Model theoretic-characterization of predicate intuitionistic formulas

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Abstract. Notions of asimulation and k-asimulation introduced in [Olkhovikov 2011] are extended onto the level of predicate logic. We then prove that a first-order formula is equivalent to a standard translation of an intuitionistic predicate formula iff it is invariant with respect to k-asimulations for some k, and then that a first-order formula is equivalent to a standard translation of an intuitionistic predicate formula iff it is invariant with respect to asimulations. Finally, it is proved that a first-order formula is equivalent to a standard translation of an intuitionistic predicate formula over a class of intuitionistic models (intuitionistic models with constant domain) iff it is invariant with respect to asimulations between intuitionistic models (intuitionistic models with constant domain).

Van Benthem's well-known modal characterization theorem shows that expressive power of modal propositional logic as a fragment of first-order logic can be described via the notion of bisimulation invariance. Moreover, it is known that modal predicate logic, initially considered as an extension of first-order logic, can also be viewed as its fragment, although somewhat bigger than the fragment induced by propositional modal logic. Expressive power of modal predicate logic, from this vantage point, is described by the notion of world-object bisimulation which appears to be a rather direct combination of bisimulation and partial isomorphism (see, e. g. [Van Benthem 2010, p. 124, Theorem 21]).

Although intuitionistic logic has been treated as a fragment of modal logic for quite a long while, results analogous to propositional and predicate version of Van Benthem's modal characterization theorem were not obtained for it until recently. In [Olkhovikov 2011] we filled this gap for intuitionistic propositional logic. In this paper we introduced the notion of asimulation and its parametrized version, k-asimulation, and showed that they can be used to characterize expressive power of intuitionistic propositional logic in much the same way bisimulation and k-bisimulation are used to characterize modal propositional logic. In this paper we do the same job for intuitionistic predicate logic without identity.

The layout of the paper is as follows. Starting from some notational conventions and preliminary remarks in section 1, we then define a predicate version of k-asimulation and move on to the proof of a 'parametrized' version of model-theoretic characterization of intuitionistic predicate logic in section 2. Then, in section 3, we introduce the predicate version of asimulation and prove the full unparametrized counterpart to Theorem 21 of [Van Benthem 2010]. In section 4 we discuss possibilities of restriction of the latter result to special subclasses of first-order models and the final sections contains some conclusions, and mentions possible directions of further research.

1 Preliminaries

We take $\mathbb{N}$ to be the set of natural numbers *without* 0. A formula is a formula of classical predicate logic with identity whose predicate letters are in a vocabulary $\Sigma = \{ R^2, E^2 \} \cup \{ P^n_m \mid n, m \in \mathbb{N} \}$, where the upper subscript denotes the arity of the letter, so 0-ary predicate letters or propositional letters are not allowed. We refer to formulas with Greek letters distinct from α and β, and to sets of formulas with uppercase Greek letters distinct from Σ and Θ. We refer to variables with letters w, x, y, z, sometimes using primes or subscripts. If φ is a formula, then we associate with it the following finite vocabulary $\Sigma_\varphi \subseteq \Sigma$ such that $\Sigma_\varphi = \{ R^2, E^2 \} \cup \{ P^j_i \mid P^j_i \text{ occurs in } \varphi \}$. More generally, we refer with Θ to an arbitrary subset of Σ such that $R^2, E^2 \in \Theta$. If ψ is a formula and every predicate letter occurring in ψ is in Θ, then we call ψ a Θ-formula.

We refer to sequence $x_1, \ldots, x_n$ of any objects as $\bar{x}_n$. We denote ordered pair of ordered n-tuple $(\bar{x}_n)$ and ordered m-tuple $(\bar{y}_m)$ by $(\bar{x}_n; \bar{y}_m)$. We identify ordered 1-tuple with its only member. We denote the ordered 0-tuple by Λ. If all free variables of a formula φ (set of formulas Γ) are among $\bar{x}_n$, we write $\varphi(\bar{x}_n)$ $(\Gamma(\bar{x}_n))$.

For a binary relation S and any objects s, t we abbreviate the fact that $sSt \wedge tSs$ by $s\hat{S}t$.

We will denote models of classical predicate logic by letters M, N or α, β. We refer to the domain of a model M by $D(M)$. For $n \geq 0$ by an n-ary evaluation Θ-point we mean a sequence $(M, a, \bar{b}_n)$ such that M is a Θ-model and $(a, \bar{b}_n)$ is a sequence of elements of $D(M)$. If $(M, a, \bar{b}_n)$ is an n-ary evaluation point then we say that $\varphi(x, \bar{w}_n)$ is true at $(M, a, \bar{b}_n)$ and write $M, a, \bar{b}_n \models \varphi(x, \bar{w}_n)$ iff for any variable assignment f in M such that $f(x) = a$, $f(w_i) = b_i$ for any $1 \leq i \leq n$ we have $M, f \models \varphi(x, \bar{w}_n)$. It follows from this convention that truth of a formula $\varphi(x, \bar{w}_n)$ at an n-ary evaluation point is to some extent independent of a choice of its free variables.

An intuitionistic formula is a formula of intuitionistic predicate logic without identity. Propositional (i. e. 0-ary predicate) letters are allowed. We refer to intuitionistic formulas with letters i, j, k, possibly with primes or subscripts. Their variables are represented in the same way as in formulas. We assume a standard Kripke semantics for intuitionistic predicate logic where in a given world a predicate letter might be true only for some tuples of objects present in this world.

If x is an individual variable in a first-order language, then by a standard x-translation of intuitionistic formulas into formulas we mean the following map ST defined by induction on the complexity of the corresponding intuitionistic formula. First we assume some map of intuitionistic predicate letters into classical ones which correlates with each n-ary intuitionistic predicate letter P an $(n+1)$-ary classical predicate letter P' distinct from R^2, E^2. We assume that this correlation is surjective, that is, that every predicate letter in Σ distinct from R^2, E^2 is standard translation of an intuitionistic predicate letter. Then our induction goes as follows:

$$
\begin{aligned}
&ST(P(\bar{w}_n), x) = P'(x, \bar{w}_n);\\
&ST(\bot, x) = (x \neq x);\\
&ST(i(\bar{w}_n) \wedge j(\bar{w}_n), x) = ST(i(\bar{w}_n), x) \wedge ST(j(\bar{w}_n), x);\\
&ST(i(\bar{w}_n) \vee j(\bar{w}_n), x) = ST(i(\bar{w}_n), x) \vee ST(j(\bar{w}_n), x);\\
&ST(i(\bar{w}_n) \to j(\bar{w}_n), x) = \forall y(R(x, y) \to (ST(i(\bar{w}_n), y) \to ST(j(\bar{w}_n), y)));\\
&ST(\exists w' i(\bar{w}_n, w'), x) = \exists w'(E(x, w') \wedge ST(i(\bar{w}_n, w'), x));\\
&ST(\forall w' i(\bar{w}_n, w'), x) = \forall y w'((R(x, y) \wedge E(y, w')) \to ST(i(\bar{w}_n, w'), y)).
\end{aligned}
$$

Standard conditions are imposed on the variables $x, y, \bar{w}_n, w'$.

By degree of a formula we mean the greatest number of nested quantifiers occurring in it. A degree of a formula φ is denoted by $r(\varphi)$. Its formal definition by induction on the complexity of φ goes as follows:

$$
\begin{aligned}
&r(\varphi) = 0 && \text{for atomic } \varphi\\
&r(\neg\varphi) = r(\varphi) && \\
&r(\varphi \circ \psi) = max(r(\varphi), r(\psi)) && \text{for } \circ \in \{\wedge, \vee, \to\}\\
&r(Qx\varphi) = r(\varphi) + 1 && \text{for } Q \in \{\forall, \exists\}
\end{aligned}
$$

If $k \in \mathbb{N}$ and $\varphi(x, \bar{w}_n)$ is a Θ-formula such that $r(\varphi) \leq k$, then φ is a $(\Theta, (x, \bar{w}_n), k)$-formula.

2 Characterization of intuitionistic predicate formulas via k-asimulations

We begin with extending our previous notion of k-asimulation to cover the general case of predicate logic.

