Opinion Diffusion and Campaigning on Society Graphs

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Abstract

We study the effects of campaigning, where the society is partitioned into voter clusters and a diffusion process propagates opinions in a network connecting the clusters. Our model is very powerful and can incorporate many campaigning actions, various partitions of the society into clusters, and very general diffusion processes. Perhaps surprisingly, we show that computing the cheapest campaign for rigging a given election can usually be done efficiently, even with arbitrarily-many voters. Moreover, we report on certain computational simulations.

1 Introduction

The introduction of online social networks to modern politics has thoroughly changed how political campaigns are run, as currently it is practical to influence selected individuals or groups of individuals on a scale not possible before. Political campaigns now routinely use these networks to attempt to sway elections in their favor, for instance, by targeting segments of voters with fake news [1, 25], by organizing fund-raising activities, and by running vote suppression campaigns [16]. Indeed, the use of social media in election campaigns is so ubiquitous that there already are hundreds of studies regarding their use (as a piece of evidence in this respect, we point the reader to Jungherr's overview of over a hundred Twitter-focused studies [30]). To be efficient, campaigners would like to factor-in the nuances of how each voter behaves and how beliefs diffuse in the underlying social graph. Yet, doing so is challenging for at least two reasons: On the one hand, it is difficult—albeit not impossible—to obtain such a fine-grained understanding of the social-media users and to prepare the right content for them. On the other hand, as shown by Wilder and Vorobeychik [53] and by Bredereck and Elkind [7], finding optimal strategies to affect the election results is computationally challenging even for the most basic voting rules. To circumvent their intractability results, Wilder and Vorobeychik [53] designed appropriate approximation algorithms, whereas Bredereck and Elkind [7] considered very restricted types of social networks. Our goal also is to address the computational difficulty of manipulating elections through targeting particular groups of voters, but we take a very different approach. First, instead of considering networks of individuals, we assume that the network is over clusters of like-minded voters. Second, we provide fixed-parameter tractable (FPT) algorithms parameterized by the number of candidates and the number of these clusters. As a consequence, the running times of our algorithms scale smoothly with the precision to which we model the social network.

Specifically, in our model, an external agent with limited funds observes an election and wants to ensure that a certain candidate wins. To this end, he or she can alter the preferences of some voters, e.g., by bribing them or by convincing them through some sort of a targeted campaign (we assume the ordinal election model, where each voter ranks the candidates from the most to the least appealing one, and we model campaigning actions—at least in our basic model—as shift bribery [20, 18, 24]). Then, voters' opinions diffuse through a social network and, eventually, the election result is established.

The crucial part of our model is that the social network is not over individual voters, but over clusters of voters. Each cluster might correspond to a group of voters who share the same preferences and, possibly, some other features. For example, for each given preference order we may have three clusters, containing the voters who share this preference order and are, respectively, young, middle-aged, or senior. The clusters are connected in the network and only voters in connected clusters may influence each other (the edges in our network correspond to the fact that voters from various clusters interact with each other; e.g., like-minded voters of the same age may visit the same blogs and read each other's opinions there). We refer to the networks of our type as *society graphs* (we follow the model of Knop et al. [37]). We consider a diffusion model where voters change their mind based on the most popular preferences of the voters to which they are connected, as well as its various generalizations (indeed, our FPT algorithms can work with a very broad family of diffusion models that can be expressed via linear programs of a certain kind).

Our main contributions are as follows:

- We provide the model of elections over society graphs, parametrized by both the
 voting rule and the diffusion process. Throughout most of the paper we focus on
 a simple variant of the our model, but we also discuss a very general notion of
 ILP-expressible diffusion processes.
- 2. We provide an FPT algorithm for our problem parametrized by the number of candidates. We present the algorithm for one of the simplest variants of our model, with the basic diffusion process and voters clustered by their preference order only, but we also argue how it can be extended in numerous ways (we focus on the simplest setting mostly for the sake of clarity). In particular, as the algorithm is based on expressing the problem via integer linear programming, it works for all ILP-expressible voting rules and all ILP-expressible diffusion processes. Our algorithm runs in exponential time with respect to the number of candidates, but in logarithmic time with respect to the number of voters, so it is particularly well-suited for the case of political elections.

3. We test our algorithm experimentally. It turns out that for modern ILP solvers it is still quite challenging (although usable for at least some realistic settings). Thus we provide two heuristics algorithms that are much faster, but whose outputs are not guaranteed to be optimal.

We note that our model and our society graphs in particular form a generalization of the standard model, where each voter is a node in the social network. Indeed, we can simply assume that each cluster contains a single individual. The advantage of our approach is that we can seamlessly move between more and less fine-grained views of the society and its interactions. As our model can capture many natural social behaviors within the clusters, a plethora of bribery actions, and various diffusion processes, we believe it to be quite powerful; indeed, we demonstrate its power by highlighting many modeling possibilities within it.

Yet, our model—or, more specifically, our approach—has one drawback: Our diffusion models are completely deterministic in that the effect of the diffusion is always exectly the same for given initial conditions. If one prefers to have a model with some sort of stochastic behavior, then there is a natural work-around for this issue. For example, Wilder and Vorobeychik [53] argue that by sampling several possible "diffusion scenarios" (they use the Independent Cascade model, so in their case it means sampling which edges of the network indeed propagate the influence) one can get a very good approximation of the behavior of a stochastic diffusion process. Since the same approach can be used in our case, we find it sufficient to consider deterministic diffusion models (and, indeed, deterministic models are commonly studied in the literature [44, 43, 9, 2]).

The paper is organized as follows. First, in Section 2 we discuss related work regarding algorithmic results in the areas of social networks and election bribery. Then, we describe our model in Section 3 and analyze the properties of our basic diffusion process in Section 4. In Section 5, we provide our main algorithmic results, while in Section 6 we discuss possible extensions of the basic model and how our algorithm can be adapted for them. We conclude by discussing experimental results in Section 7 and by discussing possible future work in Section 8.

2 Related Work

We study the possibility of manipulating election outcomes under the assumption that the voters' views propagate (or, diffuse) throughout an underlying social network. Below we present related work regarding social networks, bribery in elections, and a few results related by technique.

2.1 Diffusion in Social Networks

The two papers most closely related to our work, one due to Wilder and Vorobeychik [53] and one due to Bredereck and Elkind [7], study very similar issues, but differ in several important modeling choices. Foremost, we consider the model of society graphs, where the nodes of the social network represent clusters of voters, whereas both Wilder and Vorobeychik and Bredereck and Elkind consider the more typical model where each vertex represents a single individual. Furthermore, both Wilder and Vorobeychik and Bredereck and Elkind focus on the simple Plurality voting rule, whereas we consider a wide spectrum of rules (indeed, all rules that are ILP-expressible, in essence including all commonly studied rules). Regarding the diffusion process, Wilder and Vorobeychik consider the Independent Cascade model (ICM), and Bredereck and Elkind consider the Linear Threshold model (LTM).

It is worth exploring the differences between ICM, LTM, and our diffusion model. In ICM and LTM, the underlying idea is that a message is spreading from multiple seed nodes, and this message has a certain "directionality"—its intent is to activate nodes (which can be understood as, e.g., convincing them of some fact). So, these two models can be seen as modeling an intentional act of campaigning or influencing. In contrast, our diffusion model seeks to describe what happens as a result of mutual influence (or peer pressure) between voters. Indeed, in our model the voters observe the other ones—typically, those close to them in the network—and change their preferences accordingly. As a consequence, in our model a voter may change his preference order arbitrarily, if faced with appropriate pressure. In contrast, classic variants of ICM and LTM only capture binary influence—either a node is influenced or it is not (which is why influence maximization can be viewed in terms of manipulating a Plurality election over two candidates). Yet, we mention that recently Corò et al. [15] proposed a variant of LTM, called Linear Threshold Ranking, LTR, where the level of influence is quantified in a more fine-grained way (with stronger infuence, a designated candidate may be shifted up by more positions; interestingly, this is quite related to shift bribery, which we also consider).

Another difference between our diffusion model and ICM, LTM, and LTR is determinism: our model is deterministic while the other models are stochastic. There are a few reasonable ways to turn our model into a stochastic one, and we believe that our algorithmic results may carry over using the trick of considering several "diffusion scenarios" (in the fashion of Wilder and Vorobeychik), but such discussion is beyond the scope of this paper. For a general, broad discussion of diffusion processes, we point the reader to the book of Shakarian et al. [47]).

The main technical difference between our work and the papers of Wilder and Vorobeychik, and Bredereck and Elkind, is in how we deal with computational intractability: Wilder and Vorobeychik provide approximation algorithms (but also MILP formulations), Bredereck and Elkind consider very restricted classes of social networks, and we design fixed-parameter tractable algorithms, parametrized by the number of candidates and the number of voter clusters. Our main algorithm proceeds by forming and solving an ILP instance and, in this sense, it is similar to the MILP approach of Wilder and Vorobeychik. Our ILP, however is very different and, in particular, incorporates very different tricks.

Another paper that is very closely related to our work is that of Silva [49], where the author studies a similar bribery model, but for cardinal preferences expressed as numbers between 0 and 1 (e.g., corresponding to the level of support for the current government). In his model, these values can be modified through bribery actions, after which a certain dynamic process (i.e., a form of diffusion) propagates them over the network. Silva focuses on a Twitter-like network, where edges are directed and influ-

ence can only go in one way, whereas we—as well as Wilder and Vorobeychik [53] and Elkind and Bredereck [7]—consider more Facebook-like networks, where the interactions are two-way. (Note that we nevertheless later show how to also handle directed edges in our model; see Section 6.3.) Similarly to us, Silva uses integer linear programming to compute the solutions to his problems.

