On the Behaviour of Coalgebras with Side Effects and Algebras with Effectful Iteration

Stefan Milius[2,⋆], Jiří Adámek[1,⋆⋆], and Henning Urbat[2,⋆⋆⋆]

[1] Department of Mathematics, Faculty of Electrical Engineering, Czech Technical University in Prague
[2] Lehrstuhl für Theoretische Informatik, Friedrich-Alexander-Universität Erlangen-Nürnberg

Abstract For every finitary monad T on sets and every endofunctor F on the category of T-algebras we introduce the concept of an ffg-Elgot algebra for F, that is, an algebra admitting coherent solutions for finite systems of recursive equations with effects represented by the monad T. The goal is to study the existence and construction of free ffg-Elgot algebras. To this end, we investigate the locally ffg fixed point φF, i.e. the colimit of all F-coalgebras with free finitely generated carrier, which is shown to be the initial ffg-Elgot algebra. This is the technical foundation for our main result: the category of ffg-Elgot algebras is monadic over the category of T-algebras.

1 Introduction

Terminal coalgebras yield a fully abstract domain of behavior for a given kind of state-based systems whose transition type is described by an endofunctor F. Often one is mainly interested in the study of the semantics of *finite* coalgebras. For instance, regular languages are the behaviors of finite deterministic automata, while the terminal coalgebra of the corresponding functor is formed by *all* formal languages. For endofunctors on sets, the *rational fixed point* introduced by Adámek, Milius and Velebil [7] yields a fully abstract domain of behavior for finite coalgebras. However, in recent years there has been a lot of interest in studying coalgebras over more general categories than sets. In particular, categories of algebras for a (finitary) monad T on sets are a paradigmatic setting; they are used, for instance, in the generalized determinization framework of Silva et al. [49] and yield *coalgebraic language equivalence* [16] as a semantic equivalence of coalgebraic systems with side effects modelled by the monad T. In the category $\mathscr{C}$ of T-algebras, several notions of 'finite' object are natural to consider, and each yields an ensuing notion of 'finite' coalgebra: (1) free objects on finitely

⋆ Supported by Deutsche Forschungsgemeinschaft (DFG) under projects MI 717/5-2 and MI 717/7-1 and as part of the Research and Training Group 2475 "Cybercrime and Forensic Computing" (393541319/GRK2475/1-2019)
⋆⋆ Supported by the Grant agency of the Czech Republic under the grant 19-00902S
⋆⋆⋆ Supported by Deutsche Forschungsgemeinschaft (DFG) under project SCHR 1118/8-2

many generators (*ffg* objects) yield precisely the coalgebras that are the target of generalized determinization; (2) finitely presentable (*fp*) objects are the ones that can be presented by finitely many generators and relations and yield the rational fixed point; and (3) finitely generated (*fg*) objects, which are the ones presented by finitely many generators (but possibly infinitely many relations). Taking the colimits of all coalgebras with ffg, fp, and fg carriers, respectively, yields three coalgebras φF, ϱF and ϑF which, under suitable assumptions on F, are all fixed points of F [7, 39, 54]. Our present paper is devoted to studying the fixed point φF, which we call the *locally ffg fixed point* of F. For a finitary endofunctor F preserving surjective and non-empty injective morphisms in $\mathscr{C}$, the three fixed points are related to each other and the terminal coalgebra νF as follows:

$$\varphi F \twoheadrightarrow \varrho F \twoheadrightarrow \vartheta F \rightarrowtail \nu F, \tag{1.1}$$

where $\twoheadrightarrow$ denotes a quotient coalgebra and $\rightarrowtail$ a subcoalgebra. The three right-hand fixed points are characterized by a universal property both as a coalgebra and (when inverting their coalgebra structure) as an algebra [7, 36, 39]; see [54] for one uniform proof. We recall this in more detail in Section 2.4.

The main contribution of the present paper is a new characterization of the locally ffg fixed point φF by a universal property as an algebra. As already observed by Urbat [54], as a coalgebra, φF does not satisfy the expected finality property since coalgebra homomorphisms from coalgebras with ffg carrier into φF may fail to be unique. A simple initiality property of φF as an algebra was recently established by Milius [38, Theorem 4.4]: φF is the initial ffg-Bloom algebra for F, where an *ffg-Bloom algebra* is an F-algebra equipped with an operation that assigns to every F-coalgebra carried by an ffg object a coalgebra-to-algebra morphism subject to a functoriality property. Equivalently, the ffg-Bloom algebras for F form the slice category $\varphi F/\mathsf{Alg}\,F$ [38, Proposition 4.5]. Here we introduce the notion of an *ffg-Elgot algebra* (Section 4), which is an algebra for F equipped with an operation that allows to take solutions of *effectful iterative equations* (see Remark 4.5) subject to two natural axioms. These axioms are inspired by and closely related to the axioms of (ordinary) Elgot algebras [6], which we recall in Section 3. We then prove that φF is the initial ffg-Elgot algebra (Theorem 4.11), which strengthens the previous initiality result.

In addition, we study the construction of *free* ffg-Elgot algebras. In the case of ordinary Elgot algebras, it was shown [6] that the rational fixed point $\varrho(F(-)+Y)$ is a free Elgot algebra on Y. In addition, the category of Elgot algebras is the Eilenberg-Moore category for the corresponding monad on $\mathscr{C}$. In the present paper, we prove that free ffg-Elgot algebras exist on every object Y of $\mathscr{C}$. But is it true that the free ffg-Elgot algebra on Y is $\varphi(F(-) + Y)$? We do not know the answer for arbitrary objects Y, but if Y is a free T-algebra (on a possibly infinite set of generators), the answer is affirmative (Theorem 4.24).

Finally, we prove that the category of ffg-Elgot algebras is monadic over $\mathscr{C}$, i.e. ffg-Elgot algebras are precisely the Eilenberg-Moore algebras for the monad that assigns to a given object Y of $\mathscr{C}$ its free ffg-Elgot algebra (Theorem 4.26).

This paper is a revised and extended version of our conference paper [5] containing full proofs.

Related Work and History. While our new notion of an ffg-Elgot algebra is directly based on the previous notion of Elgot algebra [6], studying operators taking solutions of recursive equation systems and their properties goes back a long way. The most well-known examples of such structures are probably the iteration theories of Bloom and Ésik [15] whose work is based on Elgot's seminal work [22] on the semantics of iterative specifications. Algebras for iteration were first studied by Nelson [44] (see also Tiuryn [53] for a related concept). Our work grows out of the coalgebraic approach to the semantics of iteration which started with Moss' work [43] on parametric corecursion. Independently, and almost at the same time, it was also realized by Ghani et al. [28, 29] and Aczel et al. [1, 2] that final coalgebras for parametrized functors $F(-) + Y$ give rise to a monad, whose structure generalizes substitution of infinite trees over a signature. Later it was shown by Milius [36] that one can approach this monad through algebras with unique solutions of recursive equations. The monad arising from the parametrized rational fixed points $\varrho(F(-) + Y)$ was introduced in [7] based on a category-theoretic generalization of Nelson's notion of iterative algebra. This generalizes Courcelle's regular trees [18] and their substitution. The monad of free ffg-Elgot algebras is a new example of a monad arising from parametrized coalgebras.

Outline of the Paper. We begin in Section 2 by recalling a number of preliminaries, e.g. on varieties and 'finite' objects in such categories. This material might be skipped by readers who are familiar with it. We also recall background on the four fixed points in (1.1), and, as a first highlight, we present in Proposition 2.5 an example of the locally ffg fixed point φF in a setting where the other three are trivial.

Section 3 is a brief recap on Elgot algebras and so can be skipped by expert readers who have seen them before.

The concept of ffg-Elgot algebras is introduced in Section 4. Readers who would like to see the connection of ffg-Elgot algebras to effectful iterative equations should jump right to Remark 4.5, where this connection is explained. The main technical results of our paper then follow as already explained. First, Theorem 4.11 shows that φF is the initial ffg-Elgot algebra. Second, Theorem 4.12 establishes, for a free object Y of our base variety $\mathscr{C}$, a one-to-one correspondence of pairs consisting of an ffg-Elgot algebra A for F and a morphism $Y \to A$ with ffg-Elgot algebras for $F(-) + Y$. This result turns out to be a key ingredient of the construction of free ffg-Elgot algebras from coalgebras for $F(-) + Y$ (see Construction 4.23 and Theorem 4.24) for a free object Y. Monadicity of ffg-Elgot algebras is etablished in Section 4.4.

We conclude the paper in Section 5.

Finally, in the short appendix a technical result concerning the construction of φF is presented.

Acknowedgements. We would like to thank the anonymous reviewers whose suggestions helped us to improve our paper.

2 Preliminaries

2.1 Varieties and 'Finite' Algebras

Throughout the paper we will work with a (finitary, many-sorted) variety $\mathscr{C}$ of algebras and an endofunctor F on it. Equivalently, $\mathscr{C}$ is the category of Eilenberg-Moore algebras for a finitary monad T on the category Set^S of S-sorted sets [11]. We will speak about objects of $\mathscr{C}$ (rather than algebras for T) and reserve the word 'algebra' for algebras for F. All the 'usual' categories of algebraic structures and their homomorphisms are varieties: monoids, (semi-)groups, rings, vector spaces over a fixed field, modules for a (semi-)ring, positive convex algebras, join-semilattices, Boolean algebras, distributive lattices, and many others. In each case, the corresponding monad T assigns to a set the free object on it, e.g. $TX = X^*$ for monoids, the finite power-set monad $T = \mathcal{P}_{\mathsf{f}}$ for join-semilattices, and the subdistribution monad $\mathscr{D}$ for positive convex algebras, etc.

As mentioned in the introduction, every variety $\mathscr{C}$ of algebras comes with three natural notions of 'finite' objects, each of which admits a neat category-theoretic characterization (see [11]):

Finitely presentable objects (fp objects, for short) can be presented by finitely many generators and relations. An object X is fp iff the covariant hom-functor $\mathscr{C}(X,-)\colon \mathscr{C} \to Set$ is *finitary*, i.e. it preserves filtered colimits. Recall that a category $\mathscr{D}$ is filtered if every finite subcategory has a cocone in $\mathscr{D}$, and a diagram is filtered if its scheme is a filtered category. We denote by $\mathscr{C}_{\mathsf{fp}}$ the full subcategory of $\mathscr{C}$ given by all fp objects. In our proofs we will use the well-known fact that every object X is the filtered colimit of the canonical diagram $\mathscr{C}_{\mathsf{fp}}/X \to \mathscr{C}$, i.e. objects in the diagram scheme are morphisms $P \to X$ in $\mathscr{C}$ with P fp.

Finitely generated objects (fg objects, for short) are presented by finitely many generators but, possibly, infinitely many relations. An object X is fg iff $\mathscr{C}(X,-)$ preserves filtered colimits with monic connecting morphisms. Hence, every fp object is fg but not conversely. In fact, the fg objects are precisely the (regular) quotients of the fp objects [11, Proposition 5.22].

Free finitely generated objects (ffg objects, for short) are the objects (TX_0, μ_{X_0}) where X_0 is a finite S-sorted set (i.e. the coproduct of all components X_s, $s \in S$ is finite). An object X is a split quotient of an ffg object iff $\mathscr{C}(X,-)$ preserves *sifted* colimits [11, Corollary 5.14]. Recall from [11] that sifted colimits are more general than filtered colimits: a sifted colimit is a colimit of a diagram $D\colon \mathscr{D} \to \mathscr{C}$ whose diagram scheme $\mathscr{D}$ is a sifted category, which means that finite products commute with colimits over $\mathscr{D}$ in Set. More precisely, $\mathscr{D}$ is sifted iff given any diagram $D\colon \mathscr{D} \times \mathscr{J} \to \mathsf{Set}$, where $\mathscr{J}$ is a finite discrete category, the canonical map

$$\operatorname*{colim}_{d\in\mathscr{D}} \Big(\prod_{j\in\mathscr{J}} D(d,j) \Big) \to \prod_{j\in\mathscr{J}} (\operatorname*{colim}_{d\in\mathscr{D}} D(d,j))$$

is an isomorphism. For instance, every filtered category and every category with finite coproducts is sifted [11, Example 2.16].

The category $\mathscr{C}$ is cocomplete and the forgetful functor $\mathscr{C} \to \mathsf{Set}^S$ preserves and reflects sifted colimits, that is, sifted colimits in $\mathscr{C}$ are formed on the level of underlying sets [11, Proposition 2.5].

Remark 2.1. A finitely cocomplete category has sifted colimits if and only if it has filtered colimits and reflexive coequalizers, i.e. coequalizers of parallel pairs of epimorphisms with a joint splitting. Moreover a functor preserves sifted colimits if and only if it preserves filtered colimits and reflexive coequalizers [10].

We denote by $\mathscr{C}_{\mathsf{ffg}}$ the full subcategory of ffg objects of $\mathscr{C}$. Analogously to the fact that every object of $\mathscr{C}$ is a filtered colimit of fp objects, every object X is a sifted colimit of the canonical diagram $\mathscr{C}_{\mathsf{ffg}}/X \to \mathscr{C}$; this follows from [11, Proposition 5.17].

2.2 Relation between the object classes.

We already mentioned that every fp object is fg (but not conversely, in general). Clearly, every ffg object is fp, but not conversely in general (e.g. consider any fp monoid which is not of the form X^* for some finite set X). So, in general, we have full embeddings

$$\mathscr{C}_{\mathsf{ffg}} \overset{\neq}{\hookrightarrow} \mathscr{C}_{\mathsf{fp}} \overset{\neq}{\hookrightarrow} \mathscr{C}_{\mathsf{fg}}.$$

In rare cases, all three object classes coincide; e.g. in Set (considered as a variety) and the category of vector spaces over a field.

The equation $\mathscr{C}_{\mathsf{fg}} = \mathscr{C}_{\mathsf{fp}}$ holds true, for example, for all locally finite varieties (i.e. where ffg objects are carried by finite sets), e.g. Boolean algebras, distributive lattices or join-semilattices); for positively convex algebras [51], commutative monoids [27, 46], abelian groups, and more generally, in any category of (semi-)modules for a semiring $\mathbb{S}$ that is *Noetherian* in the sense of Ésik and Maletti [24]. That means that every subsemimodule of an fg semimodule is fg itself. For example, the following semirings are Noetherian: every finite semiring, every field, every principal ideal domain such as the ring of integers and therefore every finitely generated commutative ring by Hilbert's Basis Theorem. The tropical semiring $(\mathbb{N} \cup \{\infty\}, \min, +, \infty, 0)$ is *not* Noetherian [23]. The usual semiring of natural numbers is not Noetherian either, but for the category of $\mathbb{N}$-semimodules (= commutative monoids), $\mathscr{C}_{\mathsf{fp}} = \mathscr{C}_{\mathsf{fg}}$ still holds.

2.3 Functors and Liftings

We will consider coalgebras for functors F on the variety $\mathscr{C}$. In many cases F is a *lifting* of a functor on many-sorted sets, i.e. there is a functor $F_0 \colon \mathsf{Set}^S \to \mathsf{Set}^S$ such that the square below commutes, where $U \colon \mathscr{C} \to \mathsf{Set}^S$ denotes the forgetful functor.

$$\begin{tikzcd} \mathscr{C} \arrow[r, "F"] \arrow[d, "U"'] & \mathscr{C} \arrow[d, "U"] \\ \mathsf{Set}^S \arrow[r, "F_0"'] & \mathsf{Set}^S \end{tikzcd}$$

It is well-known [12,31] that liftings of a given functor F_0 on Set^S to $\mathscr{C}$, the variety given by the monad (T, η, μ), are in bijective correspondence with distributive laws of that monad over the functor F_0. This means natural transformations $\lambda\colon TF_0 \to F_0T$ such that the following two diagrams commute:

$$\begin{tikzcd} F_0 \arrow[d, "\eta F_0"'] \arrow[dr, "F_0\eta"] & \\ TF_0 \arrow[r, "\lambda"'] & F_0T \end{tikzcd} \qquad \begin{tikzcd} TTF_0 \arrow[r, "T\lambda"] \arrow[d, "\mu F_0"'] & TF_0T \arrow[r, "\lambda T"] & F_0TT \arrow[d, "F_0\mu"] \\ TF_0 \arrow[rr, "\lambda"'] & & F_0T \end{tikzcd}$$

Given a distributive law λ of T over F_0, the corresponding lifting F assigns to a T-algebra (A, a) the T-algebra $(F_0A, F_0a \cdot \lambda_A)$. It was observed by Turi and Plotkin [45] that a final coalgebra for F_0 lifts to a final coalgebra for the lifting F. Indeed, denoting by $\xi\colon \nu F_0 \to F_0(\nu F_0)$ the final coalgebra for F_0, we obtain a canonical T-algebra structure on νF_0 by corecursion, i.e. as the unique coalgebra homomorphism $a\colon T(\nu F_0) \to \nu F_0$ in the diagram below:

$$\begin{tikzcd} T(\nu F_0) \arrow[r, "T\xi"] \arrow[d, "a"'] & TF_0(\nu F_0) \arrow[r, "\lambda_{\nu F_0}"] & F_0T(\nu F_0) \arrow[d, "F_0a"] \\ \nu F_0 \arrow[rr, "\xi"'] & & F_0(\nu F_0) \end{tikzcd}$$

It is easy to verify that a is an Eilenberg-Moore algebra and that this turns νF_0 into the final coalgebra for the lifting F. Note that the above square expresses that $(\nu F_0, a, \xi)$ is a λ-*bialgebra*, and it is the final one [45].

Coalgebras for lifted functors are significant because the targets of *finite* coalgebras X under *generalized determinization* [49] are precisely those coalgebras for the lifting F that are carried by ffg objects (TX, μ_X). In more detail, generalized determinization is the process of turning a given coalgebra $c\colon X \to F_0TX$ in Set^S into a coalgebra for the lifting F: one uses the freeness of TX and the fact that FTX is a T-algebra to extend c to a T-algebra homomorphism $c^*\colon TX \to FTX$. The *coalgebraic language semantics* [16] of (X, c) is then the final semantics of c^*. A classical instance of this is the language semantics of non-deterministic automata considered as coalgebras $X \to \{0,1\} \times (\mathcal{P}_\mathsf{f} X)^\Sigma$; here the generalized determinization with $T = \mathcal{P}_\mathsf{f}$ and $F_0 = \{0,1\} \times X^\Sigma$ on Set is the well-known subset construction turning a non-deterministic automaton into a deterministic one.

2.4 Four Fixed Points

Fixed points of a functor F are (co)algebras whose structure is invertible. Let us now consider a finitary endofunctor $F\colon \mathscr{C} \to \mathscr{C}$ on our variety. Then F has a terminal coalgebra [3, Theorem 6.10], which we denote by νF. Its coalgebra structure $\nu F \to F(\nu F)$ is an isomorphism by Lambek's lemma [34], and so νF is a fixed point of F. The terminal coalgebra νF is fully abstract w.r.t. behavioural equivalence: given F-coalgebras (X, c) and (Y, d), two states $x \in X$ and $y \in Y$ are called *behavioural equivalent* if there exists a pair of coalgebra homomorphisms $f\colon (X, c) \to (Z, e)$ and $g\colon (Y, d) \to (Z, e)$ such that $f(x) = g(y)$. Behavioural equivalence instantiates to well-known notions of indistinguishability of system states, e.g. for $F = \mathcal{P}_{\mathsf{f}}$, it is strong bisimilarity of states in finitely branching transitions systems, and for $FX = \{0, 1\} \times X^{\Sigma}$ it yields the language equivalence of states in deterministic automata. One can show that two states are behaviourally equivalent if and only if they are identified under the unique coalgebra homomorphisms into νF.

There are three further fixed points of F obtained from 'finite' coalgebras, where 'finite' can mean each of the three notions discussed in Subsection 2.1. More precisely, denote by

$$\mathsf{Coalg}\, F$$

the category of all F-coalgebras. We consider its full subcategories given by all coalgebras with fp, fg, and ffg carriers, respectively, and we denote them as shown below:

$$\mathsf{Coalg}_{\mathsf{ffg}}\, F \hookrightarrow \mathsf{Coalg}_{\mathsf{fp}}\, F \hookrightarrow \mathsf{Coalg}_{\mathsf{fg}}\, F \hookrightarrow \mathsf{Coalg}\, F.$$

Since the three subcategories above are essentially small, we can form coalgebras as the colimits of the above inclusions as follows:

$$\begin{aligned} \varphi F &= \operatorname{colim}(\mathsf{Coalg}_{\mathsf{ffg}}\, F \hookrightarrow \mathsf{Coalg}\, F), \\ \vartheta F &= \operatorname{colim}(\mathsf{Coalg}_{\mathsf{fg}}\, F \hookrightarrow \mathsf{Coalg}\, F), \\ \varrho F &= \operatorname{colim}(\mathsf{Coalg}_{\mathsf{fp}}\, F \hookrightarrow \mathsf{Coalg}\, F). \end{aligned}$$

Note that the latter two colimits are filtered; in fact, $\mathsf{Coalg}_{\mathsf{fg}}\, F$ and $\mathsf{Coalg}_{\mathsf{fp}}\, F$ are clearly closed under finite colimits in $\mathsf{Coalg}\, F$, whence they are filtered categories. The first colimit is a sifted colimit since its diagram scheme $\mathsf{Coalg}_{\mathsf{ffg}}\, F$ is closed under finite coproducts [38, Lemma 3.6]. In what follows, the objects of $\mathsf{Coalg}_{\mathsf{ffg}}\, F$ are called *ffg-coalgebras*.

We now discuss the three coalgebras above in more detail.

The rational fixed point is the coalgebra ϱF. This is a fixed point as proved by Adámek, Milius and Velebil [7]. In addition, ϱF is characterized by a universal property both as a coalgebra and as an algebra:

(1) As a coalgebra, ϱF is the terminal *locally finitely presentable* (lfp) coalgebra, where a coalgebra is called lfp if it is a filtered colimit of a diagram formed by coalgebras from $\mathsf{Coalg}_{\mathsf{fp}}\, F$ [37].

(2) As an algebra, ϱF is the initial iterative algebra for F.

