# GRAPH POLYNOMIALS: FROM RECURSIVE DEFINITIONS TO SUBSET EXPANSION FORMULAS 

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#### Abstract

Many graph polynomials, such as the Tutte polynomial, the interlace polynomial and the matching polynomial, have both a recursive definition and a defining subset expansion formula. In this paper we present a general, logic-based framework which gives a precise meaning to recursive definitions of graph polynomials. We then prove that in this framework every recursive definition of a graph polynomial can be converted into a subset expansion formula.


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## 1. Introduction

Graph polynomials are functions from the class of graphs $\mathcal{G}$ into some polynomial ring $\mathcal{R}$ which are invariant under graph isomorphisms. In recent years an abundance of graph polynomials have been studied. Among the most prominent examples we have the multivariate Tutte polynomial, BR99, Sok05, the interlace polynomial, ABS04a, ABS04b, AvdH04 which is really the Martin polynomial, cf. EM98, Cou, the matching polynomial and its relatives, HL72, LP86, GR01, and the cover polynomial for directed graphs CG95. Older graph polynomials, treated in monographs such as Big93, God93, Bol99, GR01, Die05, are the characteristic polynomial, CDS95, the chromatic polynomial, DKT05, and the original Tutte polynomial, Bol99. A general program for the comparative study of graph polynomials was outlined in Mak06, Mak07.

Graph polynomials are usually defined either recursively or explicitely by a subset expansion formula. In the case of the polynomial of the Pott's model $Z(G, q, v)$, a bivariate graph polynomial closely related to the Tutte polynomial, both definitions are easily explained.

Let $G=(V, E)$ be a (multi-)graph. Let $A \subseteq E$ be a subset of edges. We denote by $k(A)$ the number of connected components in the spanning subgraph $(V, A)$. The definition of the Pott's model using a subset expansion formula is given by

$$
\begin{equation*}
Z(G, q, v)=\sum_{A \subseteq E} q^{k(A)} v^{|A|} . \tag{1}
\end{equation*}
$$

The general subset expansion formula of a graph polynomial $P(G, \bar{X})$ now takes the form

$$
\begin{equation*}
P(G, \bar{X})=\sum_{\bar{A}:\langle G, \bar{A}\rangle \in \mathcal{C}} X_{1}^{f_{1}(G, \bar{A})} \cdot \ldots \cdot X_{n}^{f_{n}(G, \bar{A})} \tag{2}
\end{equation*}
$$

where $\bar{A}=\left(A_{1}, \ldots, A_{\ell}\right)$ are relations on $V(G)$ of arity $\rho(i)$, in other words $A_{i} \subseteq V(G)^{\rho(i)}$, the summation ranges over over a family $\mathcal{C}$ of structures of the form $\left\langle G, A_{1}, \ldots, A_{\ell}\right\rangle$, and the exponent $f_{i}(G, \bar{A})$ of the indeterminate $X_{i}$ is a function from $\mathcal{C}$ into $\mathbb{N}$. We refer to the right hand side of (2) as a subset expansion expression.
$Z(G, q, v)$ can also be defined recursively. It satisfies the initial conditions $Z\left(E_{1}\right)=q$ and $Z(\emptyset)=1$, and satisfies a linear recurrence relation

$$
\begin{align*}
Z(G, q, v) & =v \cdot Z\left(G_{/ e}, q, v\right)+Z\left(G_{-e}, q, v\right) \\
Z\left(G_{1} \sqcup G_{2}, q, v\right) & =Z\left(G_{1}, q, v\right) \cdot Z\left(G_{2}, q, v\right) \tag{3}
\end{align*}
$$

$\sqcup$ denotes the the disjoint union of two graphs, and for $e \in E$, the graph $G_{-e}$ is obtained from $G$ by deleting the edge $e$, and $G_{/ e}$ is obtained from $G$ by contracting the edge $e$. To show that $Z(G, q, v)$ is well-defined using the recurrence relation 3 one chooses an ordering of the edges and shows that the resulting polynomial does not depend on the particular choice of the ordering.

In the case of the Tutte polynomial it is a bit more complicated, as the recursion involves case distinction depending on whether the elimitated edge is a bridge, a loop or none of these. These conditions can be formulated as guards.

For most prominent graph polynomials, such as the chromatic polynomial, the Tutte polynomial, the interlace polynomial, and the cover polynomial for directed graphs, there exist both a recursive definition using a linear recurrence relation and a subset expansion formula. In each case the author proposes the two definitions and proves their equivalence.

In this paper we show how to convert a definition using a linear recurrence relation into a subset expansion formula. For this to make sense we define an appropriate framework. A special case of subset expansion formulas is the notion of a graph polynomial definable in Second Order Logic SOL, introduced first Mak04] and further studied in Mak07,

[^1]KMZ08. The exact definitions are given in Section 2.1 Roughly speaking, SOL-definable graph polynomials arise when in the subset expansion formula the class $\mathcal{C}$ is required to be definable in SOL, and similar conditions are imposed on the exponents of the indeterminates.

The recursive definition given above relies on the fact that every graph can be reduced, using edge deletion and edge contraction, to a set of isolated vertices. In a last step the isolated vertices are removed one by one. Using a fixed ordering of the edges and vertices, one can evaluate the recurrence relation. Finally one has to show that this evaluation does not depend on the ordering of the edges, provided the that in that ordering the vertices appear after all the edges.

In general, the two operations, edge deletion and contraction, will be replaced by a finite set of SOL-definable transductions $T_{1}, \ldots, T_{\ell}$, which decrease the size of the graph, and which depend on a fixed number of vertices or edges, the contexts, rather than just on a single edge. For certain orderings of the vertices and edges, this allows us to define a deconstruction tree of the graph $G$.

The recursive definition now takes the form

$$
\begin{equation*}
P(G)=\sum_{i \in\{1, \ldots, \ell\}} \sigma_{i} \cdot P\left(T_{i}[G, \vec{x}]\right) \tag{4}
\end{equation*}
$$

where $\vec{x}$ is the context and $\sigma_{i}$ are the coefficients of the recursion. Furthermore, the recurrence relation is linear in $P\left(T_{i}[G, \vec{x}]\right)$. It can be evaluated using the deconstruction tree. To assure that this defines a unique graph polynomial one has to show that the evaluation is independent of the ordering. The exact definitions are given in Section 4

Our main result, Theorem 5.1 now states that, indeed, every order invariant definition of a graph polynomial $P$ using a linear recurrence relation can be converted into a definition of $P$ as a SOL-definable graph polynomial. It seems that the converse is not true, but we have not been able to prove this.

In Section 7 we discuss a graph polynomial introduced in [NW9], which is provably not a SOL-definable graph polynomial. It is defined by a subset expansion formula, where the exponents $f_{i}(G, \bar{A})$ depend on $i$, which is not allowed in our definition of SOL-definable graph polynomials.

The choice of SOL is rather pragmatic. It makes exposition clear and covers all the examples from the literature. The logic SOL could be replaced by the weaker Fixed Point Logic FPL or by extensions of SOL, as they are used in Finite Model Theory, cf. EF95]. The polynomial introduced in NW99 would still be an example without recursive definition as long as the exponents $f_{i}(G, \bar{A})$ are not allowed to depend on $i$.

The paper is organized as follows. In Section 2 we collect the background material for Second Order Logic. In Section 3 we give a rigorous definition of SOL-definable graph polynomials and collect their basic properties. In Section 4 we present our general framework for recursive definitions of graph polynomials, and discuss examples in detail. In Section 5 we state and prove our main theorem. In Section 6 we show two derivations of subset expansion formulas, for the universal edge elimination polynomial and the cover polynomial, using the technique of the proof of Theorem 5.1] These derivations give the subset expansion formulas known in the literature. In Section 7 we discuss a polynomial which is given by a subset expansion formula but has no recursive definition in our sense. Finally, in Section 8 we draw conclusions and discuss further research.

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## 2. Logic and Translation Schemes

In this section we give a rather detailed definition of $\mathbf{S O L}$ and the formalism of translation schemes, because the notational technicalities are needed in our further exposition.

A vocabulary $\tau$ is a finite set of relation symbols, function symbols and constants. It can be many-sorted. In this paper, we shall only deal with vocabularies which do not contain any function symbols. $\tau$-structures are interpretations of vocabularies. Sorts are mapped into non-empty sets - the sort universes. Relation symbols are mapped into relations over the sorts according to their specified arities. Constant symbols are mapped onto elements of the corresponding sort-universes. We denote the set of all $\tau$-structures by $\operatorname{Str}(\tau)$. For a $\tau$-structure $\mathcal{M}$, we denote its universe by $A^{\mathcal{M}}$, or, in short, $A$, if the $\tau$-structure is clear from the context. For a logic $\mathcal{L}, \mathcal{L}(\tau)$ denotes the set of $\tau$-formulas in $\mathcal{L}$.
2.1. Second Order Logic (SOL). We denote relation symbols by bold-face letters, and their interpretation by the corresponding roman-face letter.
Definition 2.1 (Variables).
(i) $v_{i}$ for each $i \in \mathbb{N}$. These are individual variables $\left(\mathbf{V A R}_{1}\right)$.
(ii) $U_{r, i}$ for each $r, i \in \mathbb{N}, r \geq 1$. These are relation variables $\left(\mathbf{V A R}_{2}\right)$. $r$ is the arity of $U_{r, i}$.
We denote the set of variables by VAR.
Given a non-empty finite set $A$, an $A$-interpretation is a map

$$
I_{A}: \mathbf{V A R} \rightarrow A \cup \bigcup_{r} \mathbb{P}\left(A^{r}\right)
$$

such that $I_{A}\left(v_{i}\right) \in A$ and $I_{A}\left(U_{r, i}\right) \subseteq A^{r}$.

We define term $t$ and formula $\phi$ inductively, and associate with them a set of first and second-order free variables denoted by free $(t)$, free $(\phi)$ respectively.
Definition 2.2 ( $\tau$-term). A $\tau$-term is of the form $v$ or $c$ where $v$ is a variable and $c$ is some constant in $\tau$. free $(v)=\{v\}$, free $(c)=\emptyset$.
Definition 2.3 (Atomic formulas).
Atomic formulas are of the form
(i) $\left(t_{1} \simeq t_{2}\right)$ where $t_{1}, t_{2}$ are $\tau$-terms, and free $\left(t_{1} \simeq t_{2}\right)=$ free $\left(t_{1}\right) \cup$ free $\left(t_{2}\right)$.
(ii) $\phi$ of the form $U_{r, j}\left(t_{1}, t_{2}, \ldots, t_{r}\right)$ where $U_{r, j}$ is a relation variable, and $t_{1}, t_{2}, \ldots, t_{r}$ are $\tau$-terms, and free $(\phi)=\left\{U_{r, j}\right\} \cup \bigcup_{i=1}^{r}$ free $\left(t_{i}\right)$.
(iii) $\phi$ of the form $R\left(t_{1}, t_{2}, \ldots, t_{r}\right)$ where $R \in \tau$ is a relation, and $t_{1}, t_{2}, \ldots, t_{r}$ are $\tau$-terms, and free $(\phi)=\bigcup_{i=1}^{r}$ free $\left(t_{i}\right)$.
We now define inductively the set of $S O L$-formulas SOL.

Definition 2.4 (SOL formulas).
(i) Atomic formulas $\phi$ are in SOL with free $(\phi)$ as defined before.
(ii) If $\phi_{1}$ and $\phi_{2}$ are in SOL then $\phi$ of the form $\left(\phi_{1} \vee \phi_{2}\right)$, $\left(\phi_{1} \wedge \phi_{2}\right)$ or $\left(\phi_{1} \rightarrow \phi_{2}\right)$ is in SOL with free $(\phi)=$ free $\left(\phi_{1}\right) \cup$ free $\left(\phi_{2}\right)$.
(iii) If $\phi_{1}$ is in SOL then $\phi=\neg \phi_{1}$ is in SOL with free $(\phi)=$ free $\left(\phi_{1}\right)$.
(iv) If $\phi_{1}$ is in SOL then $\phi$ of the form $\exists v_{j} \phi, \forall v_{j} \phi$, is in SOL with free $(\phi)=$ free $\left(\phi_{1}\right)-\left\{v_{j}\right\}$.
(v) If $\phi_{1}$ is in SOL then $\phi$ of the form $\exists U_{r, j} \phi$ or $\forall U_{r, j} \phi$ is in SOL with free $(\phi)=$ free $\left(\phi_{1}\right)-\left\{U_{r, j}\right\}$.

### 2.2. Translation schemes and deconstruction schemes.

Definition 2.5 (Translation scheme $\Phi$ ). Let $\tau=\left\{Q_{1}, \ldots, Q_{k}\right\}$ and $\sigma=\left\{R_{1}, \ldots, R_{m}\right\}$ be two vocabularies and $\rho\left(R_{i}\right)\left(\rho\left(Q_{i}\right)\right)$ be the arity of $R_{i}\left(Q_{i}\right)$. Let $\mathcal{L}$ be a fragment of $S O L$, such as FOL, MSOL, $\exists M S O L, F P L$ (Fixed Point Logic), etc.

A tuple of $\mathcal{L}(\tau)$ formulae $\Phi=\left\langle\phi, \psi_{1}, \ldots, \psi_{m}\right\rangle$ such that $\phi$ has exactly one free first order variable and each $\psi_{i}$ has $\rho\left(R_{i}\right)$ distinct free first order variables is a $\tau-\sigma$-translation scheme.

In this paper we use only translation schemes in which $\phi$ has exactly one free variable. Such translation schemes are called non-vectorized.