Definition 1. *Let $(M, a, \bar{b}_n)$, $(N, c, \bar{d}_n)$ be two n-ary evaluation Θ-points. A binary relation*

$$A \subseteq \bigcup_{m \geq 1, l \geq 0} (((D(M)^m \times D(M)^l) \times (D(N)^m \times D(N)^l)) \cup ((D(N)^m \times D(N)^l) \times (D(M)^m \times D(M)^l))),$$

is called $\langle (M, a, \bar{b}_n), (N, c, \bar{d}_n) \rangle_k$-asimulation iff $(a; \bar{b}_n) A (c; \bar{d}_n)$ and for any $\alpha, \beta \in \{M, N\}$, any $(\bar{a'}_m, a'; \bar{b'}_l) \in D(\alpha)^{m+1} \times D(\alpha)^l$, $(\bar{c'}_m, c'; \bar{d'}_l) \in D(\beta)^{m+1} \times D(\beta)^l$, whenever we have $(\bar{a'}_m, a'; \bar{b'}_l) A (\bar{c'}_m, c'; \bar{d'}_l)$, the following conditions hold:

$$\forall P \in \Theta \setminus \{ R^2, E^2 \}(\alpha, a', \bar{b'}_l \models P(x, \bar{w}_l) \Rightarrow \beta, c', \bar{d'}_l \models P(x, \bar{w}_l)) \quad (1)$$

$$(m + l < n + k \wedge c'' \in D(\beta) \wedge c' R^\beta c'') \Rightarrow$$
$$\Rightarrow \exists a'' \in D(\alpha)(a' R^\alpha a'' \wedge (\bar{c'}_m, c', c''; \bar{d'}_l)\hat{A}(\bar{a'}_m, a', a''; \bar{b'}_l)); \quad (2)$$

$$(m + l < n + k \wedge b'' \in D(\alpha) \wedge E^\alpha(a', b'')) \Rightarrow$$
$$\Rightarrow \exists d'' \in D(\beta)(E^\beta(c', d'') \wedge (\bar{a'}_m, a'; \bar{b'}_l, b'')A(\bar{c'}_m, c'; \bar{d'}_l, d'')); \quad (3)$$

$$(m + l + 1 < n + k \wedge c'', d'' \in D(\beta) \wedge c' R^\beta c'' \wedge E^\beta(c'', d'')) \Rightarrow$$
$$\Rightarrow \exists a'', b'' \in D(\alpha)(a' R^\alpha a'' \wedge E^\alpha(a'', b'') \wedge (\bar{a'}_m, a', a''; \bar{b'}_l, b'')A(\bar{c'}_m, c', c''; \bar{d'}_l, d'')). \quad (4)$$

Lemma 1. *Let $\varphi(x, \bar{w}_n) = ST(i(\bar{w}_n), x)$ for some intuitionistic formula $i(\bar{w}_n)$, and let $r(\varphi) = k$. Let $\Sigma_\varphi \subseteq \Theta$, let $(M, t, \bar{u}_s)$, $(N, t', \bar{u'}_s)$ be two s-ary evaluation Θ-points, and let A be an $\langle (M, t, \bar{u}_s), (N, t', \bar{u'}_s)\rangle_p$-asimulation. Then*

$$\forall \alpha, \beta \in \{ M, N \} \forall (\bar{a}_m, a; \bar{b}_n) \in (D(\alpha)^{m+1} \times D(\alpha)^n) \forall (\bar{c}_m, c; \bar{d}_n) \in (D(\beta)^{m+1} \times D(\beta)^n)$$
$$(((\bar{a}_m, a; \bar{b}_n)A(\bar{c}_m, c; \bar{d}_n) \wedge m + n + k \leq p + s \wedge \alpha, a, \bar{b}_n \models \varphi(x, \bar{w}_n) \Rightarrow \beta, c, \bar{d}_n \models \varphi(x, \bar{w}_n)).$$

Proof. We proceed by induction on the complexity of i. In what follows we will abbreviate the induction hypothesis by IH.

Basis. Let $i(\bar{w}_n) = P(\bar{w}_n)$. Then $\varphi(x, \bar{w}_n) = P'(x, \bar{w}_n)$ and we reason as follows:

$(\bar{a}_m, a; \bar{b}_n)A(\bar{c}_m, c; \bar{d}_n)$	(premise)	(5)
$\alpha, a, \bar{b}_n \models P'(x, \bar{w}_n)$	(premise)	(6)
$P' \in \Theta \setminus \{ R^2, E^2 \}$	(by $\Sigma_\varphi \subseteq \Sigma'$)	(7)
$\forall Q \in \Theta \setminus \{ R^2, E^2 \}(\alpha, a, \bar{b}_n \models Q(x, \bar{w}_n) \Rightarrow \beta, c, \bar{d}_n \models Q(x, \bar{w}_n))$	(from (5) by (1))	(8)
$\alpha, a, \bar{b}_n \models P'(x, \bar{w}_n) \Rightarrow \beta, b, \bar{d}_n \models P'(x, \bar{w}_n)$	(from (7) and (8))	(9)
$\beta, c, \bar{d}_n \models P'(x, \bar{w}_n)$	(from (6) and (9))	(10)

The case $i = \perp$ is obvious.

Induction step.

Case 1. Let $i(\bar{w}_n) = j(\bar{w}_n) \wedge k(\bar{w}_n)$. Then $\varphi(x, \bar{w}_n) = ST(j(\bar{w}_n), x) \wedge ST(k(\bar{w}_n), x)$ and we reason as follows:

$(\bar{a}_m, a; \bar{b}_n)A(\bar{c}_m, c; \bar{d}_n)$	(premise)	(11)
$\alpha, a, \bar{b}_n \models ST(j(\bar{w}_n), x) \wedge ST(k(\bar{w}_n), x)$	(premise)	(12)
$m + n + r(ST(j(\bar{w}_n), x) \wedge ST(k(\bar{w}_n), x)) \leq p + s$	(premise)	(13)
$r(ST(j(\bar{w}_n), x)) \leq r(ST(j(\bar{w}_n), x) \wedge ST(k(\bar{w}_n), x))$	(by df of r)	(14)
$r(ST(k(\bar{w}_n), x)) \leq r(ST(j(\bar{w}_n), x) \wedge ST(k(\bar{w}_n), x))$	(by df of r)	(15)
$\alpha, a, \bar{b}_n \models ST(j(\bar{w}_n), x)$	(from (12))	(16)
$\alpha, a, \bar{b}_n \models ST(k(\bar{w}_n), x)$	(from (12))	(17)
$m + n + r(ST(j(\bar{w}_n), x)) \leq p + s$	(from (13) and (14))	(18)
$m + n + r(ST(k(\bar{w}_n), x)) \leq p + s$	(from (13) and (15))	(19)
$\beta, c, \bar{d}_n \models ST(j(\bar{w}_n), x)$	(from (11), (16) and (18) by IH)	(20)
$\beta, c, \bar{d}_n \models ST(k(\bar{w}_n), x)$	(from (11), (17) and (19) by IH)	(21)
$\beta, c, \bar{d}_n \models ST(j(\bar{w}_n), x) \wedge ST(k(\bar{w}_n), x)$	(from (20) and (21))	(22)

Case 2. Let $i(\bar{w}_n) = j(\bar{w}_n) \vee k(\bar{w}_n)$. Then $\varphi(x, \bar{w}_n) = ST(j(\bar{w}_n), x) \vee ST(k(\bar{w}_n), x)$ and we have then $\alpha, a, \bar{b}_n \models ST(j(\bar{w}_n), x) \vee ST(k(\bar{w}_n), x)$. Assume, without a loss of generality, that $\alpha, a, \bar{b}_n \models ST(j(\bar{w}_n), x)$. Then we reason as follows:

$\alpha, a, \bar{b}_n \models ST(j(\bar{w}_n), x)$	(premise)	(23)
$(\bar{a}_m, a; \bar{b}_n)A(\bar{c}_m, c; \bar{d}_n)$	(premise)	(24)
$m + n + r(ST(j(\bar{w}_n), x) \vee ST(k(\bar{w}_n), x)) \leq p + s$	(premise)	(25)
$r(ST(j(\bar{w}_n), x)) \leq r(ST(j(\bar{w}_n), x) \vee ST(k(\bar{w}_n), x))$	(by df of r)	(26)
$m + n + r(ST(j(\bar{w}_n), x)) \leq p + s$	(from (25) and (26))	(27)
$\beta, c, \bar{d}_n \models ST(j(\bar{w}_n), x)$	(from (23), (24) and (27) by IH)	(28)
$\beta, c, \bar{d}_n \models ST(j(\bar{w}_n), x) \vee ST(k(\bar{w}_n), x)$	(from (28))	(29)

Case 3. Let $i(\bar{w}_n) = j(\bar{w}_n) \to k(\bar{w}_n)$. Then

$$\varphi(x, \bar{w}_n) = \forall y(R(x, y) \to (ST(j(\bar{w}_n), y) \to ST(k(\bar{w}_n), y))).$$

Let

$$\alpha, a, \bar{b}_n \models \forall y(R(x, y) \to (ST(j(\bar{w}_n), y) \to ST(k(\bar{w}_n), y))),$$

and let

$$\beta, c, \bar{d}_n \models \exists y(R(x, y) \wedge (ST(j(\bar{w}_n), y) \wedge \neg ST(k(\bar{w}_n), y))).$$

This means that we can choose a $c' \in D(\beta)$ such that $cR^{\beta}c'$ and $\beta, c', \bar{d}_n \models ST(j(\bar{w}_n), y) \wedge \neg ST(k(\bar{w}_n), y)$.