An *iterative algebra* is an F-algebra $a\colon FA \to A$ such that every *fp-equation*, i.e. a morphism $e\colon X \to FX + A$ with X fp, has a unique *solution* in A. The latter means that there exists a unique morphism $e^\dagger$ such that the following square commutes[3]:

$$\begin{tikzcd} X \arrow[r, "e^\dagger"] \arrow[d, "e"'] & A \\ FX + A \arrow[r, "Fe^\dagger + A"'] & FA + A \arrow[u, "{[a,A]}"'] \end{tikzcd} \tag{2.1}$$

This notion is a categorical generalization of iterative Σ-algebras for a single-sorted signature Σ originally introduced by Nelson [44]; see also Tiuryn [53] for a closely related concept.

The locally finite fixed point is the coalgebra ϑF. This coalgebra was recently introduced and studied by Milius, Pattinson and Wißmann [39, 40] for a finitary endofunctor F preserving non-empty monos. They proved ϑF to be a fixed point of F and characterized by two universal properties analogous to the rational fixed point:

(1) As a coalgebra, ϑF is the terminal *locally finitely generated* (lfg) coalgebra, where a coalgebra is called lfg if it is a colimit of a directed diagram of coalgebras in $\mathsf{Coalg}_{\mathsf{fg}}\, F$.

(2) As an algebra, ϑF is the initial fg-iterative algebra for F, where fg-iterative is simply the variation of iterative above where the domain object of $e\colon X \to FX+A$ is required to be fg in lieu of fp.

Moreover, ϑF is always a subcoalgebra of νF [40, Theorem 3.10] and thus fully abstract w.r.t. behavioral equivalence.

The locally ffg fixed point is the coalgebra φF. Recently, Urbat [54] has proved that φF is indeed a fixed point of F, provided that F preserves sifted colimits. Actually, in *loc. cit.* the coalgebra φF is defined to be the colimit of all F-coalgebras whose carrier is a split quotient of an ffg object. However, this is the same colimit as above, as we prove in the Appendix.

Moreover, *loc. cit.* provides a general framework that allows to prove uniformly that all four coalgebras ϱF, ϱF, ϑF and νF are fixed points. In addition, a uniform proof of the universal properties of ϱF, ϑF and νF is given.

Somewhat surprisingly, the coalgebra φF fails to have the finality property w.r.t. to coalgebras in $\mathsf{Coalg}_{\mathsf{ffg}}\, F$: Urbat [54, Example 4.12] gives such a counter-example, see Section 2.5 below. This also shows that φF cannot have a universal property as some kind of iterative algebra (i.e. where solutions are unique).

[3] Note that in a diagram we usually denote identity morphisms simply by the (co)domain object.

Relations between the Fixed Points. Recall that a *quotient* of a coalgebra is represented by a coalgebra homomorphism carried by a regular epimorphism (= surjective algebra morphism) in $\mathscr{C}$. Suppose we have a finitary functor F on $\mathscr{C}$ preserving surjective morphisms and non-empty injective ones.[4] Then the subcoalgebra ϑF of νF is a quotient of ϱF, which in turn is a quotient of φF [38, 40]; see (1.1):

$$\varphi F \twoheadrightarrow \varrho F \twoheadrightarrow \vartheta F \rightarrowtail \nu F.$$

Whenever $\mathscr{C}_{\mathsf{fp}} = \mathscr{C}_{\mathsf{fg}}$, we clearly have $\mathsf{Coalg}_{\mathsf{fp}}\, F = \mathsf{Coalg}_{\mathsf{fg}}\, F$ and hence $\varrho F \cong \vartheta F$ (i.e. ϱF is fully abstract w.r.t. behavioral equivalence). If $\mathscr{C}_{\mathsf{fp}} = \mathscr{C}_{\mathsf{fg}} = \mathscr{C}_{\mathsf{ffg}}$, ϱF and ϑF coincide with φF as well. Moreover, Milius [38] introduced the notion of a *proper* functor (generalizing the notion of a proper semiring of Ésik and Maletti [23]) and proved that a functor F is proper if and only if the three fixed points coincide, i.e. the picture above collapses to $\varphi F \cong \varrho F \cong \vartheta F \hookrightarrow \nu F$. Loc. cit. also shows that on a variety $\mathscr{C}$ where fg objects are closed under taking kernel pairs, every endofunctor mapping kernel pairs to weak pullbacks in Set is proper [38, Proposition 5.10].[5]

Instances of the three fixed points φF, ϑF and ϱF have mostly been considered for proper functors (where the three are the same, e.g. for functors on Set), or else on algebraic categories where $\mathscr{C}_{\mathsf{fp}} = \mathscr{C}_{\mathsf{fg}}$ (where $\varrho F \cong \vartheta F$, i.e. the rational and locally finite fixed points coincide). We shall see in Section 2.5 that φF can be different from ϱF and ϑF (even when the latter two are isomorphic). Before that we illustrate the relationship of ϱF and ϑF to νF by a number of well-known important examples:

Examples 2.2. (1) For the set functor $FX = \{0,1\} \times X^{\Sigma}$, whose coalgebras are deterministic automata with the input alphabet Σ, the terminal coalgebra is formed by all formal languages on Σ and the three fixed points are formed by all regular languages.

(2) For a signature $\Sigma = (\Sigma_n)_{n<\omega}$ of operation symbols with prescribed arity we have the associated polynomial endofunctor on Set given by $F_\Sigma X = \coprod_{n<\omega} \Sigma_n \times X^n$. Its terminal coalgebra is carried by the set of all (finite and infinite) Σ-trees, i.e. rooted and ordered trees where each node with n-children is labelled by an n-ary operation symbol. The three fixed points are all equiv to the subcoalgebra given by *rational* (or regular [18]) Σ-trees, i.e. those Σ-trees that have only finitely many different subtrees (up to isomorphism) This characterization is due to Ginali [30]. For example, for the signature Σ formed by a binary operation symbol $*$ and a constant c the following infinite Σ-tree (here written as an infinite term) is rational:

$$c * (c * (c * \cdots)));$$

[4] These are mild assumptions; e.g. if $\mathscr{C}$ is single-sorted and F a lifting of a set functor, then these conditions are fulfilled.

[5] Note that these conditions are fulfilled in particular by every locally finite variety and every category of semirings for a Noetherian semiring and any lifted endofunctor whose underlying Set functor preserves weak pullbacks.

in fact, up to isomorphism its only subtrees are the whole tree and the single-node tree labelled by c).

(3) Consider the endofunctor $FX = \mathbb{S} \times X^{\Sigma}$ on the category of semimodules for the semiring $\mathbb{S}$. The fixed point ϑF, which is isomorphic to ϱF if $\mathbb{S}$ is Noetherian, is formed by all formal power series (i.e. elements of $\mathbb{S}^{\Sigma^*}$) recognizable by finite $\mathbb{S}$-weighted automata. From the Kleene-Schützenberger theorem [48] (see also [14]) it follows that these are, equivalently, the *rational* formal power-series.

(4) For $FX = k \times X$ on Set the terminal coalgebra is carried by the set k^ω of all streams on k, and the three fixed points are equal; they are formed by all eventually periodic streams (also called lassos). If k is a field, and we consider F as a functor on vector spaces over k, we obtain rational streams [47].

(5) Recall [20] that a *positively convex algebra* is a set X equipped with finite convex sum operations. This means that for every n and $p_1, \dots, p_n \in [0,1]$ with $\sum_{i=1}^n p_i \le 1$ we have an n-ary operation assigning to $x_1, \dots, x_n \in X$ an element $\boxplus_{i=1}^{n} p_i x_i$ subject to the following axioms:

(a) $\boxplus_{i=1}^{n} p_i^k x_i = x_k$ whenever $p_k^k = 1$ and $p_i^k = 0$ for $i \neq k$, and

(b) $\boxplus_{i=1}^{n} p_i \left(\boxplus_{j=1}^{k} q_{i,j} x_j \right) = \boxplus_{j=1}^{k} \left(\sum_{i=1}^{n} p_i q_{i,j} \right) x_j$.

For $n = 1$ we write the convex sum operation for $p \in [0,1]$ simply as px. Positively convex algebras together with maps preserving convex sums in the obvious sense form the category PCA. Note that PCA is (isomorphic to) the Eilenberg-Moore category for the monad $\mathcal{D}$ of finitely supported subprobability distributions on sets.

Sokolova and Woracek [52] have recently proved that the functor $FX = [0,1] \times X^{\Sigma}$ and its subfunctor $\hat{F}$ mapping a set X to the set of all pairs (o, f) in $[0,1] \times X^{\Sigma}$ satisfying

$$\forall s \in \Sigma : \exists p_s \in [0,1], x_s \in X : o + \sum_{s \in \Sigma} p_s \le 1, f(s) = p_s x_s$$

are proper functors on PCA. Hence, for those functors our three fixed points coincide. In particular, the latter functor $\hat{F}$ is used to capture the complete trace semantics of generative probabilistic transition systems [50]. Hence, for $\hat{F}$, our three fixed points collect precisely the probabilistic traces of finite such systems.

(6) Given an alphabet Σ, for the functor $FX = \{0,1\} \times X^{\Sigma}$ on the category of idempotent semirings the locally finite fixed point ϑF is formed by all context-free languages [40]. Descriptions of ϱF and φF are unknown in this case.

More generally, consider first the category of associative $\mathbb{S}$-algebras for the commutative semiring $\mathbb{S}$, i.e. $\mathbb{S}$-semimodules equipped with an additional monoid structure such that multiplication is an $\mathbb{S}$-semimodule morphism in each of its arguments. This is the Eilenberg-Moore category for the monad $\mathbb{S}\langle - \rangle$ assigning to each set X the set of $\mathbb{S}$-polynomials of over X, i.e. functions $X^* \to \mathbb{S}$ with finite support. This is not quite the category $\mathscr{C}$, but one considers Σ-pointed $\mathbb{S}$-algebras, where Σ is an input alphabet, i.e. $\mathbb{S}$-algebras A equipped with a map $\Sigma \to A$. The corresponding monad is $\mathbb{S}\langle - + \Sigma\rangle$. The terminal coalgebra for the functor $FX = \mathbb{S} \times X^{\Sigma}$ on $\mathscr{C}$ is again carried by the set of all formal power series over Σ, and the locally finite fixed point ϑF is formed by all constructively $\mathbb{S}$-algebraic formal power-series [39]. (The original definition of those power-series goes back to Fliess [26], see also [21]; an equivalent coalgebraic characterization was first provided by Winter et al. [55].)

Remark 2.3. The rational fixed point ϱF and the locally finite one, ϑF, are defined and studied more generally than in the present setting, namely for finitary functors F on a locally finitely presentable category $\mathscr{C}$ (see Adámek and Rosický [9] for an introduction to locally presentable categories); see [7, 37] for ϱF and [39, 40] for ϑF.

The following are instances of ϱF and ϑF for F on a locally finitely presentable category $\mathscr{C}$:

(1) Consider the functor category $\mathsf{Set}^{\mathcal{F}}$, where $\mathcal{F}$ is the category of finite sets and maps and denote by $V\colon \mathcal{F} \hookrightarrow \mathsf{Set}$ is the full embedding. Further, consider the endofunctor $FX = V + X \times X + \delta(X)$ with $\delta(X)(n) = X(n+1)$. This is a paradigmatic example of a functor arising from a *binding signature* for which initial semantics was studied by Fiore et al. [25].

The final coalgebra νF is carried by the presheaf of all λ-trees modulo α-equivalence [8]. In fact, the functor νF assigns to n the set of all (finite and infinite) λ-trees in n free variables (note that such a tree may have infinitely many bound variables). Moreover, ϱF is carried by the rational λ-trees, where an α-equivalence class is called *rational* if it contains at least one λ-tree which has (up to isomorphism) only finitely many different subtrees (see *op. cit.*).

The coalgebra of all λ-trees with finitely many free variables modulo α-equivalence also appears as the final coalgebra for a very similar functor on the category of nominal sets [33]. Moreover, the rational λ-trees form its rational fixed point [42]. Similarly for any functor on nominal sets arising from a binding signature [33, 41].

(2) Courcelle's algebraic trees [18] occur as a locally finite fixed point. In more detail, fix a polynomial functor $H_\Sigma\colon \mathsf{Set} \to \mathsf{Set}$ and consider the category $\mathscr{C} = H_\Sigma/\mathsf{Mnd_f}(\mathsf{Set})$ of H_Σ-pointed finitary monads M on Set, i.e. those equipped with a natural transformation $H_\Sigma \to M$. The assignment $M \mapsto H_\Sigma M + \mathsf{Id}$ provides an endofunctor $F\colon \mathscr{C} \to \mathscr{C}$ whose terminal coalgebra is carried by the monad T_Σ assigning to a set X the set of all Σ-trees over X. The locally finite fixed point ϑF is the monad A_Σ of algebraic Σ-trees [39]. Note that in this category $\mathscr{C}$, fp and fg objects do not coincide. Hence, it is unclear whether ϑF and ϱF are isomorphic.

In the setting of general locally finitely presentable categories, there is no analogy to φF, of course.

2.5 A Nontrivial Example of the Locally ffg Fixed Point

We now present a new example where only φF is interesting whereas the other three fixed points are trivial.

We consider the monad T on Set whose algebras are the algebras with one unary operation u (with no equation):

$$TX = \mathbb{N} \times X \quad \text{with} \quad u(n,x) = (n+1,x).$$

The unit η and multiplication μ of this monad are given by $\eta_X(x) = (0,x)$ and $\mu_X(n,(m,x)) = (n+m,x)$. Since TX is the free algebra with one unary operation on X, its elements (n,x) correspond to terms $u^n(x)$. Let F be the identity functor Id on the category $\mathscr{C} = \mathsf{Set}^T$. The final coalgebra for Id is lifted from Set: it is the trivial algebra on 1 with id_1 as its coalgebra structure. Since 1 is clearly finitely presented by one generator x and the relation $u(x) = x$, both of the diagrams $\mathsf{Coalg}_{\mathsf{fp}}\,\mathsf{Id}$ and $\mathsf{Coalg}_{\mathsf{fg}}\,\mathsf{Id}$ have a terminal object. This is then their colimit, whence $\varrho\mathsf{Id} \cong \vartheta\mathsf{Id} \cong 1$.

However, $\varphi\mathsf{Id}$ is non-trivial and interesting. An ffg-coalgebra $TX \xrightarrow{\gamma} TX$ may be viewed (by restricting it to its generators in X) as obtained by generalized determinization of an FT-coalgebra with $F = \mathsf{Id}$ on Set, i.e. a map $X \xrightarrow{\langle o,\delta\rangle} \mathbb{N}\times X$ that we call *stream coalgebra*. Given a state $x \in X$, we call the sequence of natural numbers

$$(\, o(x), o(\delta(x)), o(\delta^2(x)), \dots\,)$$

the *stream generated by* x. Since the set X is finite, this stream is eventually periodic, i.e. of the form $s = s_0 s_1^\omega$ for finite lists s_0 and s_1 of natural numbers. (Here $(-)^\omega$ means infinite iteration.) Two eventually periodic streams $s = s_0 s_1^\omega$ and $t = t_0 t_1^\omega$ with $s_1 = (s_{1,0}, \dots, s_{1,p-1})$ and $t_1 = (t_{1,0}, \dots, t_{1,q-1})$ are called *equivalent* if one has

$$q \cdot \sum_{i<p} s_{1,i} = p \cdot \sum_{j<q} t_{1,j}, \tag{2.2}$$

i.e. the two lists s_1 and t_1 have the same arithmetic mean (or, equivalently, the entries of the two lists s_1^q and t_1^p of length $p \cdot q$ have the same sum). For instance, the streams

$$s = (1,2,7,4)(1,3,2)^\omega = (1,2,7,4,1,3,2,1,3,2,1,3,2,\dots)$$

and

$$t = (5,6)(0,4)^\omega = (5,6,0,4,0,4,0,4,0,4,\dots)$$

are equivalent. Note that the above notion of equivalence is well-defined, i.e. not depending on the choice of the finite lists s_0, s_1 and t_0, t_1 in the representation of s and t. In fact, given alternative representations $s = \overline{s}_0\overline{s}_1^\omega$ and $t = \overline{t}_0\overline{t}_1^\omega$ with $\overline{s}_1 = (\overline{s}_{1,0}, \dots, \overline{s}_{\overline{p}-1})$ and $\overline{t}_1 = (\overline{t}_{1,0}, \dots, \overline{t}_{1,\overline{q}-1})$, the lists $s_1^{\overline{p}}$ and $\overline{s}_1^p$ are equal up to cyclic shift, as are the lists $t_1^{\overline{q}}$ and $\overline{t}_1^q$. Therefore from (2.2) it follows that

$$\overline{q} \cdot q \cdot p \cdot \sum_{i<\overline{p}} \overline{s}_{1,i} = \overline{q} \cdot q \cdot \overline{p} \cdot \sum_{i<p} s_{1,i} = \overline{q} \cdot \overline{p} \cdot p \cdot \sum_{j<q} t_{1,j} = \overline{p} \cdot p \cdot q \cdot \sum_{j<\overline{q}} \overline{t}_{1,j}.$$

Dividing by $p \cdot q$ yields the required result:

$$\overline{q} \cdot \sum_{i<\overline{p}} \overline{s}_{1,i} = \overline{p} \cdot \sum_{j<\overline{q}} \overline{t}_{1,j}.$$

Remark 2.4. (1) In the proof of Proposition 2.5 further below we use the following well-known fact about colimits of sets. For every diagram $D\colon \mathscr{D} \to \mathsf{Set}$, a cocone $c_i\colon Di \to C$ ($i \in \mathscr{D}$) is a colimit iff (a) the colimit injections c_i are jointly surjective, i.e. $C = \bigcup c_i[Di]$, and (b) given $c_i(x) = c_j(y)$ for some pair $x \in Di, y \in Dj$, there exists a zig-zag of morphisms of $\mathscr{D}$ whose D-image connects x and y.

(2) Moreover, if D is a filtered diagram, then condition (b) can be substituted by the condition that when two elements $x, y \in Di$ are merged by c_i then they are also merged by $Dh\colon Di \to Dj$ for some morphism $h\colon i \to j$ of $\mathscr{D}$.

Proposition 2.5. *The coalgebra $\varphi\mathsf{Id}$ is carried by the set of equivalence classes (cf.* (2.2)*) of eventually periodic streams.*

In more detail, the unary operation and the coalgebra structure are both given by $\mathsf{id}\colon \varphi\mathsf{Id} \to \varphi\mathsf{Id}$, and for every Id-coalgebra (TX, γ_X) with X finite, the colimit injection $\gamma_X^\sharp\colon TX \to \varphi\mathsf{Id}$ maps $(m, x) \in TX$ to the equivalence class of the stream generated by x.

Proof. (1) We first show that the above morphisms $(-)^\sharp$ form a cocone. Given an ffg-coalgebra (TX, γ_X) for Id and elements $(m, x), (n, y) \in TX$ with $\gamma_X(m, x) = (n, y)$, the stream generated by y is the tail of the stream generated by x, and thus the two streams are equivalent. This shows that $\gamma_X^\sharp$ is a coalgebra homomorphism.

To show that the morphisms $(-)^\sharp$ form a cocone, suppose that $h\colon (TX, \gamma_X) \to (TY, \gamma_Y)$ is a homomorphism in $\mathsf{Coalg}_{\mathsf{ffg}}\,\mathsf{Id}$, and let $(m, x) \in TX$ and $(n, y) \in TY$ with $h(m, x) = (n, y)$ be given. We need to show that the streams generated by x and y are equivalent. Denote by

$$(m_j, x_j) := \gamma_X^j(m, x) \quad \text{and} \quad (n_j, y_j) := \gamma_Y^j(n, y) \qquad (j = 0, 1, 2, \ldots) \tag{2.3}$$

the states reached from (m, x) and (n, y), resp., after j steps. Since h is a coalgebra homomorphism, one has $h(m_j, x_j) = (n_j, y_j)$ for all j. Since X is finite, there exist natural numbers $k \geq 0$ and $p > 0$ with $x_k = x_{k+p}$. Then the eventually periodic stream generated by x is given by

$$(m_1 - m_0, m_2 - m_1, \ldots, m_k - m_{k-1})(m_{k+1} - m_k, \ldots, m_{k+p} - m_{k+p-1})^\omega$$

Since $h(m_k, x_k) = (n_k, y_k)$ and $h(m_{k+p}, x_{k+p}) = (n_{k+p}, y_{k+p})$, one has $y_k = y_{k+p}$, which implies that y generates the stream

$$(n_1 - n_0, n_2 - n_1, \ldots, n_k - n_{k-1})(n_{k+1} - n_k, \ldots, n_{k+p} - n_{k+p-1})^\omega$$

To show that the streams generated by x and y are equivalent, it suffices to verify that $m_{k+p} - m_k = n_{k+p} - n_k$, as this entails that

$$p \cdot \sum_{i<p} m_{k+i+1} - m_{k+i} = p \cdot (m_{k+p} - m_k) = p \cdot (n_{k+p} - n_k) = p \cdot \sum_{i<p} n_{k+i+1} - n_{k+i}.$$

To prove the desired equation, we compute

$$\begin{aligned}(n_{k+p}, y_{k+p}) &= h(m_{k+p}, x_{k+p})\\ &= h(m_{k+p}, x_k)\\ &= h(m_{k+p} - m_k + m_k, x_k)\\ &= (m_{k+p} - m_k + n_k, y_k)\end{aligned}$$

where the last equality uses that $h(m_k, x_k) = (n_k, y_k)$ and that h is a morphism of $\mathscr{C}$. This implies $n_{k+p} = m_{k+p} - m_k + n_k$.