In our case $\{x: \phi(x)\} \subset A$ holds. Such translation schemes are called relativized.
We now define the transduction which is the semantic map associated with $\Phi$.
Definition 2.6 (The induced transduction $\Phi^{\star}$ ). Given a $\tau-\sigma$-translation scheme $\Phi$, the function $\Phi^{\star}: \operatorname{Str}(\tau) \rightarrow \operatorname{Str}(\sigma)$ is a (partial) function from $\tau$-structures to $\sigma$-structures. $\Phi^{\star}[\mathcal{M}]$ is defined by:
(i) the universe of $\Phi^{\star}[\mathcal{M}]$ is the set

$$
A^{\Phi^{\star}[\mathcal{M}]}=\{a \in A: \mathcal{M} \models \phi(a)\}
$$

(ii) the interpretation of $R_{i}$ in $\Phi^{\star}[\mathcal{M}]$ is the set

$$
R_{i}^{\Phi^{\star}[\mathcal{M}]}=\left\{\bar{a} \in\left(A^{\Phi^{\star}[\mathcal{M}]}\right)^{\rho\left(R_{i}\right)}: \mathcal{M} \models \psi_{i}(\bar{a})\right\} .
$$

Next we define the syntactic map associated with $\Phi$, the translation.
Definition 2.7 (The induced translation $\left.\Phi^{\sharp}\right)$. Given a $\tau-\sigma$-translation scheme $\Phi$ we define a function $\Phi^{\sharp}: \mathcal{L}(\sigma) \rightarrow \mathcal{L}(\tau)$ from $\mathcal{L}(\sigma)$-formulae to $\mathcal{L}(\tau)$-formulae inductively as follows:
(i) For $R_{i} \in \sigma$ with $\rho\left(R_{i}\right)=m$ and $\theta=R_{i}\left(x_{1}, \ldots, x_{m}\right)$, we put

$$
\Phi^{\sharp}(\theta)=\left(\psi_{i}\left(x_{1}, \ldots, x_{m}\right) \wedge \bigwedge_{j=1}^{m} \phi\left(x_{j}\right)\right)
$$

(ii) This also works for equality and relation variables $U$ instead of relation symbols $R$.
(iii) For the boolean connectives, the translation distributes, i.e.
(iii.a) if $\theta=\left(\theta_{1} \vee \theta_{2}\right)$ then $\Phi^{\sharp}(\theta)=\left(\Phi^{\sharp}\left(\theta_{1}\right) \vee \Phi^{\sharp}\left(\theta_{2}\right)\right)$
(iii.b) if $\theta=\neg \theta_{1}$ then $\Phi^{\sharp}(\theta)=\Phi^{\sharp}\left(\neg \theta_{1}\right)$
(iii.c) similarly for $\wedge$ and $\rightarrow$.
(iv) For the existential quantifier, we use relativization to $\phi$ :

If $\theta=\exists y \theta_{1}$, we put

$$
\Phi^{\sharp}(\theta)=\exists y\left(\phi(y) \wedge \Phi^{\sharp}\left(\theta_{1}\right)(y)\right) .
$$

(v) For the universal quantifier, we also use relativization to $\phi$ : If $\theta=\forall y \theta_{1}$, we put

$$
\Phi^{\sharp}(\theta)=\forall y\left(\phi(y) \rightarrow \Phi^{\sharp}\left(\theta_{1}\right)(y)\right) .
$$

This concludes the inductive definition for first order logic FOL.
(vi) For second order quantification of variables $V$ of arity $\ell$ and $a$ vector $\bar{a}$ of length $\ell$ of first order variables or constants, we translate $\theta=\exists V\left(\theta_{1}(V)\right)$ by treating $V$ as a relation symbol above $A$ and put

$$
\Phi^{\sharp}(\theta)=\exists V\left(\forall \bar{v}\left[V(\bar{v}) \rightarrow\left(\bigwedge_{i=1}^{\ell} \phi\left(v_{i}\right)\right)\right] \wedge \Phi^{\sharp}\left(\theta_{1}\right)(V)\right)
$$



Figure 1. A diagram of translation scheme $\Phi$
(vii) For $\theta=\forall V\left(\theta_{1}(V)\right), \rho(V)=\ell$ the relativization yields:

$$
\Phi^{\sharp}(\theta)=\forall V\left(\left[\forall \bar{v}\left(V(\bar{v}) \rightarrow \bigwedge_{i=1}^{\ell} \phi\left(v_{i}\right)\right)\right] \rightarrow \Phi^{\sharp}\left(\theta_{1}\right)(V)\right)
$$

Next we present the well known fundamental property of translation schemes Mak04.
Theorem 2.8 (Fundamental Property).
Let $\Phi=\left\langle\phi, \psi_{1}, \ldots, \psi_{m}\right\rangle$ be a $(\tau-\sigma)$-translation scheme in a logic $\mathcal{L}$. Then the transduction $\Phi^{\star}$ and the translation $\Phi^{\sharp}$ are linked in $\mathcal{L}$. In other words, given $\mathcal{M}$ be a $\tau$-structure and $\theta$ be a $\mathcal{L}(\sigma)$-formula
then

$$
\mathcal{M} \models \Phi^{\sharp}(\theta) \Leftrightarrow \Phi^{\star}(\mathcal{M}) \models \theta
$$

The property is illustrated in Figure 1

Proposition 2.9. Mak04 Let $\Phi$ be a $\tau-\sigma$-translation scheme which is either in $S O L$ or in $M S O L$.
(i) If $\Phi$ is in $M S O L$ and non-vectorized, and $\theta$ is in $M S O L$ then $\Phi^{\sharp}(\theta)$ is in $M S O L$
(ii) If $\Phi$ is of quantifier rank $q$ and has $p$ parameters, and $\theta$ is a $\sigma$-formula of quantifier rank $r$, then the quantifier rank of $\Phi^{\sharp}(\theta)$ is bounded by $r+q+p$.

## 3. SOL-polynomials

SOL-polynomial expressions are expressions the interpretation of which are graph polynomials. We define SOL-polynomial expressions inductively.
3.1. SOL-polynomial expressions. Let the domain $\mathcal{R}$ be a commutative semi-ring, which contains the semi-ring of the integers $\mathbb{N}$. For our discussion it is sufficient for $\mathcal{R}$ to be $\mathbb{N}, \mathbb{Z}$ or polynomials over these, but the definitions generalize. Our polynomials have a fixed set of indeterminates $\mathcal{I}$. We denote the indeterminates by capital letters $X, Y, \ldots$ We distinguish them from the variables of SOL which we denote by lowercase letters $v, u, e, x, \ldots$
Definition 3.1 (SOL-monomial expressions). We first define the SOL-monomial expressions inductively.
(i) $a \in \mathcal{R}$ is a SOL-monomial expression, and free $(a)=\emptyset$.
(ii) Given a logical formula $\varphi, \operatorname{tv}(\varphi)$ is a SOL-monomial expression. $\operatorname{tv}(\varphi)$ stands for the truth value of the formula $\varphi$.
(iii) For a finite product $M=\prod_{i=1}^{r} t_{i}$ of monomial expressions $t_{i}, M$ is a SOLmonomial expression, and free $(M)=\bigcup_{i=1}^{r}$ free $\left(t_{i}\right)$.
(iv) Let $\phi(\bar{a}, \bar{b}, \bar{U})$ be a $\tau \cup\{\bar{a}, \bar{b}, \bar{U}\}$-formula in $\mathbf{S O L}$, where $\bar{a}=\left(a_{1}, \ldots, a_{m}\right)$ is a finite sequence of constant symbols not in $\tau, \bar{b}$ is a sequence of free individual variables, and $\bar{U}$ is a sequence of free relation variables. Let $t(\bar{a}, \bar{b}, \bar{U})$ be a SOL-monomial expression. Then

$$
M(\bar{b}, \bar{U})=\prod_{\bar{a}: \phi(\bar{a}, \bar{b}, \bar{U})} t(\bar{a}, \bar{b}, \bar{U})
$$

is a SOL-monomial expression and free $(M)=$ free $(t) \cup$ free $(\phi) \backslash\{\bar{a}\}$. Thus, $\Pi$ is a binding operator which binds $\bar{a}$.
Definition 3.2 (SOL-polynomial expressions). The SOL-polynomial expressions are defined inductively:
(i) SOL-monomial expressions are SOL-polynomial expressions.
(ii) For a finite sum $S=\sum_{i=1}^{r} t_{i}$ of SOL-polynomial expressions $t_{i}, S$ is a SOLpolynomial expression, and free $(S)=\bigcup_{i=1}^{r}$ free $\left(t_{i}\right)$.
(iii) Let $\phi(\bar{a}, \bar{b}, \bar{U})$ be a $\tau \cup\{\bar{a}, \bar{b}, \bar{U}\}$-formula in SOL where $\bar{a}=\left(a_{1}, \ldots, a_{m}\right)$ is a finite sequence of constant symbols not in $\tau, \bar{b}$ is a sequence of free individual variables, and $\bar{U}$ is a sequence of free relation variables. Let $t(\bar{a}, \bar{b}, \bar{U})$ be a SOL-polynomial expression. Then

$$
S(\bar{b}, \bar{U})=\sum_{\bar{a}: \phi(\bar{a}, \bar{b}, \bar{U})} t(\bar{a}, \bar{b}, \bar{U})
$$

is a SOL-polynomial expression and free $(P)=$ free $(t) \cup$ free $(\phi) \backslash\{\bar{a}\}$. Thus, $\sum$ is a binding operator which binds $\bar{a}$.
(iv) Let $\phi(\bar{W}, \bar{b}, \bar{U})$ be a $\tau \cup\{\bar{W}, \bar{b}, \bar{U}\}$-formula in SOL where $\bar{W}=\left(W_{1}, \ldots, W_{m}\right)$ is a finite sequence of relation symbols not in $\tau, \bar{b}$ is a sequence of free individual variables, and $\bar{U}$ is a sequence of free relation variables. Let $t(\bar{W}, \bar{b}, \bar{U})$ be a SOL-polynomial expression. Then

$$
S(\bar{b}, \bar{U})=\sum_{\bar{W}: \phi(\bar{W}, \bar{b}, \bar{U})} t(\bar{W}, \bar{b}, \bar{U})
$$

is a SOL-polynomial expression and
free $(P)=$ free $(t) \cup$ free $(\phi) \backslash\{\bar{W}\}$. Again, $\sum$ is a binding operator which binds $\bar{W}$.

Note that our definition of SOL-polynomial expressions is the normal form definition as it appears for example in KMZ08. We use only the normal form in this paper.

From our definitions the following is obvious.
Proposition 3.3. Every SOL-polynomial expression is also a subset expansion expression, where $\mathcal{C}$ is SOL-definable.

### 3.2. Interpretations of SOL-polynomial expressions.

Let $G$ be a graph and $z$ be an assignment of variables to elements of the graph. The interpretation $e(S, G, z)$ of a SOL-polynomial expression $S$ will be an element in the polynomial ring $\mathcal{R}$. We shall associate with each SOL-polynomial expression $S$ a graph polynomial $S^{*}$ defined by $S^{*}(G)=e(S, G, z)$. We shall say that $P(G, \bar{X})$ is a SOLpolynomial if there is a SOL-polynomial expression $S$ such that for all graphs $G$ we have $P(G, \bar{X})=S^{*}(G)$.

We now proceed with the precise definitions.

Definition 3.4 (Variable assignment).
(i) Given a $\tau$-structure $\mathcal{M}$ with domain $A^{\mathcal{M}}$, an assignment $z$ is an $A^{\mathcal{M}}$-interpretation of VAR.
(ii) We denote the set of all assignments above by $\operatorname{Ass}(\mathcal{M})$.
(iii) Let $z_{1}$ and $z_{2}$ be two assignments in $\operatorname{Ass}(\mathcal{M})$. Let $v \in \operatorname{VAR}$ be a variable. We write $z_{1}=v z_{2}$ if for every variable $u \neq v$ we have that $z_{1}(u)=z_{2}(u)$.
Our notation naturally extends to vectors of variables.
Definition 3.5 (Interpretation of SOL-monomial expressions). Given a $\tau$-structure $\mathcal{M}$ and an assignment $z \in \operatorname{Ass}(\mathcal{M})$, the interpretation $e(S, \mathcal{M}, z)$ of a SOL-monomial expression $S$ is defined as follows:
(i) If $S=a \in \mathcal{R}, e(S, \mathcal{M}, z)=a$.
(ii) Given a logical formula $\varphi$,

$$
e(\operatorname{tv}(\varphi), \mathcal{M}, z)= \begin{cases}1^{\mathcal{R}} & \text { if } \mathcal{M}, z \models \varphi \\ 0^{\mathcal{R}} & \text { otherwise }\end{cases}
$$

(iii) For a finite product $S=\prod_{i=1}^{r} t_{i}$ of monomials $t_{i}$,

$$
e(S, \mathcal{M}, z)=\prod_{i=1}^{r} e\left(t_{i}, \mathcal{M}, z\right)
$$

(iv) If $S(\bar{b}, \bar{U})=\prod_{\bar{a}: \phi(\bar{a}, \bar{b}, \bar{U})} t(\bar{a}, \bar{b}, \bar{U})$ then

$$
e(S(\bar{b}, \bar{U}), \mathcal{M}, z)=\prod_{\substack{z_{1} \text { s.t. } z_{1}={ }_{\bar{a}} z \text { and } \\ \mathcal{M}, z_{1} \models \phi(\bar{a}, \bar{b}, \bar{U})}} e\left(t(\bar{a}, \bar{b}, \bar{U}), \mathcal{M}, z_{1}\right) .
$$

We call the expression $S$ a short product as the number of elements in the product is polynomial in the size of the universe of $\mathcal{M}$.