More broadly, our work is closely related to the stream of papers on the interface between social choice and social networks, which includes the problems of recovering ground truth [12, 45], the problems of iterative voting [52, 50], various issues related to liquid democracy [4], certain forms of multiwinner voting [51], and many others (for a more detailed discussion, see the overview of Grandi et al. [28]). In particular, our work is quite closely related to that of Brill et al. [9], who study the diffusion of ordinal preferences through a social network. In their network, each vertex (i.e., each individual) swaps two candidates if the majority of its neighbors ranks them differently than him. This is quite similar to our basic model, where connected clusters of voters have identical preference orders, up to a single swap. The two main differences are that, on the one hand, our network is more restricted and, on the other hand, we allow for the initial modification of some of the preference orders, whereas Brill et al. [9] focus only on the diffusion. Similarly, our work is related to that of Botan et al. [5], where the authors consider a diffusion process of preferences expressed as Boolean propositions. In a similar vein, Christoff and Grossi [11] study the convergence of binary opinions over a network.

Our work is also naturally connected to the stream of work on influence maximization in social networks (see, e.g., the papers of Kempe et al. [34, 35] and Chen et al. [10] as well as many follow-up ones). In these works, the goal is to choose a set of nodes in a social network so that if we pass some information to them and wait for the diffusion process to converge, then as many nodes as possible will have received our information (the work of Bredereck and Elkind [7] can be understood in these terms as well; alternatively, influence maximization can also be seen in terms of manipulating Plurality voting).

2.2 Bribery in Elections

Stepping away from social networks, our work belongs to the broad stream of papers on the complexity of manipulating elections. For a general overview of this topic, we point the reader to the surveys of Conitzer and Walsh [13] and Faliszewski and Rothe [24]; here we will discuss the few most related papers.

We model campaigning actions via the shift bribery problem, which itself is a special case of the swap bribery problem; both introduced by Elkind et al. [20, 18]. Briefly put, in the shift bribery problem we are given an election, where each voter ranks all the candidates from the most to the least appealing one, and, upon paying a required price, we can ask some of the voters to shift a certain candidate p higher. Our goal is to ensure that p wins, but without exceeding a given budget. Swap bribery generalizes shift bribery by allowing swaps of any adjacent candidates, and not just p, with those who preceed him or her. Both shift bribery and swap bribery are NP-hard for many natural voting rules (including, e.g., Borda, Copeland, Maximin, and various elimination-based rules [41]), but shift bribery is generally easier to deal with. For

example, Elkind et al. [20, 18] provided approximation algorithms for shift bribery under several voting rules (recently strengthened by Faliszewski et al. [23]), Bredereck et al. [8] and Zhou and Guo [54] provided several FPT algorithms, and Elkind et al. [19] gave polynomial-time algorithms for several structured preference domains. For the case of swap bribery, Dorn and Schlotter [17] provided a careful analysis for the case of approval voting, whereas Knop et al. [36] gave a general FPT algorithm parameterized by the number of candidates. Both problems were also studied in the destructive setting, where the goal is to prevent a given candidate from being a winner [31, 48]. Interestingly, in this case the problem is often efficiently solvable. Nonetheless, we do not consider the destructive setting in our work (we do not expect the tractability results to carry over to our model due to the computational complexity implied by the diffusion process).

We stress that our choice of shift bribery as a model for campaigning bears only limited significance regarding the complexity of our problem. Indeed, we could have used full-fledged swap bribery or the classic bribery problem of Faliszewski at al. [21] or any other problem from the bribery family [24] and—as long as it were expressible within an ILP program—the complexity of our algorithm would stay intact.

2.3 Other Related Work

Finally, we mention that the idea of using society graphs and clusters of voters of a given type was heavily inspired by the work of Knop et al. [37], who studied a very general form of manipulating elections. In their work, an election is represented as a society vector (s_1, \ldots, s_n) , where each s_i specifies how many voters of type i there are (e.g., how many voters have the i-th possible preference order). The goal is to find a minimum-cost transformation of a society vector to one satisfying a given condition, provided a certain cost measure for transforming the society. Our work extends the approach of Knop et al. [37] to include the social network in the election. The idea of types was also considered, e.g., by Izsak et al. [29], but for the case of candidates.

3 Formal Model and Combinatorial Problem

In this section we present a very basic variant of our model, where voter types correspond to preference orders, edges exist between two orders that can be obtained by a single swap of adjacent candidates, the diffusion process is done in a simple, particular way, and the bribery actions are limited. Later, in Section 6, we discuss various generalizations of our approach, by considering arbitrary voter types, arbitrary bribery actions, and generalized diffusion processes. Yet, the basic model will allow us to develop intuitions, prove strong hardness results, and present our tractability results clearly. For $n \in \mathbb{N}$, by [n] we mean the set $\{1, \ldots, n\}$.

3.1 Elections and Voting Rules

We consider ordinal elections held with n voters, expressing preferences over m candidates $C = \{c_1, \dots, c_m\}$, where the *preference order* of a voter is a linear order over

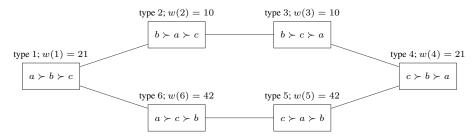


Figure 1: A society graph with three candidates and six types (corresponding to the six possible preference orders on those three candidates). In this graph there are, e.g., 42 voters of type 6, each with preference order $a \succ c \succ b$; this graph corresponds to a society $\mathbf{w} = [21, 10, 10, 21, 42, 42]$.

C. A voting rule \mathcal{R} is a function taking an election as input and returning a set of tied winners. A candidate winning under \mathcal{R} for a given election is called an \mathcal{R} -winner of the election. As an example, under the *Plurality* rule the candidates ranked first most frequently win, and under the *Borda* rule, for each position i each voter gives m-i points to the candidate ranked there; the candidates with the highest total number of points win.

3.2 Voter Types, Societies, and Society Graphs

For the time being, we let the preference order of a voter be her *type*. Thus, there are $\tau \leq m!$ types present in a given election and we order them arbitrarily so that we can speak of "the j-th type" for a given $j \in [\tau]$. By the *weight* of voters with type j, denoted either w_j or w(j), as is more convenient, we mean the number of voters of type j. Sometimes we represent an election as a vector $\mathbf{w} \in \mathbb{N}^{\tau}$, whose j-th entry represents the weight of type j. We refer to such vectors as *societies*. (in this we follow the model of Knop et al. [37]).

As we are interested in diffusion processes operating on the voter types, we associate a given election with a vertex-weighted graph $G=(V,\mathbf{w},E)$, termed the *society graph*. The society graph contains τ vertices, where τ is the number of types in the election (specifically, $V=\{v_1,\ldots,v_\tau\}$, where vertex v_j corresponds to voter type j, and its weight w_j is equal to the number of voters of that type in the given election). There is an edge between vertices v_j and $v_{j'}$ if the preference orders corresponding to types j and j' differ by the ordering of a single pair of adjacent candidates (in other words, if it is possible to transform one into the other with a single swap of two consecutive candidates). We show an example of a society graph in Figure 1.

¹Graphs of this form are quite popular in the study of permutations. Regarding their use in the context of elections, we point out, e.g., to the work of Puppe and Slinko [46]. Recall that later we will also consider other graphs.

3.3 Diffusion of Preferences

Given a society graph (which encodes a given election), we consider two variants of the diffusion process, namely asynchronous and synchronous. In the asynchronous variant, in each step of the process some vertex v of the society graph G is picked and, then, the following occurs (we do not specify which vertex is selected and, as we will see in Example 1 below, different orders of selecting the vertices may lead to different outcomes of the process). We consider the closed neighborhood N[v] of v in G and check whether there is a neighbor x of v for which $w_x > 1/2 \sum_{u \in N[v]} w_u$; that is, a neighbor whose weight exceeds the sum of the weights of all other vertices in the closed neighborhood of v. If such a neighbor x exists, then we add the current weight w_v of v to that of v and change the weight v0 be 0. Intuitively, the voters of type represented at v1 look at all the voters with similar or identical preferences and if there is a majority support among these voters for some preference order, then they switch to it. In the synchronous variant we proceed in the same way, but simultanously for all vertices. The diffusion process halts whenever it stabilizes.

Example 1. Consider the society graph depicted in Figure 1 and asynchronous diffusion. Assume that we first choose type 3. As type 3 has as neighbors types 2 and 4, together there are 41 voters of these types, and 21 of them have preference order $c \succ b \succ a$. So, the 10 voters with type 3 move to have type 4. If we then select type 4, type 6, and then type 2, then the diffusion converges with 115 voters of type 5 (with preference order $c \succ a \succ b$) and 31 voters of type 1 (with preference order $a \succ b \succ c$); thus, Plurality selects c. However, if we select first type 2, then 1, then 5, and then 3, then we reach convergence with 115 voters of type 6 (with preference order $a \succ c \succ b$) and 31 voters of type 5 (with preference order $c \succ b \succ a$); thus, Plurality selects a. This shows that the asynchronous diffusion process can lead to different outcomes, depending on the order in which vertices are considered.

Let us now consider the same society graph and synchronous diffusion. After the first round, we have 10 voters of type 1 (voters of type 2 moved to have type 1, whereas original type 1 voters moved to have type 6), no voters of types 2 and 3, 10 voters of type 4, 63 voters of type 5, and 63 voters of type 6. After the next round there are 73 voters of type 5 and 73 voters of type 6. No further changes are possible and the process converges; Plurality selects a and c as two tied winners.

3.4 Bribery in Society Graphs

Besides issues related to the diffusion of preferences, we are mainly interested in understanding the possibility of manipulating election outcomes. Thus we assume that there is an external briber who has some budget and, using this budget, can affect the original preference orders of some voters (i.e., the preference orders they have prior to the diffusion). Specifically, in a single bribery action the briber chooses a single voter and, at unit cost, shifts the briber's preferred candidate p up by one position in this voter's preference order (in effect, changing this voter's type; see the work of Elkind et al. [20, 18] and Bredereck et al. [6] for a detailed discussion of shift bribery and its various cost models). The briber performs as many bribery actions as he wants, up to the budget limit, and then the diffusion process takes place. The goal of the briber is to

have his preferred candidate p win the resulting election (under a given, predetermined voting rule). Formally, we are interested in the following general problem.