(2) We prove that the cocone $(-)^\sharp$ is a colimit cocone. Since sifted colimits in $\mathsf{Coalg}\,\mathsf{Id}$ are formed as in $\mathscr{C}$ and thus as in Set, we can apply Remark 2.4: we will show that (a) the morphisms $\gamma_X^\sharp$ are jointly surjective and (b) given ffg-coalgebras (TX, γ_X) and (TY, γ_Y) and two states $(m, x) \in TX$ and $(n, y) \in TY$ merged by $\gamma_X^\sharp$ and $\gamma_Y^\sharp$, there exists a zig-zag in $\mathsf{Coalg}_{\mathsf{ffg}}\,\mathsf{Id}$ connecting the two states. Statement (a) is clear because finite stream coalgebras generate precisely the eventually periodic streams. For (b), we adapt the argument of the first part of our proof and continue to use the notation (2.3). Since X and Y are finite, there exist natural numbers $k \geq 0$ and $p > 0$ with $x_k = x_{k+p}$ and $y_k = y_{k+p}$. As the streams generated by x and y are equivalent, one has $m_{k+p} - m_k = n_{k+p} - n_k$. Consider the ffg-coalgebra (TZ, γ_Z) with $Z = \{z_0, z_1, \dots, z_{k+p-1}\}$, and γ_Z defined on the generators by

$$\gamma_Z(z_j) = (0, z_{j+1}) \;\; (j < k + p - 1) \quad \text{and} \quad \gamma_Z(z_{k+p-1}) = (m_{k+p} - m_k, z_k).$$

Form the morphisms $g\colon TZ \to TX$ and $h\colon TZ \to TX$ given on generators by

$$g(z_j) = (m_j, x_j) \quad \text{and} \quad h(z_j) = (n_j, y_j) \qquad (j < k + p).$$

Then g is a coalgebra homomorphism. Indeed, for $j < k + p - 1$ we have

$$\begin{aligned}g(\gamma_Z(z_j)) &= g(0, z_{j+1}) && (\text{def. } \gamma_Z)\\ &= (m_{j+1}, x_{j+1}) && (\text{def. } g)\\ &= \gamma_X(m_j, x_j) && (\text{def. } m_{j+1},\ x_{j+1})\\ &= \gamma_X(g(z_j)) && (\text{def. } g)\end{aligned}$$

and moreover

$$\begin{aligned}g(\gamma_Z(z_{k+p-1})) &= g(m_{k+p} - m_k, z_k) && (\text{def. } \gamma_Z)\\ &= (m_{k+p} - m_k + m_k, x_k) && (\text{def. } g)\\ &= (m_{k+p}, x_{k+p})\\ &= \gamma_X(m_{k+p-1}, x_{k+p-1}) && (\text{def. } m_{k+p},\ x_{k+1})\\ &= \gamma_X(g(z_{k+p-1})). && (\text{def. } g)\end{aligned}$$

Analogously for h. Thus we have constructed a zig-zag

$$(TX, \gamma_X) \xleftarrow{g} (TZ, \gamma_Z) \xrightarrow{h} (TY, \gamma_Y)$$

in $\mathsf{Coalg}_{\mathsf{ffg}}\,\mathsf{Id}$ connecting (m, x) and (n, y), as required. $\square$

Observe that every non-empty ffg-coalgebra (TX, γ_X) admits infinitely many coalgebra homomorphisms into $\varphi\mathsf{Id}$. For instance, any constant map into $\varphi\mathsf{Id}$ is one. This shows that, in general, the coalgebra φF is not final w.r.t. the ffg-coalgebras.

3 Recap: Elgot Algebras

In this section we briefly recall the notion of an Elgot algebra [6] and some key results in order to contrast this with our subsequent development of ffg-Elgot algebras in Section 4. Throughout this section we assume the endofunctor $F\colon \mathscr{C} \to \mathscr{C}$ to be finitary.

Recall from Section 2.4 that an *fp-equation* is a morphism

$$e\colon X \to FX + A,$$

where X is an fp object (of variables) and A an arbitrary object of *parameters*.

Furthermore, if A carries the structure of an F-algebra $a\colon FA \to A$, then a *solution* of e in A is a morphism $e^\dagger\colon X \to A$ such that the square (2.1) commutes.

Notation 3.1. We use the following notation for fp-equations:

(1) Given an fp-equation $e\colon X \to FX + A$ and a morphism $h\colon A \to B$ we have an fp-equation

$$h \bullet e = (\, X \xrightarrow{e} FX + A \xrightarrow{FX+h} FX + B\,).$$

(2) Given a pair of fp-equations $e\colon X \to FX + Y$ and $f\colon Y \to FY + Z$ we combine them into the following fp-equation

$$e \blacksquare f = (\, X + Y \xrightarrow{[e,\mathsf{inr}]} FX + Y \xrightarrow{FX+f} FX + FY + Z \xrightarrow{\mathsf{can}+Z} F(X+Y) + Z\,),$$

where $\mathsf{can} = [F\mathsf{inl}, F\mathsf{inr}]\colon FX+FY \to F(X+Y)$ denotes the canonical morphism.

Definition 3.2 [6]. An *Elgot algebra* is a triple $(A, a, \dagger)$ where (A, a) is an F-algebra and $\dagger$ is an operation

$$\frac{e\colon X \to FX + A}{e^\dagger\colon X \to A}$$

assigning to every fp-equation in A a solution, subject to the following two conditions:

(1) *Weak Functoriality.* Given a pair of fp-equations $e\colon X \to FX+Z$ and $f\colon Y \to FY + Z$, where Z is an fp object, and a coalgebra homomorphism $m\colon X \to Y$ for $F(-) + Z$, then for every morphism $h\colon Z \to A$ we have $(h \bullet f)^\dagger \cdot m = (h \bullet e)^\dagger$:

$$\begin{tikzcd} X \arrow[r, "e"] \arrow[d, "m"'] & FX + Z \arrow[d, "Fm+Z"] \\ Y \arrow[r, "f"'] & FY + Z \end{tikzcd} \quad\Longrightarrow\quad \begin{tikzcd} X \arrow[dr, "(h\bullet e)^\dagger"] \arrow[dd, "m"'] & \\ & A. \\ Y \arrow[ur, "(h\bullet f)^\dagger"'] & \end{tikzcd}$$

(2) *Compositionality.* For every pair of fp-equations $e\colon X \to FX + Y$ and $f\colon Y \to FY + A$ we have

$$(f^\dagger \bullet e)^\dagger = (X \xrightarrow{\mathsf{inl}} X + Y \xrightarrow{(e \,\blacksquare\, f)^\dagger} A).$$

Remark 3.3. Later we will need the following properties of $\bullet$ and $\blacksquare$:

(1) $t \bullet (s \bullet e) = (t \cdot s) \bullet e$ for every $e\colon X \to FX + A$, $s\colon A \to B$ and $t\colon B \to C$;

(2) $s \bullet (e \,\blacksquare\, f) = e \,\blacksquare\, (s \bullet f)$ for every $e\colon X \to FX + Y$, $f\colon Y \to FY + A$ and $s\colon A \to B$;

(3) $(e \,\blacksquare\, f) \,\blacksquare\, g = (\mathsf{inl} \bullet e) \,\blacksquare\, (f \,\blacksquare\, g)$ for every $e\colon X \to FX + Y$, $f\colon Y \to FY + Z$ and $g\colon Z \to FZ + V$.

For the proof of the first two see [6, Remark 4.6]. The remaining one is easy to prove by considering the three coproduct components of $X + Y + Z$ separately. We leave this as an exercise for the reader.

Note that, in lieu of weak functoriality, $\dagger$ was previously required to satisfy (full) functoriality [6]; this states that for every pair of fp-equations $e\colon X \to FX + A$, $f\colon Y \to FY + A$ and a coalgebra homomorphism $m\colon (X, e) \to (Y, f)$ we have $f^\dagger \cdot m = e^\dagger\colon X \to A$. However, this makes no difference:

Lemma 3.4. *Functoriality and Weak Functoriality are equivalent properties of* $\dagger$.

Proof. Functoriality clearly implies Weak Functoriality. In order to prove the converse, let $e\colon X \to FX + A$, $f\colon Y \to FY + A$ be fp-equations, and let $m\colon (X, e) \to (Y, f)$ be a coalgebra morphism. Given an algebra (A, a), write A as the filtered colimit of its canonical diagram $\mathscr{C}_{\mathsf{fp}}/A$ (cf. Section 2.1). The functor $FX + (-)$ preserves filtered colimits, and so $FX + A$ is the filtered colimit of the diagram formed by all morphisms $FX + h\colon FX + Z \to FX + A$, where h ranges over $\mathscr{C}_{\mathsf{fp}}/A$. Since X is fp, the morphism $e\colon X \to FX + A$ factors through one of these morphisms, i.e. there exists a morphism $h\colon Z \to A$ with Z fp and $e'\colon X \to FX + Z$ such that $e = h \bullet e'$:

$$\begin{tikzcd} X \arrow[r, "e"] \arrow[dr, "e'"'] & FX + A \\ & FX + Z \arrow[u, "FX+h"'] \end{tikzcd}$$

Similarly, we have a factorization of $f\colon Y \to FY + A$, and by filteredness of the diagram $\mathscr{C}_{\mathsf{fp}}/A \to \mathscr{C}$, we can assume that the same $h\colon Z \to A$ is used. Thus a morphism $f'\colon Y \to FY + Z$ is given such that $h \bullet f' = (FY + h) \cdot f' = f$. We do not claim that m is a coalgebra homomorphism from (X, e') to (Y, f'). However, the corresponding equation holds when postcomposed by the colimit injection

$FY + h$:

$$\begin{aligned}(FX + h) \cdot (Fm + Z) \cdot e' &= (Fm + A) \cdot (FX + h) \cdot e' \\ &= (Fm + A) \cdot e \\ &= f \cdot m \\ &= (FY + h) \cdot f' \cdot m.\end{aligned}$$

By Remark 2.4(2), there exists a morphism $h' \colon Z' \to A$ with Z' fp and a connecting morphism $z \colon Z \to Z'$ in $\mathscr{C}_{\mathsf{fp}}/A$, i.e. z satisfies $h' \cdot z = h$, such that $FY + z$ merges $(Fm + Z) \cdot e'$ and $f' \cdot m$. It follows that m is a coalgebra homomorphism from $z \bullet e'$ to $z \bullet f'$. Indeed, in the following diagram

$$\begin{tikzcd} X \arrow[r, "e'"] \arrow[d, "m"'] \arrow[rr, bend left, "z\bullet e'"] & FX + Z \arrow[r, "FX+z"] \arrow[d, "Fm+Z"] & FX + Z' \arrow[d, "Fm+Z'"] \\ Y \arrow[r, "f'"'] \arrow[rr, bend right, "z\bullet f'"'] & FY + Z \arrow[r, "FY+z"'] & FY + Z' \end{tikzcd}$$

the left-hand square commutes when postcomposed with $FY + z$; thus, since the upper and lower parts as well as the right-hand square commute, so does the outside, as desired. By Weak Functoriality, we thus conclude

$$\begin{aligned}f^\dagger \cdot m = (h \bullet f')^\dagger \cdot m = ((h' \cdot z) \bullet f')^\dagger \cdot m = (h' \bullet (z \bullet f'))^\dagger \cdot m \\ = (h' \bullet (z \bullet e'))^\dagger = ((h' \cdot z) \bullet e')^\dagger = (h \bullet e')^\dagger = e^\dagger.\end{aligned}$$ ☐

Examples 3.5. Let us recall a few examples of Elgot algebras [6].

(1) Iterative F-algebras (cf. Section 2.4): the operation $\dagger$ assigning to every equation its unique solution satisfies Compositionality and (Weak) Functoriality, see [6, 2.15–2.19]. It follows that ϱF, ϑF and νF are Elgot algebras.

(2) Cpo enrichable algebras. Recall that a *complete partial order* (*cpo*, for short) is a partially ordered set having joins of ω-chains. Cpos form a category CPO whose morphisms are the *continuous* functions, i.e. functions preserving joins of ω-chains. Let $F_0 \colon \mathsf{Set} \to \mathsf{Set}$ be a functor having a *locally continuous* lifting $F \colon \mathsf{CPO} \to \mathsf{CPO}$, i.e. a lifting such that the derived mappings $\mathsf{CPO}(X, Y) \to \mathsf{CPO}(FX, FY)$ are continuous for all cpos X and Y. (For example, every polynomial functor F_Σ associated to the signature Σ has a lifting to CPO.)

Suppose further that $a \colon FA \to A$ is an algebra where A is a cpo with a least element $\perp$ and a is continuous. Then A is an Elgot algebra w.r.t. the operation $\dagger$ assigning to an fp-equation its least solution. More precisely, given an fp-equation $e \colon X \to F_0X + A$ (in Set), consider X as a cpo with discrete order. Then we obtain the following continuous endomap on $\mathsf{CPO}(X, A)$, the cpo of continuous functions from X to A:

$$h \mapsto [a, A] \cdot (Fh + A) \cdot e$$

(cf. (2.1)), and we let $e^\dagger$ be its least fixed point (which exists by Kleene's fixed point theorem). For details see [6, 3.5–3.8].

(3) CMS enrichable algebras. A related example is based on *complete metric spaces*, i.e. metric spaces in which every Cauchy sequence has a limit. Here one considers the category CMS of complete metric spaces with distances in $[0,1]$ and non-expanding maps, i.e. maps $f\colon X \to Y$ such that for every $x, x' \in X$ one has $d_Y(fx, fx') \le d_X(x, x')$. Note that for two complete metric spaces X and Y the set of non-expanding maps $\mathsf{CMS}(X,Y)$ forms a complete metric space with the supremum metric

$$d_{X,Y}(f,g) = \sup_{x\in X} d_Y(f(x), g(x)).$$

Let $F_0\colon \mathsf{Set} \to \mathsf{Set}$ be a functor having a *locally contracting* lifting to CMS, i.e. a lifting $F\colon \mathsf{CMS} \to \mathsf{CMS}$ for which there exists some $\varepsilon < 1$ such that for all $f, g\colon X \to Y$ in CMS one has

$$d_{X,Y}(f,g) \le \varepsilon d_{FX,FY}(Ff, Fg).$$

(Again, polynomial set functors have locally contracting liftings to CMS.)

Now suppose that $a\colon FA \to A$ is a non-empty algebra such that A carries a complete metric space and a is a non-expanding map. Then A is iterative, whence an Elgot algebra. In fact, for every equation $e\colon X \to FX + A$ consider X as a discrete metric space (i.e. all distances are 1) and consider the endofunction on $\mathsf{CMS}(X,A)$ given by

$$h \mapsto [a, A] \cdot (Fh + A) \cdot e,$$

which is ε-contracting for the ε above. Then, by Banach's fixed point theorem, this function has a unique fixed point, viz. the unique solution of e. For details see [6, 2.8–2.11].

(4) As a concrete instance of the previous point one can obtain fractals as solutions of equations. For example, let A be the set of closed subsets of the unit interval $[0,1]$ equipped with the following binary operation:

$$(C, C') \mapsto \frac{1}{3}C \cup \left(\frac{1}{3}C' + \frac{2}{3}\right),$$

where $\frac{1}{3}C = \{\frac{1}{3}c \mid c \in C\}$ etc. Then A is an algebra for $F_0X = X \times X$ on Set, and this F_0 has the locally contracting lifting $F(X,d) = (X \times X, \frac{1}{3}d_{\max})$, where $d_{\max}$ denotes the usual maximum metric on the cartesian product. One sees that A is an algebra for F when equipped with the so-called Hausdorff metric. Hence, it is an Elgot algebra. For example, let $X = \{x\}$ and let $e\colon X \to FX + A$ be given by $e(x) = (x, x)$. Then $e^\dagger(x)$ is the well-known Cantor set.

We have already mentioned in Section 2.4 that the rational fixed point ϱF is an initial iterative F-algebra. Moreover, for every object Y, the rational fixed point $\varrho(F(-) + Y)$ is a free iterative algebra on Y. Thus, the object assignment $Y \mapsto \varrho(F(-) + Y)$ yields a monad R on $\mathscr{C}$.

Theorem 3.6 ([6]). *The category of Eilenberg-Moore algebras for the monad R is isomorphic to the category of Elgot algebras for F.*

Thus, in particular, $\varrho(F(-) + Y)$ is not only a free iterative algebra, but it is also a free Elgot algebra on Y, whence ϱF is the initial Elgot algebra.

4 FFG-Elgot Algebras

The rest of our paper is devoted to studying the fixed point φF, the colimit of all ffg-coalgebras for F, in its own right and establish a universal property of it as an algebra. Recall that by a variety $\mathscr{C}$ we mean a finitary, many sorted variety. That is, $\mathscr{C}$ is (isomorphic to) the category of Eilenberg-Moore algebras for a finitary monad T on Set^S, where S is a set of sorts.

Assumption 4.1. *Throughout the rest of the paper we assume that $\mathscr{C}$ is a variety of algebras and that $F\colon \mathscr{C} \to \mathscr{C}$ is an endofunctor preserving sifted colimits.*

Examples 4.2. (1) For the monad T representing $\mathscr{C}$, all functors that are liftings of a finitary functor F_0 on Set^S (via a distributive law of T over F_0) preserve sifted colimits. Indeed, finitary functors $F_0 : \mathsf{Set}^S \to \mathsf{Set}^S$ preserve them [11, Proposition 6.30]. Since the forgetful functor $U\colon \mathscr{C} \to \mathsf{Set}^S$ preserves and reflects sifted colimits, it follows that every lifting of F_0 preserves sifted colimits, too.

The following examples are not liftings of set functors.

(2) The functor $FX = X + X$, where $+$ denotes the coproduct of $\mathscr{C}$, preserves sifted colimits. More generally, every coproduct of sifted-colimit preserving functors preserves them too. Similarly for finite products of sifted-colimit preserving functors. Thus, all polynomial functors on $\mathscr{C}$ preserve sifted colimits.

(3) Let $\mathscr{C}$ be an *entropic* variety (see e.g. [19]) aka *commutative* variety (see e.g. [35]), i.e. such that the usual tensor product $\otimes$ (representing bimorphisms) makes it a symmetric monoidal closed category. (Examples include sets, vector spaces, join-semilattices, or abelian groups.) Then the functor $FX = X \otimes X$ preserves sifted colimits. To see this, it suffices to show that (a) F is finitary and (b) it preserves reflexive coequalizers (see Remark 2.1). First note that since $\mathscr{C}$ is symmetric monoidal closed, we know that each functor $X \otimes -$ and $- \otimes X$ is a left adjoint and therefore preserves all colimits.

Ad (a). Suppose that $D : \mathscr{D} \to \mathscr{C}$ is a filtered diagram with colimit injections $a_d : Dd \to A$ for $d \in \mathscr{D}$. We need to prove that all $a_d \otimes a_d : Dd \otimes Dd \to A \otimes A$ form a colimit cocone. That is, for every morphism $f : X \to A \otimes A$ with X fp, (i) there exists some $d \in \mathscr{D}$ and $g : X \to Dd \otimes Dd$ with $(a_d \otimes a_d) \cdot g = f$ and (ii) given $g, h : X \to Dd \otimes Dd$ that yield f in this way, there exists a morphism $m : d \to d'$ in $\mathscr{D}$ such that $Dm \otimes Dm$ merges g and h [4, Lemma 2.6].

To prove (i), we use that $- \otimes A$ is finitary to obtain some $d \in \mathscr{D}$ and $f' : X \to A \otimes Dd$ with $(A \otimes a_d) \cdot f' = f$. Now use that $Dd \otimes -$ is finitary to obtain $d' \in \mathscr{D}$ and $f'' : X \to Dd \otimes Dd'$ with $(Dd \otimes a_{d'}) \cdot f'' = f'$. Since $\mathscr{D}$ is filtered, we can choose morphisms $m : d \to \bar{d}$ and $n : d' \to \bar{d}$ in $\mathscr{D}$. Let $g = (Dm \otimes Dn) \cdot f''$. Then we have

$$\begin{aligned}(a_{\bar{d}} \otimes a_{\bar{d}}) \cdot g = (a_{\bar{d}} \otimes a_{\bar{d}}) \cdot (Dm \otimes Dn) \cdot f'' &= (a_d \otimes a_{d'}) \cdot f'' \\ = (a_d \otimes A) \cdot (Dd \otimes a_{d'}) \cdot f'' &= (a_d \otimes A) \cdot f' = f\end{aligned}$$

as desired.

For (ii), use first that $- \otimes A$ is finitary and choose some morphism $o : d \to d'$ such that

$$(Do \otimes A) \cdot ((Dd \otimes a_d) \cdot g) = (Do \otimes A) \cdot ((Dd \otimes a_d) \cdot h) .$$

It follows that $(Dd' \otimes a_d)$ merges $(Do \otimes Dd) \cdot g$ and $(Do \otimes Dd) \cdot h$. Now use that $Dd' \otimes -$ is finitary and choose a morphism $p : d \to d''$ in $\mathscr{D}$ such that $(Dd' \otimes Dp)$ also merges those two morphisms. Finally, use that $\mathscr{D}$ is filtered to choose two morphisms $q : d' \to \bar{d}$ and $r : d'' \to \bar{d}$ such that $q \cdot o = r \cdot p$, and let us call this last morphism $m : d \to \bar{d}$. Then $Dm \otimes Dm$ merges g and h:

$$\begin{aligned}(Dm \otimes Dm) \cdot g &= (D(q \cdot o) \otimes D(r \cdot p)) \cdot g = (Dq \otimes Dr) \cdot (Do \otimes Dp) \cdot g \\ &= (Dq \otimes Dr) \cdot (Dd' \otimes Dp) \cdot (Do \otimes Dd) \cdot g \\ &= (Dq \otimes Dr) \cdot (Dd' \otimes Dp) \cdot (Do \otimes Dd) \cdot h \\ &= (Dm \otimes Dm) \cdot h.\end{aligned}$$

Ad (b). Let $f, g : A \to B$ be a (not necessarily reflexive) pair, and let $c : B \to C$ be its coequalizer. Use that all functors $- \otimes X$ and $X \otimes -$ preserve coequalizers to see that in the following diagram, whose parts commute in the obvious way, all rows and columns are coequalizers:

$$\begin{tikzcd}
A \otimes A \arrow[r, shift left, "f\otimes A"] \arrow[r, shift right, "g\otimes A"'] \arrow[d, shift right, "A\otimes g"'] \arrow[d, shift left, "A\otimes f"] & B \otimes A \arrow[r, "c\otimes A"] \arrow[d, shift right, "B\otimes g"'] \arrow[d, shift left, "B\otimes f"] & C \otimes A \arrow[d, shift right, "C\otimes g"'] \arrow[d, shift left, "C\otimes f"] \\
A \otimes B \arrow[r, shift left, "f\otimes B"] \arrow[r, shift right, "g\otimes B"'] \arrow[d, "A\otimes c"] & B \otimes B \arrow[r, "c\otimes B"] \arrow[d, "B\otimes c"] & C \otimes B \arrow[d, "C\otimes c"] \\
A \otimes C \arrow[r, shift left, "f\otimes C"] \arrow[r, shift right, "g\otimes C"'] & B \otimes C \arrow[r, "c\otimes C"] & C \otimes C
\end{tikzcd}$$

By the '3-by-3 lemma' [32, Lemma 0.17], it follows that the diagonal yields a coequalizer too, i.e. $c \otimes c$ is a coequalizer of the pair $f \otimes f, g \otimes g$, as desired.