The degree of the polynomial $e(S, \mathcal{M}, z)$, is polynomially bounded by the size of $\mathcal{M}$.
Definition 3.6 (Interpretation of SOL-polynomial expressions). Given a $\tau$-structure $\mathcal{M}$ and an assignment $z \in \operatorname{Ass}(\mathcal{M})$, the meaning function $e(S, \mathcal{M}, z)$ of a SOL-polynomial expression $S$ is defined as follows:
(i) For a finite sum $S=\sum_{i=1}^{r} t_{i}$ of SOL-polynomial expressions $t_{i}$, $e(S, \mathcal{M}, z)=\sum_{i=1}^{r} e\left(t_{i}, \mathcal{M}, z\right)$.
(ii) If $S(\bar{b}, \bar{U})=\sum_{\bar{a}: \phi(\bar{a}, \bar{b}, \bar{U})} t(\bar{a}, \bar{b}, \bar{U})$ then

$$
e(S(\bar{b}, \bar{U}), \mathcal{M}, z)=\sum_{\substack{z_{1} \text { s.t. } z_{1}=\bar{a} z \text { and } \\ \mathcal{M}, z_{1} \models \phi(\bar{a}, \bar{b}, \bar{U})}} e\left(t(\bar{a}, \bar{b}, \bar{U}), \mathcal{M}, z_{1}\right)
$$

We call the expression $S$ a short sum as the number of summands in the sum is polynomially bounded in the size of the universe of $\mathcal{M}$.
(iii) If $S(\bar{b}, \bar{U})=\sum_{\bar{W}: \phi(\bar{W}, \bar{b}, \bar{U})} t(\bar{W}, \bar{b}, \bar{U})$ then

$$
e(S(\bar{b}, \bar{U}), \mathcal{M}, z)=\sum_{\substack{z_{1} \text { s.t. } z_{1} \bar{W}_{\bar{W}} z \text { and } \\ \mathcal{M}, z_{1} \models \phi(\bar{W}, \bar{b}, \bar{U})}} e\left(t(\bar{W}, \bar{b}, \bar{U}), \mathcal{M}, z_{1}\right) .
$$

We call such a sum $S$ a long sum as the number of addends in the sum can be exponential in the size of the universe of $\mathcal{M}$.
(iv) A SOL-polynomial expression $S$ is short if it does not contain any long sums as subexpressions.

With these definition we have
Proposition 3.7. Let $S$ be an SOL-polynomial expression. Let $S^{*}$ be defined by $S^{*}(G)=$ $e(S, G, z)$. Then there is a graph polynomial $P(G, \bar{X})$ such that for all graphs $G$ we have $P(G, \bar{X})=S^{*}(G)$.

We say that $P(G, \bar{X})$ is a SOL-polynomial if there is a SOL-polynomial expression $S$ such that $P(G, \bar{X})=S^{*}(G)$.

### 3.3. Examples.

In the following section we represent graphs using one of the following two vocabularies: $\tau_{\text {graph }(1)}=\{E\}$ and $\tau_{\text {graph }(2)}=\{N\}$. For vocabulary $\tau_{\operatorname{graph}(1)}$, the universe of the graph is the set of its vertices, $A=V$, and $R=E \subseteq V^{2}$ is the relation that represents the edges. For $\tau_{\text {graph(2) }}$, the universe consists of both vertices and edges, $A=V \cup E$, and $R=N \subseteq V \times E$ relates vertices to adjacent edges.

Below are some formulas we need for many of the examples below. All the formulas are in $\operatorname{SOL}\left(\tau_{\operatorname{graph}(1)}\right)$ or $\mathbf{S O L}\left(\tau_{\operatorname{graph}(2)}\right)$ logic. We denote by $x, y, s, t, u, v, z$ the $\mathbf{V A R}_{1}$ variables, by $A, B, F, S, U, W$ the $\mathbf{V A R}_{2}$ variables and by $X, Y, Z$ the indeterminants in $\mathcal{I}$. For any formula $f$ :

$$
\exists^{k} x(f(x))=\exists x_{1} \cdots \exists x_{k}\left(\bigwedge_{i \neq j} x_{i} \neq x_{j} \wedge \bigwedge_{i=1}^{k} f\left(x_{i}\right) \wedge \forall y\left(\left(\bigwedge_{i=1}^{k} y \neq x_{i}\right) \rightarrow \neg f(y)\right)\right)
$$

For $D \subseteq A(G)$ and $S \subseteq E(G)$, Touching $(D, S)$ expresses the set of vertices or edges in $D$ which are adjacent to at least one edge from $S$, $\operatorname{Cycle}(S)$ is valid iff $S$ forms a cycle in $G$, and Connected ${ }_{S}(u, v)$ expresses that $u$ is connected to $v$ through the edges in $S$. These formulas take different form over vocabularies $\tau_{\operatorname{graph}(1)}$ and $\tau_{\operatorname{graph}(2)}$. Over the vocabulary $\tau_{\text {graph(1) }} S$ is a symmetric relation, and then:

$$
\begin{gathered}
\text { Touching }(D, S)=\{v: v \in D \wedge \exists u(S(v, u))\} \\
\operatorname{Cycle}(S)=\forall u, v \in{\operatorname{Touching}(V, S)\left[\exists \exists^{2} y(S(u, y)) \wedge \text { Connected }_{S}(u, v)\right]}_{\text {Connected }_{S}(s, t)=}(s=t) \vee \exists U[U(s) \wedge U(t) \wedge \forall x[U(x) \rightarrow \exists y(y \neq x \wedge S(x, y))] \wedge \\
\\
\quad \neg \exists W[W(s) \wedge \neg W(t) \wedge \forall x[(W(x) \rightarrow \\
\\
\\
(U(x) \wedge \forall y((S(x, y) \wedge U(y)) \rightarrow W(y)))]] .
\end{gathered}
$$

This formula expresses the fact that there is no subset $W \subsetneq U$ which contains $s$, does not contain $t$, and such that for each vertex $x \in W$ all the neighbors of $x$ in $U$ are also on $W$ i.e., $W$ is a $S$-closed subset of $U$ which separates $s$ from $t$.

For the cases we use $\tau_{\operatorname{graph}(2)}\left(A^{G}=V \cup E\right)$, we define shorthand formulas to identify an element of the universe to be an edge or a vertex respectively: $P_{E}(x)=\exists y(R(y, x)), P_{V}(x)=$ $x \in A \wedge \neg P_{E}(x)$,

Over the vocabulary $\tau_{\operatorname{graph}(2)} S$ is a subset $S \subseteq\left\{x: P_{E}(x)\right\}$, and then:

$$
\begin{aligned}
\text { Touching }_{(D, S)}(D) \\
\begin{aligned}
& \text { Cycle }(S)=\forall u, v \in \text { Touching }(V, S)\left[\exists^{2} e(S(e) \wedge N(u, e)) \wedge \text { Connected }_{S}(u, v)\right] \\
& \text { Connected }_{S}(s, t)=(s=t) \vee \exists U[\forall e(U(e) \rightarrow S(e)) \wedge \\
& \forall v\left[\left((v=s \vee v=t) \rightarrow\left(U(s) \vee \exists^{1} e(U(e) \wedge N(v, e))\right)\right) \wedge\right. \\
&\left(\left(P_{V}(v) \wedge v \neq s \wedge v \neq t\right) \rightarrow\right. \\
&\left.\left.\left(\neg \exists e(U(e) \wedge N(v, e)) \vee \exists^{2} e(U(e) \wedge N(v, e))\right)\right)\right] .
\end{aligned}
\end{aligned}
$$

This formula expresses the fact that there is a subset $U \subseteq S$ which contains a direct path from $s$ to $t$.

We also define $\operatorname{LastInComp}(D, S)$ to be the set of elements in $D$ each of which is the last one by a given order $O$ in its component defined by the edges in $S$. Formally:

$$
\begin{equation*}
\operatorname{LastInComp}(D, S) \doteq \forall x \in D \forall y\left[\left(\text { Connected }_{S}(x, y) \wedge x \neq y\right) \rightarrow x \succ_{0} y\right] \tag{5}
\end{equation*}
$$

Example 3.8 (Matching polynomial). There are different versions of the matching polynomial discussed in the literature (cf. [HL72, LP86, GR01]), for example matching generating polynomial $g(G, \lambda)=\sum_{i=0}^{n} a_{i} \lambda^{i}$ and matching defect polynomial $\mu(G, \lambda)=$ $\sum_{i=0}^{n}(-1)^{i} a_{i} \lambda^{n-2 i}$, where $n=|V|$ and $a_{i}$ is the number of $i$-matchings in $G$. We shall use the bivariate version that incorporates the both above:

$$
\begin{equation*}
M(G, X, Y)=\sum_{i=0}^{n} a_{i} X^{n-2 i} Y^{i} \tag{6}
\end{equation*}
$$

Note that using the formulas defined above, if $F$ is a matching in $G$ then $i=|F|$ and $n-2 i=\mid V \backslash$ Touching $(V, F)$. This formula expressed as a $\operatorname{SOL}\left(\tau_{\operatorname{graph}(2)}\right)$-polynomial expression is:

$$
\begin{equation*}
M(G, X, Y)=\sum_{F: \operatorname{Matching}(F)}\left[\prod_{v: P_{V}(v) \wedge \neg(v \in \operatorname{Touching}(V, F))} X\right] \cdot\left[\prod_{e: e \in F} Y\right] \tag{7}
\end{equation*}
$$

where

$$
\operatorname{Matching}(F)=\forall e_{1}, e_{2} \in F\left[P_{E}\left(e_{1}\right) \wedge\left(e_{1} \neq e_{2}\right) \rightarrow \neg \exists v\left(N\left(v, e_{1}\right) \wedge N\left(v, e_{2}\right)\right)\right] .
$$

Example 3.9 (Tutte polynomial). The classical two-variable Tutte polynomial satisfies a subset expansion formula using spanning forests (cf. for example B.Bollobás [Bol99]). Given a graph $G=\langle V \sqcup E, R\rangle, O$ an ordering of $E$, and $F \subseteq E$ a spanning forest of $G$, i.e., each component of $(V, F)$ is a spanning tree of a component of $G$. An edge $e \in F$ is internally active (for $F, O$ ) if it is the first edge in the set $\operatorname{Cut}_{F}(e)=\left\{e^{\prime} \in E\right.$ : $F-\{e\} \cup\left\{e^{\prime}\right\}$ is a spanning forest $\}$. An edge $e \in E-F$ is externally active (for $F, O$ ) if it is the first edge in the unique cycle Cycle $_{F}(e)$ of $F \cup\{e\}$.

For graphs $G$ with edge ordering $O$ the Tutte polynomial satisfies

$$
\begin{equation*}
T(G, X, Y)=\sum_{F} X^{i} Y^{j} \tag{8}
\end{equation*}
$$

where the sum is over all spanning forests of $G$ and $i(j)$ is the number of internally (externally) active edges of $F$ with respect to $O$. Furthermore, this is independent of the ordering $O$.

Let $F \subset E(V)$ be a spanning forest of $G$, i.e. $F$ contains no cycles and any connected component by $E(G)$ is also connected by $F$ :

SpanningForest $_{G}(F)=\neg \exists U[U \subseteq F \wedge C y c l e(U)] \wedge \forall v, u\left[\right.$ Connected $\left._{E}(v, u) \leftrightarrow \operatorname{Connected}_{F}(v, u)\right]$
The cycle of $e \notin F$ is a set of edges $Z_{F}(e)$ such that:

$$
e \in Z_{F}(e) \wedge\left(Z_{F}(e) \subseteq F \cup\{e\}\right) \wedge C y \operatorname{cle}\left(Z_{F}(e)\right)
$$

The cut defined by $e \in F$ is a set of edges $U_{F}(e)$ such that:

$$
U_{F}(e)=\left\{e^{\prime}: \text { SpanningForest }_{G}\left((F \backslash\{e\}) \cup\left\{e^{\prime}\right\}\right)\right\}
$$

Then, formula 8 expressed as a $\mathbf{S O L}\left(\tau_{\operatorname{graph}(2)}\right)$-polynomial expression is:

$$
\begin{align*}
T(G, X, Y)= & \sum_{F: \text { SpanningForest } G_{G}(F)} \tag{9}
\end{align*} \quad\left[\prod_{e: \forall e^{\prime}\left(\left(e^{\prime} \in U_{F}(e) \wedge e \neq e^{\prime}\right) \rightarrow e \prec o e^{\prime}\right)} X\right] .
$$

Example 3.10 (The polynomial of the Pott's model). This is a version of the Tutte polynomial used by A.Sokal Sok05, known as the (bivariate) partition function of the Pott's model:

$$
\begin{equation*}
Z(G, q, v)=\sum_{A \subseteq E} q^{k(A)} v^{|A|} \tag{10}
\end{equation*}
$$

Note that $k(A)=|\operatorname{LastInComp}(V, A)|$. Formula 10 expressed as a $\mathbf{S O L}\left(\tau_{\operatorname{graph}(2)}\right)$ polynomial expression is:

$$
\begin{equation*}
Z(G, q, v)=\sum_{A: A \subseteq E}\left[\prod_{v: v \in \operatorname{LastInComp}(V, A)} q\right] \cdot\left[\prod_{e: e \in A} v\right] . \tag{11}
\end{equation*}
$$

3.4. Properties of SOL-definable polynomials. The following is taken from KMZ08.

## Proposition 3.11.

(i) If we write an SOL-definable polynomial as a sum of monomials, then the coefficients of the monomials are in $\mathbb{N}$.
(ii) Let $M$ be an SOL-definable monomial viewed as a polynomial. Then $M$ is a product of a finite number $s$ of terms of the form $\prod_{\bar{a}:\langle\mathcal{M}, \bar{a}\rangle \models \phi_{i}} t_{i}$, where $i \in[s]$, $t_{i} \in \mathbb{N} \cup \mathcal{I}$ and $\phi_{i} \in \mathbf{S O L}$.
(iii) The product of two $\mathbf{S O L}(\tau)$-definable polynomials is again a $\mathbf{S O L}(\tau)$-definable polynomial.
(iv) The sum of two $\mathbf{S O L}(\tau)$-definable polynomials is again a $\mathbf{S O L}(\tau)$-definable polynomial.
(v) Let $\Phi(\mathcal{A}, \bar{X})$ be a SOL-definable monomial and $P: \operatorname{Str}(\tau) \rightarrow \mathbb{N}[\bar{X}]$ be of form

$$
P(\mathcal{M}, \bar{X})=\sum_{\bar{R}:\langle\mathcal{M}, \bar{R}\rangle \models \chi_{R}} \prod_{\bar{b}:\langle\mathcal{M}, \bar{R}, \bar{b}\rangle \models \psi} \sum_{\bar{a}:\langle\mathcal{M}, \bar{R}, \bar{a}, \bar{b}\rangle \models \phi} \Phi(\langle\mathcal{M}, R, \bar{a}, \bar{b}\rangle, \bar{X}) .
$$

Then $P(\mathcal{M}, \bar{X})$ is a SOL-definable polynomial.
3.5. Combinatorial polynomials. In the examples we need the fact that some combinatorial polynomials are indeed SOL-definable polynomials. The question which combinatorial function can be written as SOL-definable polynomials is beyong the scope of this paper, and is the topic of T. Kotek's thesis Kot10.