 \mathcal{R} -Bribery in society graphs (\mathcal{R} -BSG)

Input: A society graph G (given directly as a graph), a preferred candidate p, and a budget b.

Question: Are there at most b (unit-cost, shift-) bribery actions, such that after performing them on G and then running the diffusion process, p is an R-winner of the resulting election?

Corresponding to the synchronous and asynchronous diffusion processes, we consider both sync-R-BSG and async-R-BSG problems. For the asynchronous diffusion, we further consider the optimistic and pessimistic variants of the problem. In the former, we ask whether the briber's preferred candidate wins for some order of the diffusion steps. In the latter, we require that p wins for every order of diffusion steps that leads to convergence.

Remark 1. The input to \mathcal{R} -BSG is a labeled graph with weighted vertices, a preferred candidate p, and a budget b. Thus the size of the input encoding is linear in the number of voter types and only logarithmic in the number of voters.

4 Convergence and Diffusion Order

Before we tackle the \mathcal{R} -BSG problem, we first show that our diffusion processes always converge, but the complexity of deciding if a particular candidate may become a winner due to the diffusion may be NP-hard.

For the synchronous case, convergence follows by arguing that in each diffusion step at least one vertex loses its weight completely, and a vertex of weight zero never increases its weight. In consequence, we have that the number of synchronous diffusion steps is bounded by the number of voter types. The asynchronous case is even simpler, but requires appropriate terminology: If a diffusion step does not change the society graph (e.g., due to the choice of the vertex) then we call it *redundant*. We say that a sequence of non-redundant diffusion steps is *irredundant*. A *maximal irredundant sequence* is an irredundant sequence after executing which all remaining steps are redundant.

Proposition 1. For each society graph G, the synchronous diffusion process converges in at most τ steps. The asynchronous diffusion process converges if the sequence of diffusion steps contains a maximal irredundant sequence as a subsequence. The length of a maximal irredundant sequence is bounded by τ .

Proof. We consider the asynchronous case first. Consider some diffusion step. At this point either no vertex changes any further, or at least one vertex, if chosen for the next diffusion step, would have its weight reduced to zero. Since no weight-zero vertex can ever increase its weight (by the definition of the diffusion step), it follows that every irredundant sequence consists of at most τ steps. By definition, if a sequence contains a maximal irredundant subsequence, it produces the same graph as this subsequence.

For the synchronous case, it suffices to show that after every diffusion step (prior to convergence), the number of vertices with non-zero weight decreases. Consider a diffusion step before convergence. There is some vertex v, which is to be assimilated into one of its neighbors, u. If no other neighbor of v is to be assimilated by v in this step, then we are done: The number of vertices with non-zero weight will decrease by at least one after this diffusion step. Perhaps, however, there is some neighbor v' of v that is to be assimilated by v in the current step. It must be that $v' \neq u$, as we require a strict majority for a vertex to be assimilated by one of its neighbors. If no neighbor of v' is assimilated by v', then we are done (by the same token as before, we see that the number of weight-zero vertices will increase). Otherwise, there is some neighbor v'' of v' which is to be assimilated by v'. By following this logic exhaustively, either we reach a vertex whose weight is to decrease to zero, or some vertex repeats. However, the latter is impossible as, by definition of the diffusion process, the weights of the vertices that we encounter form a decreasing sequence. Thus the number of weight-zero vertices increases after each diffusion step and the claim follows.

For synchronous diffusion, the final society graph is defined uniquely and so is the outcome of the election (for a given voting rule). This is not the case for asynchronous diffusion. Indeed, in Example 1 we have seen that two different diffusion orders may lead to two different society graphs. In the next theorem we show a stronger statement, namely that the problem of deciding whether a given candidate may be a Plurality winner after asynchronous diffusion is NP-hard.²

Theorem 1. Given a society graph G and a preferred candidate p, deciding whether there is an order of asynchronous diffusion steps that results in p being a Plurality winner in the converged election is NP-hard.

Proof. We provide a reduction from the NP-complete problem CUBIC VERTEX COVER [27]. In this problem we are given a graph G, where each vertex has degree at most three, and an integer k (sometimes authors assume that the degree of the vertices in the graph is exactly three, but it is not needed in our case). We ask whether it is possible to select at most k vertices so that each edge touches at least one of the selected vertices.

Consider an instance of CUBIC VERTEX COVER that consists of a graph G and an integer k. We write $V(G) = \{v_1, \ldots, v_n\}$ to denote the set of G's vertices and $E(G) = \{e_1, \ldots, e_m\}$ to denote the set of its edges. We build the following election (and the associated society graph). We let the candidate set be $C = \{c, d, e, p\} \cup A \cup B$, where $A = \{a_1, \ldots, a_n\}$ and $B = \{b_1, \ldots, b_n\}$. The role of the candidates in sets A and B is to encode subsets of V(G). We sometimes use the following convention to describe the nodes of the society graph: Given three candidates $x, y, z \in \{d, e, p\}$, some subset S of V(G), and an integer w, by w/xyzS we mean a node of the society

²The proof of Theorem 1 presented below is different than the one included in the conference version of this paper. The latter had a technical flaw due to which the society graph used in the construction was not implementable. The proof presented below is based on a different construction and fixes this issue. As an added benefit, it uses weights whose values are polynomially bounded with respect to the size of the input instance (i.e., it shows that the problem is strongly NP-hard, whereas the previous proof was showing weak NP-hardness).

graph with weight w and preference order obtained from:

$$x \succ y \succ z \succ a_1 \succ b_1 \succ a_2 \succ b_2 \succ \cdots \succ a_n \succ b_n \succ c$$

by swapping for each $v_i \in S$ candidate a_i with candidate b_i (intuitively, candidates from the set $A \cup B$ create a signature that puts our preference order at sufficient swap distance from other votes that rank candidates x, y, z in the same way).

To form the society graph, let T be some large positive integer (to be specified later), let X=10, let Y=100, and let Z=1000. The society graph consists of two parts. The first part contains three isolated vertices:

- 1. A vertex with candidate c ranked first and with weight T.
- 2. A vertex with candidate p ranked first and with weight T (2X + 9)|E(G)|.
- 3. A vertex with candidate d ranked first and with weight T (Y + Z)|V(G)| kX.

By analyzing the second part of the society graph, it will become clear that it is easy to make sure that these vertices are indeed isolated (for the first one, it suffices to rank c on the first place and to rank all the other candidates arbitrarily; for the latter two it suffices to rank the requested candidate on top, followed first by c and then all the other candidates arbitrarily).

The second part of the society graph encodes the graph G. We build it as follows:

1. For each vertex $v_i \in V(G)$, we form three nodes, $v_i^{(1)}$, $v_i^{(2)}$, and $v_i^{(3)}$, specified as follows (recall our convention for describing the nodes):

$$v_i^{(3)} : Z/dpe\{v_i\}, \qquad v_i^{(2)} : Y/dep\{v_i\}, \qquad v_i^{(1)} : X/edp\{v_i\}.$$

Note that these vertices form a path $v_i^{(3)} - v_i^{(2)} - v_i^{(1)}$ and that there are no edges between society-graph nodes associated with different vertices $v_i, v_j \in V(G)$.

2. For each edge $e_t = \{v_i, v_i\} \in E(G)$ we form three nodes:

$$e_t^{(1)} \colon 1/edp\{v_i,v_j\}, \quad e_t^{(2)} \colon X + 3/epd\{v_i,v_j\}, \quad e_t^{(3)} \colon X + 5/ped\{v_i,v_j\}.$$

These vertices form a path $e_t^{(1)}$ – $e_t^{(2)}$ – $e_t^{(3)}$ and there are no edges between society-graph nodes associated with different edges from E(G). However, there is an edge between $e_t^{(1)}$ and $v_i^{(1)}$, and between $e_t^{(1)}$ and $v_j^{(1)}$.

The society graph does not contain any edges aside from those explicitly mentioned in the construction above. We set T to be the square of the sum of the weights of the nodes from the second part of the society graph (intuitively, T is simply a large number). This completes the construction. See Figure 2 for an example with a small input graph.

Prior to the diffusion, the candidates from sets A and B have score 0 (and their score never increases), whereas the remaining candidates have the following scores:

1. candidate c has score T (this is currently the highest score and it cannot change; any candidate that becomes a winner after the diffusion has to reach at least score T),

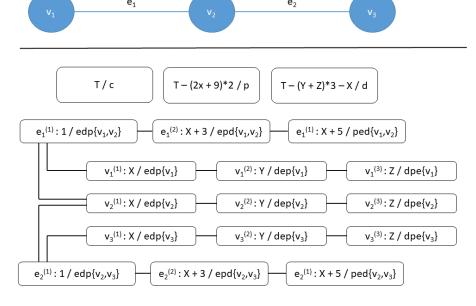


Figure 2: An example for the reduction described in the proof of Theorem 1. On top, we show a graph containing 3 vertices and 2 edges. At the bottom, we show the constructed society graph.

- 2. candidate d has score T kX,
- 3. candidate p has score T (X + 4)|E(G)|, and
- 4. candidate e has score much below T (and this score cannot reach T even due to the diffusion).

Let us now argue that if there is a set S of at most k vertices from V(G) such that each edge in E(G) touches at least one vertex from S, then there is a diffusion order that leads to p being a Plurality winner of the given election. Indeed, one such diffusion order proceeds as follows:

- 1. First, for each $v_i \in S$, society-graph node $v_i^{(1)}$ is assimilated into $v_i^{(2)}$.
- 2. Then, for each $j \in [n]$, society-graph node $v_j^{(2)}$ is assimilated into $v_j^{(3)}$. Note that, after this happens, there are no more nodes of the form $v_i^{(\ell)}$ that can be assimilated into any other node.
- 3. Next, for each $t \in [m]$, node $e_t^{(1)}$ is assimilated into $e_t^{(2)}$ (this is possible because, as S is a vertex cover, now each $e_t^{(1)}$ is connected to at most one node of the form $v_i^{(1)}$, with weight X, and to exactly one node $e_t^{(2)}$, with weight X+3; as the total weight that $e_t^{(1)}$ sees is 2X+4, its weight can move to $e_t^{(2)}$).