(4) Combining the previous argument with induction, we see that sifted-colimit preserving functors on an entropic variety $\mathscr{C}$ are stable under finite tensor products. Thus, all tensor-polynomial functors on $\mathscr{C}$ preserve sifted colimits.

Under our assumptions we know that φF is a fixed point of F [54], and we will henceforth denote the inverse of its coalgebra structure by $t\colon F(\varphi F) \to \varphi F$.

The following is a variation of Definition 3.2 where the variable objects X are now restricted to be ffg objects:

Definition 4.3. By an *ffg-equation* is meant a morphism $e\colon X \to FX + A$ where X is an ffg object (of *variables*) and A an arbitrary object (of *parameters*). An *ffg-Elgot algebra* is a triple $(A, a, \dagger)$ where (A, a) is an F-algebra and $\dagger$ is an operation

$$\frac{e\colon X \to FX + A}{e^\dagger\colon X \to A}$$

assigning to every ffg-equation in A a solution (cf. (2.1)) and satisfying Weak Functoriality 3.2(1) and Compositionality 3.2(2) with X, Y and Z restricted to ffg objects.

Remark 4.4. (1) Note that in categories where fp objects are ffg, e.g. in the category of sets or vector spaces, (ordinary) Elgot algebras and ffg-Elgot algebras are the same concept. However, in the present setting this may not be the case.

(2) Since fp-equations have variable objects X such that $\mathscr{C}(X, -)$ preserves filtered colimits, one could expect that ffg-equations will have X as those objects for which $\mathscr{C}(X, -)$ preserves sifted colimits. Indeed, that would yield the same colimit φF, as we prove in the Appendix.

(3) We do not know whether, for ffg-Elgot algebras, Weak Functoriality implies Functoriality. The proofs of our main results (in particular Proposition 4.8 and Theorem 4.12) do not work when Weak Functoriality is replaced by Functoriality.

Remark 4.5. In the case where $F\colon \mathsf{Set}^T \to \mathsf{Set}^T$ is a lifting of a functor $F_0\colon \mathsf{Set} \to \mathsf{Set}$ (via a distributive law λ), an F-algebra is given by a set A equipped with both a T-algebra structure $\alpha\colon TA \to A$ and an F_0-algebra structure $a\colon F_0A \to A$ such that a is a T-algebra homomorphism, i.e. one has $\alpha \cdot Ta = a \cdot F\alpha \cdot \lambda_A$. Morphisms of F-algebras are those maps that are both T-algebra and F_0-algebra homomorphisms. Now one may think of ffg-equations and their solutions as modelling *effectful iteration*. Indeed, let X_0 be a finite set of variables and consider any map

$$e_0\colon X_0 \to T(F_0X_0 + A).$$

This may be regarded as a system of recursive equations with variables from X_0 and parameters in A, where for every recursive call a side effect in T might happen. If (A, α, a) is an F-algebra, a solution of such a recursive system should assign to each variable in X_0 an element of A, i.e. we have a map $e_0^\dagger\colon X_0 \to A$, such that the square below commutes (here we write $+$ for disjoint union):

$$\begin{tikzcd}
X_0 \arrow[rr, "e_0^\dagger"] \arrow[dd, "e_0"'] & & A \\
& & TA \arrow[u, "\alpha"'] \\
T(F_0X_0 + A) \arrow[rr, "T(F_0e_0^\dagger + A)"'] & & T(F_0A + A) \arrow[u, "{T[a,A]}"']
\end{tikzcd}$$

Indeed, from e_0 we may form the map

$$\overline{e} = (X_0 \xrightarrow{e_0} T(F_0X_0 + A) \xrightarrow{\cong} TF_0X_0 \oplus TA \xrightarrow{\lambda_X \oplus \alpha} FTX_0 \oplus A),$$

where $\oplus$ denotes the coproduct in $\mathscr{C}$, which may be different from disjoint union. Then its unique extension $TX_0 \to FTX_0 \oplus A$ to a T-algebra morphism is an ffg-equation, and a solution $TX_0 \to A$ of this in the sense of Definition 4.3 is precisely the same as an extension of a solution for e_0 in the above sense.

Construction 4.6. We aim at proving that φF is the initial ffg-Elgot algebra. For that we first construct a solution $e^\dagger \colon X \to \varphi F$ for every ffg-equation $e \colon X \to FX + \varphi F$. Recall that $\varphi F = \operatorname{colim} D$ for the inclusion $D \colon \mathsf{Coalg}_{\mathsf{ffg}} F \rightarrowtail \mathsf{Coalg}\, F$ and denote the colimit injections by $c^\sharp : C \to \varphi F$ for every ffg-coalgebra (C, c). Thus $FX + \varphi F = \operatorname{colim}(FX + D)$ with colimit injections $FX + c^\sharp$. Since X is an ffg-object, this sifted colimit is preserved by $\mathscr{C}(X, -)$. Thus, the diagram

$$\hat{D} \colon \mathsf{Coalg}_{\mathsf{ffg}} F \to \mathsf{Set}, \quad (C \xrightarrow{c} FC) \mapsto \mathscr{C}(X, FX + C)$$

has

$$\operatorname{colim} \hat{D} = \mathscr{C}(X, FX + \varphi F)$$

with colimit injections given by postcomposition with $FX + c^\sharp$.

By Remark 2.4(1), every ffg-equation $e \colon X \to FX + \varphi F$ thus factorizes through one of the colimit injections $FX + c^\sharp$, i.e. for some ffg-coalgebra $c \colon C \to FC$ and $w \colon X \to FX + C$ we have the commutative triangle below:

$$\begin{tikzcd} X \arrow[r, "e"] \arrow[dr, "w"'] & FX + \varphi F \\ & FX + C \arrow[u, "FX + c^\sharp"'] \end{tikzcd} \tag{4.1}$$

We see that w is an ffg-equation. We combine it with the ffg-equation c (having the initial object 0 as parameter, see Definition 4.3) to $w \mathbin{\blacksquare} c \colon X + C \to F(X + C)$, which is an object of $\mathsf{Coalg}_{\mathsf{ffg}} F$. Finally, we put

$$e^\dagger = (\, X \xrightarrow{\mathsf{inl}} X + C \xrightarrow{(w \,\blacksquare\, c)^\sharp} \varphi F\,). \tag{4.2}$$

We prove below that $e^\dagger$ is indeed a solution of e in the algebra φF (cf. (2.1)) and verify some properties used later.

Lemma 4.7. *The definition of $e^\dagger$ in (4.2) is independent of the choice of the factorization (4.1), and $e^\dagger$ is a solution of e in φF.*

Proof. (1) We first show the independence: given another ffg-coalgebra $\overline{c} \colon \overline{C} \to F\overline{C}$ and a factorization $e = (FX + \overline{c}^\sharp) \cdot \overline{w}$, we prove

$$(w \mathbin{\blacksquare} c)^\sharp \cdot \mathsf{inl} = (\overline{w} \mathbin{\blacksquare} \overline{c})^\sharp \cdot \mathsf{inl}. \tag{4.3}$$

Recall the category $\mathsf{el}\,\hat{D}$ of elements of $\hat{D}$: its objects are triples (C, c, w) where $(C, c) \in \mathsf{Coalg}_{\mathsf{ffg}}\, F$ and $w \in \hat{D}(C, c)$, i.e. $w\colon X \to FX + C$, and a morphism into $(\overline{C}, \overline{c}, \overline{w})$ is a coalgebra homomorphism $h\colon (C, c) \to (\overline{C}, \overline{c})$ with $(FX + h) \cdot w = \overline{w}$.

Given two factorizations $(FX + c^\sharp) \cdot w = e = (FX + \overline{c}^\sharp) \cdot \overline{w}$, we thus see that the colimit injection $FX + c^\sharp$ takes the element w to the same value to which the colimit injection $FX + \overline{c}^\sharp$ takes $\overline{w}$. This implies that w and $\overline{w}$ lie in the same connected component of $\mathsf{el}\,\hat{D}$. Therefore it suffices to prove (4.3) under the assumption that a morphism h from w to $\overline{w}$ exists in $\mathsf{el}\,\hat{D}$: then that equation holds in the whole connected component. Thus, we have the following commutative diagram:

$$\begin{tikzcd}
 & X \arrow[dl, "w"'] \arrow[dr, "\overline{w}"] & \\
FX + C \arrow[rr, "FX+h"] \arrow[d, "FX+c"'] & & FX + \overline{C} \arrow[d, "FX+\overline{c}"] \\
FX + FC \arrow[rr, "FX+Fh"'] & & FX + F\overline{C}
\end{tikzcd}$$

It follows that $X + h$ is a coalgebra homomorphism from $w \blacksquare c$ to $\overline{w} \blacksquare \overline{c}$. Indeed, in the following diagram

$$\begin{tikzcd}
X + C \arrow[r, "{[w,\mathsf{inr}]}"] \arrow[d, "X+h"'] \arrow[rrr, bend left, "w \blacksquare c"] & FX + C \arrow[r, "FX+c"] \arrow[d, "FX+h"'] & FX + FC \arrow[r, "\mathsf{can}"] \arrow[d, "FX+Fh"'] & F(X + C) \arrow[d, "F(X+h)"] \\
X + \overline{C} \arrow[r, "{[\overline{w},\mathsf{inr}]}"'] \arrow[rrr, bend right, "\overline{w} \blacksquare \overline{c}"'] & FX + \overline{C} \arrow[r, "FX+\overline{c}"'] & FX + F\overline{C} \arrow[r, "\mathsf{can}"'] & F(X + \overline{C})
\end{tikzcd}$$

the left-hand square and the middle one commute by the preceding diagram, and the right-hand square commutes trivially. Since the colimit injections $(-)^\sharp$ form a compatible family, we obtain $(w \blacksquare c)^\sharp = (\overline{w} \blacksquare \overline{c})^\sharp \cdot (X + h)$. Precomposed with inl this yields the desired equation (4.3).

(2) We show that $e^\dagger$ is a solution of e in φF.

(2a) First note that the following triangle commutes:

$$\begin{tikzcd}
C \arrow[r, "c^\sharp"] \arrow[d, "\mathsf{inr}"'] & \varphi F \\
X + C \arrow[ur, "(w \blacksquare c)^\sharp"'] &
\end{tikzcd} \tag{4.4}$$

To this end, we just need to verify that inr is a morphism in $\mathsf{Coalg}_{\mathsf{ffg}}\, F$ from (C, c) to $(X + C, w \blacksquare c)$, which is established by the commutative diagram below:

$$\begin{tikzcd}
C \arrow[rrr, "c"] \arrow[dr, "\mathsf{inr}"] \arrow[dd, "\mathsf{inr}"'] & & & FC \arrow[dl, "\mathsf{inr}"'] \arrow[dd, "F\mathsf{inr}"] \\
& FX + C \arrow[r, "FX+c"] & FX + FC \arrow[dr, "\mathsf{can}"] & \\
X + C \arrow[ur, "{[w,\mathsf{inr}]}"] \arrow[rrr, "w \blacksquare c"'] & & & F(X + C)
\end{tikzcd}$$

(2b) The commutative triangle (4.4) together with $(w \blacksquare c)^\sharp \cdot \mathsf{inl} = e^\dagger$ yield the following commutative triangle:

$$\begin{tikzcd}
FX + FC \arrow[r, "{[Fe^\dagger, Fc^\sharp]}"] \arrow[d, "\mathsf{can}"'] & F(\varphi F) \\
F(X + C) \arrow[ur, "F(w \blacksquare c)^\sharp"'] &
\end{tikzcd} \tag{4.5}$$

We conclude that the following diagram

$$\begin{tikzcd}
X \arrow[rr, "e^\dagger"] \arrow[d, "w"'] \arrow[dr, "\mathsf{inl}"] & & \varphi F \\
FX + C \arrow[ddd, "FX+c"'] & X + C \arrow[ur, "(w \blacksquare c)^\sharp"'] \arrow[d, "w \blacksquare c"] & \\
& F(X + C) \arrow[dr, "F(w \blacksquare c)^\sharp"] & \\
& & \\
FX + FC \arrow[uur, "\mathsf{can}"'] \arrow[rr, "{[Fe^\dagger, Fc^\sharp]}"'] & & F(\varphi F) \arrow[uuuu, "t"']
\end{tikzcd} \tag{4.6}$$

commutes: the left-hand part follows from the definition of $w \blacksquare c$, the upper one is the definition of $e^\dagger$, the right-hand one uses that $(w \blacksquare c)^\sharp$ is a coalgebra homomorphism, and the lower one is the triangle (4.5).

We are ready to prove that $e^\dagger$ is a solution of e, which means that the outside of the following diagram commutes:

$$
\begin{tikzcd}
X \arrow[rr, "e^\dagger"] \arrow[d, "w"'] \arrow[dd, bend right=60, "e"'] & & \varphi F \\
FX + C \arrow[r, "FX+c"] \arrow[d, "FX+c^\sharp"'] & FX + FC \arrow[r, "{[Fe^\dagger, Fc^\sharp]}"] \arrow[d, "FX+Fc^\sharp"'] & F(\varphi F) \arrow[u, "t"'] \\
FX + \varphi F \arrow[rr, bend right=30, "Fe^\dagger + \varphi F"'] & FX + F(\varphi F) \arrow[l, "FX+t"] \arrow[r, "Fe^\dagger + t"'] \arrow[ur, "{[Fe^\dagger, F(\varphi F)]}"'] & F(\varphi F) + \varphi F \arrow[uu, bend right=60, "{[t, \varphi F]}"']
\end{tikzcd}
$$

The upper part has just been established in (4.6). The left-hand part commutes by (4.1), the lower left-hand square commutes because $c^\sharp$ is a coalgebra homomorphism, and the three remaining parts commute trivially. $\square$

Proposition 4.8. *The algebra $t\colon F(\varphi F) \to \varphi F$ together with the solution operator $\dagger$ from Construction 4.6 is an ffg-Elgot algebra.*

Proof. Weak Functoriality. Suppose that the commutative square below and a morphism $h\colon Z \to \varphi F$ are given, where X, Y, and Z are ffg objects.

$$
\begin{tikzcd}
X \arrow[r, "e"] \arrow[d, "m"'] & FX + Z \arrow[d, "Fm+Z"] \\
Y \arrow[r, "f"'] & FY + Z
\end{tikzcd}
$$

Since Z is ffg, the morphism h factorizes through the colimit injection $c^\sharp$ of some coalgebra $c\colon C \to FC$ in $\mathsf{Coalg}_{\mathsf{ffg}}\, F$ as in the triangle below:

$$
\begin{tikzcd}
Z \arrow[r, "h"] \arrow[dr, "v_0"'] & \varphi F \\
& C \arrow[u, "c^\sharp"']
\end{tikzcd}
$$

Form the two ffg-equations

$$
v = v_0 \bullet e\colon X \to FX + C \quad \text{and} \quad w = v_0 \bullet f\colon Y \to FY + C,
$$

and observe that the following diagram commutes:

$$
\begin{tikzcd}
X \arrow[r, "e"] \arrow[d, "m"'] \arrow[rr, bend left=30, "v"] & FX + Z \arrow[r, "FX+v_0"] \arrow[d, "Fm+Z"] & FX + C \arrow[d, "Fm+C"] \\
Y \arrow[r, "f"'] \arrow[rr, bend right=30, "w"'] & FY + Z \arrow[r, "FY+v_0"'] & FY + C
\end{tikzcd}
$$

Consequently, in the following diagram

$$\begin{tikzcd} X+C \arrow[r, "{[v,\mathsf{inr}]}"] \arrow[d, "m+C"'] \arrow[rrr, bend left, "v \blacksquare c"] & FX+C \arrow[r, "FX+c"] \arrow[d, "Fm+FC"'] & FX+FC \arrow[r, "\mathsf{can}"] \arrow[d, "Fm+FC"] & F(X+C) \arrow[d, "F(m+C)"] \\ Y+C \arrow[r, "{[w,\mathsf{inr}]}"'] \arrow[rrr, bend right, "w \blacksquare c"'] & FY+C \arrow[r, "FY+c"'] & FY+FC \arrow[r, "\mathsf{can}"'] & F(Y+C) \end{tikzcd}$$

the left-hand square commutes. The other parts are clearly commutative, and thus we see that $m+C$ is a coalgebra homomorphism from $v \blacksquare c$ to $w \blacksquare c$. Therefore

$$(v \blacksquare c)^\sharp = (w \blacksquare c)^\sharp \cdot (m+C),$$

which yields the desired equation $(h \bullet f)^\dagger \cdot m = (h \bullet e)^\dagger$, as shown by the commutative diagram below:

$$\begin{tikzcd} X \arrow[r, "\mathsf{inl}"] \arrow[d, "m"'] \arrow[rr, bend left, "(h\bullet e)^\dagger"] & X+C \arrow[d, "m+C"] \arrow[dr, "(v \blacksquare c)^\sharp"] & \\ & & \varphi F \\ Y \arrow[r, "\mathsf{inl}"'] \arrow[rr, bend right, "(h\bullet f)^\dagger"'] & Y+C \arrow[ur, "(w \blacksquare c)^\sharp"'] & \end{tikzcd}$$

Compositionality.

(1) Suppose that two ffg-equations $e\colon X \to FX+Y$ and $f\colon Y \to FY+\varphi F$ are given, and factorize f through some colimit injection $FY+c^\sharp$ of $FY+C$:

$$\begin{tikzcd} Y \arrow[r, "f"] \arrow[dr, "v"'] & FY+\varphi F \\ & FY+C \arrow[u, "FY+c^\sharp"'] \end{tikzcd}$$

Then, by the definition of $\dagger$, we have

$$f^\dagger = (v \blacksquare c)^\sharp \cdot \mathsf{inl}.$$

This implies that the ffg-equation $f^\dagger \bullet e\colon X \to FX+\varphi F$ factorizes as follows:

$$\begin{tikzcd} X \arrow[r, "e"] \arrow[drr, "\mathsf{inl}\bullet e"'] \arrow[rr, bend left, "f^\dagger\bullet e"] & FX+Y \arrow[r, "FX+f^\dagger"] \arrow[dr, "FX+\mathsf{inl}"] & FX+\varphi F \\ & & FX+Y+C \arrow[u, "FX+(v \blacksquare c)^\sharp"'] \end{tikzcd}$$

Thus, by the definition of † again, the solution $(f^\dagger \bullet e)^\dagger\colon X \to \varphi F$ of $f^\dagger \bullet e$ is given by the coproduct injection $\mathsf{inl}\colon X \to X + Y + C$ followed by the colimit injection

$$[(\mathsf{inl} \bullet e) \,\blacksquare\, (v \,\blacksquare\, c)]^\sharp\colon X + Y + C \to \varphi F.$$

By Remark 3.3(3) the last morphism is equal to $[e \,\blacksquare\, (v \,\blacksquare\, c)]^\sharp$, thus we obtain:

$$(f^\dagger \bullet e)^\dagger = (X \xrightarrow{\mathsf{inl}} X + Y + C \xrightarrow{[e \,\blacksquare\, (v \,\blacksquare\, c)]^\sharp} \varphi F).$$

(2) The equation $e \,\blacksquare\, f\colon X + Y \to F(X+Y) + \varphi F$ factorizes as follows:

$$\begin{tikzcd}
X+Y \arrow[r, "{[e,\mathsf{inl}]}"] \arrow[rrr, bend left, "e \,\blacksquare\, f"] \arrow[ddrrr, bend right, "e \,\blacksquare\, v"'] & FX+Y \arrow[r, "FX+f"] \arrow[dr, "FX+v"'] & FX+FY+\varphi F \arrow[r, "\mathsf{can}+\varphi F"] & F(X+Y)+\varphi F \\
 & & FX+FY+C \arrow[u, "FX+FY+c^\sharp"'] \arrow[dr, "\mathsf{can}+C"] & \\
 & & & F(X+Y)+C \arrow[uu, "F(X+Y)+c^\sharp"']
\end{tikzcd}$$

Therefore, by the definition of †, we have

$$(e \,\blacksquare\, f)^\dagger = (X + Y \xrightarrow{\mathsf{inl}} X + Y + C \xrightarrow{[(e \,\blacksquare\, v) \,\blacksquare\, c]^\sharp} \varphi F).$$

Precomposing this with the coproduct injection $\mathsf{inl}\colon X \to X + Y$ proves the desired equality

$$(e \,\blacksquare\, f)^\dagger \cdot \mathsf{inl} = [(e \,\blacksquare\, v) \,\blacksquare\, c]^\sharp \cdot \mathsf{inl} = (f^\dagger \bullet e)^\dagger. \qquad \square$$

Definition 4.9. A *morphism of ffg-Elgot algebras* from $(A, a, \dagger)$ to $(B, b, \ddagger)$ is a morphism $h\colon A \to B$ in $\mathscr{C}$ *preserving solutions*, i.e. for every ffg-equation $e\colon X \to FX + A$ we have

$$(h \bullet e)^\ddagger = h \cdot e^\dagger.$$

Identity morphisms are clearly ffg-Elgot algebra morphisms, and morphisms of ffg-Elgot algebra compose. Therefore ffg-Elgot algebras form a category, which we denote by

$$\mathsf{ffg\text{-}Elgot}\, F.$$

Lemma 4.10. *Morphisms of ffg-Elgot algebras are F-algebra homomorphisms.*

Proof. This is completely analogous to the proof of [6, Lemma 4.2]. The only small modification is needed at the beginning of the proof as follows:

Let $\mathscr{C}_{\mathsf{ffg}}/A$ be the slice category of all arrows $q\colon X \to A$ with X ffg. Since $\mathscr{C}$ is a variety, A is the sifted colimit of the diagram $D_A\colon \mathscr{C}_{\mathsf{ffg}}/A \to \mathscr{C}$ given by $(q\colon X \to A) \mapsto X$.

The remainder of the proof is identical. $\square$

Note that the converse of the above lemma fails in general. In fact, [6, Example 4.4] exhibits an (ffg-)Elgot algebra for the identity functor on Set and an algebra homomorphism on it which is not solution-preserving.

Theorem 4.11. *The triple $(\varphi F, t, \dagger)$ is the initial ffg-Elgot algebra for F.*

Proof. Let $(A, a, \ddagger)$ be an ffg-Elgot algebra. For the initial object 0 we denote by $i_A\colon 0 \to A$ the unique morphism.