The following are all SOL-definable polynomials. We denote by $\operatorname{card}_{\mathcal{M}, \bar{v}}(\varphi(\bar{v}))$ the number of $\bar{v}$ 's in $\mathcal{M}$ that satisfy $\varphi$.

Cardinality, I:: The cardinality of a definable set $\operatorname{card}_{\mathcal{M}, \bar{v}}(\varphi(\bar{v}))=\sum_{\bar{v}: \varphi(\bar{v})} 1$ is an evaluation of a SOL-definable polynomial.
Cardinality, II:: The cardinality as the exponent in a monomial $X^{\operatorname{card} \mathcal{M}_{\mathcal{V}}(\varphi(\bar{v}))}=\prod_{\bar{v}: \varphi(\bar{v})} X$ is an SOL-definable polynomial.
Factorials:: The factorial of the cardinality of a definable set is an instance of a SOL-definable polynomial:
$\operatorname{card}_{\mathcal{M}, \bar{v}}(\varphi(\bar{v}))!=\sum_{\pi: F u n c 1 t o 1(\pi,\{\bar{v}: \varphi(\bar{v})\},\{\bar{v}: \varphi(\bar{v})\})} 1$,
where Funclot $(\pi, A, B)$ says that $\pi$ is a one-to-one function from relation $A$ to relation $B$ :

$$
\begin{aligned}
\text { Func1to1 }(\pi, A, B)= & \forall \bar{v} \forall \bar{u}[\pi(\bar{v}, \bar{u}) \rightarrow[\bar{v} \in A \wedge \bar{u} \in B \wedge \\
& \neg \exists \bar{w}((\bar{w} \neq \bar{v} \wedge \pi(\bar{w}, \bar{u})) \vee(\bar{w} \neq \bar{u} \wedge \pi(\bar{v}, \bar{w})))]] .
\end{aligned}
$$

Falling factorial:: The falling factorial

$$
(X)_{\operatorname{card} d_{\mathcal{M}, \bar{v}}(\varphi(\bar{v}))}=X \cdot(X-1) \cdot \ldots \cdot\left(X-\operatorname{car}_{\mathcal{M}, \bar{v}}(\varphi(\bar{v}))\right.
$$

is not an SOL-definable polynomial, because it contains negative terms, which contradicts Proposition 3.11 However, if the underlying structure has a linear order, then it is an evaluation of an SOL-definable polynomial. We write

$$
(X)_{\operatorname{card} d_{\mathcal{M}, \bar{v}}(\varphi(\bar{v}))}=\prod_{\bar{a}: \varphi} X-\operatorname{card}_{\mathcal{M}, \bar{v}}\left(\varphi_{<\bar{a}}(\bar{v})\right)
$$

where $\varphi_{<\bar{a}}$ is the formula ( $\left.\varphi(\bar{v}) \wedge \bar{v}<\bar{a}\right)$ and $\bar{v}<\bar{a}$ is shorthand for the lexicographical order of tuples of vertices.

## 4. Deconstruction of a Signed graph and its valuation

In the following section we use the notation $\tau_{\text {graph(1) }}$ and $\tau_{\text {graph(2) }}$ for graph vocabularies as defined in Subsection 3.3 The definitions below are applicable for either vocabulary.

### 4.1. Deconstruction trees. Let $\tau \in\left\{\tau_{\text {graph }(1)}, \tau_{\text {graph }(2)}\right\}$.

Definition 4.1 (Context). Given a graph $G$ and $\vec{x} \in A^{m}, m \in \mathbb{N}$, a vector of elements of $G$, we call $\vec{x}$ an $m$-context. Given a vocabulary $\tau$ we denote by $\tau_{m}$ the vocabulary $\tau$ augmented by $m$ constant symbols interpreted by the $m$-context $\vec{x}$. We denote by $\mathcal{G}_{m}$ the collection of graphs $\langle G, \vec{x}\rangle$ with an $m$-context.

We now equip the graph $\langle G, \vec{x}\rangle$ with a linear ordering of its $m$-tuples.
Definition 4.2 (Context ordering VALORD ${ }_{m}$ ). Let $\tau_{m}^{o}=\tau \cup\left\{a_{1}, \ldots, a_{m}, O\right\}$ where the $a_{i}$ 's are constants symbols and $O$ is a $2 m$-ary relation symbol. Let $\phi_{\text {ord }} \in \mathbf{S O L}\left(\tau_{m}^{o}\right)$. The class VALORD $_{m}$ consists of $\tau_{m}$-structures such that
(i) $\langle G, \vec{x}, O\rangle \in$ VALORD iff $\langle G, \vec{x}, O\rangle \models \phi_{\text {ord }}$.
(ii) The interpretation of $O$ is a linear ordering of the m-tuples of $G$.
(iii) For every $\langle G, \vec{x}\rangle$ there is an $O \subset A^{2 m}$ with $\langle G, \vec{x}, O\rangle \models \phi_{\text {ord }}$.
(iv) $\vec{x}$ is the first element in the ordering $O$ of $\langle G, \vec{x}, O\rangle$

We denote by $\bar{G}$ strutures of the form $\langle G, \vec{x}, O\rangle$, by $A(\bar{G})$ the universe of $\bar{G}$, by $R(\bar{G})$ the graph relation of $\bar{G}$, and by $c(\bar{G})$ the context of $\bar{G}$, and by $O(\bar{G})$ the context ordering of $\bar{G}$.

Definition 4.3 (SOL-Deconstruction Scheme). Let $\Phi$ be a $\tau_{m}^{o}-\tau_{m}^{o}$-translation scheme. $\Phi$ is a SOL-deconstruction scheme along VALORD, if
(i) $A^{\Phi^{\star}[\bar{G}]} \subsetneq A$;
(ii) at least one element $x_{i}$ of $\vec{x}$ is deleted, i.e., $x_{i} \notin A^{\Phi^{\star}[\bar{G}]}$;
(iii) $O^{\Phi^{\star}[\bar{G}]}=\left.O\right|_{A^{\Phi \star}[\bar{G}]}$;
(iv) If $\bar{G} \in$ VALORD then $\Phi^{\star}[\bar{G}] \in$ VALORD;

In this case we call $\Phi^{\star}$ a SOL-deconstruction along VALORD, or simply a deconstruction, if VALORD is clear from the context.

Definition 4.4 (Guarded SOL-Deconstruction Scheme). A guarded SOL-deconstruction is a pair $(T, \varphi)$, such that $T$ is a SOL-deconstruction scheme and $\varphi$ is a $\mathbf{S O L}\left(\tau_{m}^{o}\right)$-formula, and such that $\Phi^{\star}(\bar{G})$ is a non-empty structure for each $\bar{G}$ which satisfies $\varphi$.

## Remark 4.5.

(i) Note that the formulas in $\Phi$ and the formula $\varphi$ may have up to $m$ additional free individual variables for the m-context.
(ii) We say that the guarded SOL-deconstruction $(T, \varphi)$ is enabled on a graph $\bar{G}$ if $\bar{G} \models \varphi$.
(iii) One could have incorporated the guard in the definition of $\Phi$, but this is not suitable here, because we want to refer to the guard $\varphi$ explicitly.

A SOL-deconstruction tree for a graph $G$ with an $m$-context $\vec{x}$ and for a set of guarded deconstructions $\left\{\left(T_{1}, \varphi_{1}\right), \ldots,\left(T_{\ell}, \varphi_{\ell}\right)\right\}$ is a tree each internal node of which is labeled by a graph with an $m$-context. The arc from a node labeled with $\left\langle G_{1}, \vec{x}_{1}\right\rangle$ to its child labeled with $\left\langle G_{2}, \vec{x}_{2}\right\rangle$ respectively, is labeled with a guarded deconstruction $\left(T_{i}, \varphi_{i}\right)$ such that $\left\langle G_{1}, \vec{x}_{1}\right\rangle \models \varphi_{i}$ and $G_{2}=T_{i}^{*}\left[G_{1}, \vec{x}_{1}\right]$. Additionally we require that for each internal node labeled with $\langle G, \vec{x}\rangle$ and each guarded deconstruction enabled on $\langle G, \vec{x}\rangle$ there is an outgoing arc labeled by it. Furthermore, each leaf of the deconstruction tree is labeled by the empty graph. With full noational details this looks as follows.

Definition 4.6 (SOL-Deconstruction tree along VALORD). Given a graph $\bar{G} \in$ VALORD over $\tau_{m}^{o}$ and given a set of guarded SOL-definable deconstructions schemes $\left\{\left(T_{i}, \varphi_{i}\right)\right\},(i=$ $1, \ldots, l)$, we define a SOL-deconstruction tree $\Gamma=\Gamma(\bar{G})$ along VALORD as follows:
(i) We have $\ell$ partial functions $f_{i}, i \leq \ell$, denoting the $\ell$ child relations.
(ii) The root of $\Gamma, r$, is a node marked by $\bar{G}$.
(iii) Each internal node $n$ of $\Gamma$ is marked by a graph $\bar{G}_{n}$.
(iv) The child $f_{i}(n)$ of an internal node $n$ marked with a non-empty graph $\bar{G}_{n}$ is marked with $T_{i}^{\star}\left(\bar{G}_{n}\right)$, where $T_{i}^{\star}\left(\bar{G}_{n}\right)$ is enabled and not empty.
(v) If $T_{i}^{\star}\left(\bar{G}_{n}\right)$ is not enabled $f_{i}(n)$ is undefined.
(vi) Each leaf in $\Gamma$ is marked by the empty graph.

With this definition we have
Proposition 4.7. For every set of guarded SOL-deconstructions $\mathcal{T}=\left\{\left(T_{i}, \varphi_{i}\right): i \leq \ell\right\}$ acting on VALORD defined by $\varphi_{\text {ord }}$, and for every $\bar{G} \in$ VALORD there is at most one SOL-deconstruction tree $\Gamma(\bar{x})$.

We denote by

$$
\operatorname{enabled}(\vec{x})=\bigvee_{i=1}^{l} \varphi_{i}(\vec{x})
$$

and call the formula $\operatorname{enabled}(\vec{x})$ the deconstruction enabling formula. Note that the labeling of each internal node $n$ in the deconstruction tree must satisfy $\bar{G}_{n} \models$ enabled.

The graph $\bar{G}_{n}$ associated with the node $n$ is called the world view of $n$. We denote the subtree of $\Gamma$ rooted at an internal node $n$ by $\Gamma_{n}=\Gamma_{n}\left(\bar{G}_{n}\right)$.
4.2. The linear recurrence relation. The recursive definition of a graph polynomial $P$ tells us how to compute $P(\bar{G})$ from $T_{i}^{*}(\bar{G})$. The linear recurrence relation we have in mind takes the form

$$
\begin{equation*}
\text { rec }: P(\bar{G})=\sum_{i: \bar{G} \models \varphi_{i}} \sigma_{i}(\bar{G}) \cdot P\left(T_{i}^{*}(\bar{G})\right) \tag{12}
\end{equation*}
$$

where $\varphi_{i}$ is the guard of $T_{i}$. We still have to specify what the coefficients $\sigma_{i}(\bar{G})$ are allowed to be.

Definition 4.8 (Coefficients of the linear recurrence relation). Let $\left\{\sigma_{i}:\right.$ VALORD $\mapsto$ $\mathcal{R}\},(i=1, \ldots, l)$ be a set of mappings such that each $\sigma_{i}$ is a map associated with $T_{i}$ which maps a graph with an m-context into an element of $\mathcal{R}$. Furthermore we require that $\sigma_{i}(\bar{G})$ is given by a short SOL-polynomial expression.
4.3. Valuation of a deconstruction tree. Given a deconstruction tree $\Gamma(G)$ we want to assign to $\Gamma(G)$ a value in $\mathcal{R}$.

Given a graph $G$, a deconstruction tree $\Gamma(G)$ of $G$ and coefficients $\left\{\sigma_{i}\right\}$, we compute the deconstruction tree valuation by applying the formula below to each internal node $n$ of $\Gamma(G)$ :

$$
\begin{equation*}
P\left(\bar{G}_{n}\right)=\sum_{\substack{i \in\{1, \ldots, l\} \\ \text { s.t. } \bar{G}_{n} \models \varphi_{i}}} \sigma_{i}\left(\bar{G}_{n}\right) \cdot P\left(T_{i}^{\star}\left(\bar{G}_{n}\right)\right) \tag{13}
\end{equation*}
$$

If $n$ is a leaf we define $P\left(G_{n}, \vec{x}_{n}\right)=1^{\mathcal{R}}$. This computation is well defined for every ordered graph with a context $\bar{G}$, but the computation may depend on the underlying order of the contexts.