4. Finally, for each $t \in [m]$, node $e^{(2)}$, currently with weight X+4, is assimilated into $e_t^{(3)}$, who (prior to this) has weight X+5. Note that, after this happens, no further diffusion steps are possible. Moreover, p and c have score T, d has score at most T, and e has score below T. Thus p is a winner.

Let us now consider the other direction, i.e., the case that p can become a winner of our election for some diffusion order. For this to happen, p has to reach score at least T and this is possible only if for each $t \in [m]$, the weight from nodes $e_t^{(1)}$ and $e_t^{(2)}$ is assimilated into the node $e_t^{(3)}$, which ranks p on top (note that no other nodes can transfer their weight to nodes that rank p on top). However, for a given $t \in [m]$, the weight from $e_t^{(1)}$ can reach $e_t^{(3)}$ only by first being assimilated by $e_t^{(2)}$. Yet, prior to the diffusion, $e_t^{(1)}$ sees total weight equal to 3X+4 and cannot be immediately assimilated by any of its neighbors $(e_t^{(1)}$ sees its own weight of 1, weight X+3 of $e_t^{(2)}$ and weight X+X of the two nodes $v_t^{(1)}$ and $v_t^{(1)}$, such that edge e_t connects v_t and v_t . Thus, for each $e_t=\{v_t,v_t\}\in E(G)$, at least one of $v_t^{(1)}$, $v_t^{(1)}$ has to be assimilated into, respectively, $v_t^{(2)}$ or $v_t^{(2)}$. However, each such assimilation increases the score of d by X. If this happens for more than k nodes, then the score of d exceeds d and d does not become a winner. In other words, if d becomes a winner, then there is a set of d vertices in d that touch all of the edges from d that there is a vertex cover of size d in our input graph.

We have phrased Theorem 1 to speak of the Plurality rule as it is the simplest, yet most widely used voting rule. Nonetheless, similar results hold for many other voting rules. Indeed, it would be rather remarkable if there were a natural voting rule for which it did not hold.

5 Complexity of Manipulating Society Graphs

In this section we present our main theoretical results. Briefly put, \mathcal{R} -BSG is intractable (both in the synchrounous and asynchronous variants) for nearly all natural voting rules, but is fixed-parameter tractable with respect to the number of candidates.

5.1 General Intractability of BSG

As $\mathcal{R}\text{-BSG}$ is, in essence, a variant of the Shift Bribery problem [20], it naturally inherits most of its hardness results. The difference between $\mathcal{R}\text{-BSG}$ and standard Shift Bribery is that the former involves the diffusion process after the bribery. In the reduction below, we "turn off" this diffusion by ensuring that every voter from a Shift Bribery instance forms an isolated vertex in the society graph (as the diffusion anyhow does not happen in isolated graphs, our proof works for both the synchronous and asynchronous cases). We show the reduction for the Borda rule, but then we argue that it is also applicable for many other rules.

Proposition 2. Borda-BSG is NP-hard for both the synchronous and the asynchronous cases.

Proof. An instance of Borda-Shift Bribery with Unit Costs (Borda-SB for short) consists of an election with candidate set $C = \{c_1, \ldots, c_m\}$, voters v_1, \ldots, v_n , a distinguished candidate $p \in C$, and budget b. We ask if it is possible to ensure that p becomes a Borda winner of this election by performing b unit-cost shift bribery actions. Borda-SB is well-known to be NP-hard [6, Proposition 3].

Given an instance of Borda-SB, we create an instance of Borda-BSG. The idea is to alter the instance so that, even after any set of bribery actions, the swap distance between each two voters would be at least two; this will ensure that each voter is a singleton in the society graph and will prevent diffusion from happening. As Borda-BSG is, in essence, Borda-SB with diffusion, the result will follow. To this end, our instance of Borda-BSG is the same as the input Borda-SB distance, but with the following two changes:

- 1. We introduce two sets of additional dummy candidates, $D = \{d_1, \dots, d_n\}$ and $E = \{e_1, \dots, e_n\}$.
- 2. We set the preference orders of the voters as follows. For each voter v_i , let $\operatorname{pref}(v_i)$ mean v_i 's original preference order regarding the candidates c_1,\ldots,c_m . We extend this preference order to be: $\operatorname{pref}(v_i) \succ d_i \succ D \setminus \{d_i\} \succ e_i \succ E \setminus \{e_i\}$, where by $D \setminus \{d_i\}$ and $E \setminus \{e_i\}$ we mean, respectively, the preference orders $d_1 \succ d_2 \succ \cdots \succ d_n$ and $e_1 \succ e_2 \succ \cdots \succ e_n$ with candidates d_i and e_i removed.

One can verify that in the resulting instance of Borda-BSG each two voters have preference orders that are at swap distance at least two and, thus, each voter is a singleton in the society graph associated with the instance. Moreover, the dummy vertices do not change the winner. As a consequece, the correctness of the reduction follows.

Remark 2. The above proof works for every voting rule (i) for which Shift Bribery with unit costs is NP-hard and (ii) whose results do not change after we add some candidates that the voters rank last. Such rules include, e.g., Copeland or Maximin [6] and, indeed, both conditions are commonly satisfied (yet, Plurality is an example of a rule that fails the first criterion, and Veto is an example of a rule that fails the second one).

5.2 Voting Rules and Integer Linear Programs

In the next section we show that the BSG problem is fixed-parameter tractable for the parametrization by the number of candidates. Our algorithm is based on solving an integer linear program (ILP), and we use the notion of *ILP-expressible voting rules* to capture the class of rules for which the algorithm is applicable.

Intuitively, we say that a voting rule \mathcal{R} is ILP-expressible if the problem of deciding whether a given candidate p is an election winner can be expressed as a problem of testing whether a certain integer linear program has a feasible solution. We require that the number of variables and constraints in this program is a function of the number of candidates and voter types in the election, and that the election is specified through variables that represent the number of voters of different types. Formally, we have the following definition.

Definition 1 (**ILP-expressible voting rule**). Let E be an election with candidate set C and with a collection of n voters. The preferences of the voters are encoded as a society $\mathbf{w} \in \mathbb{N}^{\tau}$, where the j-th entry of \mathbf{w} indicates how many voters of the j-th type are present in the election. Let $p \in C$ be a distinguished candidate. A voting rule \mathcal{R} is ILP-expressible if there exists a computable function f and integers τ' , $r \leq f(\tau + |C|)$, a matrix $W \in \mathbb{Z}^{r \times (\tau + \tau')}$, and a vector $\mathbf{b} \in \mathbb{Z}^r$ such that (a) W and b are computable in FPT time with respect to $m + \tau$, and (b) it holds that p is a winner of the election, i.e., $p \in \mathcal{R}(E)$, if and only if:

$$\exists \mathbf{x} \in \mathbb{Z}^{\tau'} : W \cdot [\mathbf{w}, \mathbf{x}] \le \mathbf{b},$$

where $[\mathbf{w}, \mathbf{x}] \in \mathbb{Z}^{\tau + \tau'}$ is the column vector obtained by concatenating \mathbf{w} and \mathbf{x} .

The class of ILP-expressible voting rules is quite large. Indeed, it includess all scoring rules, all C1 rules (these are rules depending only on the majority graph of the input election), Bucklin, STV, Kemeny, and many others. In the two examples below we show arguments regarding the Borda rule and the STV rule.

Example 2 (Borda is ILP-expressible). We use the same notation as in Definition 1. Additionally, for a candidate $c \in C$ and voter type $j \in [\tau]$, by rank(c, j) we mean the position on which c is ranked by the type-j voters. To show that Borda is ILP-expressible, we describe a collection of linear inequalities (defining the matrix W), which are satisfied exactly if the Borda score of p is at least as large as the score of every other candidate:

$$\sum_{\substack{i \in [m], j \in [\tau] \\ \operatorname{rank}(c,j) = i}} (m-i)w_j \leq \sum_{\substack{i \in [m], j \in [\tau] \\ \operatorname{rank}(p,j) = i}} (m-i)w_j \quad \forall c \in C, c \neq p.$$

Remark 3. For our next example, as well as for some further arguments in the paper, we will need the ability to express a disjunction of several inequalities as an integer linear program. We now discuss how to achieve this (we mention that such tricks are well-known regarding ILPs [3, Section 7.4]). For the sake of simplicity, let x_1, x_2, x_3 be three ILP variables and let us consider four linear inequalities (values $a_{i,j}$ and b_i are constants):

$$\begin{aligned} a_{1,1}x_1 + a_{1,2}x_2 + a_{1,3}x_3 &\leq b_1, \\ a_{2,1}x_1 + a_{2,2}x_2 + a_{2,3}x_3 &\leq b_2, \\ a_{3,1}x_1 + a_{3,2}x_2 + a_{3,3}x_3 &\leq b_3, \\ a_{4,1}x_1 + a_{4,2}x_2 + a_{4,3}x_3 &\leq b_4. \end{aligned}$$

Additionally, we assume that x_1 , x_2 , and x_3 come from some bounded domain (this is always the case for the variables that we use in our ILP programs, and can be guaran-

teed in general by standard techniques³). We would like to express the fact that:

(the first inequality is satisfied) ∨ (the second inequality is satisfied) ∨ (the last two inequalities are satisfied).