We now reason as follows:

$$\beta, c', \bar{d}_n \models ST(j(\bar{w}_n), y) \wedge \neg ST(k(\bar{w}_n), y) \quad \text{(by choice of } c'\text{)} \quad (30)$$
$$c' \in D(\beta) \wedge cR^\beta c' \quad \text{(by choice of } c'\text{)} \quad (31)$$
$$(\bar{a}_m, a; \bar{b}_n)A(\bar{c}_m, c; \bar{d}_n) \quad \text{(premise)} \quad (32)$$
$$m + n + r(\varphi(x, \bar{w}_n)) \leq p + s \quad \text{(premise)} \quad (33)$$
$$r(\varphi(x, \bar{w}_n))) \geq 1 \quad \text{(by df of } r\text{)} \quad (34)$$
$$m + n < p + s \quad \text{(from (33) and (34))} \quad (35)$$
$$\exists a' \in D(\alpha)(aR^\alpha a' \wedge (\bar{c}_m, c, c'; \bar{d}_n)\hat{A}(\bar{a}_m, a, a'; \bar{b}_n)) \quad \text{(from (31), (32) and (35) by(2))} \quad (36)$$

Now choose an a' for which (36) is satisfied; we add the premises following from our choice of a' and continue our reasoning as follows:

$$a' \in D(\alpha) \wedge aR^\alpha a' \quad \text{(by choice of } a'\text{)} \quad (37)$$
$$(\bar{c}_m, c, c'; \bar{d}_n)A(\bar{a}_m, a, a'; \bar{b}_n) \quad \text{(by choice of } a'\text{)} \quad (38)$$
$$(\bar{a}_m, a, a'; \bar{b}_n)A(\bar{c}_m, c, c'; \bar{d}_n) \quad \text{(by choice of } a'\text{)} \quad (39)$$
$$r(ST(j(\bar{w}_n), y)) \leq r(\varphi(x, \bar{w}_n)) - 1 \quad \text{(by df of } r\text{)} \quad (40)$$
$$r(ST(k(\bar{w}_n), y)) \leq r(\varphi(x, \bar{w}_n)) - 1 \quad \text{(by df of } r\text{)} \quad (41)$$
$$m + 1 + n + r(ST(j(\bar{w}_n), y)) \leq p + s \quad \text{(from (33) and (40))} \quad (42)$$
$$m + 1 + n + r(ST(k(\bar{w}_n), y)) \leq p + s \quad \text{(from (33) and (41))} \quad (43)$$
$$\alpha, a', \bar{b}_n \models ST(j(\bar{w}_n), x) \quad \text{(from (30), (38), (42) by IH)} \quad (44)$$
$$\alpha, a', \bar{b}_n \models \neg ST(k(\bar{w}_n), x) \quad \text{(from (30), (39), (43) by IH)} \quad (45)$$
$$\alpha, a', \bar{b}_n \models ST(j(\bar{w}_n), y) \wedge \neg ST(k(\bar{w}_n), y) \quad \text{(from (44), (45))} \quad (46)$$
$$\alpha, a, \bar{b}_n \models \exists y(R(x, y) \wedge (ST(j(\bar{w}_n), y) \wedge \neg ST(k(\bar{w}_n), y))) \quad \text{(from (37) and (46))} \quad (47)$$

The last line contradicts our initial assumption that

$$\alpha, a, \bar{b}_n \models \forall y(R(x, y) \to (ST(j(\bar{w}_n), y) \to ST(k(\bar{w}_n), y))),$$

Case 4. Let $i(\bar{w}_n) = \exists w' j(\bar{w}_n, w')$. Then

$$\varphi(x, \bar{w}_n) = \exists w'(E(x, w') \wedge ST(j(\bar{w}_n, w'), x)).$$

Let $\alpha, a, \bar{b}_n \models \exists w'(E(x, w') \wedge ST(j(\bar{w}_n, w'), x))$. This means that we can choose a $b' \in D(\alpha)$ such that $aE^\alpha b'$ and $\alpha, a, \bar{b}_n, b' \models ST(j(\bar{w}_n, w'), x)$. We now reason as follows:

$$\alpha, a, \bar{b}_n, b' \models ST(j(\bar{w}_n, w'), x) \quad \text{(by choice of } b'\text{)} \tag{48}$$
$$b' \in D(\alpha) \wedge E^{\alpha}(a, b') \quad \text{(by choice of } b'\text{)} \tag{49}$$
$$(\bar{a}_m, a; \bar{b}_n)A(\bar{c}_m, c; \bar{d}_n) \quad \text{(premise)} \tag{50}$$
$$m + n + r(\varphi(x, \bar{w}_n)) \leq p + s \quad \text{(premise)} \tag{51}$$
$$r(\varphi(x, \bar{w}_n)) \geq 1 \quad \text{(by df of } r\text{)} \tag{52}$$
$$m + n < p + s \quad \text{(from (51) and (52))} \tag{53}$$
$$\exists d' \in D(\beta)(E^{\beta}(c, d') \wedge (\bar{a}_m, a; \bar{b}_n, b')A(\bar{c}_m, c; \bar{d}_n, d')) \quad \text{(from (49), (50) and (53) by (3))} \tag{54}$$

Now choose a d' for which (54) is satisfied; we add the premises following from our choice of d' and continue our reasoning as follows:

$$d' \in D(\beta) \wedge E^{\beta}(c, d') \quad \text{(by choice of } d'\text{)} \tag{55}$$
$$(\bar{a}_m, a; \bar{b}_n, b')A(\bar{c}_m, c; \bar{d}_n, d') \quad \text{(by choice of } d'\text{)} \tag{56}$$
$$r(ST(j(\bar{w}_n, w'), x)) = r(\varphi(x, \bar{w}_n)) - 1 \quad \text{(by df of } r\text{)} \tag{57}$$
$$m + n + 1 + r(ST(j(\bar{w}_n, w'), x)) \leq p + s \quad \text{(from (51) and (57))} \tag{58}$$
$$\beta, c, \bar{d}_n, d' \models ST(j(\bar{w}_n, w'), x) \quad \text{(from (48), (56), (58) by IH)} \tag{59}$$
$$\beta, c, \bar{d}_n \models \exists w'(E(x, w') \wedge ST(j(\bar{w}_n, w'), x)) \quad \text{(from (55) and (59))} \tag{60}$$

Case 5. Let $i(\bar{w}_n) = \forall w' j(\bar{w}_n, w')$. Then

$$\varphi(x, \bar{w}_n) = \forall y w'((R(x, y) \wedge E(y, w')) \to ST(j(\bar{w}_n, w'), y)).$$

Let

$$\alpha, a, \bar{b}_n \models \forall y w'((R(x, y) \wedge E(y, w')) \to ST(j(\bar{w}_n, w'), y)),$$

and let

$$\beta, c, \bar{d}_n \models \exists y w'((R(x, y) \wedge E(y, w')) \wedge \neg ST(j(\bar{w}_n, w'), y)).$$