### 4.4. Well defined recursive definition.

Definition 4.9. A recursive definition of a graph polynomial $P$ is given by a triple ( $\mathcal{T}$, rec, $\left.\varphi_{\text {ord }}\right)$, where
(i) $\mathcal{T}=\left\{\left(T_{i}, \varphi_{i}\right): i \leq \ell\right\}$ is a finite family of guarded SOL-destruction schemes acting on VALORD defined by $\varphi_{\text {ord }}$, and
(ii)

$$
\text { rec }: P(\bar{G})=\sum_{i: \bar{G} \models \varphi_{i}} \sigma_{i}(\bar{G}) \cdot P\left(T_{i}^{*}(\bar{G})\right)
$$

is a linear recurrence relation.
For the recursive definition ( $\mathcal{T}$, rec, $\varphi_{\text {ord }}$ ) of a graph polynomial $P$ to be well defined we need several conditions to be satisfied.

Definition 4.10. A triple ( $\mathcal{T}$, rec, $\varphi_{\text {ord }}$ ) is SOL-feasible for $P$ if the following conditions are satisfied.
(i) VALORD is SOL-definable by a SOL-formula $\varphi_{\text {ord }}$.
(ii) Every graph $G \in \mathcal{G}_{m}$ has an expansion $\bar{G}=\langle G, \vec{x}, O\rangle$ with an order $O$ such that $\langle G, \vec{x}, O\rangle \models \varphi_{\text {ord }}$, i.e., such that $\langle G, \vec{x}, O\rangle \in$ VALORD.
(iii) Every graph $\bar{G} \in$ VALORD has a SOL-deconstruction tree $\Gamma(\bar{G})$.
(iv) Given two orders $O_{1}$ and $O_{2}$ on $G$ and the corresponding deconstruction trees $\Gamma\left(G, O_{1}\right), \Gamma\left(G, O_{2}\right)$ we have $P\left(\Gamma\left(G, O_{1}\right)\right)=P\left(\Gamma\left(G, O_{2}\right)\right)$.

Proposition 4.11. Given a SOL-feasible triple ( $\mathcal{T}$, rec, VALORD), there is a unique graph invariant $P$ such that for all ordered graphs $\langle G, O\rangle \in$ VALORD

$$
P(G)=P(\Gamma(G, O))
$$

Note that we can replace the logic SOL in the definitions of this section by other logics used in finite model theory, say Fixed Point Logic FPL, Monadic Second Order Logic MSOL, etc. Such logics are defined in detail in, say EF95. The choice of SOL here is a choice of convenience. In Section 8 we shall return to the use of other logics.

### 4.5. Examples.

In all the examples below, the universe of $G$ is $A^{G}=V \cup E$, the context is monadic $(m=1)$ and we take VALORD ${ }_{1}$ to be defined by $\phi_{\text {ord }}=\forall x, y\left[\left(P_{E}(x) \wedge P_{V}(y)\right) \rightarrow x \prec o y\right]$, i.e., we require the edges in $G$ to come before the vertices in the order $O$.

Example 4.12 (Matching polynomial). The bivariate matching polynomial (cf. for example HL72, LP86, GR01) is defined by

$$
M(G, X, Y)=\sum_{i=0}^{n} a_{i} X^{n-2 i} Y^{i}
$$

Alternatively, it can be also defined by a linear recurrence relation as follows. The initial conditions are $M\left(E_{1}\right)=X$ and $M(\emptyset)=1$. Additionally, it satisfies the recurrence
relations

$$
\begin{align*}
M(G) & =M\left(G_{-e}\right)+Y \cdot M\left(G_{\dagger e}\right) \\
M\left(G_{1} \oplus G_{2}\right) & =M\left(G_{1}\right) \cdot M\left(G_{2}\right) \tag{14}
\end{align*}
$$

Here $M\left(G_{\dagger e}\right)$ is the graph obtained from $G$ by deleting the edge $e=(u, v)$ together with the vertices $u$ and $v$ and all the edges incident with $u$ and $v$.

To express this defintion within our framework, we take $A^{G}=V \cup E$ and $R=N \subseteq$ $V \times E$ is the adjacency relation between vertices and edges. We define shorthand formulas to identify an item of the universe to be edge or vertex respectively: $P_{E}(x)=$ $\exists y(R(y, x)), P_{V}(x)=x \in A \wedge \neg P_{E}(x)$, and a formula which captures the universe elements which are removed during the extraction of an edge $x$ :

$$
\operatorname{Extracted}(x, y)=[y=x \vee R(y, x) \vee \exists u(R(u, x) \wedge R(u, y))]
$$

The following table summarizes the formulas for the recursive definition of the matching polynomial.

|  | Action <br> $i$ |  | $T_{i}[G, x]$ | $T_{i}[G, x]$ |  |
| :--- | :--- | :--- | :--- | :--- | :--- |
| 1 | $G_{-v}$ | $\varphi_{i}(x)$ | $P_{V}(x) \wedge \neg \exists y(R(x, y))$ | $y \neq x$ | $\psi_{i}(y, z)$ |

Note that in this case, enabled $(G, x)$ does not contain the case of $P_{V}(x) \wedge \exists y(R(x, y))$, therefore not for every order $O$ there exists a valid fixed order deconstruction tree with order $O$. However, any order $O$ in which all the edges come before all the vertices, defines a valid fixed order deconstruction tree.

Example 4.13 (Tutte polynomial). The Tutte polynomial is defined (cf. for example Bol99, BR99) by the initial conditions $T\left(E_{1}\right)=1$ and $T(\emptyset)=1$ and has linear recurrence relation:

$$
T(G, X, Y)= \begin{cases}X \cdot T\left(G_{-e}, X, Y\right) & \text { if e is a bridge } \\ Y \cdot T\left(G_{-e}, X, Y\right) & \text { if e is a loop } \\ T\left(G_{/ e}, X, Y\right)+T\left(G_{-e}, X, Y\right) & \text { otherwise }\end{cases}
$$

(15) $T\left(G_{1} \oplus G_{2}, X, Y\right)=T\left(G_{1}, X, Y\right) \cdot T\left(G_{2}, X, Y\right)$
where a bridge is an edge removing which separates its connected component to two connected components.

As in the case of matching polynomial we define $A^{G}=V \cup E, R=N \subseteq V \times E$, $P_{E}(x)=\exists y(R(y, x))$ and $P_{V}(x)=x \in A \wedge \neg P_{E}(x)$ In addition we define the next shorthand formulas:
For any formula $f$ :

$$
\exists^{k} x(f(x))=\exists x_{1} \cdots \exists x_{k}\left(\bigwedge_{i \neq j} x_{i} \neq x_{j} \wedge \bigwedge_{i=1}^{k} f\left(x_{i}\right) \wedge \forall y\left(\left(\bigwedge_{i=1}^{k} y \neq x_{i}\right) \rightarrow \neg f(y)\right)\right)
$$

For any two monadic relations $U$ and $W$ :

$$
U \subseteq W \equiv \forall x(U(x) \rightarrow W(x))
$$

We define formulas to express an edge being a bridge, a loop, or none of these, respectively:

$$
\begin{gathered}
\operatorname{Bridge}(x)=\quad P_{E}(x) \wedge \exists y, z[y \neq z \wedge R(y, x) \wedge R(z, x) \wedge \\
\neg \exists U\left(U \subseteq P_{E} \wedge \neg U(x) \wedge \exists u_{1}, u_{2}[ \right. \\
U\left(u_{1}\right) \wedge U\left(u_{2}\right) \wedge R\left(y, u_{1}\right) \wedge R\left(x, u_{2}\right) \wedge \\
\forall u_{3}\left[\left(P_{V}\left(u_{3}\right) \wedge u_{3} \neq y \wedge u_{3} \neq z\right) \rightarrow\right. \\
\left.\left.\left.\quad\left(\neg \exists e_{1}\left(U\left(e_{1}\right) \wedge R\left(u_{3}, e_{1}\right)\right) \vee\left(\exists^{2} e_{2}\left(U\left(e_{2}\right) \wedge R\left(u_{3}, e_{2}\right)\right)\right)\right)\right]\right)\right] \\
\operatorname{Loop}(x)=P_{E}(x) \wedge \exists^{1} y(R(y, x)) \\
\operatorname{None}(x)=P_{E}(x) \wedge \neg \operatorname{Bridge}(x) \wedge \neg \operatorname{Loop}(x)
\end{gathered}
$$

In the case of contraction of edge $x$ we remove the edge and the smaller one (by order $O)$ of its end vertices $u, v$. The remaining end vertex $v$ becomes adjacent to all the edges which entered either of $u, v$. To describe this we need the next formulas:

$$
\begin{aligned}
\operatorname{EdgeEnds}(x, u, v) & =R(u, x) \wedge R(v, x) \wedge u \prec_{O} v \\
\operatorname{Left}(x, u) & =P_{E}(x) \wedge \exists v(\operatorname{EdgeEnds}(x, u, v)) \\
\operatorname{Right}(x, v) & =P_{E}(x) \wedge \exists u(\operatorname{EdgeEnds}(x, u, v))
\end{aligned}
$$

The resulting adjacency relation is:

$$
\psi_{\text {Contract }}(x, y, z)=\exists u, v[E d g e \operatorname{Ends}(x, u, v) \wedge(R(y, z) \vee(y=v \wedge R(u, z))]
$$

The following table summarizes the formulas for the recursive definition of the Tutte polynomial.

|  | Action |  | $T_{i}[G, x]$ | $T_{i}[G, x]$ |  |
| :--- | :--- | :--- | :--- | :--- | :--- |
| $i$ | type | $\varphi_{i}(x)$ | $\phi_{i}(y)$ | $\psi_{i}(y, z)$ | $\sigma_{i}(x)$ |
| 1 | $G_{-e}$ | $\operatorname{Bridge}(x)$ | $y \neq x$ | $R(y, z)$ | $X$ |
| 2 | $G_{-e}$ | $\operatorname{Loop}(x)$ | $y \neq x$ | $R(y, z)$ | $Y$ |
| 3 | $G_{/ e}$ | $\operatorname{None}(x)$ | $\neg \operatorname{Left}(x, y)$ | $\psi_{\text {Contract }}(x, y, z)$ | 1 |
| 4 | $G_{-e}$ | $\operatorname{None}(x)$ | $y \neq x$ | $R(y, z)$ | 1 |
| 5 | $G_{-v}$ | $P_{V}(x) \wedge \neg \exists y(R(x, y))$ | $y \neq x$ | $R(y, z)$ | 1 |

Example 4.14 (Pott's model). The polynomial $Z(G, q, v)$, called the Pott's model, is defined (cf. for example Sok05) by the initial conditions $Z\left(E_{1}\right)=q$ and $Z(\emptyset)=1$, and satisfies the linear recurrence relation

$$
\begin{align*}
Z(G, q, v) & =v \cdot Z\left(G_{/ e}, q, v\right)+Z\left(G_{-e}, q, v\right) \\
Z\left(G_{1} \sqcup G_{2}, q, v\right) & =Z\left(G_{1}, q, v\right) \cdot Z\left(G_{2}, q, v\right) \tag{16}
\end{align*}
$$

Again we define $A^{G}=V \cup E, R=N \subseteq V \times E, P_{E}(x)=\exists y(R(y, x))$ and $P_{V}(x)=x \in$ $A \wedge \neg P_{E}(x)$ We also borrow the definition of $\psi_{\text {Contract }}(x, y, z)$ from the Tutte polynomial.

The following table summarizes the formulas for the recursive definition for the Pott's model.

|  | Action |  | $T_{i}[G, x]$ | $T_{i}[G, x]$ |  |
| :--- | :--- | :--- | :--- | :--- | :--- |
| $i$ | type | $\varphi_{i}(x)$ | $\phi_{i}(y)$ | $\psi_{i}(y, z)$ | $\sigma_{i}(x)$ |
| 1 | $G_{-v}$ | $P_{V}(x)$ | $P_{V}(x) \wedge \neg \exists y(R(x, y))$ | $R(y, z)$ | $q$ |
| 2 | $G_{/ e}$ | $P_{E}(x)$ | $\neg \operatorname{Left}(x, y)$ | $\psi_{\text {Contract }}(x, y, z)$ | $v$ |
| 3 | $G_{-e}$ | $P_{E}(x)$ | $y \neq x$ | $R(y, z) \wedge z \neq x$ | 1 |

## 5. Main result

We now can state and prove our main result.
Theorem 5.1. Let the triple ( $\mathcal{T}$, rec, $\varphi_{\text {ord }}$ ) be SOL-feasible defining a graph polynomial $P$. Then there exists a SOL-polynomial expression $S$ such that for every $\bar{G} \models \varphi_{\text {ord }}$, and for every $z, P(\Gamma(\bar{G}))=e(S, \bar{G}, z)$.
The following lemma, schematically represented by Figure 2 will be useful for the proof of the theorem:

Lemma 5.2. Let $\Phi_{1}=\left\langle\phi_{1}, \psi_{1}\right\rangle, \Phi_{2}=\left\langle\phi_{2}, \psi_{2}\right\rangle$ be translation schemes on graphs. Let $G_{1}=\Phi_{1}(G), G_{2}=\Phi_{2}\left(G_{1}\right)$, where $G, G_{1}, G_{2}$ are graphs over the same vocabulary. Then there exists a translation scheme $\Phi_{3}=\Phi_{1}^{\sharp}\left(\Phi_{2}\right)=\left\langle\Phi_{1}^{\sharp}\left(\phi_{2}\right), \Phi_{1}^{\sharp}\left(\psi_{2}\right)\right\rangle$ such that $G_{2}=\Phi_{3}(G)$.