(1) We obtain a cocone of the diagram

$$\mathsf{Coalg}_{\mathsf{ffg}}\, F \rightarrowtail \mathsf{Coalg}\, F \xrightarrow{U} \mathscr{C},$$

where U is the forgetful functor, as follows: to every ffg-coalgebra $c\colon C \to FC$ assign the solution

$$(i_A \bullet c)^{\ddagger}\colon C \to A$$

of the ffg-equation $i_A \bullet c\colon C \to FC + A$. Indeed, given a coalgebra homomorphism $m\colon (C, c) \to (C', c')$ in $\mathsf{Coalg}_{\mathsf{ffg}}\, F$, Weak Functoriality applied to $h = i_A$ yields

$$(i_A \bullet c)^{\ddagger} = (C \xrightarrow{m} C' \xrightarrow{(i_A \bullet c')^{\ddagger}} A).$$

Since φF is the colimit of the embedding $\mathsf{Coalg}_{\mathsf{ffg}}\, F \rightarrowtail \mathsf{Coalg}\, F$ and since U preserves colimits, there exists a unique morphism $h\colon \varphi F \to A$ in $\mathscr{C}$ such that the following triangles

$$\begin{tikzcd} C \arrow[d, "c^{\sharp}"'] \arrow[dr, "(i_A \bullet c)^{\ddagger}"] & \\ \varphi F \arrow[r, dashed, "h"'] & A \end{tikzcd}$$

commute for all ffg-coalgebras $c\colon C \to FC$.

(2) We prove that h is solution-preserving. Given an ffg-equation $e\colon X \to FX + \varphi F$, factorize e through one of the colimit injections $FX + c^{\sharp}$ of $FX + \varphi F$:

$$\begin{tikzcd} X \arrow[r, "e"] \arrow[dr, "v"'] & FX + \varphi F \\ & FX + C \arrow[u, "FX + c^{\sharp}"'] \end{tikzcd}$$

Since $e = c^{\sharp} \bullet v$, Remark 3.3(1) and the definition of h yield

$$(h \bullet e)^{\ddagger} = [h \bullet (c^{\sharp} \bullet v)]^{\ddagger} = [(h \cdot c^{\sharp}) \bullet v]^{\ddagger} = [(i_A \bullet c)^{\ddagger} \bullet v]^{\ddagger}.$$

The last morphism is, due to Compositionality, equal to

$$[v \,\blacksquare (i_A \bullet c)]^{\ddagger} \cdot \mathsf{inl}.$$

Thus, it remains to verify that $h \cdot e^{\dagger}$ is the same morphism. From $e = c^{\sharp} \bullet v$ the definition of $\dagger$ yields $e^{\dagger} = (v \,\blacksquare\, c)^{\sharp} \cdot \mathsf{inl}$ and we get

$$h \cdot e^{\dagger} = h \cdot (v \,\blacksquare\, c)^{\sharp} \cdot \mathsf{inl} = [i_A \bullet (v \,\blacksquare\, c)]^{\ddagger} \cdot \mathsf{inl} = [v \,\blacksquare (i_A \bullet c)]^{\ddagger} \cdot \mathsf{inl},$$

where the last step uses Remark 3.3(2).

(3) It remains to prove the uniqueness of h. Thus suppose that another solution-preserving morphism $g\colon \varphi F \to A$ is given. It is sufficient to prove

$$g \cdot c^\sharp = h \cdot c^\sharp \quad \text{for all ffg-coalgebras } c\colon C \to FC.$$

Form the ffg-equation $i_{\varphi F} \bullet c = \mathsf{inl} \cdot c\colon C \to FC + \varphi F$. Then it is easy to verify that the left coproduct injection $\mathsf{inl}\colon C \to C + C$ is a coalgebra homomorphism from (C, c) to $(C + C, \overline{c})$ where $\overline{c} = (\mathsf{inl} \cdot c) \blacksquare c$. Therefore, the compatibility of the colimit injections $(-)^\sharp$ yields $c^\sharp = \overline{c}^\sharp \cdot \mathsf{inl}$. Now $i_{\varphi F} \bullet c$ factorizes as follows:

$$\begin{tikzcd} C \arrow[r, "i_{\varphi F} \bullet c"] \arrow[dr, "\mathsf{inl} \cdot c"'] & FC + \varphi F \\ & FC + C \arrow[u, "FC + c^\sharp"'] \end{tikzcd}$$

Therefore the definition of $\dagger$ yields

$$(i_{\varphi F} \bullet c)^\dagger = ((\mathsf{inl} \cdot c) \blacksquare c)^\sharp \cdot \mathsf{inl} = \overline{c}^\sharp \cdot \mathsf{inl} = c^\sharp.$$

Since g preserves solutions, using Remark 3.3(1), and that $g \cdot i_{\varphi F} = i_A\colon 0 \to A$, we thus get

$$g \cdot c^\sharp = g \cdot (i_{\varphi F} \bullet c)^\dagger = (g \bullet (i_{\varphi F} \bullet c))^\ddagger = ((g \cdot i_{\varphi F}) \bullet c))^\ddagger = (i_A \bullet c)^\ddagger = h \cdot c^\sharp$$

as required. This concludes the proof. $\square$

The following result is the key to constructing free ffg-Elgot algebras. In the case where $\mathscr{C}_{\mathsf{ffg}} = \mathscr{C}_{\mathsf{fp}}$, hence where ffg-Elgot algebras agree with ordinary ones, we thus obtain a new result about ordinary Elgot algebras.

Theorem 4.12. *Let $a\colon FA \to A$ be an F-algebra, Y a free object of $\mathscr{C}$, and $h\colon Y \to A$ a morphism. Then there is a bijective correspondence between*

(1) *solution operators $\dagger$ such that $(A, a, \dagger)$ is an ffg-Elgot algebra for F, and*

(2) *solution operators $\ddagger$ such that $(A, [a, h], \ddagger)$ is an ffg-Elgot algebra for $F(-)+Y$.*

Remark 4.13. The correspondence is given as follows:

(1) For every ffg-Elgot algebra $(A, a, \dagger)$ for F, we define a solution operator $\ddagger$ w.r.t. $F(-) + Y$ as follows. Given $e\colon X \to FX + Y + A$, put

$$e_h = (X \xrightarrow{e} FX + Y + A \xrightarrow{FX+[h,A]} FX + A) \tag{4.7}$$

and

$$e^\ddagger \;:=\; e_h^\dagger.$$

(2) Conversely, for every ffg-Elgot algebra $(A, [a, h], \ddagger)$ for $F(-) + Y$, we define a solution operator $\dagger$ w.r.t. F as follows. Given an ffg-equation $e\colon X \to FX + A$, put

$$\overline{e} = (X \xrightarrow{e} FX + A \xrightarrow{[\mathsf{inl},\mathsf{inr}]} FX + Y + A) \tag{4.8}$$

and

$$e^\dagger \;:=\; \overline{e}^\ddagger.$$

We will show that these two constructions are mutually inverse and yield the desired bijective correspondence.

In the next two subsections we will present the proof of Theorem 4.12. We will establish this result in two steps: first we prove it for ffg objects Y and then, using the first step, for arbitrary free objects. Readers who would like to skip the proof on first reading could jump straight to Section 4.3.

4.1 Proof of Theorem 4.12 for the case where Y is an ffg object

Suppose that Y is an ffg object.

(1) We prove that $(A, [a,h], \ddagger)$ is an ffg-Elgot algebra whenever $(A, a, \dagger)$ is.

(1a) Given an ffg-equation $e\colon X \to FX + Y + A$, then $e^\ddagger$ is a solution, as shown by the diagram below:

$$\begin{tikzcd}
X \arrow[rrr, "e^\ddagger = e_h^\dagger"] \arrow[dr, "e_h"] \arrow[dd, "e"'] & & & A \\
& FX + A \arrow[r, "Fe_h^\dagger + A"] & FA + A \arrow[ur, "{[a,A]}"'] & \\
FX + Y + A \arrow[ur, "{FX+[h,A]}"'] \arrow[rrr, "Fe^\ddagger + Y + A"'] & & & FA + Y + A \arrow[ul, "{FA+[h,A]}"] \arrow[uu, "{[[a,h],A]}"']
\end{tikzcd}$$

(1b) $\ddagger$ is weakly functorial. Suppose that a commutative square

$$\begin{tikzcd}
X \arrow[r, "e"] \arrow[d, "m"'] & FX + Y + Z \arrow[d, "Fm + Y + Z"] \\
X' \arrow[r, "f"'] & FX' + Y + Z
\end{tikzcd}$$

and a morphism $g\colon Z \to A$ are given where X, X' and Z are ffg objects. We need to prove

$$(g \bullet e)^\ddagger = (g \bullet f)^\ddagger \cdot m.$$

From the following diagram:

$$\begin{tikzcd}
X \arrow[r, "e"] & FX + Y + Z \arrow[r, "FX+Y+g"] \arrow[rr, bend right, "{FX+[h,g]}"'] & FX + Y + A \arrow[r, "{FX+[h,A]}"] & FX + A
\end{tikzcd}$$

we deduce

$$(g \bullet e)_h = [h,g] \bullet e.$$

Here, by abuse of notation, $\bullet$ is used both for F and $F(-) + Y$. Analogously,

$$(g \bullet f)_h = [h,g] \bullet f.$$

Since $\dagger$ is weakly functorial, we get

$$([h,g] \bullet e)^\dagger = ([h,g] \bullet f)^\dagger \cdot m$$

and therefore

$$(g \bullet e)^\ddagger = (g \bullet e)^\dagger_h = ([h,g] \bullet e)^\dagger = ([h,g] \bullet f)^\dagger \cdot m = (g \bullet f)^\dagger_h \cdot m = (g \bullet f)^\ddagger \cdot m.$$

(1c) $\ddagger$ is compositional. Given ffg-equations for $F(-)+Y$

$$e\colon X \to FX + Y + Z \quad \text{and} \quad f\colon Z \to FZ + Y + A,$$

we are to prove

$$(f^\ddagger \bullet e)^\ddagger = (e \blacksquare f)^\ddagger \cdot \mathsf{inl}.$$

Express A as a sifted colimit $a_i\colon A_i \to A$ ($i \in I$) of ffg objects. Then also the morphisms $FZ + Y + a_i\colon FZ + Y + A_i \to FZ + Y + A$ form a sifted colimit cocone, and since Z is an ffg object, f factorizes through one of them:

$$\begin{tikzcd}
Z \arrow[r, "f"] \arrow[dr, "f_0"'] & FZ + Y + A \\
 & FZ + Y + A_i \arrow[u, "FZ+Y+a_i"']
\end{tikzcd}$$

Define ffg-equations $\hat{f}$ and $\hat{f}_0$ by the commutative diagrams below (where inm denotes the middle coproduct injection):

$$\begin{tikzcd}
Y \arrow[r, "h"] \arrow[d, "\mathsf{inl}"'] & A \arrow[d, "\mathsf{inr}"] \\
Y + Z \arrow[r, "\hat{f}"] & F(Y+Z) + A \\
Z \arrow[u, "\mathsf{inr}"] \arrow[r, "f"'] & FZ + Y + A \arrow[u, "F\mathsf{inr}+[h,A]"']
\end{tikzcd}
\qquad
\begin{tikzcd}
Y \arrow[dr, bend left, "\mathsf{inm}"] \arrow[d, "\mathsf{inl}"'] & \\
Y + Z \arrow[r, "\hat{f}_0"] & F(Y+Z) + Y + A_i \\
Z \arrow[u, "\mathsf{inr}"] \arrow[r, "f_0"'] & FZ + Y + A_i \arrow[u, "F\mathsf{inr}+Y+A_i"']
\end{tikzcd}$$

Since $\dagger$ is compositional, we have

$$(e \blacksquare \hat{f})^\dagger \cdot \mathsf{inl} = (\hat{f}^\dagger \bullet e)^\dagger.$$

We now verify that $[\mathsf{inl},\mathsf{inr}]\colon X + Z \to X + Y + Z$ is a coalgebra homomorphism from $e \blacksquare f_0$ to $e \blacksquare \hat{f}_0$. (Here we again use $\blacksquare$ for both F and $F(-)+Y$.) This is shown by the commutative diagram below, where can in the upper row is w.r.t. $F(-)+Y$, and in the lower row it is w.r.t. F:

$$\begin{tikzcd}
X{+}Z \arrow[r, "{[e,\mathsf{inr}]}"] \arrow[d, "{[\mathsf{inl},\mathsf{inr}]}"'] \arrow[rrr, bend left, "e \blacksquare f_0"] & FX{+}Y{+}Z \arrow[r, "FX+Y+f_0"] \arrow[d, equal] & FX{+}Y{+}FZ{+}Y{+}A_i \arrow[r, "\mathsf{can}+A_i"] & F(X{+}Z){+}Y{+}A_i \arrow[d, "F{[\mathsf{inl},\mathsf{inr}]}+Y+A_i"'] \\
X{+}Y{+}Z \arrow[r, "{[e,\mathsf{inr}]}"'] \arrow[rrr, bend right, "e \blacksquare \hat{f}_0"'] & FX{+}Y{+}Z \arrow[r, "FX+\hat{f}_0"'] & FX{+}F(Y{+}Z){+}Y{+}A_i \arrow[r, "\mathsf{can}+Y+A_i"'] & F(X{+}Y{+}Z){+}Y{+}A_i
\end{tikzcd}$$

Moreover, we have

$$[h, a_i] \bullet (e \blacksquare f_0) = (e \blacksquare f)_h$$

as shown by the following computation:

$$\begin{aligned}
[h, a_i] \bullet (e \blacksquare f_0) &= ([h, A] \cdot (Y + a_i)) \bullet (e \blacksquare f_0) \\
&= [h, A] \bullet ((Y + a_i) \bullet (e \blacksquare f_0)) && \text{Remark 3.3(1)} \\
&= [h, A] \bullet (e \blacksquare ((Y + a_i) \bullet f_0)) && \text{Remark 3.3(2)} \\
&= [h, A] \bullet (e \blacksquare f) && \text{def. } f_0 \\
&= (e \blacksquare f)_h && \text{def. } (-)_h.
\end{aligned}$$

Analogously,

$$[h, a_i] \bullet (e \blacksquare \hat{f}_0) = e \blacksquare \hat{f}.$$

Since † is weakly functorial, we get

$$(e \blacksquare f)_h^\dagger = (e \blacksquare \hat{f})^\dagger \cdot [\mathsf{inl}, \mathsf{inr}]. \tag{4.9}$$

We apply the Weak Functoriality of † also the to lower square of the diagram defining $\hat{f}_0$ and to $[h, a_i]$ in lieu of h and use that $[h, a_i] \cdot \hat{f}_0 = \hat{f}$ to obtain

$$([h, a_i] \bullet f_0)^\dagger = ([h, a_i] \bullet \hat{f}_0)^\dagger \cdot \mathsf{inr} = \hat{f}^\dagger \cdot \mathsf{inr}.$$

This implies that

$$\hat{f}^\dagger \cdot \mathsf{inr} = f_h^\dagger$$

since, using Remark 3.3(2),

$$\hat{f}^\dagger \cdot \mathsf{inr} = ([h, a_i] \bullet f_0)^\dagger = ([h, A] \bullet ((Y + a_i) \bullet f_0))^\dagger = ([h, A] \bullet f)^\dagger = f_h^\dagger.$$

We conclude

$$\hat{f}^\dagger = [h, f_h^\dagger] \colon Y + Z \to A \tag{4.10}$$

since the left-hand component $\hat{f}^\dagger \cdot \mathsf{inl} = h$ follows from the fact that $\hat{f}^\dagger$ is a solution of $\hat{f}$:

$$\begin{tikzcd}
Y + Z \arrow[rr, "\hat{f}^\dagger"] \arrow[dddd, "\hat{f}"'] & & A \\
& Y \arrow[ul, "\mathsf{inl}"'] \arrow[ur, "h"] \arrow[d, "h"'] & \\
& A \arrow[uur, equal] \arrow[ddl, "\mathsf{inr}"'] \arrow[ddr, "\mathsf{inr}"] & \\
& & \\
F(Y + Z) + A \arrow[rr, "F\hat{f}^\dagger + A"'] & & FA + A \arrow[uuuu, "{[a, A]}"']
\end{tikzcd}$$

Thus, we conclude the proof with the following computation:

$$
\begin{aligned}
(f^\ddagger \bullet e)^\ddagger &= (f_h^\dagger \bullet e)_h^\dagger && \text{def. } \ddagger \\
&= ([h, f_h^\dagger] \bullet e)^\dagger && \text{def. } (-)_h \\
&= (\hat{f}^\dagger \bullet e)^\dagger && (4.10) \\
&= (e \blacksquare \hat{f})^\dagger \cdot \mathsf{inl} && \text{compositionality of } \dagger \\
&= (e \blacksquare f)_h^\dagger \cdot \mathsf{inl} && (4.9) \\
&= (e \blacksquare f)^\ddagger \cdot \mathsf{inl} && \text{def. } \ddagger
\end{aligned}
$$

(2) For every ffg-Elgot algebra $(A, [a, h]^\ddagger)$ for $F(-) + Y$, we prove that $(A, a, \dagger)$ with $e^\dagger := \overline{e}^\ddagger$ is an ffg-Elgot algebra for F.

(2a) $e^\dagger$ is a solution of $e\colon X \to FX + A$:

$$
\begin{tikzcd}
X \arrow[r, "e^\dagger = \overline{e}^\ddagger"] \arrow[d, "\overline{e}"] \arrow[dd, bend right=60, "e"'] & A \\
FX + Y + A \arrow[r, "F\overline{e}^\ddagger + Y + A"'] & FA + Y + A \arrow[u, "{[[a,h],A]}"'] \\
FX + A \arrow[u, "FX + \mathsf{inr}"] \arrow[r, "Fe^\dagger + A"'] & FA + A \arrow[u, "{[\mathsf{inl},\mathsf{inr}]}"'] \arrow[uu, bend right=60, "{[a,A]}"']
\end{tikzcd}
$$

Indeed, the upper square commutes since $\overline{e}^\ddagger$ is a solution of $\overline{e}$, and for the lower one recall that $e^\dagger = \overline{e}^\ddagger$.

(2b) $\dagger$ is weakly functorial. Given a coalgebra homomorphism m from $e\colon X \to FX + Z$ to $f\colon X' \to FX' + Z$ and a morphism $h\colon Z \to A$ where X, X', and Z are ffg objects, we need to prove $(h \bullet e)^\dagger = (h \bullet f)^\dagger \cdot m$. From the following diagram we see that m is also a coalgebra homomorphism for $F(-) + Y + Z$ from $\overline{e}$ to $\overline{f}$:

$$
\begin{tikzcd}
X \arrow[r, "e"] \arrow[d, "m"'] \arrow[rr, bend left, "\overline{e}"] & FX + Z \arrow[r, "{[\mathsf{inl},\mathsf{inr}]}"] \arrow[d, "Fm + Z"] & FX + Y + Z \arrow[d, "Fm + Y + Z"] \\
X' \arrow[r, "f"'] \arrow[rr, bend right, "\overline{f}"'] & FX' + Z \arrow[r, "{[\mathsf{inl},\mathsf{inl}]}"'] & FX' + Y + Z
\end{tikzcd}
$$

Hence, Weak Functoriality of $\ddagger$ yields

$$(h \bullet \overline{e})^\ddagger = (h \bullet \overline{f})^\ddagger \cdot m.$$

This implies the desired equality since

$$\overline{h \bullet e} = h \bullet \overline{e} \tag{4.11}$$

(and analogously for f) due to the following diagram:

$$\begin{tikzcd}
X \arrow[r, "h\bullet e"] \arrow[dr, "e"'] \arrow[rr, bend left, "\overline{h\bullet e}"] \arrow[drr, bend right=50, "\overline{e}"'] & FX+A \arrow[r, "{[\mathsf{inl},\mathsf{inr}]}"] & FX+Y+A \\
& FX+Z \arrow[u, "FX+h"'] \arrow[r, "{[\mathsf{inl},\mathsf{inr}]}"'] & FX+Y+Z \arrow[u, "FX+Y+h"']
\end{tikzcd}$$

(2c) ‡ is compositional. Given ffg-equations $e\colon X \to FX+Z$ and $f\colon Z \to FZ+A$, we need to prove $(f^\dagger \bullet e)^\dagger = (e \blacksquare f)^\dagger \cdot \mathsf{inl}$. We first observe that

$$\overline{e} \blacksquare \overline{f} = \overline{e \blacksquare f}. \tag{4.12}$$

This follows from the diagram below (where can on the right-hand arrow is w.r.t. $F(-)+Y$ and can in the middle of the diagram w.r.t. F):

$$\begin{tikzcd}[column sep=small]
X+Z \arrow[r, "{[e,\mathsf{inr}]}"] \arrow[ddrr, "e\blacksquare f"'] \arrow[rr, bend left, "{[\overline{e},\mathsf{inr}]}"] \arrow[ddrrrr, bend right, "\overline{e\blacksquare f}"'] & FX+Z \arrow[r, "{[\mathsf{inl},\mathsf{inr}]}"] \arrow[dr, "FX+f"'] & FX+Y+Z \arrow[r, "FX+Y+f"] \arrow[rr, bend left, "FX+Y+\overline{f}"] & FX+Y+FZ+A \arrow[r, "{FX+Y+[\mathsf{inl},\mathsf{inr}]}"] & FX+Y+FZ+Y+A \arrow[dd, "\mathsf{can}+A"] \\
& & FX+FZ+A \arrow[ur, "{[\mathsf{inl},\mathsf{inr}]+A}"'] \arrow[d, "\mathsf{can}+A"] & & \\
& & F(X+Z)+A \arrow[rr, "{[\mathsf{inl},\mathsf{inr}]}"'] & & F(X+Z)+Y+A
\end{tikzcd}$$

Note that the upper path composed with $\mathsf{can}+A$ yields $\overline{e \blacksquare f}$. The proof of compositionality now easily follows:

$$\begin{aligned}
(f^\dagger \bullet e)^\dagger &= \left(\overline{\overline{f}^\ddagger \bullet e}\right)^\ddagger && \text{def. } \dagger \\
&= (\overline{f}^\ddagger \bullet \overline{e})^\ddagger && \text{by (4.11)} \\
&= (\overline{e} \blacksquare \overline{f})^\dagger \cdot \mathsf{inl} && \ddagger \text{ compositional} \\
&= (\overline{e \blacksquare f})^\ddagger \cdot \mathsf{inl} && \text{by (4.12)} \\
&= (e \blacksquare f)^\dagger \cdot \mathsf{inl} && \text{def. } \dagger
\end{aligned}$$

(3) We prove that the two passages (1) and (2) in Remark 4.13 are mutually inverse.