The latter fact means that we can choose some $c', d' \in D(\beta)$ such that $cR^{\beta}c'$, $E^{\beta}(c', d')$, and $\beta, c', \bar{d}_n, d' \models \neg ST(j(\bar{w}_n, w'), y)$. We now reason as follows:

$$\beta, c', \bar{d}_n, d' \models \neg ST(j(\bar{w}_n, w'), y) \quad \text{(by choice of } c', d'\text{)} \tag{61}$$
$$c' \in D(\beta) \wedge cR^{\beta}c' \quad \text{(by choice of } c'\text{)} \tag{62}$$
$$d' \in D(\beta) \wedge E^{\beta}(c', d') \quad \text{(by choice of } c', d'\text{)} \tag{63}$$
$$(\bar{a}_m, a; \bar{b}_n)A(\bar{c}_m, c; \bar{d}_n) \quad \text{(premise)} \tag{64}$$
$$m + n + r(\varphi(x, \bar{w}_n)) \leq p + s \quad \text{(premise)} \tag{65}$$
$$r(\varphi(x, \bar{w}_n))) \geq 2 \quad \text{(by df of } r\text{)} \tag{66}$$
$$m + n + 1 < p + s \quad \text{(from (65) and (66))} \tag{67}$$

$$\exists a'b' \in D(aR^{\alpha}a' \wedge E^{\alpha}(a', b') \wedge (\bar{a}_m, a, a'; \bar{b}_n, b')A(\bar{c}_m, c, c'; \bar{d}_n, d')) \tag{68}$$
$$\text{(from (62), (63), (64) and (67) by(4))}$$

Then choose $a', b' \in D(\alpha)$ for which (68) is satisfied. We add the premises following from our choice of a', b' and continue our reasoning as follows:

$a' \in D(\alpha) \wedge aR^{\alpha}a'$	(by choice of a')	(69)
$b' \in D(\alpha) \wedge E^{\alpha}(a', b')$	(by choice of a', b')	(70)
$(\bar{a}_m, a, a'; \bar{b}_n, b')A(\bar{c}_m, c, c'; \bar{d}_n, d')$	(by choice of a', b')	(71)
$r(\neg ST(j(\bar{w}_n, w'), y)) = r(\varphi(x, \bar{w}_n)) - 2$	(by df of r)	(72)
$m + 1 + n + 1 + r(\neg ST(j(\bar{w}_n, w'), y)) \leq p + s$	(from (65) and (72))	(73)
$\alpha, a', \bar{b}_n, b' \models \neg ST(j(\bar{w}_n, w'), y)$	(from (61), (71), (73) by IH)	(74)
$\alpha, a, \bar{b}_n \models \exists yw'((R(x, y) \wedge E(y, w')) \wedge \neg ST(j(\bar{w}_n, w'), y))$	(from (69), (70) and (74))	(75)

The last line contradicts our initial assumption that

$$\alpha, a, \bar{b}_n \models \forall yw'((R(x, y) \wedge E(y, w')) \rightarrow ST(j(\bar{w}_n, w'), y)).$$

□

Definition 2. *A formula $\varphi(x, \bar{w}_n)$ is invariant with respect to k-asimulations iff for any Θ such that $\Sigma_\varphi \subseteq \Theta$, any two n-ary evaluation Θ-points $(M, a, \bar{b}_n)$ and $(N, c, \bar{d}_n)$, if there exists a $\langle (M, a, \bar{b}_n), (N, c, \bar{d}_n)\rangle_k$-asimulation A and $M, a, \bar{b}_n \models \varphi(x, \bar{w}_n)$, then $N, c, \bar{d}_n \models \varphi(x, \bar{w}_n)$.*

Corollary 1. *If $\varphi(x, \bar{w}_n)$ is a standard x-translation of an intuitionistic formula and $r(\varphi) = k$, then $\varphi(x, \bar{w}_n)$ is invariant with respect to k-asimulations.*

Corollary 1 immediately follows from Lemma 1 setting $\alpha = M$, $\beta = N$, $m = 0$, $p = k$, $t = a$, $\bar{u}_s = \bar{b}_n$, $t' = c$, $\bar{u'}_s = \bar{d}_n$.

Before we state and prove our main result, we need to mention a fact from classical model theory of first-order logic.

Lemma 2. *For any finite Θ and any natural n, k there are, up to logical equivalence, only finitely many $(\Theta, (x, \bar{w}_n), k)$-formulas.*

This fact is proved as Lemma 3.4 in [Ebbinghaus et al. 1984, pp. 189–190].

Definition 3. *Let $\varphi(x, \bar{w}_n)$ be a formula. A conjunction of $(\Sigma_\varphi, (x, \bar{w}_n), k)$-formulas $\Psi(x, \bar{w}_n)$ is called a complete $(\varphi, (x, \bar{w}_n), k)$-conjunction iff (1) every conjunct in $\Psi(x)$ is a standard x-translation of an intuitionistic formula; and (2) there is an n-ary evaluation point $(M, a, \bar{b}_n)$ such that $M, a, \bar{b}_n \models \Psi(x, \bar{w}_n) \wedge \varphi(x, \bar{w}_n)$ and for any $(\Sigma_\varphi, (x, \bar{w}_n), k)$-formula $\psi(x, \bar{w}_n)$, if $\psi(x, \bar{w}_n)$ is a standard x-translation of an intuitionistic formula and $M, a, \bar{b}_n \models \psi(x, \bar{w}_n)$, then $\Psi(x, \bar{w}_n) \models \psi(x, \bar{w}_n)$.*

Lemma 3. *For any formula $\varphi(x, \bar{w}_n)$, any natural k, and any n-ary evaluation point $(M, a, \bar{b}_n)$ such that $M, a, \bar{b}_n \models \varphi(x, \bar{w}_n)$ there is a complete $(\varphi, (x, \bar{w}_n), k)$-conjunction $\Psi(x, \bar{w}_n)$ such that $M, a, \bar{b}_n \models \Psi(x, \bar{w}_n) \wedge \varphi(x, \bar{w}_n)$.*

Proof. Let $\{\,\psi_1(x,\bar{w}_n)\ldots,\psi_n(x,\bar{w}_n),\ldots\}$ be the set of all $(\Sigma_\varphi,(x,\bar{w}_n),k)$-formulas that are standard x-translations of intuitionistic formulas true at $(M,a,\bar{b}_n)$. This set is non-empty since $ST(\bot\to\bot,x)$ will be true at $(M,a,\bar{b}_n)$. Due to Lemma 2, we can choose in this set a non-empty finite subset $\{\,\psi_{i_1}(x,\bar{w}_n)\ldots,\psi_{i_n}(x,\bar{w}_n)\,\}$ such that any formula from the bigger set is logically equivalent to (and hence follows from) a formula in this subset. Therefore, every formula in the bigger set follows from $\psi_{i_1}(x,\bar{w}_n)\wedge\ldots,\wedge\psi_{i_n}(x,\bar{w}_n)$ and we also have $M,a,\bar{b}_n\models\psi_{i_1}(x,\bar{w}_n)\wedge\ldots,\wedge\psi_{i_n}(x,\bar{w}_n)$, therefore, $\psi_{i_1}(x,\bar{w}_n)\wedge\ldots,\wedge\psi_{i_n}(x,\bar{w}_n)$ is a complete $(\varphi,(x,\bar{w}_n),k)$-conjunction. □

Lemma 4. *For any formula $\varphi(x,\bar{w}_n)$ and any natural k there are, up to logical equivalence, only finitely many complete $(\varphi,(x,\bar{w}_n),k)$-conjunctions.*

Proof. It suffices to observe that for any formula $\varphi(x,\bar{w}_n)$ and any natural k, a complete $(\varphi,(x,\bar{w}_n),k)$-conjunction is a $(\Sigma_\varphi,(x,\bar{w}_n),k)$-formula. Our lemma then follows from Lemma 2. □

In what follows we adopt the following notation for the fact that for any sequence $(x,\bar{w}_n)$ of variables all $(\Sigma_\varphi,(x,\bar{w}_n),k)$-formulas that are standard translations of intuitionistic formulas true at $(M,a,\bar{b}_n)$, are also true at $(N,c,\bar{d}_n)$:

$$(M,a,\bar{b}_n)\leq_{\varphi,n,k}(N,c,\bar{d}_n).$$

Theorem 1. *Let $r(\varphi(x,\bar{w}_n))=k$ and let $\varphi(x,\bar{w}_n)$ be invariant with respect to k-asimulations. Then $\varphi(x,\bar{w}_n)$ is equivalent to a standard x-translation of an intuitionistic formula.*

Proof. We may assume that both $\varphi(x)$ and $\neg\varphi(x)$ are satisfiable, since both $\bot$ and $\top$ are obviously invariant with respect to k-asimulations and we have, for example, the following valid formulas:

$$\bot\leftrightarrow ST(\bot,x),\top\leftrightarrow ST(\bot\to\bot,x).$$

We may also assume that there are two complete $(\varphi,(x,\bar{w}_n),k+2)$-conjunctions $\Psi(x,\bar{w}_n),\Psi'(x,\bar{w}_n)$ such that $\Psi'(x,\bar{w}_n)\models\Psi(x,\bar{w}_n)$, and both formulas $\Psi(x,\bar{w}_n)\wedge\varphi(x,\bar{w}_n)$ and $\Psi'(x,\bar{w}_n)\wedge\neg\varphi(x,\bar{w}_n)$ are satisfiable.