## Proof:

By definition of $\Phi_{2}$, we have

$$
\begin{aligned}
& A\left(G_{2}\right)=A^{\Phi_{2}^{\star}\left[G_{1}\right]}=\left\{a \in A\left(G_{1}\right): G_{1} \models \phi_{2}(a)\right\} \\
& R\left(G_{2}\right)=R^{\Phi_{2}^{\star}\left[G_{1}\right]}=\left\{\vec{a} \in A\left(G_{2}\right)^{2}: G_{1} \models \psi_{2}(\vec{a})\right\}
\end{aligned}
$$

By the fundamental property (Theorem [2.8), because $G_{1}=\Phi_{1}^{\star}[G]$, we have

$$
\begin{aligned}
\forall a \in A\left(G_{1}\right)\left(G_{1} \models \phi_{2}(a) \leftrightarrow G \models\left[\Phi_{1}^{\sharp}\left(\phi_{2}\right)\right](a)\right) \\
\forall \vec{a} \in A\left(G_{1}\right)^{2}\left(G_{1} \models \psi_{2}(\vec{a}) \leftrightarrow G \models\left[\Phi_{1}^{\sharp}\left(\psi_{2}\right)\right](\vec{a})\right)
\end{aligned}
$$

This is equivalent to

$$
\begin{gathered}
\forall a \in A(G)\left(G_{1} \models\left(\phi_{2}(a) \wedge a \in A\left(G_{1}\right)\right) \leftrightarrow G \models\left[\Phi_{1}^{\sharp}\left(\phi_{2}\right)\right](a)\right) \\
\forall \vec{a} \in A(G)^{2}\left(G_{1} \models\left(\psi_{2}(\vec{a}) \wedge \vec{a} \in A\left(G_{1}\right)^{2}\right) \leftrightarrow G \models\left[\Phi_{1}^{\sharp}\left(\psi_{2}\right)\right](\vec{a})\right)
\end{gathered}
$$

because if $A\left(G_{1}\right) \neq A(G)$ then $\Phi_{1}^{\sharp}$ relativizes $\phi_{2}, \psi_{2}$ to accept only $a \in A\left(G_{1}\right)$. Thus we can take $\Phi_{3}=\Phi_{1}^{\sharp}\left(\Phi_{2}\right)=\left\langle\Phi_{1}^{\sharp}\left(\phi_{2}\right), \Phi_{1}^{\sharp}\left(\psi_{2}\right)\right\rangle$
Q.E.D.


Figure 2. Translation scheme composition
Now let us prove Theorem 5.1

## Proof:

The proof is constructive. The formula will simulate the iterative application of the reduction formula on some deconstruction tree $\Upsilon=\Upsilon(\bar{G})$. The recursive definition ( $\mathcal{T}$, rec, $\varphi_{\text {ord }}$ ) is SOL-feasible and therefore is invariant in the deconstruction tree, thus without loss of generality we can take $\Upsilon$ to be some fixed order deconstruction tree with a SOL-feasible order $O$. Note that the actual order of contexts in a branch $b$ is a sub-order $O_{b}$ of $O$. A context $\vec{x} \in A^{m}$ might be omitted from $O_{b}$ because the deconstructions performed along
$b$ prior to the node marked by $\vec{x}$ might have deleted an element of $\vec{x}$. This would make it impossible to use $\vec{x}$ as a context of any deconstruction.

The SOL-polynomial expression we define, $S$, is a sum of the valuations of all the branches of $\Upsilon$. Each branch $b$ is uniquely defined by the sequence of deconstructions ( $T_{i^{-}}$ s) performed along the branch. We define the vector of marks, $\vec{U}=\left(U_{1}, \ldots, U_{l}\right)$, which mark each context $\vec{x}$ according to the deconstruction performed at the node of $\Upsilon$ marked by $\vec{x}$. Note that not all the contexts are covered by $U_{i}$-s. Only the contexts that were not omitted from $O_{b}$ will be covered, as only at the nodes marked by these contexts a deconstruction was performed. We mark the rest of the contexts by $D$. Note also that the arity of each $U_{i}$, and of $D$, is $m$ - the cardinality of the contexts.

As follows from Definition [12] the valuation of the branch $b$ is the product of the elementary valuations $\sigma_{i}(\vec{x})$ applied at each node $n$ marked by the context $\vec{x}$ such that $T_{i}^{\star}$ is applied at $n$, i.e., in our notation,

$$
\prod_{i=1}^{l} \prod_{\vec{x}: U_{i}(\vec{x})} \sigma_{i}(\vec{x})
$$

The SOL-polynomial expression $S$ is now defined as follows:

$$
\begin{equation*}
S=\sum_{\vec{U}, D: \Psi(\vec{U}, D, O)} \prod_{i=1}^{l} \prod_{\vec{x}: U_{i}(\vec{x})} \sigma_{i}(\vec{x}) \tag{17}
\end{equation*}
$$

Where $\Psi$ is

$$
\begin{align*}
& \Psi(\vec{U}, D, O)= \\
& \quad \operatorname{Disjoint}(\vec{U}, D) \wedge \operatorname{Cover}\left(\left(D \cup \bigcup U_{i}\right), A^{m}\right) \wedge \\
& \exists B \exists Q[ \\
& \quad \forall \overrightarrow{x_{0}}\left[\text { first }\left(\overrightarrow{x_{0}}\right) \rightarrow\right.  \tag{18}\\
& \left.\quad \forall u \forall v\left(B\left(\overrightarrow{x_{0}}, u\right) \wedge\left(R(u, v) \leftrightarrow Q\left(\overrightarrow{x_{0}}, u, v\right)\right)\right)\right] \wedge \\
& \forall \overrightarrow{x_{1}} \forall \overrightarrow{x_{2}}\left(\overrightarrow{x_{2}}=\operatorname{next}_{O}\left(\overrightarrow{x_{1}}\right) \rightarrow\right. \\
& \left.\left.\quad \text { ChangeWorldView }\left(\vec{U}, D, B, Q, \overrightarrow{x_{1}}, \overrightarrow{x_{2}}\right)\right)\right]
\end{align*}
$$

The predicate $\operatorname{Disjoint}(\vec{U}, D)$ means that the relations $U_{1}, \ldots, U_{l}, D$ are disjoint, and $\operatorname{Cover}\left(\left(D \cup \bigcup U_{i}\right), A^{m}\right)$, meaning that each element of $A^{m}$ (i.e., each context) is marked either by $D$ or by some $U_{i}$. We use $B \subseteq A^{m+1}$ and $Q \subseteq A^{m+2}$ to encode the world view of the nodes of $\Upsilon$. Below we show that for a node $n$ on the branch $b$ which is marked by the context $\vec{x}, B, Q$ satisfy $A\left(G_{n}\right)=\{v: B(\vec{x}, v)\}$ and $R\left(G_{n}\right)=\{(v, u): Q(\vec{x}, v, u)\}$. If a context $\vec{x}$ is the first context in $O\left(x_{0}\right)$, then no deconstruction has been performed prior to the node marked by $x$. Thus the world view of $x$ should be the original graph $G$. Otherwise, there exists a context $x_{1}$ which is an immediate predecessor of $x$ in $O$. Then the world view of $x$ can be derived from the world view of $x_{1}$, and the connection between these world views is described by the formula ChangeWorldView $\left(\vec{U}, D, B, Q, \overrightarrow{x_{1}}, \vec{x}\right)$

In order to define ChangeWorldView, the following definitions will be used:
For relations $B_{1}, Q_{1}$ such that $\rho\left(B_{1}\right)=1, \rho\left(Q_{1}\right)=2$ we define the translation scheme $\Phi_{B_{1}, Q_{1}}=\left\langle B_{1}, Q_{1}\right\rangle$. For two relations $R_{1}, R_{2}$ of the same arity $\ell$ we overload the equality symbol to denote $R_{1}=R_{2} \Leftrightarrow \forall u_{1} \ldots \forall u_{l}\left(R_{1}\left(u_{1}, \ldots, u_{l}\right) \leftrightarrow R_{2}\left(u_{1}, \ldots, u_{l}\right)\right)$.

$$
\begin{align*}
& \text { ChangeWorldView }\left(\vec{U}, D, B, Q, \overrightarrow{x_{1}}, \overrightarrow{x_{2}}\right)= \\
& \exists B_{1} \exists B_{2} \exists Q_{1} \exists Q_{2}[ \\
& \forall u \forall v\left(\left(B_{1}(u) \leftrightarrow B\left(\overrightarrow{x_{1}}, u\right)\right) \wedge\left(B_{2}(u) \leftrightarrow B\left(\overrightarrow{x_{2}}, u\right)\right) \wedge\right. \\
& \left.\quad\left(Q_{1}(u, v) \leftrightarrow Q\left(\overrightarrow{x_{1}}, u, v\right)\right) \wedge\left(Q_{2}(u, v) \leftrightarrow Q\left(\overrightarrow{x_{2}}, u, v\right)\right)\right) \wedge \\
& \bigwedge_{i=1}^{l}\left(U _ { i } ( \vec { x _ { 1 } } ) \rightarrow \left[\Phi_{B_{1}, Q_{1}}^{\sharp}\left[\varphi_{i}\right]\left(\overrightarrow{x_{1}}\right) \wedge\right.\right.  \tag{19}\\
& B_{2}=A^{\Phi_{3}^{\star}\left[G, \overrightarrow{\left.x_{1}\right]}\right]} \wedge \\
& \left.\left.\quad Q_{2}=R^{\Phi_{3}^{\star}\left[G, \overrightarrow{x_{1}}\right]}\right]\right) \wedge \\
& \left.\left(D\left(\overrightarrow{x_{1}}\right) \rightarrow\left[\left(\exists j \neg B_{1}\left(\overrightarrow{x_{1}}[j]\right)\right) \wedge B_{1}=B_{2} \wedge Q_{1}=Q_{2}\right]\right) \quad\right]
\end{align*}
$$

where $\rho\left(B_{1}\right)=\rho\left(B_{2}\right)=1, \rho\left(Q_{1}\right)=\rho\left(Q_{2}\right)=2$ and $\Phi_{3}=\Phi_{B_{1}, Q_{1}}^{\sharp}\left[T_{i}\right]$.
In accordance with the role of $B$ and $Q$, the first part of the formula defines the relations $B_{i}, Q_{i}$ to comprise the world view of the context $x_{i}$.

The second part of the formula treats the case when the context $\overrightarrow{x_{1}}$ is marked by some $U_{i}$, i.e., the case when the deconstruction $T_{i}^{\star}$ was applied at the node $n_{1}$ marked by $\overrightarrow{x_{1}}$. To make the application of $T_{i}^{\star}$ at $G_{n_{1}}$ possible, $G_{n_{1}} \models \varphi_{i}\left(\overrightarrow{x_{1}}\right)$ should hold. We need to find a formula $\widetilde{\varphi_{i}}$ such that $G \models \widetilde{\varphi_{i}}\left(\overrightarrow{x_{1}}\right)$ iff $G_{n_{1}} \models \varphi_{i}\left(\overrightarrow{x_{1}}\right)$. $B_{1}, Q_{1}$ comprise the world view of $x_{1}, G_{n_{1}}$. Thus by definition of $\Phi_{B_{1}, Q_{1}}=\left\langle B_{1}, Q_{1}\right\rangle$, we have that $\Phi_{B_{1}, Q_{1}}$ is a translation scheme translating $G$ to $G_{n_{1}}$. Then, by Theorem 2.8, $G_{n_{1}} \models \varphi_{i}\left(\overrightarrow{x_{1}}\right)$ iff $G \models \Phi_{B_{1}, Q_{1}}^{\sharp}\left[\varphi_{i}\right]\left(\overrightarrow{x_{1}}\right)$, taking $\widetilde{\varphi_{i}}=\Phi_{B_{1}, Q_{1}}^{\sharp}\left[\varphi_{i}\right]$.

The world view of $x_{2}, G_{n_{2}}$, is the result of application of $T_{i}^{\star}$ to $G_{n_{1}}$, and is comprised of $B_{2}, Q_{2}$. Using Lemma 5.2 applied to $\Phi_{1}=\Phi_{B_{1}, Q_{1}}$ and $\Phi_{2}=T_{i}$, we obtain that $\Phi_{B_{2}, Q_{2}}=\Phi_{B_{1}, Q_{1}}^{\sharp}\left[T_{i}\right]$.

The last part of the formula treats the case when the context $\overrightarrow{x_{1}}$ (or part of it) is already deleted by deconstructions applied to contexts which precede it in $O$. Therefore it should be marked by $D$. No deconstruction is applied to $\overrightarrow{x_{1}}$, thus the world view of $\overrightarrow{x_{1}}$ and its successor, $\overrightarrow{x_{2}}$, are the same.
Q.E.D.

Note that if the coefficients $\sigma_{i}(\bar{G})$ of the recurrence relation are given by short SOLpolynomial expression then the expression $S$ defines a SOL-polynomial.

## 6. Derivations of subset expansion formulas

In this section we shall show how the proof of Theorem 5.1 can be applied to obtain a subset expansion formula for the universal edge elimination polynomial AGM08, and the cover polynomial CG95.

### 6.1. The universal edge elimnation polynomial.

The universal edge elimination polynomial $\xi(G, X, Y, Z)$ is a generalization of both the Matching and the Pott's model, and is recursively defined in AGM08.