(3a) The fact that (2) followed by (1) yields the identity is easy to see since for every ffg-equation $e\colon X \to FX + A$ for F, we have

$$\overline{e}_h = e,$$

as shown by the commutative diagram below:

$$\begin{tikzcd}
X \arrow[r, "e"] \arrow[rr, bend left, "\overline{e}"] \arrow[drr, bend right, "e"'] & FX + A \arrow[r, "\mathsf{inl}+A"] \arrow[dr, equal] & FX + Y + A \arrow[d, "FX+{[h,A]}"] \\
& & FX + A
\end{tikzcd}$$

(3b) In order to show that (1) followed by (2) is the identity, we prove for every ffg-equation $e\colon X \to FX + Y + A$ that $\overline{(e_h)}^{\ddagger} = e^{\ddagger}$. (We do *not* claim that $\overline{(e_h)} = e$.) Express A as a sifted colimit $a_i\colon A_j \to A$ ($j \in J$) of ffg objects. Then also the morphisms $FX + Y + a_i\colon FX + Y + A_i \to FX + Y + A$ form a sifted colimit cocone, and since X is an ffg object, there exists $j \in J$ and a morphism e_0 such that the following triangle commutes:

$$\begin{tikzcd}
X \arrow[r, "e"] \arrow[dr, "e_0"'] & FX + Y + A \\
& FX + Y + A_j \arrow[u, "FX+Y+a_j"']
\end{tikzcd}$$

Consider the ffg-equation

$$f = (Y + A_j \xrightarrow{\mathsf{inr}} F(Y + A_j) + Y + A_j \xrightarrow{F(Y+A_j)+Y+a_j} F(Y + A_j) + Y + A).$$

(Note that $f = a_j \bullet f_0$ for $f_0 = \mathsf{inr}$.) We have that

$$f^{\ddagger} = (Y + A_j \xrightarrow{Y+a_j} Y + A \xrightarrow{[h,A]} A) \tag{4.13}$$

as demonstrated by the diagram below:

$$\begin{tikzcd}
Y + A_j \arrow[rr, "f^{\ddagger}"] \arrow[d, "Y+a_j"] \arrow[dd, bend right=60, "f"'] & & A \\
Y + A \arrow[urr, "{[h,A]}"'] \arrow[d, "\mathsf{inr}"] \arrow[drr, "\mathsf{inr}"'] & & \\
F(Y + A_j) + Y + A \arrow[rr, "Ff^{\ddagger}+Y+A"'] & & FA + Y + A \arrow[uu, "{[[a,h],A]}"']
\end{tikzcd}$$

We also have that

$$\overline{(e_h)} = f^{\ddagger} \bullet e_0', \tag{4.14}$$

where

$$e_0' = (X \xrightarrow{e_0} FX + Y + A_j \xrightarrow{\mathsf{inl}+Y+A_j} FX + Y + Y + A).$$

Indeed, the following diagram commutes using (4.13) for the right-hand part and (4.7), (4.8) for the lower left-hand one:

$$
\begin{tikzcd}
X \arrow[r, "e_0"] \arrow[dr, "e"] \arrow[ddr, "e_h"'] \arrow[ddrr, bend right, "\overline{(e_h)}"'] & FX+Y+A_j \arrow[r, "\mathsf{inl}+Y+A_j"] \arrow[d, "FX+Y+a_j"] & FX+Y+Y+A_j \arrow[d, "FX+Y+Y+a_j"'] \arrow[dd, bend left, "FX+Y+f^{\ddagger}"] \\
& FX+Y+A \arrow[r, "\mathsf{inl}+Y+A"] \arrow[d, "FX+[h,A]"] & FX+Y+Y+A \arrow[d, "FX+Y+[h,A]"'] \\
& FX+A \arrow[r, "{[\mathsf{inl},\mathsf{inr}]}"'] & FX+Y+A
\end{tikzcd}
$$

Finally, we have a coalgebra homomorphism inl from e_0 to $e_0' \blacksquare f_0$:

$$
\begin{tikzcd}
X \arrow[r, "\mathsf{inl}"] \arrow[dr, "e_0"] \arrow[ddddd, "e_0"'] & X+Y+A_j \arrow[d, "{[e_0,\mathsf{inr}]}"] \\
& FX+Y+A_j \arrow[d, "\mathsf{inl}+Y+A_j"] \\
& FX+Y+Y+A_j \arrow[d, "FX+Y+f_0"] \\
& FX+Y+F(Y+A_j)+Y+A_j \arrow[dd, "\mathsf{can}+A_j"] \\
& \\
FX+Y+A_j \arrow[uuur, bend left, "\mathsf{inl}+Y+A_j"] \arrow[uur, "\mathsf{inl}+Y+A_j"] \arrow[r, "F\mathsf{inl}+Y+A_j"'] & F(X+Y+A_j)+Y+A_j
\end{tikzcd}
$$

Thus, we obtain

$$
\begin{aligned}
e^{\ddagger} &= (a_j \bullet e_0)^{\ddagger} & & \\
&= (a_j \bullet (e_0' \blacksquare f_0))^{\ddagger} \cdot \mathsf{inl} & & \ddagger \text{ weakly functorial} \\
&= (e_0' \blacksquare (a_j \bullet f_0))^{\ddagger} \cdot \mathsf{inl} & & \text{Remark 3.3(2)} \\
&= (e_0' \blacksquare f)^{\ddagger} \cdot \mathsf{inl} & & \text{def. } f_0 \\
&= (f^{\ddagger} \bullet e_0')^{\ddagger} & & \ddagger \text{ compositional} \\
&= \overline{(e_h)}^{\ddagger} & & \text{by (4.14).}
\end{aligned}
$$

This concludes the proof.

□

4.2 Proof of Theorem 4.12 for an arbitrary free object

Now assume that Y is an arbitrary free object of $\mathscr{C}$. We shall reduce this case to the previous situation using filtered colimits.

Notation 4.14. Fix an F-algebra $a\colon FA \to A$ and a morphism $h\colon Y \to A$. Since in every variety $\mathscr{C}$ the free functor (left adjoint to the forgetful functor from $\mathscr{C}$ to Set^S) preserves colimits, we can express the free object Y as a colimit of a filtered diagram D_Y of ffg objects Y_i:

$$Y = \operatorname{colim} Y_i \qquad \text{with injections } y_i\colon Y_i \to Y \quad (i \in I).$$

Definition 4.15. By a *compatible family of ffg-Elgot algebras* is meant a family

$$(A, [a, h_i], (-)^{\dagger,i}) \qquad (\text{for } i \in I) \tag{4.15}$$

of ffg-Elgot algebras for the functors $F(-) + Y_i$ such that for every connecting morphism $y_{ij}\colon Y_i \to Y_j$ of the diagram D_Y and every ffg-equation $e\colon X \to FX + Y_i + A$, one has

$$((FX + y_{ij} + A) \cdot e)^{\dagger,j} = e^{\dagger,i}.$$

To establish Theorem 4.12, we prove the following more refined result:

Theorem 4.16. *For every F-algebra (A, a) there is a bijective correspondence between*

(1) *solution operations $\dagger$ such that $(A, a, \dagger)$ is an ffg-Elgot algebra for F,*

(2) *families of solution operations $(-)^{\dagger,i}$ such that $(A, [a, h_i], (-)^{\dagger,i})$ $(i \in I)$ is a compatible family of ffg-Elgot algebras, and*

(3) *solution operations $\ddagger$ such that $(A, [a, h], \ddagger)$ is an ffg-Elgot algebra for $F(-) + Y$.*

The proof is split into four lemmas.

Lemma 4.17. *Let $(A, a, \dagger)$ be an ffg-Elgot algebra. Every cocone $h_i\colon Y_i \to A$ $(i \in I)$ induces a compatible family of ffg-Elgot algebras $(A, [a, h_i], (-)^{\dagger,i})$ with solution operations given by*

$$e^{\dagger,i} = (X \xrightarrow{e} FX + Y_i + A \xrightarrow{FX + [h_i, A]} FX + A)^{\dagger}.$$

Proof. By part (1) in Subsection 4.1, $(A, [a, h_i], (-)^{\dagger,i})$ is an ffg-Elgot algebra for every $i \in I$. For compatibility, let $e\colon X \to FX + Y_i + A$ be an ffg-equation and let $y_{ij}\colon Y_i \to Y_j$ be a connecting morphism of D_Y. Then the triangle below commutes:

$$\begin{tikzcd}
FX + Y_i + A \arrow[r, "FX+[h_i,A]"] \arrow[d, "FX+y_{ij}+A"'] & FX + A \\
FX + Y_j + A \arrow[ur, "FX+[h_j,A]"']
\end{tikzcd}$$

Therefore

$$\begin{aligned}
((FX + y_{ij} + A) \cdot e)^{\dagger,j} &= ((FX + [h_j, A]) \cdot (FX + y_{ij} + A) \cdot e)^{\dagger} \\
&= ((FX + [h_i, A]) \cdot e)^{\dagger} \\
&= e^{\dagger,i}
\end{aligned}$$

Here the first equation is the definition of $(-)^{\dagger,j}$, the second one follows from the above commutative triangle, and the last one is the definition of $(-)^{\dagger,i}$. $\square$

Lemma 4.18. *Suppose that a compatible family* (4.15) *of ffg-Elgot algebras is given. Then for every ffg equation* $e\colon X \to FX + A$ *the morphism*

$$e^\dagger = (X \xrightarrow{e} FX + A \xrightarrow{[\mathsf{inl},\mathsf{inr}]} FX + Y_i + A)^{\dagger,i}$$

is independent of the choice of i. *Moreover,* $(A, a, \dagger)$ *is an ffg-Elgot algebra for* F, *and the morphisms* h_i $(i \in I)$ *form a cocone of the diagram* D_Y.

Proof. (1) By part (2) in Subsection 4.1, we know that $(A, a, \dagger)$ is an ffg-Elgot algebra. Let us verify that $\dagger$ is independent of the choice of i. Given $i, j \in I$, choose $k \in I$ and connecting morphisms $y_{ik}\colon Y_i \to Y_k$ and $y_{jk}\colon Y_j \to Y_k$, using that D_Y is filtered. Then the following diagram commutes:

$$\begin{tikzcd}
 & FX + Y_i + A \arrow[d, "FX+y_{ik}+A"] \\
FX + A \arrow[ur, "{[\mathsf{inl},\mathsf{inr}]}"] \arrow[r, "{[\mathsf{inl},\mathsf{inr}]}"] \arrow[dr, "FX+\mathsf{inr}"'] & FX + Y_k + A \\
 & FX + Y_j + A \arrow[u, "FX+y_{jk}+A"']
\end{tikzcd}$$

Therefore, by compatibility of the family (4.15), one has

$$(X \xrightarrow{e} FX+A \xrightarrow{[\mathsf{inl},\mathsf{inr}]} FX+Y_i+A)^{\dagger,i} = (X \xrightarrow{e} FX+A \xrightarrow{[\mathsf{inl},\mathsf{inr}]} FX+Y_j+A)^{\dagger,j},$$

as required.

(2) Next, we show that for every $i \in I$ the ffg-equation $Y_i \xrightarrow{\mathsf{inm}} FY_i + Y_i + A$ has the solution $\mathsf{inm}^{\dagger,i} = h_i$:

$$\begin{tikzcd}
Y_i \arrow[r, "\mathsf{inm}^{\dagger,i}"] \arrow[d, "\mathsf{inm}"'] \arrow[dr, "\mathsf{inm}"] & A \\
FY_i + Y_i + A \arrow[r, "F\mathsf{inm}^{\dagger,i}+Y_i+A"'] & FA + Y_i + A \arrow[u, "{[a,h_i,A]}"']
\end{tikzcd}$$

Here the outside commutes by the definition of a solution, and the lower triangle commutes trivially. Therefore the upper triangle commutes, showing that $h_i = \mathsf{inm}^{\dagger,i}$.

(3) Finally, we prove that the h_i's form a cocone. Suppose that a connecting morphism $y_{ij}\colon Y_i \to Y_j$ is given, and consider the following commutative diagram

(here $i_A\colon 0 \to A$ denotes the unique morphism from the initial object to A):

$$
\begin{tikzcd}
 & & & FY_i + Y_i + A \arrow[d, "FY_i+y_{ij}+A"] \\
Y_i \arrow[uurrr, phantom] \arrow[urrr, bend left, "\mathsf{inm}"] \arrow[r, "\mathsf{inm}"] \arrow[d, "y_{ij}"] & FY_i + Y_i + 0 \arrow[r, "FY_i+y_{ij}+0"] & FY_i + Y_j + 0 \arrow[r, "FY_i+Y_j+i_A"] \arrow[d, "Fy_{ij}+Y_j+0"] & FY_i + Y_j + A \arrow[d, "Fy_{ij}+Y_j+A"] \\
Y_j \arrow[rr, "\mathsf{inm}"'] \arrow[rrr, bend right, "\mathsf{inm}"'] & & FY_j + Y_j + 0 \arrow[r, "FY_j+Y_j+i_A"'] & FY_j + Y_j + A
\end{tikzcd}
$$

Then we get

$$
\begin{aligned}
h_i &= \mathsf{inm}^{\dagger,i} \\
&= (i_A \bullet ((FY_i + y_{ij} + 0) \cdot \mathsf{inm}))^{\dagger,j} \\
&= (i_A \bullet \mathsf{inm})^{\dagger,j} \cdot y_{ij} \\
&= \mathsf{inm}^{\dagger,j} \cdot y_{ij} \\
&= h_j \cdot y_{ij}
\end{aligned}
$$

Here the first equation follows from part (2) above, the second one follows from the upper part of the above diagram and compatibility, the third one follows from the central part of the diagram via Weak Functoriality of $(-)^{\dagger,j}$, the fourth one is the lower part of the diagram, and the last equation is again part (2). ◻

Lemma 4.19. *Every ffg-Elgot algebra $(A, [a,h], \ddagger)$ for $F(-) + Y$ induces the following compatible family of ffg-Elgot algebras: $(A, [a, h_i], (-)^{\dagger,i})$ ($i \in I$), where $h_i = h \cdot y_i$ and the solution operations are given by*

$$e^{\dagger,i} = (X \xrightarrow{e} FX + Y_i + A \xrightarrow{FX+y_i+A} FX + Y + A)^{\ddagger}.$$

Proof. (1) We first show that $(A, [a, h_i], (-)^{\dagger,i})$ is an ffg-Elgot algebra for every $i \in I$. In the following, for every ffg-equation $e\colon X \to FX + Y_i + A$, we put

$$\overline{e} = (X \xrightarrow{e} FX + Y_i + A \xrightarrow{FX+y_i+A} FX + Y + A).$$

Solution. Consider the diagram below:

$$
\begin{tikzcd}
X \arrow[r, "e^{\dagger,i}"] \arrow[d, "\overline{e}"] \arrow[dd, bend right=60, "e"'] & A \\
FX + Y + A \arrow[r, "Fe^{\ddagger,i}+Y+A"] & FA + Y + A \arrow[u, "{[a,h,A]}"'] \\
FX + Y_i + A \arrow[u, "FX+y_i+A"'] \arrow[r, "Fe^{\dagger,i}+Y_i+A"'] & FA + Y_i + A \arrow[u, "FA+y_i+A"] \arrow[uu, bend right=60, "{[a,h_i,A]}"']
\end{tikzcd}
$$

The upper part commutes because $e^{\dagger,i} = \overline{e}^{\ddagger}$ is the solution of $\overline{e}$, and the other three parts commute trivially. Therefore the outside of the diagram commutes, showing that $e^{\dagger,i}$ is a solution of e.

Weak functoriality. Suppose that two ffg-equations $e\colon X \to FX + Y_i + Z$ and $f\colon X' \to FX' + Y_i + Z$ are given together with a coalgebra homomorphism m from e to f and a morphism $g\colon Z \to A$. Then m is also a coalgebra homomorphism w.r.t. $F(-) + Y$:

$$\begin{tikzcd}
X \arrow[r, "e"] \arrow[d, "m"'] \arrow[rr, bend left, "\overline{e}"] & FX + Y_i + Z \arrow[r, "FX+y_i+A"] \arrow[d, "Fm+Y_i+A"] & FX + Y + A \arrow[d, "Fm+Y+A"] \\
X' \arrow[r, "f"'] \arrow[rr, bend right, "\overline{f}"'] & FX' + Y_i + A \arrow[r, "FX'+y_i+A"'] & FX' + Y + A
\end{tikzcd}$$

Moreover, we have

$$g \bullet \overline{e} = \overline{g \bullet e} \tag{4.16}$$

and similarly for f, due to the following diagram:

$$\begin{tikzcd}
X \arrow[r, "e"] \arrow[dr, "\overline{e}"'] \arrow[rr, bend left, "g\bullet e"] \arrow[drr, bend right, "g\bullet\overline{e}"'] & FX + Y_i + Z \arrow[r, "FX+Y_i+g"] \arrow[d, "FX+y_i+Z"] & FX + Y_i + A \arrow[d, "FX+y_i+A"] \\
 & FX + Y + Z \arrow[r, "FX+Y+g"'] & FX + Y + A
\end{tikzcd}$$

Thus, Weak Functoriality of $\dagger, i$ follows from that of $\ddagger$:

$$(g \bullet f)^{\dagger,i} \cdot m = (\overline{g \bullet f})^{\ddagger} \cdot m = (g \bullet \overline{f})^{\ddagger} \cdot m = (g \bullet \overline{e})^{\ddagger} = (\overline{g \bullet e})^{\ddagger} = (g \bullet e)^{\dagger,i}.$$

Compositionality. Using the definition of $(-)^{\dagger,i}$, one easily verifies that for two ffg-equations $e\colon X \to FX + Y_i + Z$ and $f\colon Z \to FZ + Y_i + A$ one has $\overline{f \blacksquare e} = \overline{f} \blacksquare \overline{e}$ due to the following commutative diagram:

$$\begin{tikzcd}
X{+}Z \arrow[r, "{[e,\mathsf{inr}]}"] \arrow[dr, "{[\overline{e},\mathsf{inr}]}"'] \arrow[rrrr, bend left, "f \blacksquare e"] \arrow[drrr, bend right, "\overline{f}\bullet\overline{e}"'] & FX{+}Y_i{+}Z \arrow[r, "FX+Y_i+f"] \arrow[d, "FX+y_i+Z"] & FX{+}Y_i{+}FZ{+}Y_i{+}A \arrow[r, "\mathsf{can}+A"] \arrow[d, "FX+y_i+FZ+y_i+A"] & F(X{+}Z){+}Y_i{+}Z \arrow[d, "F(X+Z)+y_i+A"] \\
 & FX{+}Y{+}A \arrow[r, "FX+Y+\overline{f}"'] & FX{+}Y{+}FZ{+}Y{+}A \arrow[r, "\mathsf{can}+A"'] & F(X{+}Z){+}Y{+}A
\end{tikzcd}$$

Thus we obtain $(f \mathbin{\blacksquare} e)^{\dagger,i} = (\overline{f \mathbin{\blacksquare} e})^{\ddagger} = (\overline{f} \mathbin{\blacksquare} \overline{e})^{\ddagger}$, and we have

$$(f^{\dagger,i} \bullet e)^{\dagger,i} = (\overline{f}^{\ddagger} \bullet e)^{\dagger,i} = \left(\overline{\overline{f}^{\ddagger} \bullet e}\right)^{\ddagger} = (\overline{f}^{\ddagger} \bullet e)^{\ddagger}.$$

Then compositionality of $\ddagger$ implies

$$(f \mathbin{\blacksquare} e)^{\dagger,i} \cdot \mathsf{inl} = (\overline{f} \mathbin{\blacksquare} \overline{e})^{\ddagger} \cdot \mathsf{inl} = (\overline{f}^{\ddagger} \bullet \overline{e})^{\ddagger} = (f^{\dagger,i} \bullet e)^{\dagger,i}.$$

(2) To prove that the given family of ffg-Elgot algebras is compatible, let $e\colon X \to FX + Y_i \to A$ be an ffg-equation and $y_{ij}\colon Y_i \to Y_j$ a connecting morphism of D_Y. Then

$$\begin{aligned}((FX + y_{ij} + A) \cdot e)^{\dagger,j} &= ((FX + y_j + A) \cdot (FX + y_{ij} + A) \cdot e)^{\ddagger} \\ &= ((FX + y_i + A) \cdot e)^{\ddagger} \\ &= e^{\dagger,i},\end{aligned}$$

where the first equation uses the definition of $(-)^{\dagger,j}$, the second one uses that y_{ij} is a connecting morphism, and the last equation uses the definition of $(-)^{\dagger,i}$. ☐

Notation 4.20. (1) By Lemma 4.18, for every compatible family (4.15) of ffg-Elgot algebras, the morphisms $h_i\colon Y_i \to A$ form a cocone and thus induce a unique morphism $h\colon Y \to A$ with $h_i = h \cdot y_i$ for all $i \in I$.

(2) For every ffg equation $e\colon X \to FX + Y + A$ there exists a factorization

$$e = (\,X \xrightarrow{e_i} FX + Y_i + A \xrightarrow{FX+y_i+A} FX + Y + A\,)$$

with $i \in I$. We put $e^{\ddagger} := e_i^{\dagger,i}$ (and prove below that this is independent of the choice of i).

Lemma 4.21. *Every compatible family* (4.15) *of ffg-Elgot algebras induces an ffg-Elgot algebra* $(A, [a, h], \ddagger)$.

Proof. We first observe that the factorization of e exists because $(FX + Y_i + A \xrightarrow{FX+y_i+A} FX + Y + A)_{i\in I}$ is a filtered colimit cocone and X, being an ffg object, is finitely presentable. Let us show that $\ddagger$ well-defined, i.e. independent of the choice of the factorization. To see this, suppose that another factorization $e = (FX + y_j + A) \cdot e_j$ is given. Since D_Y is filtered, there exists $k \in I$ and connecting morphisms $y_{ik}\colon Y_i \to Y_k$ and $y_{jk}\colon Y_j \to Y_j$ with $e_k := (FX + y_{ik} + A) \cdot e_i = (FX + y_{jk} + A) \cdot e_j$. Then compatibility of the given family of ffg-Elgot algebras shows that

$$e_i^{\dagger,i} = e_k^{\dagger,k} = e_j^{\dagger,j},$$

as required.