For suppose otherwise. Then take the set of all complete $(\varphi,(x,\bar{w}_n),k+2)$-conjunctions $\Psi(x,\bar{w}_n)$ such that the formula $\Psi(x,\bar{w}_n)\wedge\varphi(x,\bar{w}_n)$ is satisfiable. This set is non-empty, because $\varphi(x,\bar{w}_n)$ is satisfiable, and by Lemma 3, it can be satisfied only together with some complete $(\varphi,(x,\bar{w}_n),k+2)$-conjunction. Now, using Lemma 4, choose in it a finite non-empty subset $\{\,\Psi_{i_1}(x,\bar{w}_n)\ldots,\Psi_{i_n}(x,\bar{w}_n)\,\}$ such that any complete $(\varphi,(x,\bar{w}_n),k+2)$-conjunction is equivalent to an element of this subset. We can show that $\varphi(x,\bar{w}_n)$ is logically equivalent to $\Psi_{i_1}(x,\bar{w}_n)\vee\ldots\vee\Psi_{i_n}(x,\bar{w}_n)$. In fact, if $M,a,\bar{b}_n\models\varphi(x,\bar{w}_n)$ then, by Lemma 3, at least one complete $(\varphi,(x,\bar{w}_n),k+2)$-conjunction is true at $(M,a,\bar{b}_n)$ and therefore, its equivalent in $\{\,\Psi_{i_1}(x,\bar{w}_n)\ldots,\Psi_{i_n}(x,\bar{w}_n)\,\}$ is also true at $(M,a,\bar{b}_n)$, and so, finally we have $M,a,\bar{b}_n\models\Psi_{i_1}(x,\bar{w}_n)\vee\ldots,\vee\Psi_{i_n}(x,\bar{w}_n)$. In the other direction, if $M,a,\bar{b}_n\models\Psi_{i_1}(x,\bar{w}_n)\vee\ldots,\vee\Psi_{i_n}(x,\bar{w}_n)$ then for some $1\leq j\leq n$ we have $M,a,\bar{b}_n\models\Psi_{i_j}(x,\bar{w}_n)$. Then, since $\Psi_{i_j}(x,\bar{w}_n)\models\Psi_{i_j}(x,\bar{w}_n)$ and since by the choice of $\Psi_{i_j}(x,\bar{w}_n)$ the formula $\Psi_{i_j}(x,\bar{w}_n)\wedge\varphi(x,\bar{w}_n)$ is satisfiable, so, by our assumption, the formula $\Psi_{i_j}(x,\bar{w}_n)\wedge\neg\varphi(x,\bar{w}_n)$ must be unsatisfiable, and hence

$\varphi(x, \bar{w}_n)$ must follow from $\Psi_{i_j}(x, \bar{w}_n)$. But in this case we will have $M, a, \bar{b}_n \models \varphi(x, \bar{w}_n)$ as well. So $\varphi(x, \bar{w}_n)$ is logically equivalent to $\Psi_{i_1}(x, \bar{w}_n) \vee \ldots, \vee \Psi_{i_n}(x, \bar{w}_n)$ but the latter formula, being a disjunction of conjunctions of standard x-translations of intuitionistic formulas is itself a standard x-translation of an intuitionistic formula and so we are done.

If, on the other hand, one can take two complete $(\varphi, (x, \bar{w}_n), k+2)$-conjunctions $\Psi(x, \bar{w}_n), \Psi'(x, \bar{w}_n)$ such that $\Psi'(x, \bar{w}_n) \models \Psi(x, \bar{w}_n)$, and formulas $\Psi(x, \bar{w}_n) \wedge \varphi(x, \bar{w}_n)$ and $\Psi'(x, \bar{w}_n) \wedge \neg\varphi(x, \bar{w}_n)$ are satisfiable, we reason as follows.

Take any n-ary evaluation Σ_φ-point $(M, a, \bar{b}_n)$ such that both $M, a, \bar{b}_n \models \Psi(x, \bar{w}_n) \wedge \varphi(x, \bar{w}_n)$ and for any $(\Sigma_\varphi, (x, \bar{w}_n), k)$-formula $\psi(x, \bar{w}_n)$, if $\psi(x, \bar{w}_n)$ is a standard x-translation of an intuitionistic formula and $M, a, \bar{b}_n \models \psi(x, \bar{w}_n)$, then $\Psi(x, \bar{w}_n) \models \psi(x, \bar{w}_n)$. Then take any n-ary evaluation Σ_φ-point $(N, c, \bar{d}_n)$ such that $N, c, \bar{d}_n \models \Psi'(x, \bar{w}_n) \wedge \neg\varphi(x, \bar{w}_n)$.

We can construct a $\langle (M, a, \bar{b}_n), (N, c, \bar{d}_n) \rangle_k$-asimulation and thus obtain a contradiction in the following way.

Let $\alpha, \beta \in \{ M, N \}$ and let $(\bar{a'}_m, a', \bar{b'}_l) \in (D(\alpha)^{m+1} \times D(\alpha)^l)$ and $(\bar{c'}_m, c'; \bar{d'}_l) \in (D(\beta)^{m+1} \times D(\beta)^n)$. Then $(\bar{a'}_m, a'; \bar{b'}_l) A (\bar{c'}_m, c'; \bar{d'}_l)$ iff

$$m + l \leq n + k \wedge (\alpha, a', \bar{b'}_l) \leq_{\varphi, l, n+k+2-m-l} (\beta, c', \bar{d'}_l).$$

By the choice of $\Psi(x, \bar{w}_n), \Psi'(x, \bar{w}_n)$ and the independence of truth at an n-ary evaluation point from the choice of free variables in a formula we obviously have $(a; \bar{b}_n) A (c; \bar{d}_n)$. It remains to verify conditions (1)–(4) of Definition 1.

Verification of (1). Since the degree of any atomic formula is 0, and the above condition implies that $n+k+2-m-l \geq 2$, it is evident that for any $(\bar{a'}_m, a'; \bar{b'}_l) A (\bar{c'}_m, c'; \bar{d'}_l)$ and any predicate letter $P \in \Sigma_\varphi \setminus \{ R^2, E^2 \}$ we have $\alpha, a', \bar{b'}_l \models P(x, \bar{w}_l) \Rightarrow \beta, c', \bar{d'}_l \models P(x, \bar{w}_l)$.

Verification of (2). Assume then that for some $(\bar{a'}_m, a'; \bar{b'}_l) A (\bar{c'}_m, c'; \bar{d'}_l)$ such that $m + l < n + k$ there exists a $c'' \in D(\beta)$ such that $c' R^\beta c''$. In this case we will also have $m + 1 + l \leq n + k$.

