantics with Visser semantics in mind. Actually, we proved that every Visser frame can be considered as a topological frame (Corollary 3.7). Also, each Visser frame can be considered as a Veltman frame, but it is not known whether each Veltman frame can be considered as a topological frame. In this regard, we propose the following problem.
Problem 6.1. Is there a normal extension $L$ of $\mathbf{C L}$ such that $L$ is complete with respect to Veltman semantics but not with respect to our topological semantics?

While CL and some of its extensions are strongly complete with respect to our semantics, they are not with respect to Veltman and Visser semantics. This seems to be an evidence that our semantics can provide more models than these relational semantics. Then, we expect an affirmative answer to the following problem.
Problem 6.2. Is there a normal extension $L$ of $\mathbf{C L}$ such that $L$ is complete with respect to our semantics but not with respect to Veltman semantics?

Visser [20] proved that the logics ILP and ILW have finite model property with respect to Visser semantics. That is, each of these logics is determined by a class of corresponding finite Visser frames. Therefore, these logics also have finite model property with respect to our topological semantics. On the other hand, Visser also proved that IL and ILM do not have finite model property with respect to Visser semantics (See also Visser [21). Regarding this point, we propose the following problem.
Problem 6.3. Do the logics CL, CLM, IL and ILM have finite model property with respect to our topological semantics?

In order to understand the properties of axioms of $\mathbf{C L}$ and $\mathbf{I L}$ in more detail, the authors recently introduced several sublogics of them, and studied their basic characters such as completeness with respect to relational semantics and interpolation property ( $[9,11])$. We ask the following question about these sublogics.
Problem 6.4. Can we develop a topological semantics for these sublogics of $\mathbf{C L}$ and $\mathbf{I L}$ ?

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[^1]:    ${ }^{1}$ Actually, Shehtman proved that GL is strongly complete with respect to neighborhood semantics. Esakia [5] proved that for GL, neighborhood semantics and topological semantics coincide, and so we can state Shehtman's theorem as the topological strong completeness theorem of GL.