It remains to show that $(A, [a, h], \ddagger)$ is an ffg-Elgot algebra.

Solution. Consider the following diagram:

$$
\begin{tikzcd}
X \arrow[r, "e^{\ddagger}"] \arrow[d, "e"] \arrow[dd, bend right=60, "e_i"'] & A \\
FX+Y+A \arrow[r, "Fe^{\ddagger}+Y+A"] & FA+Y+A \arrow[u, "{[a,h,A]}"'] \\
FX+Y_i+A \arrow[r, "Fe^{\ddagger}+Y_i+A"'] \arrow[u, "FX+y_i+A"'] & FA+Y_i+A \arrow[u, "FA+y_i+A"] \arrow[uu, bend right=60, "{[a,h_i,A]}"']
\end{tikzcd}
$$

Its outside commutes because $e^{\ddagger} = e_i^{\dagger,i}$ and $e_i^{\dagger,i}$ is a solution of e_i. All other parts except, perhaps, the upper one commute trivially. Therefore, the upper part commutes, showing that $e^{\ddagger}$ is a solution of e.

Weak Functoriality. Suppose that we are given ffg-equations $e\colon X \to FX+Y+Z$ and $f\colon X' \to FX'+Y+Z$, where Z is an ffg object, a coalgebra homomorphism m from e to f, and a morphism $g : Z \to A$. We choose factorizations

$$
\begin{tikzcd}
X \arrow[r, "e"] \arrow[dr, "e_i"'] & FX+Y+Z \\
& FX+Y_i+Z \arrow[u, "FX+y_i+Z"']
\end{tikzcd}
\qquad
\begin{tikzcd}
X' \arrow[r, "f"] \arrow[dr, "f_i"'] & FX'+Y+Z \\
& FX'+Y_i+Z \arrow[u, "FX'+y_i+Z"']
\end{tikzcd}
$$

for some $i \in I$; note that we may choose the same i for both e and f since D_Y is filtered. Then in the following diagram the outside and all inner parts except the left-hand square commute:

$$
\begin{tikzcd}
X \arrow[r, "e_i"] \arrow[d, "m"'] \arrow[rr, bend left, "e"] & FX+Y_i+Z \arrow[r, "FX+y_i+Z"] \arrow[d, "Fm+Y_i+Z"] & FX+Y+Z \arrow[d, "Fm+Y+Z"] \\
X' \arrow[r, "f_i"'] \arrow[rr, bend right, "f"'] & FX'+Y_i+Z \arrow[r, "FX'+y_i+Z"'] & FX'+Y+Z
\end{tikzcd}
$$

Hence, it follows that the two morphisms

$$(Fm+Y_i+Z)\cdot e_i,\ f_i \cdot m\colon X \to FX'+Y_i+Z$$

are merged by the colimit injection $F\overline{X}+y_i+Z$. Since X is an ffg object and D_Y is filtered, some connecting morphism $FX+y_{ij}+Z$ with $j \in I$ merges them, too. Put

$$e_j := (FX+y_{ij}+Z)\cdot e_i \quad \text{and} \quad f_j := (F\overline{X}+y_{ij}+Z)\cdot \overline{f}_i.$$

Then the outside of the following diagram commutes:

$$\begin{tikzcd}
X \arrow[r, "e_i"] \arrow[d, "m"'] \arrow[rr, bend left, "e_j"] & FX + Y_i + Z \arrow[r, "FX+y_{ij}+Z"] \arrow[d, "Fm+Y_i+Z"] & FX + Y_j + Z \arrow[d, "Fm+Y_j+Z"] \\
X' \arrow[r, "f_i"'] \arrow[rr, bend right, "f_j"'] & FX' + Y_i + Z \arrow[r, "FX'+y_{ij}+Z"'] & FX' + Y_j + Z
\end{tikzcd}$$

Now observe that $g \bullet e$ factorizes through $g \bullet e_i$ as follows:

$$\begin{tikzcd}
X \arrow[r, "e"] \arrow[dr, "e_i"'] \arrow[rr, bend left, "g\bullet e"] \arrow[drr, bend right, "g\bullet e_i"'] & FX + Y + Z \arrow[r, "FX+Y+g"] & FX + Y + A \\
& FX + Y_i + Z \arrow[u, "FX+y_i+Z"] \arrow[r, "FX+Y_i+g"'] & FX + Y_i + A \arrow[u, "FX+y_i+A"']
\end{tikzcd}$$

Similarly for $g \bullet f$. Furthermore note that $(FX + y_{ij} + A) \cdot (g \bullet e_i) = g \bullet e_j$:

$$\begin{tikzcd}
X \arrow[r, "e_i"] \arrow[dr, "e_j"'] \arrow[rr, bend left, "g\bullet e_i"] \arrow[drr, bend right, "g\bullet e_j"'] & FX + Y_i + Z \arrow[r, "FX+Y_i+g"] \arrow[d, "FX+y_{ij}+Z"] & FX + Y_i + A \arrow[d, "FX+y_{ij}+A"] \\
& FX + Y_j + Z \arrow[r, "FX+Y_j+g"'] & FX + Y_j + A
\end{tikzcd}$$

and similarly $(FX + y_{ij} + A) \cdot (g \bullet f_i) = g \bullet f_j$. Thus, we obtain the Weak Functoriality of $\ddagger$ from that of $\dagger, i$:

$$\begin{aligned}
(g \bullet e)^{\ddagger} &= (g \bullet e_i)^{\dagger,i} && \text{def. of } \ddagger \\
&= ((FX + y_{ij} + A) \cdot (g \bullet e_i))^{\dagger,j} && \text{compatibility} \\
&= (g \bullet e_j)^{\dagger,j} \\
&= (g \bullet f_j)^{\dagger,j} \cdot m && \text{Weak Functoriality of } \dagger, j \\
&= ((FX + y_{ij} + A) \cdot (g \bullet f_i))^{\dagger,j} \cdot m \\
&= (g \bullet f_i)^{\dagger,i} \cdot m && \text{compatibility} \\
&= (g \bullet f)^{\ddagger} && \text{def. of } \ddagger
\end{aligned}$$

Compositionality. Let $e\colon X \to FX + Y + A$ and $f\colon Z \to FZ + Y + A$ be two ffg-equations. Factorize $e = (FX + y_i + A) \cdot e_i$ and $f = (FX + y_i + A) \cdot f_i$ with

$i \in I$. Then

$$\begin{aligned}(f \blacksquare e)^{\ddagger} \cdot \mathsf{inl} &= (f_i \blacksquare e_i)^{\dagger,i} \cdot \mathsf{inl} \\ &= (f_i^{\dagger,i} \bullet e_i)^{\dagger,i} \\ &= (f^{\ddagger} \bullet e_i)^{\dagger,i} \\ &= (f^{\ddagger} \bullet e)^{\ddagger}\end{aligned}$$

Here the first equation uses the definition of $\ddagger$ and the fact that $f \blacksquare e = (FX + y_i + A) \cdot (f_i \blacksquare e_i)$. The second equation is compositionality of $(-)^{\dagger,i}$, the third one uses that $f^{\ddagger} = f_i^{\dagger,i}$ by the definition of $\ddagger$, and the last equation uses the definition of $\ddagger$ and the fact that $(f^{\ddagger} \bullet e) = (FX + y_i + A) \cdot (f^{\ddagger} \bullet e_i)$. □

Proof of Theorem 4.12. In order to complete the proof of Theorem 4.16 (and therefore that of Theorem 4.12), observe that the constructions of Lemma 4.17 and 4.18 are mutually inverse; the proof is completely analogous to parts (3a) and (3b) of the proof in Subsection 4.1. Moreover, the constructions of Lemma 4.21 and 4.19 are clearly mutually inverse.

4.3 Free FFG-Elgot Algebras

We will now prove that for a free object Y of $\mathscr{C}$ the free ffg-Elgot algebra on Y is given by the locally ffg fixed point $\varphi(F(-) + Y)$. We begin with a consequence of Theorem 4.12. For the forgetful functor of ffg-Elgot algebras

$$U_F\colon \mathsf{ffg\text{-}Elgot}\, F \to \mathscr{C}$$

recall that the category $Y \downarrow U_F$ has as objects all morphisms $y\colon Y \to U_F(A, a, \dagger)$, and morphisms into $y'\colon Y \to U_F(B, b, \ddagger)$ are the solution-preserving morphisms $p\colon (A, a, \dagger) \to (B, b, \ddagger)$ with $p \cdot y = p'$. Denote by $\pi\colon Y \downarrow U_F \to \mathscr{C}$ the projection functor given by $\pi(y) = A$.

Proposition 4.22. *For every free object Y of $\mathscr{C}$ there is an isomorphism I of categories making the following triangle commutative:*

$$\begin{tikzcd} \mathsf{ffg\text{-}Elgot}(F(-)+Y) \arrow[rr, "I"] \arrow[dr, "U_{F(-)+Y}"'] & & Y \downarrow U_F \arrow[dl, "\pi"] \\ & \mathscr{C} & \end{tikzcd}$$

It is given by $(A, [a, h], \ddagger) \mapsto (h\colon Y \to U_F(A, a, \dagger))$.

Proof. Using Theorem 4.12, we just need to verify for every pair of ffg-Elgot algebras $(A, [a, h], \ddagger)$ and $(A', [a', h'], \ddagger')$ that a morphism $p\colon A \to A'$ is solution-preserving for $F(-) + Y$ iff it is solution-preserving for F and satisfies $h' = p \cdot h$.

($\Rightarrow$) If p is solution-preserving for $F(-) + Y$, then by Lemma 4.10 it is a homomorphism, i.e. $p \cdot [a, h] = [a', h'] \cdot Fp$. This implies $p \cdot h = h'$. Moreover, for every ffg-equation $e\colon X \to FX + A$ the ffg-equation

$$\overline{e} = X \xrightarrow{e} FX + A \xrightarrow{[\mathsf{inl}, \mathsf{inr}]} FX + Y + A$$

satisfies $p \cdot \overline{e}^{\ddagger} = (p \bullet \overline{e})^{\ddagger} = \overline{p \bullet e}^{\ddagger}$, using (4.11), that is, $p \cdot e^{\dagger} = (p \bullet e)^{\dagger}$.

($\Leftarrow$) If p is solution-preserving for F and $h' = p \cdot h$, then for every ffg-equation $e \colon X \to FX + Y + A$ we know that $p \cdot e_h^{\dagger} = (p \bullet e_h)^{\dagger}$ (recalling e_h from Remark 4.13(1)). In order to derive $p \cdot e^{\ddagger} = (p \bullet e)^{\ddagger}$, it remains to verify that $p \bullet e_h = (p \bullet e)_{h'}$, which follows from the following commutative diagram:

$$
\begin{tikzcd}
X \arrow[r, "e"] \arrow[dr, "e"'] \arrow[rrr, bend left, "p\bullet e_h"] \arrow[drrr, bend right=40, "p\bullet e"'] & FX+Y+A \arrow[r, "FX+{[h,A]}"] \arrow[d, equal] & FX+A \arrow[r, "FX+p"] & FX+A' \\
& FX+Y+A \arrow[rr, "FX+Y+p"'] & & FX+Y+A' \arrow[u, "FX+{[h',A]}"']
\end{tikzcd}
$$

$\square$

Construction 4.23. Given an object Y of $\mathscr{C}$, we denote by ΦY the colimit of all ffg-coalgebras for $F(-) + Y$, that is, $\Phi Y = \varphi(F(-) + Y)$. Its coalgebra structure is invertible [54], and we denote by

$$t_Y : F\Phi Y \to \Phi Y \qquad \text{and} \qquad \eta_Y : Y \to \Phi Y$$

the components of its inverse.

The F-algebra $(\Phi Y, t_Y)$ is endowed with a canonical solution operation $\dagger$ defined as follows. Given an ffg-equation $e \colon X \to FX + \Phi Y$, put

$$\overline{e} = (X \xrightarrow{e} FX + \Phi Y \xrightarrow{FX+\mathsf{inl}} FX + Y + \Phi Y).$$

This ffg-equation for $F(-) + Y$ has a solution $\overline{e}^{\ddagger}$ in the ffg-Elgot algebra ΦY, and we put

$$e^{\dagger} := (X \xrightarrow{\overline{e}^{\ddagger}} \Phi Y).$$

Theorem 4.24. *For every free object Y of $\mathscr{C}$, the algebra $(\Phi Y, t_Y)$ with the solution operation $\dagger$ is a free ffg-Elgot agebra for F on Y.*

Proof. We prove that $\eta_Y : Y \to \Phi Y$ in Construction 4.23 is the universal morphism. ΦY is an ffg-Elgot algebra since, together with η_Y, it corresponds to the initial ffg-Elgot algebra $\varphi(F(-) + Y)$ under the isomorphism of Proposition 4.22. This follows from Theorem 4.11 applied to $F(-) + Y$. To verify its universal property, let $(A, a, \dagger)$ be an ffg-Elgot algebra for F and $h \colon Y \to A$ a morphism. Proposition 4.22 gives an ffg-Elgot algebra $(A, [a, h], \oplus)$ for $F(-) + Y$ with $e^{\dagger} = \overline{e}^{\oplus}$ for all ffg-equations $e \colon X \to FX + A$ (cf. Remark 4.13). Furthermore, Proposition 4.22 states that a morphism $p \colon \Phi Y \to A$ in $\mathscr{C}$ is solution-preserving w.r.t. $F(-) + Y$ if and only if it is solution-preserving w.r.t. F and satisfies $p \cdot \eta_Y = h$. Therefore, the universal property of $\eta_Y : Y \to \Phi Y$ w.r.t. F follows from the initiality of ΦY w.r.t. $F(-) + Y$. $\square$

4.4 Monadicity of FFG-Elgot Algebras

We will now prove that the forgetful functor $U_F \colon \mathsf{ffg\text{-}Elgot}\, F \to \mathscr{C}$ is monadic. This means that all ffg-Elgot algebras form an algebraic category over the given variety $\mathscr{C}$. To this end we must first establish that its forgetful functor has a left-adjoint, which assigns to every object Y of $\mathscr{C}$ a free ffg-Elgot algebra on Y. So far we have seen in Theorem 4.24 that on every free object Y we have a free ffg-Elgot algebra on Y. To extend this to arbitrary objects of $\mathscr{C}$ we will make use of the following result.

Proposition 4.25. *The forgetful functor $U_F \colon \mathsf{ffg\text{-}Elgot}\, F \to \mathscr{C}$ creates sifted colimits.*

Proof. Let $D \colon \mathscr{D} \to \mathsf{ffg\text{-}Elgot}\, F$ be a sifted diagram with objects $(A_d, a_d, (-)^{\dagger,d})$ for $d \in \mathscr{D}$. Let

$$i_d \colon A_d \to A \quad (d \in \mathscr{D})$$

be a colimit cocone of $U_F \cdot D$ in $\mathscr{C}$. Since F preserves sifted colimits, the forgetful functor from $\mathsf{Alg}\, F$ to $\mathscr{C}$ creates them, i.e. there exists a unique F-algebra structure $a \colon FA \to A$ making every i_d an F-algebra homomorphism:

$$\begin{tikzcd} FA_d \arrow[r, "a_d"] \arrow[d, "Fi_d"'] & A_d \arrow[d, "i_d"] \\ FA \arrow[r, "a"'] & A \end{tikzcd}$$

Moreover $(A, a) = \mathrm{colim}_{d \in \mathscr{D}} (A_d, a_d)$ in $\mathsf{Alg}\, F$. We need to show that there is a unique solution operation $\dagger$ on (A, a) such that $(A, a, \dagger)$ is an ffg-Elgot algebra and every i_d is solution-preserving, and moreover i_d $(d \in \mathscr{D})$ is a colimit cocone in $\mathsf{ffg\text{-}Elgot}\, F$.

(1) Uniqueness of $\dagger$. Given a solution operation $\dagger$ on the algebra (A, a) for which all i_d's are solution-preserving, then for every ffg-equation $e \colon X \to FX + A$, an explicit formula for $e^\dagger$ is given as follows: since $FX + A$ is a sifted colimit of $FX + A_d$ $(d \in \mathscr{D})$ and X is an ffg object, there exists a factorization

$$\begin{tikzcd} X \arrow[r, "e"] \arrow[dr, "e_0"'] & FX + A \\ & FX + A_d \arrow[u, "FX + i_d"'] \end{tikzcd}$$

Thus $e = i_d \bullet e_0$, which implies

$$e^\dagger = i_d \cdot e_0^{\dagger,d} \tag{4.17}$$

because i_d is solution-preserving. This shows that $\dagger$ is uniquely determined.

(2) Existence of †. The formula (4.17) defines a solution operation †; the independence of the choice of the factorization is established as in the proof of Lemma 4.7. Let us verify that $(A, a, \dagger)$ is an ffg-Elgot algebra.

Solution. $e^\dagger$ is a solution of e:

$$\begin{tikzcd}
X \arrow[r, "e_0^{\dagger,d}"] \arrow[d, "e_0"] \arrow[rr, bend left, "e^\dagger"] \arrow[dd, bend right, "e"'] & A_d \arrow[r, "i_d"] & A \\
FX + A_d \arrow[r, "Fe_0^{\dagger,d} + A_d"] \arrow[d, "FX + i_d"] & FA_d + A_d \arrow[u, "{[a_d, A_d]}"] \arrow[dr, "Fi_d + i_d"] & \\
FX + A \arrow[rr, "Fe^\dagger + A"'] & & FA + A \arrow[uu, "{[a, A]}"']
\end{tikzcd}$$

Weak Functoriality. Suppose that we are given a coalgebra homomorphism

$$\begin{tikzcd}
X \arrow[r, "e"] \arrow[d, "m"'] & FX + Z \arrow[d, "Fm + z"] \\
X' \arrow[r, "f"'] & FX' + Z
\end{tikzcd}$$

together with a morphism $h\colon Z \to A$, where X, X' and Z are ffg objects. Factorize h as in the triangle below:

$$\begin{tikzcd}
Z \arrow[r, "h"] \arrow[dr, "h'"'] & A \\
& A_d \arrow[u, "i_d"']
\end{tikzcd}$$

for some $d \in \mathscr{D}$. Then the desired equality

$$(h \bullet f)^\dagger \cdot m = (h \bullet e)^\dagger$$

is established as follows:

$$\begin{aligned}
(h \bullet f)^\dagger \cdot m &= ((i_d \cdot h') \bullet f)^\dagger \cdot m \\
&= (i_d \bullet (h' \bullet f))^\dagger \cdot m && \text{Remark 3.3(1)} \\
&= i_d \cdot (h' \bullet f)^{\dagger,d} \cdot m && \text{def. } \dagger \\
&= i_d \cdot (h' \bullet e)^{\dagger,d} && (-)^{\dagger,d} \text{ weakly funct.} \\
&= (i_d \bullet (h' \bullet e))^\dagger && \text{def. } \dagger \\
&= ((i_d \cdot h') \bullet e)^\dagger && \text{Remark 3.3(1)} \\
&= (h \bullet e)^\dagger
\end{aligned}$$

Compositionality. Given ffg-equations $e\colon X \to FX + Y$ and $f\colon Y \to FY + A$, factorize f as follows:

$$\begin{tikzcd} Y \arrow[r, "f"] \arrow[dr, "f_0"'] & FY + A \\ & FY + A_d \arrow[u, "FY + i_d"'] \end{tikzcd}$$

for some $d \in \mathscr{D}$. Then we obtain

$$\begin{aligned} (f^\dagger \bullet e)^\dagger &= ((i_d \cdot f_0^{\dagger,d}) \bullet e)^\dagger && \text{def. } \dagger \\ &= (i_d \bullet (f_0^{\dagger,d} \bullet e))^\dagger && \text{Remark 3.3(1)} \\ &= i_d \cdot (f_0^{\dagger,d} \bullet e)^{\dagger,d} && \text{def. } \dagger \\ &= i_d \cdot (e \mathbin{\blacksquare} f_0)^{\dagger,d} \cdot \mathsf{inl} && (-)^{\dagger,d} \text{ compositional} \\ &= (i_d \bullet (e \mathbin{\blacksquare} f_0))^\dagger \cdot \mathsf{inl} && \text{def. } \dagger \\ &= (e \mathbin{\blacksquare} (i_d \bullet f_0))^\dagger \cdot \mathsf{inl} && \text{Remark 3.3(2)} \\ &= (e \mathbin{\blacksquare} f)^\dagger \cdot \mathsf{inl} \end{aligned}$$

This completes the proof that $(A, a, \dagger)$ is an ffg-Elgot algebra.

(3) We prove that $(A, a, \dagger)$ is a colimit of $(A_d, a_d, (-)^{\dagger,d})$ $(d \in \mathscr{D})$. Thus suppose that an ffg-Elgot algebra $(B, b, \ddagger)$ and a cocone of solution-preserving morphisms $m_d\colon A_d \to B$ $(d \in \mathscr{D})$ are given. We need to show that the unique morphism $m\colon A \to B$ with $m \cdot i_d = m_d$ for all d is solution-preserving. To this end, suppose that $e\colon X \to FX + A$ is an ffg-equation, factorized as follows:

$$\begin{tikzcd} X \arrow[r, "e"] \arrow[dr, "e_0"'] & FX + A \\ & FX + A_d \arrow[u, "FX + i_d"'] \end{tikzcd}$$

Then we obtain

$$\begin{aligned} (m \bullet e)^\ddagger &= (m \bullet (i_d \bullet e_0))^\ddagger \\ &= ((m \cdot i_d) \bullet e_0)^\ddagger && \text{Remark 3.3(1)} \\ &= (m_d \bullet e_0)^\ddagger && \text{since } m \cdot i_d = m_d \\ &= m_d \cdot e_0^{\dagger,d} && m_d \text{ solution-preserving} \\ &= m \cdot i_d \cdot e_0^{\dagger,d} && \text{since } m \cdot i_d = m_d \\ &= m \cdot e^\dagger && \text{def. } \dagger \end{aligned}$$

This completes the proof. $\square$

Theorem 4.26. *The forgetful functor $U_F\colon \mathsf{ffg\text{-}Elgot}\,F \to \mathscr{C}$ is monadic.*

Proof. (1) U_F has a left adjoint. Indeed, for every ffg object Y we have a free ffg-Elgot algebra ΦY by Theorem 4.24, which defines the corresponding functor

$$\Phi\colon \mathscr{C}_{\mathsf{ffg}} \to \mathsf{ffg\text{-}Elgot}\, F.$$

We can extend it to a left adjoint of U_F as follows. Given an object Y of $\mathscr{C}$, express it as a sifted colimit $y_i\colon Y_i \to Y$ $(i \in I)$ of ffg objects (see Section 2.1). The image of that sifted diagram under Φ has a colimit $\mathrm{colim}_{i\in I}\, \Phi Y_i$ in the category $\mathsf{ffg\text{-}Elgot}\, F$ by Proposition 4.25. It follows immediately that this colimit is a free ffg-Elgot algebra on Y.