Then consider the following two sets:

$$\Gamma = \{ ST(i(\bar{w}_l), x) \mid ST(i(\bar{w}_l), x) \text{ is a } (\Sigma_\varphi, (x, \bar{w}_l), n+k+1-m-l)\text{-formula}, \ \beta, c'', \bar{d'}_l \models ST(i(\bar{w}_l), x) \};$$
$$\Delta = \{ ST(i(\bar{w}_l), x) \mid ST(i(\bar{w}_l), x) \text{ is a } (\Sigma_\varphi, (x, \bar{w}_l), n+k+1-m-l)\text{-formula}, \ \beta, c'', \bar{d'}_l \models \neg ST(i(\bar{w}_l), x) \}.$$

These sets are non-empty, since by our assumption we have $n + k + 1 - m - l \geq 1$. Therefore, as we have $r(ST(\bot, x)) = 0$ and $r(ST(\bot \to \bot, x)) = 1$, we will also have $ST(\bot, x) \in \Delta$ and $ST(\bot \to \bot, x) \in \Gamma$. Then, according to our Lemma 2, there are finite non-empty sets of logical equivalents for both Γ and Δ. Choosing these finite sets, we in fact choose some finite $\{ ST(i_1(\bar{w}_l), x) \ldots ST(i_t(\bar{w}_l), x) \} \subseteq \Gamma$, $\{ ST(j_1(\bar{w}_l), x) \ldots ST(j_u(\bar{w}_l), x) \} \subseteq \Delta$ such that

$$\forall \psi(x, \bar{w}_l) \in \Gamma (ST(i_1(\bar{w}_l), x) \wedge \ldots \wedge ST(i_t(\bar{w}_l), x) \models \psi(x, \bar{w}_l));$$
$$\forall \chi(x, \bar{w}_l) \in \Delta (\chi(x, \bar{w}_l) \models ST(j_1(\bar{w}_l), x) \vee \ldots \vee ST(j_u(\bar{w}_l), x)).$$

But then we obtain that the formula

$$ST((i_1(\bar{w}_l) \wedge \ldots \wedge i_t(\bar{w}_l)) \to (j_1(\bar{w}_l) \vee \ldots \vee j_u(\bar{w}_l)), x)$$

is false at $(\beta, c', \bar{d'}_l)$. In fact, c'' disproves this implication for $(\beta, c', \bar{d'}_l)$. But every formula both in $\{ ST(i_1(\bar{w}_l), x) \ldots ST(i_t(\bar{w}_l), x) \}$ and $\{ ST(j_1(\bar{w}_l), x) \ldots ST(j_u(\bar{w}_l), x) \}$

is, by their choice, a $(\Sigma_\varphi, x, n+k+1-m-l)$-formula, and so standard translation of the implication under consideration must be a $(\Sigma_\varphi, x, n+k+2-m-l)$-formula. Note, further, that by $(\bar{a'}_m, a'; \bar{b'}_l)A(\bar{c'}_m, c'; \bar{d'}_l)$ we must have

$$(\alpha, a', \bar{b'}_l) \leq_{\varphi,l,n+k+2-m-l} (\beta, c', \bar{d'}_l),$$

and therefore this implication must be false at $(\alpha, a', \bar{b'}_l)$ as well. But then take any a'' such that $a'R^\alpha a''$ and $(\alpha, a'', \bar{b'}_l)$ verifies the conjunction in the antecedent of the formula but falsifies its consequent. We must conclude then, by the choice of $\{ST(i_1(\bar{w}_l), x)\dots ST(i_t(\bar{w}_l), x)\}$, that $\alpha, a'', \bar{b'}_l \models \Gamma$ and so, by the definition of A, and given that $m+l+1 \leq n+k$, that $(\bar{c'}_m, c', c''; \bar{d'}_l)A(\bar{a'}_m, a', a''; \bar{b'}_l)$. Since, in addition, $(\alpha, a'', \bar{b'}_l)$ disproves every formula from $\{ST(j_1(\bar{w}_l), x)\dots ST(j_u(\bar{w}_l), x)\}$ then by the choice of this set we must conclude that every $(\Sigma_\varphi, x, n+k+1-m-l)$-formula that is a standard x-translation of an intuitionistic formula false at $(\beta, c', \bar{d'}_l)$ is also false at $(\alpha, a'', \bar{b'}_l)$. But then, again by the definition of A, and given the fact that $m+l+1 \leq n+k$, we must also have $(\bar{a'}_m, a', a''; \bar{b'}_l)A(\bar{c'}_m, c', c''; \bar{d'}_l)$, and so condition (2) holds.

Verification of (3). Assume then that for some $(\bar{a'}_m, a'; \bar{b'}_l)A(\bar{c'}_m, c'; \bar{d'}_l)$ such that $m+l < n+k$ there exists a $b'' \in D(\alpha)$ such that $E^\alpha(a', b'')$. In this case we will also have $m+l+1 \leq n+k$.

Then consider the following set:

$$\Gamma = \{ST(i(\bar{w}_l, w'), x) \mid ST(i, x) \text{ is a } (\Sigma_\varphi, (x, \bar{w}_l, w'), n+k+1-m-l)\text{-formula}, \beta, a', \bar{b'}_l, b'' \models ST(i(\bar{w}_l, w'), x)\}.$$

This set is non-empty, since by our assumption we have $n+k+1-m-l \geq 1$. Therefore, as we have $r(ST(\bot \to \bot, x)) = 1$, we will also have $ST(\bot \to \bot, x) \in \Gamma$. Then, according to our Lemma 2, there is a finite non-empty set of logical equivalents for Γ. Choosing this finite set, we in fact choose some finite $\{ST(i_1(\bar{w}_l, w'), x)\dots ST(i_t(\bar{w}_l, w'), x)\} \subseteq \Gamma$ such that

$$\forall\psi(x, \bar{w}_l, w') \in \Gamma(ST(i_1(\bar{w}_l, w'), x) \wedge \ldots \wedge ST(i_t(\bar{w}_l, w'), x) \models \psi(x, \bar{w}_l, w')).$$

But then we obtain that the formula

$$ST(\exists w'(i_1(\bar{w}_l, w') \wedge \ldots \wedge i_t(\bar{w}_l, w')), x)$$

is true at $(\alpha, a', \bar{b'}_l)$. Moreover, every formula in $\{ST(i_1(\bar{w}_l, w'), x)\dots ST(i_t(\bar{w}_l, w'), x)\}$ is, by their choice, a $(\Sigma_\varphi, x, n+k+1-m-l)$-formula, and so standard translation of the quantified conjunction under consideration must be a $(\Sigma_\varphi, x, n+k+2-m-l)$-formula. Since we have, by $(\bar{a'}_m, a'; \bar{b'}_l)A(\bar{c'}_m, c'; \bar{d'}_l)$, that

$$(\alpha, a', \bar{b'}_l) \leq_{\varphi,l,n+k+2-m-l} (\beta, c', \bar{d'}_l),$$

then the formula in question must be true at $(\beta, c', \bar{d'}_l)$ as well. But then take any d'' such that $E^\beta(c', d'')$ and $(\beta, c', \bar{d'}_l, d'')$ verifies a standard translation of the conjunction after the existential quantifier. We must conclude then, by the choice of $\{ST(i_1(\bar{w}_l, w'), x)\dots ST(i_t(\bar{w}_l, w'), x)\}$, that $\beta, c', \bar{d'}_l, d'' \models \Gamma$ and so, by the definition of A, and given that $m+l+1 \leq n+k$, that $(\bar{a'}_m, a'; \bar{b'}_l, b'')A(\bar{c'}_m, c'; \bar{d'}_l, d'')$.

Verification of (4). Assume then that for some $(\bar{a'}_m, a', \bar{b'}_l)A(\bar{c'}_m, c', \bar{d'}_l)$ such that $m+l+1 < n+k$ there exist some $c'', d'' \in D(\beta)$ such that $c'R^\beta c'' \wedge E^\beta(c'', d'')$, but there are no $a'', b'' \in D(\alpha)$ such that $a' R^\alpha a'' \wedge E^\alpha(a'', b'')$ and $(\bar{a'}_m, a', a''; \bar{b'}_l, b'')A(\bar{c'}_m, c', c''; \bar{d'}_l, d'')$. In this case we will have $m + 1 + l + 1 \leq n + k$.