The initial conditions are $\xi\left(E_{1}, X, Y, Z\right)=X$ and $\xi(\emptyset, X, Y, Z)=1$.
The recurrence relation is

$$
\begin{equation*}
\xi(G, X, Y, Z)=\xi\left(G_{-e}, X, Y, Z\right)+y \cdot \xi\left(G_{/ e}, X, Y, Z\right)+z \cdot \xi\left(G_{\dagger e}, X, Y, Z\right) \tag{20}
\end{equation*}
$$

$$
\xi\left(G_{1} \oplus G_{2}, X, Y, Z\right)=\xi\left(G_{1}, X, Y, Z\right) \cdot \xi\left(G_{2}, X, Y, Z\right)
$$

To express this defintion within our framework, we define $A^{G}, R, P_{E}(x), P_{V}(x)$, $\psi_{\text {Contract }}(x, y, z)$ and Extracted $(x, y)$ similarly as in Example 3.10

Table 1. Formulas for the recursive definition of $\xi(G, X, Y, Z)$

|  | Action |  | $T_{i}[G, x]$ | $T_{i}[G, x]$ |  |
| :--- | :--- | :--- | :--- | :--- | :--- |
| i | type | $\varphi_{i}(x)$ | $\phi_{i}(y)$ | $\psi_{i}(y, z)$ | $\sigma_{i}(x)$ |
| 1 | $G_{-v}$ | $P_{V}(x)$ | $P_{V}(x) \wedge \neg \exists y(R(x, y))$ | $R(y, z)$ | $X$ |
| 2 | $G_{-e}$ | $P_{E}(x)$ | $y \neq x$ | $R(y, z) \wedge z \neq x$ | 1 |
| 3 | $G_{/ e}$ | $P_{E}(x)$ | $\neg R(y, x)$ | $\psi_{\text {Contract }}(x, y, z)$ | $Y$ |
| 4 | $G_{\dagger e}$ | $P_{E}(x)$ | $\neg \operatorname{Extracted}(x, y)$ | $R(y, z)$ | $Z$ |

Substituting the formulas of Table 1 in the Equations 171819) we get a SOLpolynomial expression. This expression is a sum over the colorings $U_{1}, \ldots, U_{4}$ of $A^{G}$ of addends evaluated $\prod_{i=1}^{4} \prod_{x: U_{i}(x)} \sigma_{i}(x)=X^{\left|U_{1}\right|} \cdot Y^{\left|U_{3}\right|} \cdot Z^{\left|U_{4}\right|}$.

Let $C$ be the set of the connected components of the graph $G_{C}=\left(V(G), U_{3} \cup U_{4}\right)$. In Formula (19), for each context $x_{1}$ satisfying $U_{3}\left(x_{1}\right)$ and $x_{2}=\operatorname{next}_{O}\left(x_{1}\right)$ the contraction action on edge $x_{1}$ leaves one of its end verices. In other words, if $u, v \in V(G)$ and $\left\{\left(u, x_{1}\right),\left(v, x_{1}\right)\right\} \in R$ and $u \prec o v$ then we have $B\left(x_{1}, u\right) \wedge B\left(x_{1}, v\right)$ but $\neg B\left(x_{2}, u\right) \wedge B\left(x_{2}, v\right)$. Thus, action number $3\left(G_{/ e}\right)$ can not remove a whole connected component in $C$ from $\left\{y: B\left(x_{2}, y\right)\right\}$.

Therefore, for each component $c \in C$, actions $1\left(G_{-v}\right)$ or $4\left(G_{\dagger e}\right)$ must be used on the last vertex or edge in $c$ to eliminate whole of $c$ form $\{y: B(x, y)\}$ for some $x$ such that $U_{1}(x)$ or $U_{4}(x)$, respectively.

We divide the components in $C$ into two sets:

$$
\begin{aligned}
& C_{A}=\left\{c \in C: \exists x \in c\left(U_{1}(x)\right)\right\} \\
& C_{B}=\left\{c \in C: \exists x \in c\left(U_{4}(x)\right)\right\}
\end{aligned}
$$

and define the next edge sets:

$$
\begin{aligned}
& A=\left\{x \in E(G): \exists c\left(x \in c \in C_{A}\right)\right\} \\
& B=\left\{x \in E(G): \exists c\left(x \in c \in C_{B}\right)\right\}
\end{aligned}
$$

Recallin the definition of $\operatorname{Touching}(D, S)$ and $\operatorname{LastInComp}(D, S)$ from Section 3.3we get:

$$
\begin{aligned}
U_{1} & =\operatorname{LastIn} \operatorname{Comp}(V, A \cup B) \backslash \operatorname{Touching}(V, B)\} \\
U_{3} & =A \cup B \backslash \operatorname{LastInComp}(B, B) \\
U_{4} & =\operatorname{LastInComp}(B, B)
\end{aligned}
$$

If we rewrite Equation (17) using these terms, we get the next simple SOL-polynomial expression:

$$
\begin{align*}
\xi(G, X, Y, Z)= & \sum_{A, B: A, B \subseteq E \wedge V \operatorname{ertexDisjoint}(A, B)}\left[\prod_{v: v \in(\operatorname{LastInComp}(V, A \cup B) \backslash \operatorname{Touching}(V, B))} X\right] . \\
(21) & {\left[\prod_{e: e \in(A \cup B \backslash \operatorname{LastInComp}(B, B))}\right] \cdot\left[\prod_{e: e \in \operatorname{LastInComp}(B, B)}\right] . } \tag{21}
\end{align*}
$$

where VertexDisjoint $(A, B)=\neg \exists v \exists a \in A \exists b \in B(N(v, a) \wedge N(v, b))$.
From this one can get

$$
\begin{equation*}
\xi(G, X, Y, Z)=\sum_{(A \sqcup B) \subseteq E} X^{k(A \sqcup B)-k_{\operatorname{cov}}(B)} \cdot Y^{|A|+|B|-k_{\operatorname{cov}}(B)} \cdot Z^{k_{\operatorname{cov}}(B)} \tag{22}
\end{equation*}
$$

where by abuse of notation we use $(A \sqcup B) \subseteq E$ for summation over subsets $A, B \subseteq E$, such that the subsets of vertices $V(A)$ and $V(B)$, covered by respective subset of edges, are disjoint: $V(A) \cap V(B)=\emptyset ; k(A)$ denotes the number of spanning connected components
in $(V, A)$, and $k_{\text {cov }}(B)$ denotes the number of covered connected components, i.e. the connected components of $(V(B), B)$.

Note that $k(A \sqcup B)-k_{\text {cov }}(B)=\mid \operatorname{LastInComp}(V, A \cup B) \backslash$ Touching $(V, B)|,|A|+|B|-$ $k_{\text {cov }}(B)=|A \cup B \backslash \operatorname{LastInComp}(B, B)|$ and $k_{\text {cov }}(B)=|\operatorname{LastInComp}(B, B)|$.

Now, Equation 22 is the subset expansion formula for $\xi(G, X, Y, Z)$ presented in AGM08.

### 6.2. The cover polynomial.

The standard definition of the Cover polynomial for a directed graph $D$ is (see CG95):

$$
\begin{aligned}
C(\emptyset) & =1, \\
C\left(E_{n}\right) & =X^{\underline{n}}=X(X-1) \cdots(X-n+1), \\
C(D) & = \begin{cases}C\left(D_{-e}\right)+C\left(D_{/ e}\right) & \text { if } e \text { is a loop, } \\
C\left(D_{-e}\right)+Y \cdot C\left(D_{/ e}\right) & \text { if e is a not a loop }\end{cases}
\end{aligned}
$$

where a contraction of a directed edge $e$ is defined in the following manner:

- If the edge is a loop then it and its adjacent vertex is deleted.
- Otherwise we remove this edge, replace both its adjacent vertices by a single vertex and keep all their adjecent edges which agree with the direction of $e$. I.e., if $e=\langle u, v\rangle$ we remove them both, replace them by a new vertex $w$ and connect all edges $\langle x, w\rangle$ such that $\langle x, u\rangle \in E(D)$ and all edges $\langle w, y\rangle$ such that $\langle v, y\rangle \in E(D)$.
This polynomial is for directed graphs, we express the graph within an extended vocabulary $\tau_{\text {direct-graph(2) }}=\left\langle A, N^{O}, N^{I}\right\rangle$ where the interprestation is: $A=V \cup E$ is the universe of the graph, $N^{O} \subseteq V \times E$ is the adjacency relation for the outbound edges, and $N^{I} \subseteq E \times V$ is the one for inbound edges. The relevant shorthand formulas to identify an element of the universe to be an edge or a vertex respectively, are: $P_{E}(x)=\exists y, z\left[N^{O}(y, x) \wedge N^{I}(x, z)\right], P_{V}(x)=x \in A \wedge \neg P_{E}(x)$.

Other shorthand formulas we use:

$$
\begin{aligned}
\text { DEdgeEnds }(x, u, v) & =N^{O}(u, x) \wedge N^{I}(x, v) \\
\text { DLoop }(x) & =P_{E}(x) \wedge \exists y\left[N^{O}(y, x) \wedge N^{I}(x, y)\right] \\
\psi_{\text {Contract }}^{O}(x, y, z) & =N^{O}(y, z) \\
\psi_{\text {Contract }}^{I}(x, y, z) & =\exists u, v\left[D E d g e E n d s(x, u, v) \wedge\left(N^{I}(y, z) \vee\left(z=v \wedge N^{I}(y, u)\right)\right]\right. \\
\text { DExtracted }(x, y) & =\exists u, v\left[D E d g e E n d s(x, u, v) \wedge y \neq u \wedge \neg N^{O}(u, y) \wedge \neg N^{I}(y, v)\right] \\
\text { DLoopExtracted }(x, y) & =\neg \exists u\left[N^{O}(u, x) \wedge\left(y=u \vee N^{I}(y, u) \vee N^{O}(u, y)\right)\right]
\end{aligned}
$$

Note that $\sigma_{4}(x)$ is a SOL-definable polynomial so our main result validity is supported by the last SOL-definable polynomial property in Proposition 3.11

Substituting the formulas of Table 2in the Equations (17|18|19) we get a $\mathbf{S O L}\left(\tau_{\text {direct-graph(2) }}\right)$ polynomial expression. Note that in this case Formula (19) should be extended to represent both the realtions $N^{I}$ and $N^{O}$. This is peformed trivially by introducing $Q^{I}$ and $Q^{O}$ ternary relations into Formulas (18) and (19), instead the single $Q$ relation.

This $\mathbf{S O L}\left(\tau_{\text {direct-graph(2) }}\right)$-polynomial expression is a sum over the colorings $U_{1}, \ldots, U_{4}$ of $A^{G}$ of addends evaluated $\prod_{i=1}^{4} \prod_{x: U_{i}(x)} \sigma_{i}(x)$.

We use similar arguments as in previous section (6.1). Let $C$ be the connected components of $G_{C}=\left(V(G), U_{2} \cup U_{3}\right)$. To eliminate a component $c \in C$ from $\{y, B(x, y)\}$ for some context $y$ actions $3\left(G_{/ e}\right)$ or $4\left(G_{-v}\right)$ must be used on the last edge or vertex of $c$.

We divide the components in $C$ into two sets:

$$
\begin{aligned}
& C_{P}=\left\{c \in C: \exists x \in c\left(U_{4}(x)\right)\right\} \\
& C_{C}=\left\{c \in C: \exists x \in c\left(U_{3}(x)\right)\right\}
\end{aligned}
$$

TABLE 2. Formulas for the recursive definition of the Cover polynomial

| i | Action <br> type | $\varphi_{i}(x)$ | $\sigma_{i}(x)$ |
| :--- | :--- | :--- | :--- |
| 1 | $D_{-e}$ | $P_{E}(x)$ | 1 |
| 2 | $G_{/ e}$ | $P_{E}(x) \wedge \neg D \operatorname{Loop}(x)$ | 1 |
| 3 | $G_{/ e}$ | $D \operatorname{Loop}(x)$ | $Y$ |
| 4 | $G_{-v}$ | $\neg \exists y\left(P_{E}(y)\right)$ | $X+(-1)^{\mathcal{R}} \sum_{y: \phi_{4}(y)} 1^{\mathcal{R}}$ |


|  | Action | $T_{i}[G, x]$ | $T_{i}[G, x]$ | $T_{i}[G, x]$ |
| :--- | :--- | :--- | :--- | :--- |
| i | type | $\phi_{i}(y)$ | $\psi_{i}^{O}(y, z)$ | $\psi_{i}^{I}(y, z)$ |
| 1 | $D_{-e}$ | $y \neq x$ | $N^{O}(y, z) \wedge z \neq x$ | $N^{I}(y, z) \wedge y \neq x$ |
| 2 | $G_{/ e}$ | DExtracted $(x, y)$ | $\psi_{\text {Contract }}^{O}(x, y, z)$ | $\psi_{C \text { Contract }}^{I}(x, y, z)$ |
| 3 | $G_{/ e}$ | DLoopExtracted $(x, y)$ | $N^{O}(y, z)$ | $N^{I}(y, z)$ |
| 4 | $G_{-v}$ | $y \neq x$ | $\emptyset$ | $\emptyset$ |

Note that if for edge $x_{1}$, such that $U_{2}\left(x_{1}\right) \vee U_{3}\left(x_{1}\right)$, we have $N^{O}\left(u, x_{1}\right) \wedge N^{I}\left(x_{1}, v\right)$, then for $x_{2}=\operatorname{next}_{O}\left(x_{1}\right)\left\{y: B\left(x_{2}, y\right)\right\}$ does not contain any edges into $v$ or edges out of $u$. Therefore, each vertex in $G_{C}=\left(V(G), U_{2} \cup U_{3}\right)$ is adjecent to at most one incoming and one outgoing edge. Thus, each $c \in C$ are either a path or a cycle (a single vertex without a loop is a path or it is a cycle if it has a loop).

Let OnCycle $(v, B)=\exists U\left[U \subseteq B \wedge \exists e\left(U(e) \wedge N^{O}(v, e)\right) \wedge C y c l e(B)\right]$. If we set $B=\{x$ : $\left.U_{2}(x) \wedge U_{3}(x)\right\}$ then:

$$
\begin{align*}
U_{3} & =\left\{e \in E: \exists c \in C_{C}(\{e\}=\operatorname{LastInComp}(E, c))\right\} \\
U_{4} & =\left\{v \in V: \exists c \in C_{P}(\{v\}=\operatorname{LastInComp}(V, c))\right\}  \tag{23}\\
& =\{v \in \operatorname{LastInComp}(V, B) \wedge \operatorname{OnCycle}(v, B)\} \tag{24}
\end{align*}
$$

Note that by Equation 23 we have also $U_{3}=|\{v \in \operatorname{LastInComp}(V, B) \wedge \operatorname{OnCycle}(v, B)\}|$.
Note that in this case we need to take the definitions of $\operatorname{LastInComp}(V, A), \operatorname{Cycle}(B)$ and their subformulas with the relation $N$ replaced by $N^{I}$ or $N^{O}$ in accordance to the context.