To this end, we introduce three new variables, y_1 , y_2 , and y_3 and the following (in)equalities:

$$0 \le y_i \le 1 \quad \forall i \in [3], \qquad y_1 + y_2 + y_3 = 1.$$

This ensures that exactly one of these variables takes value 1 and the other ones take value 0. Note that Definition 1 explicitly allows using such auxiliary variables. Let T be the largest value that either of the left-hand sides of our four inequalities may take (since we know the values $a_{i,j}$ and the ranges of variables x_1 , x_2 , and x_3 , we have that the value T is easy to compute). We replace our four initial inequalities with the following (note that the last two inequalities both involve variable y_3):

$$(-T+b_1)(1-y_1) + a_{1,1}x_i + a_{1,2}x_2 + a_{1,3}x_3 \le b_1,$$

$$(-T+b_2)(1-y_2) + a_{2,1}x_i + a_{2,2}x_2 + a_{2,3}x_3 \le b_2,$$

$$(-T+b_3)(1-y_3) + a_{3,1}x_i + a_{3,2}x_2 + a_{3,3}x_3 \le b_3,$$

$$(-T+b_4)(1-y_3) + a_{4,1}x_i + a_{4,2}x_2 + a_{4,3}x_3 \le b_4.$$

By the choice of T and the constraints on the y-variables, we see that we implemented exactly the desired disjunction. As an additional benefit, we can use variables y_1 , y_2 , and y_3 to read off which disjunction clause is satisfied. While, for the sake of readability, our example regards only a few inequalities, it is clear that it can be generalized in a straightforward way.

Example 3 (STV is ILP-expressible). The Single Transferrable Vote rule (STV) is defined via the following iterative process: If some candidate is a majority winner (i.e., is ranked first by more than half of the voters), then this candidate is declared a winner. Otherwise, the candidate with the lowest Plurality score is removed from the election and the next iteration starts. If several candidates have the same lowest Plurality score, then we use lexicographic tie-breaking (in our case, where $C = \{c_1, \ldots, c_m\}$, it means removing the candidate with the lowest index).

To show that STV is ILP-expressible, we will first tackle a simpler task. Let $(c_{\pi(1)}, \ldots, c_{\pi(m')})$ be a sequence of m' $(m' \leq m)$ candidates. We say that this is an elimination order if $c_{\pi(1)}$ is removed in the first iteration, $c_{\pi(2)}$ is removed in the second interation, and so on, until $c_{\pi(m')}$ who is chosen as a winner in the m'-th iteration. We will provide a set of linear inequalities that are satisfied if and only if $(c_{\pi(1)}, \ldots, c_{\pi(m')})$ is a correct elimination order. For each $i \in [m]$ and $t \in [m']$, let $\mathrm{first}(c_i, t)$ be the set of voter types that would rank c_i on the first position if candidates $c_{\pi(1)}, \ldots, c_{\pi(t-1)}$ were deleted. To ensure that the elimination order is correct, we introduce the following inequalities:

 $^{^3}$ For example, one may solve the (potentially unbounded) continuous relaxation, obtain a continuous optimum \mathbf{x}^* , apply the proximity theorem of Cook et al. [14] to determine a box around \mathbf{x}^* which must contain an integer optimum if it exists, and then introduce lower and upper bounds corresponding to this box, hence bounding all variables.

1. For each $i \in [m'-1]$ we have to ensure that $c_{\pi(i)}$ has the lowest Plurality score among the candidates remaining in the i-th round (and that it has the lowest index among the candidates with the same Plurality score). For each $c_j \in C \setminus \{c_{\pi(1)}, \ldots, c_{\pi(i-1)}\}$ we introduce one of the following inequalities. If $j < \pi(i)$ (and, thus, $c_{\pi(i)}$ has to have strictly lower Plurality score than c_j), we have:

$$\sum_{\ell \in \operatorname{first}(c_j,i)} w_\ell < \sum_{\ell' \in \operatorname{first}(c_{\pi(i)},i)} w_{\ell'},$$

and if $j > \pi(i)$ then we have an analogous inequality, but with "<" replaced with "<."

2. For each $i \in [m'-1]$ we have to ensure that neither of the remaining candidates has majority support. Thus, for each $i \in [m'-1]$ and each $c_j \in C \setminus \{c_{\pi(1)}, \ldots, c_{\pi(i-1)}\}$ we have the inequality:

$$\sum_{\ell \in ref(c_j, i)} w_{\ell} \le 1/2 \sum_{\ell' \in [\tau]} w_{\ell'}.$$

3. We require that $c_{\pi(m)}$ is selected in the final round, so we also have inequality:

$$\sum_{\ell \in ref(c_{\pi m},m)} w_\ell > 1\!/2 \sum_{\ell' \in [\tau]} w_{\ell'}.$$

To ensure that p is an STV winner, we use the above approach to generate inequalities for every possible elimination order that ends with p, and—using the trick from Remark 3—we form their disjunction.

Remark 4. Our class of ILP-expressible rules is very similar to the class of election systems described by linear inequalities of Dorn and Schlotter [17]; their definition says that there must exist f(m) linear systems $W_i \mathbf{w} \leq \mathbf{b}_i, i \in [f(m)]$, such that p is a winner if at least one of these systems is satisfied. In other words, it means that \mathcal{R} can be described by a bounded disjunction of linear systems. Since such disjunctions can be expressed in ILPs, the rules that fit their definition also fit ours.⁴ Another related notion, called integer-linear-program implementable rules, due to Faliszewski, Hemaspaandra, and Hemaspaandra [22, Definition 6.1], is weaker than ours in that it does not allow for auxiliary variables \mathbf{x} ; without these variables it is not clear how to define, e.g., Bucklin or STV (for example, it is not clear how to implement disjunctions without auxiliary variables).

5.3 Fixed-Parameter Tractability of BSG

We prove that $\mathcal{R}\text{-BSG}$ is FPT with respect to the number m of candidates for any ILP-expressible rule. The result follows by formulating $\mathcal{R}\text{-BSG}$ as an integer linear program

⁴Note that we need the variables to be bounded to implement such a disjunction. However, as demonstrated previously, boundedness is always achievable by standard techniques and, so, it is not an issue.

and invoking Lenstra's famous result [39] (which implies that ILP is FPT with respect to the number of integer variables); it is arguably quite surprising, since, as it turns out, it is possible to encode the complete diffusion process using integer variables and linear constraints. We first prove the result for the synchronous variant, and then show how to modify it to work also for asynchronous diffusion.

Theorem 2. Synchronous R-BSG is fixed-parameter tractable with respect to the number m of candidates for every ILP-expressible voting rule R.

Proof. As a preprocessing phase, we augment the given society graph to have exactly m! vertices, one vertex for each possible preference order; to this end, we might create some vertices of weight zero.⁵ Thus, the number of types in the input is $\tau=m!$, and the number of voters of type $i, i \in [\tau]$, is w_i . Let k be the number of steps of the diffusion process; Proposition 1 says that $k \le \tau$ and, thus, we simply set $k = \tau = m!$. For $i \in [\tau]$, denote by N[i] the closed neighborhood of i (i.e., the set that includes i and all the vertices that are directly connected to it by and edge). Similarly, by $N(i) = N[i] \setminus \{i\}$ we denote i's open neighborhood. We use the Iverson bracket notation, i.e., for a logical expression F, we write [F] to mean 1 when F is true and to mean 0 when F is false.

We construct an ILP with the following variables:

- 1. For each type $i \in [\tau]$ and each diffusion step $\ell \in [k]$, we define an integer variable x_i^{ℓ} representing the number of voters of type i after ℓ diffusion steps.
- 2. For types $i, j \in [\tau]$ we define variables β_{ij} describing the bribery, where β_{ij} corresponds to the number of voters bribed from being of type i to being of type j; note that we also consider β_{ii} , the number of voters of type i which are not bribed.
- 3. For every $i,j \in [\tau]$ and $\ell \in [k]$, we define a binary variable z_{ij}^{ℓ} indicating whether in the ℓ -th step the voters of type i are being assimilated into type j (for technical reasons, we also use variables t_{ij}^{ℓ} ; see explanations below).

Let c_{ij} be the cost of bribing one voter of type i to become a voter of type j (we set c_{ij} to be ∞ if j is not reachable from i). As our aim is to minimize the cost of bribery, the objective of our ILP is to minimize $\sum_{i,j} c_{ij} \beta_{ij}$. Our ILP constraints are presented in Figure 3; note that some of them are non-linear. Below we discuss their meaning and, for the non-linear ones, we explain how they can be encoded within an ILP.

Constraints (1) and (2). These constraints are standard and express that the vector \mathbf{x}^0 describes the society after the bribery (recall that β_{ii} corresponds to non-bribed voters of type i).

Constraint (3). This constraint assigns 1 to z_{ij}^{ℓ} if the weight in type j exceeds half of the total weight of N[i] and 0 otherwise. Note that we do *not* affect z_{ii}^{ℓ} here as the

⁵This phase is needed as we want to consider bribery operations, which might in certain cases introduce new preference orders not originally present in the election. To avoid formal difficulties, we simply introduce those preference orders beforehand.

$$\sum_{j=1}^{\tau} \beta_{ij} = w_i \qquad \forall i \in [\tau]$$

$$\sum_{i=1}^{\tau} \beta_{ij} = x_i^0 \qquad \forall j \in [\tau]$$
 (2)

$$z_{ij}^{\ell} = \left[\sum_{a \in N[i]} \frac{1}{2} x_a^{\ell-1} < x_j^{\ell-1}\right] \qquad \forall j \in N(i), \ell \in [k]$$
 (3)

$$\sum_{i \in N[i]} z_{ij}^{\ell} = 1 \qquad \forall i \in [\tau] \tag{4}$$

$$t_{ij}^{\ell} = z_{ij}^{\ell} x_i^{\ell-1} \qquad \forall i, j \in [\tau], \ell \in [k]$$

$$x_i^{\ell} = \sum_{i \in N[i]} t_{ij}^{\ell} \qquad \forall j \in [\tau], \ell \in [k]$$
 (6)

$$W \cdot [\mathbf{y}, \mathbf{x}^{\mathbf{k}}] \le \mathbf{b} \tag{7}$$

Figure 3: Constraints used in the proof of Theorem 2. We omitted the simple constraints requiring that the variables are in the right domains for clarity.

constraint goes only over j in the open neighborhood N(i). Since Constraint (3) is non-linear, using the approach from Remark 3, for each i, j, and ℓ we express it as a disjunction of two inequalities:

$$\left[\sum_{a \in N[i]} \frac{1}{2} x_a^{\ell-1} < x_j^{\ell-1}\right] \vee \left[\sum_{a \in N[i]} \frac{1}{2} x_a^{\ell-1} \geq x_j^{\ell-1}\right].$$

The approach taken in Remark 3 provides us with a binary variable, which takes value 1 if the first inequality is satisfied and value 0 otherwise. We simply take z_{ij}^{ℓ} to be this variable.