Then consider the following set:

$$\Delta = \{\, ST(i(\bar{w}_l, w'), x) \mid ST(i(\bar{w}_l, w'), x) \text{ is a } (\Sigma_\varphi, (x, \bar{w}_l, w'), n + k - m - l)\text{-formula}, \beta, c'', \bar{d'}_l, d'' \models \neg ST(i(\bar{w}_l, w'), x) \,\}.$$

This set is non-empty, since by our assumption we have $n + k - m - l \geq 0$. Therefore, as we have $r(ST(\bot, x)) = 0$, we will also have $ST(\bot, x) \in \Delta$. Then, according to our Lemma 2, there is a finite non-empty set of logical equivalents for Δ. Choosing this finite set, we in fact choose some finite $\{\, ST(j_1(\bar{w}_l, w'), x) \dots ST(j_u(\bar{w}_l, w'), x) \,\} \subseteq \Delta$ such that

$$\forall \chi(x, \bar{w}_l, w') \in \Delta(\chi(x, \bar{w}_l, w') \models ST(j_1(\bar{w}_l, w'), x) \vee \ldots \vee ST(j_u(\bar{w}_l, w'), x)).$$

But then we obtain that the formula

$$ST(\forall w'(j_1(\bar{w}_l, w') \vee \ldots \vee j_u(\bar{w}_l, w')), x)$$

is false at $(\beta, c', \bar{d'}_l)$. In fact, c'', d'' jointly disprove standard translation of this universally quantified disjunction for $(\beta, c', \bar{d'}_l)$. Further, every formula in $\{\, ST(j_1(\bar{w}_l), x) \dots ST(j_u(\bar{w}_l), x) \,\}$ is, by their choice, a $(\Sigma_\varphi, x, n + k - m - l)$-formula, and so standard translation of the universally quantified disjunction under consideration must be a $(\Sigma_\varphi, x, n{+}k{+}2{-}m{-}l)$-formula. Since we have, by $(\bar{a'}_m, a'; \bar{b'}_l)A(\bar{c'}_m, c'; \bar{d'}_l)$, that

$$(\alpha, a', \bar{b'}_l) \leq_{\varphi, l, n+k+2-m-l} (\beta, c', \bar{d'}_l),$$

then the formula in question must be false at $(\alpha, a', \bar{b'}_l)$ as well. But then take any a'', b'' for which we have $a' R^\alpha a''$ and $E^\alpha(a'', b'')$ such that $(\alpha, a'', \bar{b'}_l, b'')$ falsifies standard translation of the disjunction after the quantifier. We must conclude, by the choice of $\{\, ST(j_1(\bar{w}_l, w'), x) \dots ST(j_u(\bar{w}_l, w'), x) \,\}$, that every $(\Sigma_\varphi, x, n + k - m - l)$-formula that is a standard x-translation of an intuitionistic formula false at $(\beta, c'', \bar{d'}_l, d'')$ is also false at $(\alpha, a'', \bar{b'}_l, b'')$. But then, again by the definition of A, and given the fact that $m + 1 + l + 1 \leq n + k$, we must also have $(\bar{a'}_m, a', a''; \bar{b'}_l, b'')A(\bar{c'}_m, c', c''; \bar{d'}_l, d'')$, so condition (4) is satisfied. □

Theorem 2. *A formula $\varphi(x, \bar{w}_n)$ is equivalent to a standard x-translation of an intuitionistic formula iff there exists $k \in \mathbb{N}$ such that $\varphi(x, \bar{w}_n)$ is invariant with respect to k-asimulations.*

Proof. Let $\varphi(x, \bar{w}_n)$ be equivalent to $ST(i(\bar{w}_n), x)$. Then by Corollary 1, $ST(i(\bar{w}_n), x)$ is invariant with respect to $r(ST(i(\bar{w}_n), x))$-asimulations, and, therefore, so is $\varphi(x, \bar{w}_n)$. In the other direction, let $\varphi(x, \bar{w}_n)$ be invariant with respect to k-asimulations for some k. If $k \leq r(\varphi(x, \bar{w}_n))$, then every $r(\varphi(x, \bar{w}_n))$-asimulation is a k-asimulation, therefore, $\varphi(x, \bar{w}_n)$ is invariant with respect to $r(\varphi(x, \bar{w}_n))$-asimulations, and so, by Theorem 1, $\varphi(x, \bar{w}_n)$ is logically equivalent to a standard x-translation of an intuitionistic formula. If, on the other hand, $r(\varphi(x, \bar{w}_n)) < k$, then set $l = k - r(\varphi(x, \bar{w}_n))$ and consider a sequence $\bar{y}_l$ of variables not occurring in $\varphi(x, \bar{w}_n)$. Formula $\forall \bar{y}_l \varphi(x, \bar{w}_n)$ is logically equivalent to $\varphi(x, \bar{w}_n)$, hence $\forall \bar{y}_l \varphi(x, \bar{w}_n)$ is invariant with respect to k-asimulations as well. But we have $r(\forall \bar{y}_l \varphi(x, \bar{w}_n)) = k$, so, by Theorem 1, $\forall \bar{y}_l \varphi(x, \bar{w}_n)$ is logically equivalent to a standard x-translation of an intuitionistic formula. Hence $\varphi(x, \bar{w}_n)$ is equivalent to this translation, too. □

3 The main result

We begin by introducing a somewhat simpler, unparametrized version of asimulation:

Definition 4. *Let $(M, a, \bar{b}_n)$, $(N, c, \bar{d}_n)$ be two n-ary evaluation Θ-points. A binary relation*

$$A \subseteq \bigcup_{n \geq 0}(((D(M)\times D(M)^n)\times(D(N)\times D(N)^n))\cup((D(N)\times D(N)^n)\times(D(M)\times D(M)^n))),$$

is called $\langle(M, a, \bar{b}_n),(N, c, \bar{d}_n)\rangle$-asimulation iff $(a;\bar{b}_n)A(c;\bar{d}_n)$ and for any $\alpha,\beta \in \{ M, N \}$, any $(a';\bar{b'}_l) \in D(\alpha)\times D(\alpha)^l$, $(c';\bar{d'}_l) \in D(\beta)\times D(\beta)^l$, whenever we have $(a';\bar{b'}_l)A(c';\bar{d'}_l)$, the following conditions hold:

$$\forall P \in \Theta \setminus \{ R^2, E^2 \}(\alpha, a', \bar{b'}_l \models P(x, \bar{w}_l) \Rightarrow \beta, c', \bar{d'}_l \models P(x, \bar{w}_l)) \tag{76}$$

$$(c'' \in D(\beta) \wedge c' R^\beta c'') \Rightarrow$$
$$\Rightarrow \exists a'' \in D(\alpha)(a' R^\alpha a'' \wedge (c'';\bar{d'}_l)\hat{A}(a'';\bar{b'}_l)); \tag{77}$$

$$(b'' \in D(\alpha) \wedge E^\alpha(a', b'')) \Rightarrow$$
$$\Rightarrow \exists d'' \in D(\beta)(E^\beta(c', d'') \wedge (a';\bar{b'}_l, b'')A(c';\bar{d'}_l, d'')); \tag{78}$$

$$(c'', d'' \in D(\beta) \wedge c' R^\beta c'' \wedge E^\beta(c'', d'')) \Rightarrow$$
$$\Rightarrow \exists a'', b'' \in D(\alpha)(a' R^\alpha a'' \wedge E^\alpha(a'', b'') \wedge (a'';\bar{b'}_l, b'')A(c'';\bar{d'}_l, d'')). \tag{79}$$

Lemma 5. *Let A be an $\langle(M, a, \bar{b}_n),(N, c, \bar{d}_n)\rangle$-asimulation, and let*

$$A' = \{ \langle(\bar{a'}_m, a';\bar{b'}_l),(\bar{c'}_m, c';\bar{d'}_l)\rangle \mid (a';\bar{b'}_l)A(c';\bar{d'}_l) \}.$$

Then A' is an $\langle(M, a, \bar{b}_n),(N, c, \bar{d}_n)\rangle_k$-asimulation for any $k \in \mathbb{N}$.

Proof. We obviously have $(a;\bar{b}_n)A'(c;\bar{d}_n)$, and since for any $\alpha,\beta \in \{ M, N \}$, and any $(\bar{a'}_m, a';\bar{b'}_l)$ in $D(\alpha)^{m+1} \times D(\alpha)^l$, $(\bar{c'}_m, c';\bar{d'}_l)$ in $D(\beta)^{m+1} \times D(\beta)^l$ such that $(\bar{a'}_m, a';\bar{b'}_l)A'(\bar{c'}_m, c';\bar{d'}_l)$ we have $(a';\bar{b'}_l)A(c';\bar{d'}_l)$, condition (1) for A' follows from the fulfilment of condition (76) for A. So it remains to verify that the other three conditions hold for A' for every k.

Condition (2): If $(\bar{a'}_m, a';\bar{b'}_l)A'(\bar{c'}_m, c';\bar{d'}_l)$ then $(a';\bar{b'}_l)A(c';\bar{d'}_l)$, and if, further, $c'' \in D(\beta)$ and $c' R^\beta c''$ then by condition (77) we can choose $a'' \in D(\alpha)$ such that $a' R^\alpha a''$, and $(c'';\bar{d'}_l)\hat{A}(a'';\bar{b'}_l)$. But then, by definition of A' we will also have $(\bar{c'}_m, c', c'';\bar{d'}_l)\hat{A}'(\bar{a'}_m, a', a'';\bar{b'}_l)$.