Because the context ordering VALORD ${ }_{m}$ permits only orders $O$ such that the vertices come after edges, for any choice of valid coloring $\vec{U}$ there exists a vertex $y$ such that its world view graph $\left\langle B(y, \ldots), Q^{I}(y, \ldots), Q^{O}(y, \ldots)\right\rangle=E_{k}$ for some $k$ and therefore for all $x \succ_{O} y$ we have $U_{4}(x)$ or $D(x)$. For such vertices $x$ with $U_{4}(x), \sigma_{4}(x)=X-k+1$ and in Formola 17 we get $\prod_{x: U_{i}(x)} \sigma_{i}(x)=X \underline{\left|U_{4}\right|}$. Thus, $\prod_{i=1}^{4} \prod_{x: U_{i}(x)} \sigma_{i}(x)=X \underline{\left|U_{4}\right|} \cdot Y^{\left|U_{3}\right|}$.

We denote CyclePathCover $(B)$ to be valid iff for every vertex $v$ no two edges of $B$ emanate or enter $v$ :

$$
\begin{aligned}
\text { CyclePathCover }(B)= & \forall v\left[P_{V}(v) \rightarrow \neg \exists e_{1}, e_{2}\left(e_{1} \neq e_{2} \wedge\right.\right. \\
& {\left.\left.\left[\left(N^{O}\left(v, e_{1}\right) \wedge N^{O}\left(v, e_{2}\right)\right) \vee\left(N^{I}\left(e_{1}, v\right) \wedge N^{I}\left(e_{2}, v\right)\right)\right]\right)\right] }
\end{aligned}
$$

If we rewrite Equation (17) using these terms, we get the next simple $\mathbf{S O L}\left(\tau_{\text {direct-graph(2) }}\right)$ polynomial expression:
$C(D, X, Y)=\sum_{B, L: B \subseteq E \wedge L=\operatorname{LastInComp}(V, B)}\left[(X)_{\{v: v \in L \wedge \neg \operatorname{OnCycle}(v, B)\}}\right] \cdot\left[\prod_{v: v \in L \wedge \text { OnCycle }(v, B)} Y\right]$.
where $(X)_{\{v: v \in L \wedge \neg O n C y c l e(v, B)\}}$ is a falling factorial which by the properties listed in Section 3.5 is expressible by a SOL-polynomial expression over $\mathcal{R}$ which contains $\mathbb{Z}$. Though Formula (25) is not a SOL-polynomial expression in a normal form, by Proposition 3.11, item ( ( $\mathbf{V}$ ), it is still a SOL-polynomial expression.

Formula (25) is equivalent to the one presented in CG95:

$$
\begin{equation*}
C(D, X, Y)=\sum_{i, j} c_{D}(i, j) X^{\underline{i}} Y^{j} \tag{26}
\end{equation*}
$$

where $c_{D}(i, j)$ is the number of ways of covering all the vertices of $D$ with $i$ directed paths and $j$ directed cycles (all disjoint of each other), $X^{\underline{i}}=X(X-1) \cdots(X-i+1)$ and $X^{0}=1 . c_{D}(i, j)$ is taken to be 0 when it is not defined, e.g., when $i<0$ or $j<0$.

## 7. A graph polynomial with no recurrence relation

In NW99 a graph polynomial $U(G, \bar{X}, Y)$ is introduced which generalises the Tutte polynomial, the matching polynomial, and the stability polynomial. $U(G, \bar{X}, Y)$ is defined for a graph $G=(V, E)$ as

$$
\begin{equation*}
U(G, \bar{X}, Y)=\sum_{A \subseteq E} y^{|A|-r(A)} \prod_{i=1}^{|V|} X_{i}^{s(i, A)} \tag{27}
\end{equation*}
$$

where $s(i, A)$ denotes the number of connected components of size $i$ in the spanning $\operatorname{subgraph}(V, A)$, and $r(A)=|V|-k(A)$ is the rank of $(V, A)$.

It is obtained from a graph polynomial $W_{G, w}(\bar{X}, Y)$ for weighted graphs $\langle G, w\rangle$ by setting all the weights equal 1 . For the weighted version there is a recurrence relation reminiscent of the one for the Tutte polynomial, but the edge contraction operation for an edge $e=\left(v_{1}, v_{2}\right)$, wich results in a new vertex $u$, gives $u$ the weight $w(u)=w\left(v_{1}\right)+w\left(v_{2}\right)$. For $W_{G, w}(\bar{X}, Y)$ a subset expansion formula is proven, which is equivalent to Equation (27), when all the weights are set to 1. Equation (27) is used in NW99] as the definition of the polynomial $U(G, \bar{X}, Y)$ for graphs without weights. It is noted that the recursive definition given for $W_{G, w}(\bar{X}, Y)$ does not work, as the edge contraction operation for weighted graphs, when applied to the case where all weights equal 1 , gives a graph with weight for the new vertex resulting from the contraction.

We now show, that the polynomial $U(G, \bar{X}, Y)$ is not an SOL-polynomial, and therefore has no feasible recurrence relation in our sense. To see this we note a simple property of SOL-polynomials.
Definition 7.1. Let $\bar{X}=\left(X_{1}, \ldots, X_{n}\right)$ be a set of variables, and

$$
P(G, \bar{X})=\sum \bar{A} X_{1}^{f_{1}(G, \bar{A})} \cdot \ldots \cdot X_{n}^{f_{n}(G, \bar{A})}
$$

be a subset expansion of a graph polynomial $P$. We say that $P$ is invariant under variable renaming if for all graphs $G$ and for all permutation $\sigma: \mathbb{N} \rightarrow \mathbb{N}$ we have

$$
P\left(G, X_{\sigma(1)}, \ldots, X_{\sigma(n)}\right)=\sum \bar{A} \prod_{i \leq n} X_{\sigma(i)}^{f_{\sigma(i)}(G, \bar{A})}
$$

The following is easy to see:
Proposition 7.2. Assume for

$$
P(G, \bar{X})=\sum \bar{A} X_{1}^{f_{1}(G, \bar{A})} \cdot \ldots \cdot X_{n}^{f_{n}(G, \bar{A})}
$$

that for all $i \leq n$ the exponent $f_{i}(G, \bar{A})$ of $X_{i}$ is not dependent on $i$. Then $P(G, \bar{X})$ is invariant under variable renaming. In particular, SOL-polynomials are invariant under variable renaming.

Proposition 7.3. $U(G, \bar{X}, Y)$ is not invariant under variable renaming.
Proof. Let $E_{n}$ be the graph consisting of $n$ isolated vertices. Then $s(i, A)=|A|$ if $i=1$ and $s(i, A)=0$ if $i \geq 2$. We have

$$
U\left(E_{n}, X_{1}, \ldots, X_{n}, y\right)=\sum_{A \subseteq E} y^{|A|-r(A)} \cdot X_{1}^{|A|}
$$

If we now set $\sigma(n)=n+1$ we get

$$
U\left(E_{n}, X_{2}, \ldots, X_{n+1}, Y\right)=\sum_{A \subseteq E} y^{|A|-r(A)}
$$

## Corollary 7.4.

(i) $U(G, \bar{X}, Y)$ is not a SOL-definable polynomial.
(ii) There is no feasible recursive definition of $U(G, \bar{X}, Y)$.

## 8. Conclusion and open problems

We have shown with Theorem 5.1 how to convert certain recursive definition of graph polynomials, the SOL-feasible recursive definitions, into SOL-definable subset expansion formulas, herewith generalizing many special cases from the literature, in particular the classical results for the Tutte polynomial, the interlace polynomial, and the matching polynomial. We have also explained how Theorem 5.1 was used in AGM08 to find a subset expansion formula for the universal edge elimination polynomial $\xi(G, X, Y, Z)$.

Our framework does not cover all the graph polynomials which appear in the literature. We have not discussed graph polynomials where indeterminates are indexed by elements of the graph. This occurs for example in Sok05. Our framework can be easily adapted to this situation. In this case renaming of the variables has to include also a renaming of the elements of the universe.

The weighted graph polynomial from NW99, however, is not invariant under variable renaming because the integer index of the variables carries a graph theoretic meaning. It is this feature which allows us to show that $U(G, \bar{X}, Y)$ is not SOL-definable.

We have not discussed the possibility of a converse of Theorem 5.1
Problem 1. Find a graph polynomial $P$ which is defined by a SOL-definable subset expansion formula and which is invaraint under variable renaming, but which has no SOL-feasible (linear) recurrence relation.

In our framework of SOL-feasible recursive definitions the recurrence relation is required to be linear. We chose this restriction because we did not want to generalize beyond the natural examples.

Problem 2. Are there combinatorially interesting graph polynomials defined recursively by non-linear recurrence relations?
Problem 3. Is there an analogue to Theorem 5.1 for non-linear recurrence relations?
The choice of Second Order Logic SOL as the base logic for this approach is merely pragmatical. It can be replaced by Fixed Point Logic FPL and extensions of SOL. It seems not to work for Monadic Second Order Logic MSOL. In our proof of Theorem 5.1 we have to quantify over relations which are at least ternary, even if the recursive definition is MSOL-feasible.

Problem 4. Find a sufficent condition which ensures that an MSOL-feasible recursive definition can be converted into an MSOL-definable subset expansion formula.

## References

[ABS04a] R. Arratia, B. Bollobás, and G.B. Sorkin. The interlace polynomial of a graph. Journal of Combinatorial Theory, Series B, 92:199-233, 2004.
[ABS04b] R. Arratia, B. Bollobás, and G.B. Sorkin. A two-variable interlace polynomial. Combinatorica, 24.4:567-584, 2004.
[AGM08] I. Averbouch, B. Godlin, and J.A. Makowsky. An extension of the bivariate chromatic polynomial. submitted, 2008.
[AvdH04] M. Aigner and H. van der Holst. Interlace polynomials. Linear Algebra and Applications, 377:11-30, 2004.
[Big93] N. Biggs. Algebraic Graph Theory, 2nd edition. Cambridge University Press, 1993.
[Bol99] B. Bollobás. Modern Graph Theory. Springer, 1999.
[BR99] B. Bollobás and O. Riordan. A Tutte polynomial for coloured graphs. Combinatorics, Probability and Computing, 8:45-94, 1999.
[CDS55] D.M. Cvetković, M. Doob, and H. Sachs. Spectra of Graphs. Johann Ambrosius Barth, 3rd edition, 1995.
[CG95] F.R.K. Chung and R.L. Graham. On the cover polynomial of a digraph. Journal of Combinatorial Theory, Ser. B, 65(2):273-290, 1995.
[Cou] B. Courcelle. A multivariate interlace polynomial. Preprint, December 2006.
[Die05] R. Diestel. Graph Theory. Graduate Texts in Mathematics. Springer, 3 edition, 2005.
[DKT05] F.M. Dong, K.M. Koh, and K.L. Teo. Chromatic Polynomials and Chromaticity of Graphs. World Scientific, 2005.
[EF95] H. Ebbinghaus and J. Flum. Finite Model Theory. Springer Verlag, 1995.
[EM98] J. Ellis-Monaghan. New results for the Martin polynomial. Journal of Combinatorial Theory, Series B, 74:326-352, 1998.
[God93] C.D. Godsil. Algebraic Combinatorics. Chapman and Hall, 1993.
[GR01] C. Godsil and G. Royle. Algebraic Graph Theory. Graduate Texts in Mathematics. Springer, 2001.
[HL72] C.J. Heilmann and E.H. Lieb. Theory of monomer-dymer systems. Comm. Math. Phys, 28:190-232, 1972.
[KMZ08] T. Kotek, J.A. Makowsky, and B. Zilber. On counting generalized colorings. In CSL'08, volume 5213 of Lecture Notes in Computer Science, pages xx-yy, 2008.
[Kot10] Tomer Kotek. Definability of combinatorial functions. PhD thesis, Technion - Israel Institute of Technology, Haifa, Israel, 2009-2010. In progress.
[LP86] L. Lovasz and M.D. Plummer. Matching Theory, volume 29 of Annals of Discrete Mathematics. North Holland, 1986.
[Mak04] J.A. Makowsky. Algorithmic uses of the Feferman-Vaught theorem. Annals of Pure and Applied Logic, 126.1-3:159-213, 2004.
[Mak06] J.A. Makowsky. From a zoo to a zoology: Descriptive complexity for graph polynomials. In A. Beckmann, U. Berger, B. Löwe, and J.V. Tucker, editors, Logical Approaches to Computational Barriers, Second Conference on Computability in Europe, CiE 2006, Swansea, UK, July 2006, volume 3988 of Lecture Notes in Computer Science, pages 330-341. Springer, 2006.
[Mak07] J.A. Makowsky. From a zoo to a zoology: Towards a general theory of graph polynomials. Theory of Computing Systems, online first:http://dx.doi.org/10.1017/s00224-007-9022-9, July 2007.
[NW99] S.D. Noble and D.J.A. Welsh. A weighted graph polynomial from chromatic invariants of knots. Ann. Inst. Fourier, Grenoble, 49:1057-1087, 1999.
[Sok05] A. Sokal. The multivariate Tutte polynomial (alias Potts model) for graphs and matroids. In Survey in Combinatorics, 2005, volume 327 of London Mathematical Society Lecture Notes, pages 173-226, 2005.
[Tra04] L. Traldi. A subset expansion of the coloured Tutte polynomial. Combinatorics, Probability and Computing, 13:269-275, 2004.

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[^1]:    ${ }^{1} \mathrm{~L}$. Traldi coined this term in Tra04 in the context of the colored Tutte polynomial.