(2) By Beck's Theorem (see, e.g. [17, Theorem 4.4.4]) it remains to prove that U_F creates coequalizers of U_F-split pairs of morphisms. These are pairs $f, g\colon (A, a, \dagger) \to (B, b, \ddagger)$ of morphisms of ffg-Elgot algebras such that morphisms $c\colon B \to C$, $s\colon C \to B$ and $t\colon B \to A$ in $\mathscr{C}$ are given with $c \cdot f = c \cdot g$, $c \cdot s = \mathsf{id}_C$, $g \cdot t = \mathsf{id}_B$ and $s \cdot c = f \cdot t$.

$$\begin{tikzcd}
A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \arrow[r, "c"] \arrow[l, bend left=60, "t"] & C \arrow[l, bend left=50, "s"]
\end{tikzcd}$$

Since F is a finitary functor, the forgetful functor from $\mathsf{Alg}\, F$ to $\mathscr{C}$ is monadic, see [13]. Thus, by Beck's Theorem, there is a unique structure $\gamma\colon FC \to C$ such that c is an F-algebra homomorphism from (B, b) to (C, γ); moreover, c is a coequalizer of f and g in $\mathsf{Alg}\, F$. We need to show that there is a unique solution operator $*$ for the algebra (C, γ) such that $(C, \gamma, *)$ is an ffg-Elgot algebra and c is solution-preserving, and that c is then a coequalizer of f and g in $\mathsf{ffg\text{-}Elgot}\, F$.

Given an ffg-equation $e\colon X \to FX + C$, we define

$$e^* = (X \xrightarrow{(s\bullet e)^\ddagger} B \xrightarrow{c} C).$$

Then c is solution-preserving:

$$\begin{aligned}
(c \bullet e)^* &= c \cdot (s \bullet (c \bullet e))^\ddagger && \text{def. } * \\
&= c \cdot ((s \cdot c) \bullet e)^\ddagger && \text{Remark 3.3(1)} \\
&= c \cdot ((f \cdot t) \bullet e)^\ddagger && s \cdot c = f \cdot t \\
&= c \cdot (f \bullet (t \bullet e))^\ddagger && \text{Remark 3.3(1)} \\
&= c \cdot f \cdot (t \bullet e)^\dagger && f \text{ solution-preserving} \\
&= c \cdot g \cdot (t \bullet e)^\dagger && c \cdot f = c \cdot g \\
&= c \cdot (g \bullet (t \bullet e))^\ddagger && g \text{ solution-preserving} \\
&= c \cdot ((g \cdot t) \bullet e)^\ddagger && \text{Remark 3.3(1)} \\
&= c \cdot e^\ddagger && g \cdot t = \mathsf{id}
\end{aligned}$$

We prove that $*$ satisfies the axioms of an ffg-Elgot algebra, and that it is the unique ffg-Elgot algebra structure on (C, γ) for which c is solution-preserving.

(a) e^* is a solution of e:

$$\begin{tikzcd}
X \arrow[rrr, "(s\bullet e)^\ddagger"] \arrow[rrrr, bend left, "e^*"] \arrow[dr, "s\bullet e"] \arrow[dd, "e"'] & & & B \arrow[r, "c"] & C \\
& FX+B \arrow[rr, "F(s\bullet e)^\ddagger + B"'] & & FB+B \arrow[u, "{[b,B]}"'] \arrow[dr, "Fc+c"] & \\
FX+C \arrow[ur, "FX+s"'] \arrow[rrrr, "Fe^*+C"'] & & & & FC+C \arrow[uu, "{[\gamma,C]}"']
\end{tikzcd}$$

All inner parts of this diagram commute; for the left-hand component of the right-hand part, use that c is solution-preserving and thus a homomorphism of F-algebras by Lemma 4.10.

(b) *Weak Functoriality.* Suppose that we have a coalgebra homomorphism

$$\begin{tikzcd}
X \arrow[r, "e"] \arrow[d, "m"'] & FX+Z \arrow[d, "Fm+Z"] \\
Y \arrow[r, "f"'] & FY+Z
\end{tikzcd}$$

and a morphism $h\colon Z \to C$ where X, Y and Z are ffg objects. Then

$$\begin{aligned}
(h \bullet e)^* &= c \cdot (s \bullet (h \bullet e))^\ddagger && \text{def. } * \\
&= c \cdot ((s \cdot h) \bullet e)^\ddagger && \text{Remark 3.3(1)} \\
&= c \cdot ((s \cdot h) \bullet f)^\ddagger \cdot m && \ddagger \text{ weakly functorial} \\
&= c \cdot (s \bullet (h \bullet f))^\ddagger \cdot m && \text{Remark 3.3(1)} \\
&= (h \bullet f)^* \cdot m && \text{def. } *
\end{aligned}$$

(c) *Compositionality.* Given ffg-equations $e\colon X \to FX + Y$ and $f\colon Y \to FY + C$ we compute

$$\begin{aligned}
(f^* \bullet e)^* &= ((c \cdot (s \bullet f)^\ddagger) \bullet e)^* && \text{def. } * \\
&= (c \bullet ((s \bullet f)^\ddagger \bullet e))^* && \text{Remark 3.3(1)} \\
&= c \cdot ((s \bullet f)^\ddagger \bullet e)^\ddagger && c \text{ solution-preserving} \\
&= c \cdot (e \blacksquare (s \bullet f))^\ddagger \cdot \mathsf{inl} && \ddagger \text{ compositional} \\
&= c \cdot (s \bullet (e \blacksquare f))^\ddagger \cdot \mathsf{inl} && \text{Remark 3.3(2)} \\
&= (e \blacksquare f)^* \cdot \mathsf{inl} && \text{def. } *
\end{aligned}$$

(d) We show the uniqueness of $*$. Suppose that $+$ is another solution operation for (C, γ) such that c is solution-preserving. Then

$$\begin{aligned} e^* &= c \cdot (s \bullet e)^\ddagger && \text{def. } * \\ &= (c \bullet (s \bullet e))^+ && c \text{ solution-preserving} \\ &= ((c \cdot s) \bullet e)^+ && \text{Remark 3.3(1)} \\ &= e^+ && c \cdot s = \mathsf{id} \end{aligned}$$

(e) We finally show that c is a coequalizer of f and g. Let $m\colon (B, b, \ddagger) \to (D, d, +)$ be a solution-preserving morphism with $m \cdot f = m \cdot g$. Since $\mathscr{C}$ is an (absolute) coequalizer in $\mathscr{C}$, there exists a unique morphism $h\colon C \to D$ with $h \cdot c = m$. We only need to show that it is solution-preserving. Indeed, given an ffg-equation $e\colon X \to FX + C$, we compute:

$$\begin{aligned} h \cdot e^* &= h \cdot c \cdot (s \bullet e)^\ddagger && \text{def. } * \\ &= m \cdot (s \bullet e)^\ddagger && h \cdot c = m \\ &= (m \bullet (s \bullet e))^+ && m \text{ solution-preserving} \\ &= ((m \cdot s) \bullet e)^+ && \text{Remark 3.3(1)} \\ &= ((h \cdot c \cdot s) \bullet e)^+ && h \cdot c = m \\ &= (h \cdot e)^+ && c \cdot s = \mathsf{id} \end{aligned}$$ □

Corollary 4.27. *The forgetful functor $W_F\colon$ $\mathsf{ffg\text{-}Elgot}\,F \to \mathsf{Alg}\,F$ is monadic.*

Proof. Indeed, we have a commutative triangle

$$\begin{tikzcd} \mathsf{ffg\text{-}Elgot}\,F \arrow[rr, "W_F"] \arrow[dr, "U_F"'] & & \mathsf{Alg}\,F \arrow[dl, "V_F"] \\ & \mathscr{C} & \end{tikzcd}$$

of forgetful functors, where U_F and V_F are monadic. By Proposition 4.25 we know that $\mathsf{ffg\text{-}Elgot}\,F$ has reflexive coequalizers. Thus by [17, Corollary 4.5.7 and Exercise 4.8.6], W_F is monadic, too. □

5 Conclusions and Further Work

For a functor F on a variety $\mathscr{C}$ preserving sifted colimits, the concept of an Elgot algebra [6] has a natural weakening obtained by working with iterative equations having ffg objects of variables. We call such algebras ffg-Elgot algebras. We have proved that the locally ffg fixed point φF, constructed by taking the colimit of all F-coalgebras with an ffg carrier, is the initial ffg-Elgot algebra for F. Furthermore, all free ffg-Elgot algebras exist, and the colimit of all ffg-coalgebras for $F(-) + Y$ yields a free ffg-Elgot algebra on Y, whenever Y is a free object of $\mathscr{C}$ on some (possibly infinite) set. Finally, we have proved that the forgetful functor from the category of ffg-Elgot algebras to $\mathscr{C}$ is monadic.

An open problem is giving a coalgebraic construction of free ffg-Elgot algebras over arbitrary objects Y, similarly to Construction 4.23, which only works for *free* object Y, cf. Theorem 4.24. In addition, the study of the properties of the ensuing free ffg-Elgot algebra monad is also left for the future. The monad of ordinary free Elgot algebras (cf. Section 3) was proved [6] to be the free Elgot monad on the given endofunctor F. It would be interesting to see whether the above monad of free ffg-Elgot algebras is characterized by a similar universal property.

Finally, in the current setting we have the following forgetful functors:

$$\mathsf{ffg\text{-}Elgot}\,F \to \mathsf{Alg}\,F \to \mathscr{C} \to \mathsf{Set}.$$

Each of those functors has a left-adjoint and is in fact monadic, and we have shown that the composite of the first two is monadic, too. We leave the question whether the composite of all three functors is monadic for further work.

References

1. Aczel, P., Adámek, J., Milius, S., Velebil, J.: Infinite trees and completely iterative theories: A coalgebraic view. Theoret. Comput. Sci. 300, 1–45 (2003), **Fundamental study**
2. Aczel, P., Adámek, J., Velebil, J.: A coalgebraic view of infinite trees and iteration. In: Proc. Coalgebraic Methods in Computer Science (CMCS'01). Electron. Notes Theor. Comput. Sci., vol. 44, pp. 1–26 (2001)
3. Adámek, J., Milius, S., Moss, L.S.: Fixed points of functors. J. Log. Algebr. Methods Program. 95, 41–81 (2018), https://doi.org/10.1016/j.jlamp.2017.11.003
4. Adámek, J., Milius, S., Sousa, L., Wißmann, T.: On finitary functors. Theory Appl. Categ. 34, 1134–1164 (2019)
5. Adámek, J., Milius, S., Urbat, H.: On algebras with effectful iteration. In: Cîrstea, C. (ed.) Proc. Coalgebraic Methods in Computer Science (CMCS). Lecture Notes Comput. Sci., vol. 11202, pp. 144–166. Springer (2018)
6. Adámek, J., Milius, S., Velebil, J.: Elgot algebras. Log. Methods Comput. Sci. 2(5:4), 31 pp. (2006)
7. Adámek, J., Milius, S., Velebil, J.: Iterative algebras at work. Math. Structures Comput. Sci. 16(6), 1085–1131 (2006)
8. Adámek, J., Milius, S., Velebil, J.: Semantics of higher-order recursion schemes. Log. Methods Comput. Sci. 7(1:15), 43 pp. (2011)
9. Adámek, J., Rosický, J.: Locally presentable and accessible categories. Cambridge University Press (1994)
10. Adámek, J., Rosický, J., Vitale, E.: What are sifted colimits? Theory Appl. Categ. 23, 251–260 (2010)
11. Adámek, J., Rosický, J., Vitale, E.: Algebraic Theories. Cambridge University Press (2011)
12. Applegate, H.: Acyclic models and resolvent functors. Ph.D. thesis, Columbia University (1965)

13. Barr, M.: Coequalizers and free triples. Math. Z. 116, 307–322 (1970)
14. Berstel, J., Reutenauer, C.: Rational Series and Their Languages. Springer-Verlag (1988)
15. Bloom, S.L., Ésik, Z.: Iteration Theories: the equational logic of iterative processes. EATCS Monographs on Theoretical Computer Science, Springer (1993)
16. Bonsangue, M.M., Milius, S., Silva, A.: Sound and complete axiomatizations of coalgebraic language equivalence. ACM Trans. Comput. Log. 14(1:7), 52 pp. (2013)
17. Borceux, F.: Handbook of Categorical Algebra, vol. 2. Cambridge University Press (1994)
18. Courcelle, B.: Fundamental properties of infinite trees. Theoret. Comput. Sci. 25, 95–169 (1983)
19. Davey, B.A., Davis, G.: Tensor products and entropic varieties. Algebra Universalis 21, 68–88 (1985)
20. Doberkat, E.: Eilenberg-moore algebras for stochastic relations. Inf. Comput. 204(12), 1756–1781 (2006), erratum and addendum published in *Inf. Comput.* 206(12), 1476–1484 (2008)
21. Droste, M., Kuich, W., Vogler, H. (eds.): Handbook of weighted automata. Monographs in Theoretical Computer Science, Springer (2009)
22. Elgot, C.C.: Monadic computation and iterative algebraic theories. In: Rose, H.E., Sheperdson, J.C. (eds.) Logic Colloquium '73. vol. 80, pp. 175–230. North-Holland Publishers, Amsterdam (1975)
23. Ésik, Z., Maletti, A.: Simulation vs. equivalence. In: Proc. 6th Int. Conf. Foundations of Computer Science. pp. 119–122. CSREA Press (2010)
24. Ésik, Z., Maletti, A.: Simulations of weighted tree automata. In: Proc. CIAA'11. Lecture Notes Comput. Sci., vol. 6482, pp. 321–330. Springer (2011)
25. Fiore, M., Plotkin, G.D., Turi, D.: Abstract syntax and variable binding. In: Proc. LICS'99. pp. 193–202. IEEE Press (1999)
26. Fliess, M.: Sur divers produits de séries formelles. Bulletin de la Société Mathématique de France 102, 181–191 (1974)
27. Freyd, P.: Rédei's finiteness theorem for commutative semigroups. Proc. Amer. Math. Soc. 19(4), p. 1003 (1968)
28. Ghani, N., Lüth, C., Marchi, F.D., Power, A.J.: Algebras, coalgebras, monads and comonads. In: Proc. Coalgebraic Methods in Computer Science (CMCS'01). Electron. Notes Theor. Comput. Sci., vol. 44, pp. 128–145 (2001)
29. Ghani, N., Lüth, C., Marchi, F.D., Power, A.J.: Dualizing initial algebras. Math. Structures Comput. Sci. 13(2), 349–370 (2003)
30. Ginali, S.: Regular trees and the free iterative theory. J. Comput. System Sci. 18, 228–242 (1979)
31. Johnstone, P.T.: Adjoint lifting theorems for categories of algebras. Bull. London Math. Soc. 7, 294–297 (1975)
32. Johnstone, P.T.: Topos Theory. Academic Press, London (1977)
33. Kurz, A., Petrisan, D., Severi, P., de Vries, F.J.: Nominal coalgebraic data types with applications to lambda calculus. Log. Methods Comput. Sci. 9(4:20), 51 pp. (2013)
34. Lambek, J.: A fixpoint theorem for complete categories. Math. Z. 103, 151–161 (1968)
35. Linton, F.E.J.: Autonomous equational categories. J. Math. Mech. 15, 637–642 (1966)
36. Milius, S.: Completely iterative algebras and completely iterative monads. Inform. and Comput. 196, 1–41 (2005)

37. Milius, S.: A sound and complete calculus for finite stream circuits. In: Proc. LICS'10. pp. 449–458. IEEE Computer Society (2010)
38. Milius, S.: Proper functors and fixed points for finite behaviour. Log. Methods. Comput. Sci. 14(3:22), 32 pp. (2018)
39. Milius, S., Pattinson, D., Wißmann, T.: A new foundation for finitary corecursion: The locally finite fixpoint and its properties. In: Proc. FoSSaCS'16. Lecture Notes Comput. Sci. (ARCoSS), vol. 9634, pp. 107–125. Springer (2016)
40. Milius, S., Pattinson, D., Wißmann, T.: A new foundation for finitary corecursion and iterative algebras. Inform. and Comput. 271 (2020), article 104456
41. Milius, S., Schröder, L., Wißmann, T.: Regular behaviours with names: On rational fixpoints of endofunctors on nominal sets. Appl. Categ. Structures 24(5), 663–701 (2016)
42. Milius, S., Wißmann, T.: Finitary corecursion for the infinitary lambda calculus. In: Proc. CALCO'15. LIPIcs, vol. 35, pp. 336–351. Schloss Dagstuhl (2015)
43. Moss, L.S.: Parametric corecursion. Theoret. Comput. Sci. 260(1–2), 139–163 (2001)
44. Nelson, E.: Iterative algebras. Theoret. Comput. Sci. 25, 67–94 (1983)
45. Plotkin, G.D., Turi, D.: Towards a mathematical operational semantics. In: Proc. Logic in Computer Science (LICS'97). pp. 280–291 (1997)
46. Rédei, L.: The Theory of Finitely Generated Commutative Semigroups. Pergamon, Oxford-Edinburgh-New York (1965)
47. Rutten, J.J.M.M.: Rational streams coalgebraically. Log. Methods Comput. Sci. 4(3:9), 22 pp. (2008)
48. Schützenberger, M.P.: On the definition of a family of automata. Inform. and Control 4(2–3), 275–270 (1961)
49. Silva, A., Bonchi, F., Bonsangue, M.M., Rutten, J.J.M.M.: Generalizing determinization from automata to coalgebras. Log. Methods Comput. Sci 9(1:9), 27 pp. (2013)
50. Silva, A., Sokolova, A.: Sound and complete axiomatization of trace semantics for probabilistic systems. Electr. Notes Theor. Comput. Sci. 276, 291–311 (2011), https://doi.org/10.1016/j.entcs.2011.09.027
51. Sokolova, A., Woracek, H.: Congruences of convex algebras. J. Pure Appl. Algebra 219(8), 3110–3148 (2015)
52. Sokolova, A., Woracek, H.: Proper semirings and proper convex functors. In: Baier, C., Lago, U.D. (eds.) Proc. FoSSaCS 2018. Lecture Notes Comput. Sci., vol. 10803, pp. 331–347. Springer (2018)
53. Tiuryn, J.: Unique fixed points vs. least fixed points. Theoret. Comput. Sci. 12, 229–254 (1980)
54. Urbat, H.: Finite behaviours and finitary corecursion. In: Proc. CALCO'17. LIPIcs, vol. 72, pp. 24:1–24:15. Schloss Dagstuhl (2017)
55. Winter, J., Bonsangue, M.M., Rutten, J.J.: Context-free coalgebras. J. Comput. System Sci. 81(5), 911 – 939 (2015)

A Appendix

Details on the Definition of φF (see Remark 4.4(2))

Recall [11] that an object X of $\mathscr{C}$ whose hom-functor $\mathscr{C}(X,-)$ preserves sifted colimits is called *perfectly presentable*, and that these objects are precisely the split quotients of ffg objects. Let $\mathsf{Coalg}_{\mathsf{pp}}\, F$ denote the full subcategory of coalgebras carried by perfectly presentable objects. We show that φF can be defined as the colimit of all such F-coalgebras, in symbols:

$$\varphi F = \operatorname{colim}(\mathsf{Coalg}_{\mathsf{pp}}\, F \hookrightarrow \mathsf{Coalg}\, F).$$

To this end, it suffices to prove that the inclusion functor

$$I\colon\ \mathsf{Coalg}_{\mathsf{ffg}}\, F \hookrightarrow \mathsf{Coalg}_{\mathsf{pp}}\, F$$

is cofinal. This means that

(1) for every coalgebra in $\mathsf{Coalg}_{\mathsf{pp}}\, F$ there is a homomorphism into some coalgebra in $\mathsf{Coalg}_{\mathsf{ffg}}\, F$, and

(2) for every span $(Y,d) \xleftarrow{f} (X,c) \xrightarrow{g} (Z,e)$ in the category $\mathsf{Coalg}_{\mathsf{pp}}\, F$ with codomains in $\mathsf{Coalg}_{\mathsf{ffg}}\, F$, there exists a zig-zag of morphisms in the slice category $(X,c)/\,\mathsf{Coalg}_{\mathsf{ffg}}\, F$ connecting f and g.

Proof of (1). Given an F-coalgebra $c\colon X \to FX$ with X perfectly presentable, we know that X is a split quotient of some ffg object W of $\mathscr{C}$, i.e. we have $e\colon W \twoheadrightarrow X$ and $m\colon X \rightarrowtail W$ with $e \cdot m = \mathsf{id}_X$ in $\mathscr{C}$. Put

$$w := (W \xrightarrow{e} X \xrightarrow{c} FX \xrightarrow{Fm} FW).$$

Then (W,w) is an ffg-coalgebra such that $m\colon (X,c) \rightarrowtail (W,w)$ is a coalgebra homomorphisms as desired:

$$w \cdot m = Fm \cdot c \cdot e \cdot m = Fm \cdot c.$$

Proof of (2). Now suppose we have two coalgebra homomorphisms $f\colon (X,c) \to (Y,d)$ and $g\colon (X,c) \to (Z,e)$ where X is perfectly presentable and Y and Z are ffg objects. As in the proof of (1), choose e and m and form the ffg-coalgebra (W,w). Now observe that $e\colon (W,w) \twoheadrightarrow (X,c)$ is a coalgebra homomorphism:

$$Fe \cdot w = Fe \cdot Fm \cdot c \cdot e = c \cdot e.$$

Due to $e \cdot m = \mathsf{id}_X$, we then have the following zig-zag relating f and g:

$$\begin{tikzcd} & (X,c) \arrow[dl, "f"'] \arrow[d, "m"] \arrow[dr, "g"] & \\ (Y,d) & (W,w) \arrow[l, "f\cdot e"'] \arrow[r, "g\cdot e"] & (Z,e) \end{tikzcd}$$