Constraint (4). This constraint enforces that at least one of z_{ij}^ℓ is 1, and this includes z_{ii}^ℓ ; thus, if there is no $j \in N(i)$ with weight more than half of the weight of N[i], then $z_{ii}^\ell = 1$ holds, which corresponds to i keeping its weight (i.e., voters of type i are not being assimilated into some other type).

Constraints (5) and (6). These constraints define the weights for step ℓ , given the weights from step $\ell-1$. Precisely, x_j^ℓ takes the weight of all its neighbors (including itself) for whom $z_{ij}^\ell=1$. We use the t_{ij}^ℓ variables as temporary variables that are non-zero for those i and j for which $z_{ij}^\ell=1$. Notice that Constraint (5) is non-linear but, again, can be handled using standard ILP tricks. Indeed, we can express the constraint as:

$$\left((z_{ij}^{\ell} = 1) \implies (t_{ij}^{l} = x_i^{\ell-1}) \right) \wedge \left((z_{ij}^{\ell} = 0) \implies (t_{ij}^{l} = 0) \right) .$$

Since variables z_{ij}^{ℓ} are binary and implication can be seen as a disjunction, we can further transform this into:

$$\left((z_{ij}^\ell=0)\vee(t_{ij}^l=x_i^{\ell-1})\right)\wedge\left((z_{ij}^\ell=1)\vee(t_{ij}^l=0)\right)\;.$$

We express the disjunctions as in Remark 3. Everything else is linear.

Constraint (7). This constraint corresponds to the specific voting rule being considered, with \mathbf{y} as the auxiliary variables (called \mathbf{x} in Definition 1); it is satisfied if and only if the given, preferred candidate p wins the election specified by \mathbf{x}^k (i.e., the election after the bribery and at the end of the diffusion process). The constraint can be expressed as part of an ILP due to our assumption that \mathcal{R} is ILP-expressible.

This completes the description of our ILP. As the number of variables is a function of the number of candidates, we solve it in FPT time using the algorithm of Lenstra [39].

Corollary 1. Asynchronous R-BSG is fixed-parameter tractable with respect to the number m of candidates for every ILP-expressible voting rule R, for both the optimistic and the pessimistic variants.

Proof. For the optimistic variant, we modify the ILP described above as follows. We add variables y_i^ℓ , representing whether type i is updated in the ℓ -th step. We require that $\sum_i y_i^\ell = 1$ to enforce that exactly one vertex is updated. We add variables \hat{z}_{ij}^ℓ and we want to enforce that $\hat{z}_{ij}^\ell = z_{ij}^\ell \wedge y_i^\ell$; the interpretation is that i might be assimilated only if $y_i^\ell = 1$. However, instead of directly encoding $\hat{z}_{ij}^\ell = z_{ij}^\ell \wedge y_i^\ell$, in our case it suffices to add constraints $\hat{z}_{ij}^\ell \leq z_{ij}^\ell$, $\hat{z}_{ij}^\ell \leq y_i^\ell$. Then, it suffices to replace z_{ij}^ℓ with \hat{z}_{ij}^ℓ in constraint (6). Finally, we need to enforce that the diffusion ends after at most k steps. To this end, for each pair of types i, j we add a constraint requiring that it is impossible to assimilate i into j after the k-th diffusion step (the constraint can be expressed analogously to constraint (3) and we do not spell it out explicitly).

For the pessimistic variant, notice that any sequence of diffusion steps which converges contains a maximal irredundant sequence, and irredundant sequences are of length at most τ (this follows from Proposition 1). It thus suffices to consider the set of permutations of $[\tau]$. Expressing that in none of them p is losing can be done by a long conjunction of ILPs given by constraints (3)-(7), with a clause for each sequence (note that we could have used analogous approach for the optimistic variant too, but the presented approach is more efficient).

6 Model Generalizations

Here we generalize the simple model described above and demonstrate far broader scenarios for which $\mathcal{R}\text{-BSG}$ remains fixed-parameter tractable. In particular, we consider models with arbitrary connections between voter types, models with different bribery operations and manipulative actions, and models with different diffusion processes.

6.1 Various Voter Types

Instead of partitioning the voters by preference orders, we can consider arbitrary partitions. Our motivation may be either that the partition by preference orders is too crude when there are significant differences between voters with the same preference (e.g., young and old voters are convinced by different methods), or, on the other hand, the

partition may be too fine-grained when different preference orders should nonetheless be treated identically, e.g., because of the choice of a voting rule which does not significantly distinguish them. As the number of variables in the ILP described in the proof of Theorem 2 depends only on the numbers of types and candidates, it follows that $\mathcal{R}\text{-BSG}$ remains fixed-parameter tractable with respect to the number τ of types and the number of candidates. Taken to the extreme, namely if we set each voter in a given election to constitute her own voter type, we arrive at the model of diffusion studied, e.g., by Wilder and Vorobeychik [53] or by Bredereck and Elkind [7].

6.2 Arbitrary Bribery Operations and Manipulative Actions

Our model can incorporate, e.g., all bribery operations mentioned by Faliszewski and Rothe [24]. Indeed, the constants c_{ij} used in the proof of Theorem 2 encode the cost of transforming a voter of type i into a voter of type j and can be redefined for other bribery operations.

Furthermore, following the discussion of Knop et al. [37, Section 3.2], this approach can be extended to other types of manipulative operations, such as *voter control* [24], at no asymptotical cost in terms of computational complexity. Specifically, say that some voters are active while others are latent, and there is a cost for activating a latent voter or vice versa. Accordingly, we can define two types for each preference order, one corresponding to it being active while the other corresponding to it being latent, and set the cost of "moving" a voter from the active type to the latent and/or vice versa according to the specific control operation. Notice that this only doubles the number of voter types, so the asymptotic complexity remains intact.

6.3 General Diffusion Processes

So far our society graphs had an undirected edge between two vertices if their corresponding preference orders were of swap distance one, and we considered a specific, simple diffusion process. In fact, our model can incorporate directed arcs, where a vertex would be influenced by those vertices for which it has an outgoing arc and, in particular, we do not have to be confined to connections between types associated with preferences that differ in the ranking of a single pair of candidates. Furthermore, those arcs can be weighted, representing different influence strengths (e.g., consider damping the influence of voters which are, swap distance-wise, farther). Adding weights can be done by modifying Equation (3) in a straightforward way.

Moreover, and most importantly, we can express in our model a large class of diffusion processes. The following definition is inspired by viewing the diffusion of preferences as an abstract process, in which each voter holds a local election to decide which preference order to assume. For example, the diffusion process described in Section 3 corresponds to holding an election containing the voters of swap distance at most one, and changing to the preference order of the majority, if such exists. In the definition below we use the notation from Theorem 2.

Definition 2 (ILP-expressible diffusion process). Let k be an upper bound on the number of diffusion steps, recall that for diffusion step $\ell \in [k]$, the variables $x_i^{\ell-1}$,

 $i \in [\tau]$, express the current society, and let f be a computable function. Then, an ILP-expressible diffusion process is a process such that for each $i,j \in [\tau]$ and $\ell \in [k]$, there are integers $r(i,j,\ell), \tau(i,j,\ell) \leq f(\tau)$, a matrix $D_{i,j,\ell} \in \mathbb{Z}^{r(i,j,\ell) \times \tau + \tau(i,j,\ell)}$, and a vector $\mathbf{b}_{i,j,\ell} \in \mathbb{Z}^{r(i,j,\ell)}$ such that, in the ℓ -th diffusion step, voters of type i are assimilated into type j if and only if the following formula is satisfied:

$$\exists \mathbf{x}' \in \mathbb{Z}^{\tau(i,j,\ell)} \quad D_{i,j,\ell}(\mathbf{x}',\mathbf{x}^{\ell-1}) \leq \mathbf{b}_{i,j,\ell} .$$

Our basic diffusion process corresponds to Equation (3). Another ILP-expressible diffusion process is that each voter replaces her preference order by the Kemeny ranking computed for the voters in her neighborhood (for the definition of a Kemeny ranking, see the original research paper [33] or, e.g., the chapter of Fischer et al. [26]).

Remark 5. Proposition 1 does not hold for all generalized diffusion processes, as the number of diffusion steps might not be bounded by the size of the society graph or the diffusion may never stabilize (cf. Remark 6 below). (Also, new voter types might sometimes appear as a result of diffusion steps.) Thus, the corresponding ILP to solve \mathcal{R} -BSG would have to be supplied with the number k of diffusion steps to simulate. Sometimes it is indeed plausible that an agent can estimate the number of diffusion steps to occur after the manipulative actions, e.g., when he knows the time of the election, or when it is provable (although differently than by the argument of Proposition 1) that the process stabilizes after k steps.

Theorem 3. R-BSG is fixed-parameter tractable with respect to the number of candidates, the number τ of types, and the number k of diffusion steps if both R and the diffusion process are ILP-expressible.

6.4 Exemplary Models

We conclude this section with several examples of scenarios that are captured by such generalized diffusion processes.

Example 4 (Multidimensional Societies). Consider voters of different age groups. It is plausible that the tendency to be influenced by other voters depends on age, and so we might have a voter type for each tuple of (preference order, age group), with different outgoing arcs and different diffusion conditions. Note that we can thus express, e.g., that voters may be strongly influenced by some voter groups, yet cannot be assimilated into them (e.g., a junior person can be influenced by a senior one, but this does not make him or her senior).

Remark 6 (**Periodic behavior**). As an example of how Proposition 1 might not hold (as noted in Remark 5), consider that there are three voter types (0,1,2) subdivided into young (Y) and old (O). Moreover, let's say that young people change their mind, but old people don't. Next, people of type i are influenced by people of type i+1 mod 3. Denote $w_{i,T}$ for $i \in \{0,1,2\}$ and $T \in \{Y,O\}$ the number of people of type i and age group T. Finally, say that $\min_i w_{i,O} \ge \max_i w_{i,Y}$, i.e., there are at least as many old people of each type as there are young people of any type. Consider the synchronous model. Then, in each iteration, the young people of type i move to type

 $i+1 \mod 3$, and the diffusion does not stabilize but oscillates periodically. (Similar behavior can be achieved also in the asynchronous model.)

Example 5 (Advanced diffusion model). Say there are four candidates $\{c_1,\ldots,c_4\}$, 3 age groups $\{Y,M,O\}$ (for Young, Middle-aged, and Old), and two "stubbornness" levels $\{P,S\}$ (for Persuadable and Stubborn). People are divided into types by their preference, age group, and stubbornness level. Since there are 4! = 24 possible preference orders, there are altogether $\tau = 3 \cdot 2 \cdot 24 = 144$ possible voter types. Next, we describe "coefficients of influence" which will be used to define the diffusion process. Define a function f as follows:

$$f(Y,Y) = 1.2,$$
 $f(M,Y) = 0.8,$ $f(M,M) = 1,$ $f(O,M) = 0.5,$ $f(M,O) = 0.3,$ $f(O,O) = 0.8,$ $f(O,Y) = 0,$ $f(Y,O) = 0,$ $f(Y,M) = 0.$

The meaning is that age group G_1 influences age group G_2 with a coefficient $f(G_1, G_2)$ for the persuadables, and with a coefficient $0.5 \cdot f(G_1, G_2)$ for the stubborn. For example, persuadable young people have relatively low self-esteem and weight the opinion of other young people as 1.2 higher than their own; they weigh the opinion of middle-aged people as 0.8 of their own, and they completely disregard the opinions of old people.

Denote by G(t) the age group of type t, let S(t) = 1 if t is persuadable and S(t) = 0.5 otherwise, and let $k \in \mathbb{N}$ be the number of rounds of diffusion to simulate, perhaps as an estimate of how much diffusion happens before a (global) election takes place. We assume that one type is influenced by other types to a degree that exponentially decreases with the swap distance of their preference orders, and also that it decreases inversely as the election draws closer, perhaps because voters become skeptical and more rigid in their opinions. The coefficient of influence of voter type t on voter type t' in round $\ell \in [k]$ is computed as follows:

$$c_{t,t'}^{\ell} = \begin{cases} 1/\ell \cdot 1/2^{d} \cdot S(t') \cdot f(G(t), G(t')) & \text{if } t' \neq t ,\\ 1 & \text{if } t = t' . \end{cases}$$

where d is the swap distance between the preference orders of voters of types t and t'.

Then, the diffusion process is synchronous and in each step, each voter type t' holds a "local election" defined as follows. There are $c(t,t') \cdot w_t$ many voters of type t (for all types $t \in [\tau]$), and the voting rule is Borda (note that for Borda, the definition is sensible even if the number of voters as defined is fractional). After this election, we obtain some winning ranking r (breaking ties according to the ordering of candidates as c_1, \ldots, c_4), and all voters of type t' move to a type with ranking r and the same age group and stubbornness level. The final vote is evaluated with the Plurality rule.

Now we wish to apply Theorem 3 and for that we need to show that \mathcal{R} and the diffusion process are ILP-expressible. Since $\mathcal{R} = \text{Plurality}$, we focus on the diffusion process. Let rank(i,t) be the rank of candidate i in the preference order of type t. We define the score of candidate i in the local election of type t in round ℓ as

$$s_{i,t}^{\ell} = \sum_{t' \in [\tau]} c_{t',t}^{\ell} (4 - \operatorname{rank}(i,t')) x_{t'}^{\ell-1} \ .$$

Recall that we use the Iverson bracket notation, i.e., [F] evaluates either to 1 or to 0, depending on the truth value of condition F. For each two $i, i' \in [4]$, $i \neq i'$, define:

$$i \triangleleft_t^\ell i' \equiv \left[\left(s_{i,t}^\ell > s_{i',t}^\ell \right) \vee \left(\left(s_{i,t}^\ell = s_{i',t}^\ell \right) \wedge i > i' \right) \right] \ .$$

Definition 2 requires that, for each two types t,t' and each round $\ell \in [k]$, there is a linear system that is satisfied exactly if voters of type t should be assimilated by type t'. Let t' have a preference order $c_{i_1} \leq c_{i_2} \leq c_{i_3} \leq c_{i_4}$ and let G(t) = G(t') and S(t) = S(t'). Then t should be assimilated into t' exactly if

$$(i_1 \triangleleft_t^{\ell} i_2) \wedge (i_2 \triangleleft_t^{\ell} i_3) \wedge (i_3 \triangleleft_t^{\ell} i_4)$$
,

which is a boolean combination of linear inequalities and can be rewritten into the format required by Definition 2 using the same standard tricks used in the proof of Theorem 2 (see [3, Section 7.4]). This shows that the diffusion process is ILP-expressible. Hence, if the above were generalized to m candidates, α age groups, and σ stubornness levels, then R-BSG would be fixed-parameter tractable with respect to $m + \alpha + \sigma + k$ for any ILP-expressible voting rule R.

7 Experiments

In addition to our theoretical results, we also evaluated our ILP-based algorithm experimentally. Unfortunately, it turned out that, while it can produce results for up to four candidates in a reasonable amount of time, going beyond this number is not practical. Thus, we sought heuristic algorithms instead. In particular, we designed one deterministic heuristic, a greedy algorithm, and one heuristic based on simulated annealing. Unfortunately, our other attempts were not very successful either—our heuristics often produced much more costly bribery strategies than the (optimal) ILP algorithm (even for the cases of 3 or 4 candidates) and often required even more time to complete (especially for the cases with more voters). Two possible explanations for these results are that:

- 1. Our \mathcal{R} -BSG problem is a particularly hard combinatorial problem. If this is indeed the case, then it might be a good testbed for improved ILP algorithms.
- Our heuristics are poorly designed and there is room for significant improvement and further research to obtain better ones. Indeed, we did not aim at optimizing heuristics but rather at gathering a basic feeling of the practical complexity of our problem.

In either case, our results call for further research and further analysis. Below we describe our experimental setup, the heuristics that we have tried, and how they compare to the ILP-based algorithm.

7.1 Experimental Setup

We consider elections with either 3 or 4 candidates and either 1000 or 10000 voters. In each case, we generate the voters' preferences using the impartial culture model;

i.e., by drawing the preference order of each voter uniformly at random. We used the Borda voting rule and focused on the basic variant of our problem, where voter types are equivalent to the voters' preference orders, two voter types are connected if their swap distance is one, and there is unit cost for shifting the preferred candidate by one position up in a single voter's preference order. We considered the synchronous diffusion process only. For the case of three candidates, we generated 55 elections for each combination of a heuristic algorithm and the number of voters. For the case of four candidates, we generated 10 elections for each heuristic algorithm. We ran the ILP-based algorithm for every generated election, in each setting.

7.2 Algorithms

We tested three algorithms, namely our ILP-based algorithm—described in Section 5.3—and two heuristics. The ILP-based algorithm solves the optimization variant of \mathcal{R} -BSG, i.e., it does not need the budget to be part of its input; it simply finds the lowest cost of ensuring that the preferred candidate is a winner. The heuristics, on the other hand, are phrased as decision algorithms and, thus, need the budget b to be given. We convert them to optimization algorithms using the standard approach of binary searching for the right value of b. Specifically, our binary-search method has two phases and, for a given decision algorithm \mathcal{H} , proceeds as follows:

- 1. In the *first phase*, we begin with b=1; then, we use algorithm $\mathcal H$ to see whether this budget is sufficient to succeed. If this is the case, then we halt; otherwise, we multiply b by 2 and, again, use $\mathcal H$ to check whether this b suffices. If this is the case, then we halt; otherwise, we again multiply b by 2 and repeat. The first phase continues until we have some $b=2^i$ for which $\mathcal H$ succeeds.
- 2. In the *second phase*, we perform a binary search on the values of b between $b=2^{i-1}$ and $b=2^i$; this allows us to find the minimum b^* for which \mathcal{H} succeeds; we return this b^* .

7.3 Heuristic Algorithms

We design two heuristic algorithms. One is a simple greedy algorithm whereas the other is an adaptation of the classic simulated annealing approach. Both algorithms are using the same idea regarding evaluation of partial solutions, based on the idea of *margin of victory*. Given a (partial) solution for the problem, i.e., the number of positions by which to shift the preferred candidate in the preference orders of the voters, we evaluate the quality of this solution as follows:

- 1. we implement the shifts as specified in the solution,
- 2. we run the diffusion process, and
- 3. we compute the difference between the Borda score of the preferred candidate and the Borda score of the highest-scoring opponent; this value is known as margin of victory and we interpret it as the quality of the solution.

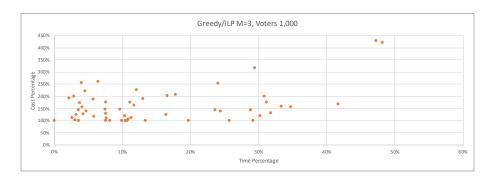


Figure 4: The performance of the greedy heuristic against ILP with respect to run time and cost with 3 candidates and 1000 voters.

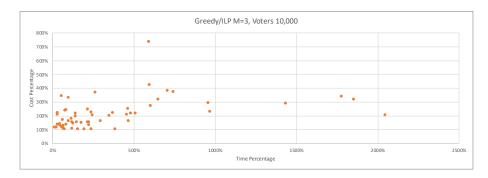


Figure 5: The performance of the greedy heuristic against ILP with respect to run time and cost with 3 candidates and 10000 voters.

Note that if the solution indeed leads to the victory of the preferred candidate then the margin of victory is non-negative. Either way, the higher it is, the better (indeed, if it is positive then we want the preferred candidate to have high advantage over the second-best candidate; if it is negative, then we want the preferred candidate to, nonetheless, be as close as possible to the current winner).

7.3.1 The Greedy Heuristic

Our greedy heuristic proceeds as follows: We maintain a solution, which is a set of bribery operations, initialized to be the empty set. We perform b iterations, where in each iteration we go over all possible bribery operations—one operation at a time—and select the operation that, when added to the partial solution, increases its quality the most. We also experimented with several other variants of this heuristic, but neither of them led to substantial changes or improvements in the performance.

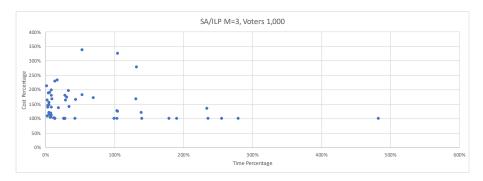


Figure 6: The performance of the SA heuristic against ILP with respect to run time and cost with 3 candidates and 1000 voters.

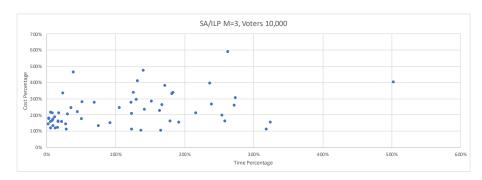


Figure 7: The performance of the SA heuristic against ILP with respect to run time and cost with 3 candidates and 10000 voters.

7.3.2 The Simulated Annealing Heuristic

In the simulated annealing heuristic (SA) we maintain a matrix A with m! rows and m columns, which represents b bribery operations, where each cell $a_{i,j}$ of the matrix represents the number of voters from type i for whom we shift the preferred candidate p up by j positions. Note that the cost of the set of bribery operations corresponding to some matrix A is the summation over the costs given by the cells, where the corresponding cost of cell $a_{i,j}$ is $a_{i,j} \cdot j$ (it costs j to shift p by j positions); for convenience, for a given A, we refer to $\sum_{i,j} a_{i,j} \cdot j$ as "the cost of A".

Initial solution We initialize A as follows: first, we set all cells of A to be 0. Then, we iterate until the cost of A is b, where in each iteration:

- 1. We choose i and j uniformly at random (i ranges from 0 to m!, while j ranges from 0 to m).
- 2. If $a_{i,j}$ is 0, then we select a value v uniformly at random between 0 and the number of voters of type i.

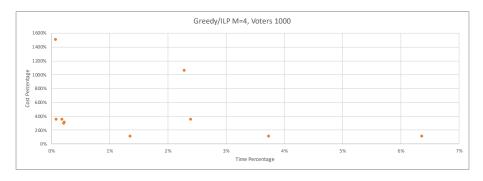


Figure 8: The performance of the greedy heuristic against ILP with respect to run time and cost with 4 candidates and 1000 voters.

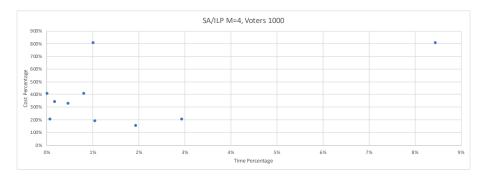


Figure 9: The performance of the SA heuristic against ILP with respect to run time and cost with 4 candidates and 1000 voters.

3. If the cost of A plus $v \cdot j$ is not greater than b, then we set $a_{i,j}$ to be v; otherwise, we leave $a_{i,j}$ to be 0 and proceed to the next iteration.

Local improvements After initializing the matrix A as described above, we proceed to the main algorithm, in which we perform T=10000 iterations, where in each iteration we aim at improving the current solution. In particular, in each iteration we pick, uniformly at random, two different voter types $i \neq k$ (note that we only choose such voter types for which at least one voter exists) and two indices j_1 and j_2 such that $a_{k,j_1}>0$, $a_{i,j_2}>0$, and proceed as follows: (1) we decrease a_{k,j_1} by one; (2) we increase a_{k,j_1-1} by one; (3) we decrease a_{i,j_2-1} by one; and (4) we increase a_{i,j_2} by one. (Intuitively, (1) and (2) "free" one unit of budget as it corresponds to taking one voter of type k that previously was shifted by j_1 positions to now be shifted by only j_1-1 positions; and, then, (3) and (4) "use" this one unit of budget as it corresponds to taking one voter of type i that previously was shifted by j_2-1 positions to now be shifted by j_2 positions. Indeed, we choose such j_1 and j_2 for which this local step is feasible.

Acceptance probability Let A be the current solution and let A' be a modified solution according to the procedure described in the previous paragraph. Next we describe under which conditions we "accept" the modification and replace A with A'. First, recall that the number of iterations is T = 1000; moreover, we have a parameter p_0 , set to be 0.2, and we maintain a parameter p_1 , initially set to be p_0 .

Now, if A' has higher quality than A, then with probability $1-p_1$ we accept it (i.e., set A=A' and reiterate); while if A' has lower quality than A, then with probability p_1 we accept it. Lastly, we replace the value of p_1 with p_1-p_0/T . This ensures that after T iterations p_1 drops down to 0, at which point we halt.

7.4 Results of the Experiments

We present the results of our experiments regarding three candidates in Figures 4 and 5 (for the greedy heuristic) and in Figures 6 and 7 (for the SA heuristic). For the case of four candidates, we show our results in Figures 8 and 9. For the case of three candidates, we show results for 1000 and 10000 voters, whereas for the case of four candidates we give results for 1000 voters only.

For the case of three candidates and 1000 voters, the greedy heuristic is significantly faster than the ILP-based algorithm, but for 10000 voters it works much more slowly. The SA heuristic tends to be slower than the ILP-based algorithm irrespective of the number of voters. However, for the case of four candidates both heuristics are much faster than the ILP solution. Unfortunately, in each of the presented cases the heuristics quite often return solutions of cost that is much higher than the optimal one (indeed, for the case of the greedy algorithm and four candidates, for one of the elections the heuristic found a solution about 15 times more expensive than the optimal one). More commonly, the heuristics provide solutions that are up to 3-4 times more expensive than the optimal ones. All in all, this means that the heuristics can hardly be seen as feasible means of solving our problem.

Nonetheless, it is interesting to also compare the running times of the heuristics depending on the number of voters. For the greedy heuristic, in Figure 4, we see that for 1000 voters the heuristic completes in at most half of the time of the ILP-based algorithm, even for the most difficult instances. Yet, for the case of 10000 voters (Figure 5) we already see that it can be up to 20 times slower than ILP (although for most instances the heuristic is at most five times slower). This is natural as the ILP-based solution scales logarithmically with the number of voters, whereas the greedy heuristic scales polynomially. On the other hand, for SA we do not see such effect as the number of iterations is fixed and its running time depends logarithmically on the number of voters (compare Figures 6, 7).

From the experiments we conclude that the problem of computing the cheapest campaign for rigging a given election by influencing a society graph either is a challenging problem, or our heuristics are poorly chosen or are poorly optimized. While we were able to prove, using certain ILP techniques, that the problem is FPT with respect to the number of candidates, there is room for future work, both theoretical and regarding algorithm engineering, to design algorithms that perform well in practice. Indeed, in our experiments we used standard local search algorithms without significant optimization, and our ILP implementation is using the standard setting of an off-the-shelf

ILP solver (Gurobi 7.5). All experiments were run on two Intel Xeon Gold 6230 - 20 Cores 125W 2.1GHz CPU Processor and 192 GB RAM.

Let us explain one possible direction that seems especially viable to us. Intuitively, the reason our ILP formulations are difficult to solve is that we are using a geometric tool (ILP) to express complex logical constraints. Recently, an alternative method for expressing ILP-definable voting rules and diffusion processes was shown via Presburger Arithmetic [38]. There exist Presburger Arithmetic solvers such as Omega [32] and TAPAS [40], and we hope that using them to solve \mathcal{R} -BSG would yield interesting results.

8 Outlook

We described a powerful model capturing various scenarios of opinion diffusion in networks, under various manipulative actions. By considering voter types and society graphs, we were able to provide quite strong tractability results. In particular, we have shown that, under certain circumstances, it is possible to find an optimal bribery scheme, taking into account various diffusion processes operating on various society graphs, for very general models. Below we discuss several research directions following from our work.

ILP Techniques. We do hope that this paper will have the side-effect of popularizing a number of ILP techniques within the area of computational social choice. While using Lenstra's algorithm to obtain FPT algorithm is already well-known, we have used a number of tricks that allow expressing more involved constraints than typically found in the ILPs used in this area (even though they are, generally, well-known in the area of mathematical programming). We hope that promoting these techniques would lead to discovering further voting-related algorithms and further transfer of knowledge between mathematical programming and computational social choice.

Generalized Diffusion Processes. We believe that our concept of a *generalized diffusion process* deserves more study. Here we mainly cared for identifying whether such generalized diffusion processes can be efficiently encoded via linear constraints, but studying their further properties, such as finding sufficient conditions for convergence, is an intriguing research direction. In particular, it would be interesting to explore connections between generalized diffusion processes and iterative voting (for more details on iterative voting, see the work of Meir et al. [42] and many papers that followed up on its ideas, in particular those of Sina et al. [50] and Tsang and Larson [52]). Under iterative voting, all the voters observe the votes currently cast by all the other voters and, in each round, they can modify their votes to obtain a more desirable outcomes. Generalized diffusion processes are capable of encoding such dynamics and, additionally, can impose restrictions on which votes the voters see (e.g., according to a given social network).

Probabilistic Models. Our model is inherently deterministic. While in the introduction we mentioned that, on the one hand, such determinism is quite common and, on the other hand, there are workarounds to simulate stochastic bahavior, it would be quite interesting to build a stochastic ingredient into the model in a way that does not require workarounds. Doing so deserves a careful study.

Our work is primarily theoretical, but we have also included an experimental component. So far, our conclusion from these experiments is that our bribery problem on society graphs is either quite challenging to solve or our approaches to solve it are too naive. Thus it is natural to seek better algorithms and to perform experiments on more realistic data, including data coming from real-life settings.

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