Condition (3): If $(\bar{a'}_m, a';\bar{b'}_l)A'(\bar{c'}_m, c';\bar{d'}_l)$ then $(a';\bar{b'}_l)A(c';\bar{d'}_l)$, and if, further, $b'' \in D(\alpha)$ and $E^\alpha(a', b'')$ then by condition (78) we can choose $d'' \in D(\beta)$ such that $E^\beta(c', d'')$, and $(a';\bar{b'}_l, b'')A(c';\bar{d'}_l, d'')$. But then, by definition of A' we will also have $(\bar{a'}_m, a';\bar{b'}_l, b'')A'(\bar{c'}_m, c';\bar{d'}_l, d'')$.

Condition (4): If $(\bar{a'}_m, a';\bar{b'}_l)A'(\bar{c'}_m, c';\bar{d'}_l)$ then $(a';\bar{b'}_l)A(c';\bar{d'}_l)$, and if, further, $c'', d'' \in D(\beta)$, $c' R^\beta c''$ and $E^\beta(c'', d'')$ then by condition (79) we can choose $a'', b'' \in D(\alpha)$ such that $a' R^\beta a''$, $E^\alpha(a'', b'')$, and $(a'';\bar{b'}_l, b'')A(c'';\bar{d'}_l, d'')$. But then, by definition of A' we will also have $(\bar{a'}_m, a', a'';\bar{b'}_l, b'')A'(\bar{c'}_m, c', c'';\bar{d'}_l, d'')$. $\square$

Definition 5. *A formula $\varphi(x, \bar{w}_n)$ is invariant with respect to asimulations iff for any Θ such that $\Sigma_\varphi \subseteq \Theta$, any n-ary evaluation Θ-points $(M, a, \bar{b}_n)$ and $(N, c, \bar{d}_n)$, if there exists an $\langle(M, a, \bar{b}_n), (N, c, \bar{d}_n)\rangle$-asimulation A and $M, a, \bar{b}_n \models \varphi(x, \bar{w}_n)$, then $N, c, \bar{d}_n \models \varphi(x, \bar{w}_n)$.*

Corollary 2. *If $\varphi(x, \bar{w}_n)$ is equivalent to a standard x-translation of an intuitionistic formula, then $\varphi(x, \bar{w}_n)$ is invariant with respect to asimulations.*

Proof. Let $\varphi(x, \bar{w}_n)$ be not invariant with respect to asimulations, and let A be an $\langle(M, a, \bar{b}_n), (N, c, \bar{d}_n)\rangle$-asimulation such that $M, a, \bar{b}_n \models \varphi(x, \bar{w}_n)$, but not $N, c, \bar{d}_n \models \varphi(x, \bar{w}_n)$. Let A' be defined as in Lemma 5. Then by this Lemma A' is an $\langle(M, a, \bar{b}_n), (N, c, \bar{d}_n)\rangle_k$-asimulation for any $k \in \mathbb{N}$. Hence, by Theorem 2, $\varphi(x, \bar{w}_n)$ cannot be equivalent to a standard x-translation of an intuitionistic formula. □

To proceed further, we need to introduce some notions and results from classical model theory. For a model M and $\bar{a}_n \in D(M)$ let $[M, \bar{a}_n]$ be the extension of M with $\bar{a}_n$ as new individual constants denoting themselves. It is easy to see that there is a simple relation between truth of a formula at a Θ-evaluation point and truth of its substitution instance in an extension of the above-mentioned kind; namely, for any Θ-model M, every Θ-formula $\varphi(\bar{y}_n, \bar{w}_m)$ and any $\bar{a}_n, \bar{b}_m \in D(M)$ it holds that:

$$[M, \bar{a}_n], \bar{b}_m \models \varphi(\bar{a}_n, \bar{w}_m) \Leftrightarrow M, \bar{a}_n, \bar{b}_m \models \varphi(\bar{y}_n, \bar{w}_m).$$

We will call a theory of M (and write $Th(M)$) the set of all first-order sentences true at M. We will call an n-type of M a set of formulas $\Gamma(\bar{w}_n)$ consistent with $Th(M)$.

Definition 6. *Let M be a Θ-model. M is ω-saturated iff for all $k \in \mathbb{N}$ and for all $\bar{a}_n \in D(M)$, every k-type $\Gamma(\bar{w}_k)$ of $[M, \bar{a}_n]$ is satisfiable in $[M, \bar{a}_n]$.*

Definition of ω-saturation normally requires satisfiability of 1-types only. However, our modification is equivalent to the more familiar version: see e.g. [Doets 1996, Lemma 4.31, p. 73].

It is known that every model can be elementarily extended to an ω-saturated model; in other words, the following lemma holds:

Lemma 6. *Let M be a Θ-model. Then there is an ω-saturated extension N of M such that for all $\bar{a}_n \in D(M)$ and every Θ-formula $\varphi(\bar{w}_n)$:*

$$M, \bar{a}_n \models \varphi(\bar{w}_n) \Leftrightarrow N, \bar{a}_n \models \varphi(\bar{w}_n).$$

The latter lemma is a trivial corollary of e.g. [Chang et al. 1973, Lemma 5.1.14, p. 216].

In what follows we adopt the following notation for the fact that for any x all Θ-formulas that are standard x-translations of intuitionistic formulas true at $(M, a, \bar{b}_n)$, are also true at $(N, c, \bar{d}_n)$:

$$(M, a, \bar{b}_n) \leq_\Theta (N, c, \bar{d}_n).$$

Lemma 7. *Let $\Theta \subseteq \Sigma$, let M, N be ω-saturated Θ-models and let $(M, a, \bar{b}_n) \leq_\Theta (N, c, \bar{d}_n)$. Then relation A such that for any $\alpha, \beta \in \{M, N\}$, any $(a'; \bar{b'}_l) \in D(\alpha) \times D(\alpha)^l$, $(c'; \bar{d'}_l) \in D(\beta) \times D(\beta)^l$*

$$(a'; \bar{b'}_l) A (c'; \bar{d'}_l) \Leftrightarrow (\alpha, a', \bar{b'}_l) \leq_\Theta (\beta, c', \bar{d'}_l)$$

is an $\langle (M, a, \bar{b}_n), (N, c, \bar{d}_n)\rangle$*-asimulation.*[1]

Proof. Throughout this proof every formula mentioned is supposed to be a Θ-formula. It is obvious that $(a; \bar{b}_n)A(c; \bar{d}_n)$, and since for any predicate letter P distinct from R^2, E^2 and variables $x, \bar{w}_n$ formula $P(x, \bar{w}_n)$ is a standard x-translation of an atomic intuitionistic formula, condition (76) is trivially satisfied for A.

To verify *condition* (77), choose any $\alpha, \beta \in \{ M, N \}$, any $(a'; \bar{b'}_l) \in D(\alpha) \times D(\alpha)^l$, $(c'; \bar{d'}_l) \in D(\beta) \times D(\beta)^l$ such that $(\alpha, a', \bar{b'}_l) \leq_\Theta (\beta, c', \bar{d'}_l)$ and choose any $c'' \in D(\beta)$ for which we have $c' R^\beta c''$.

Then choose any variables $x, \bar{w}_n$ and consider the following two sets:

$$\Gamma = \{ i(\bar{w}_l) \mid \beta, c'', \bar{d'}_l \models ST(i(\bar{w}_l), x) \};$$
$$\Delta = \{ i(\bar{w}_l) \mid \beta, c'', \bar{d'}_l \models \neg ST(i(\bar{w}_l), x) \}.$$

We have by the choice of Γ, Δ that for every finite $\Gamma' \subseteq \Gamma$ and $\Delta' \subseteq \Delta$ the formula $ST(\bigwedge(\Gamma') \to \bigvee(\Delta'), x)$ is disproved by c'' for $(\beta, c', \bar{d'}_l)$. So, by our premise that $(\alpha, a', \bar{b'}_l) \leq_\Theta (\beta, c', \bar{d'}_l)$, the standard translation of every such implication must be false at $(\alpha, a', \bar{b'}_l)$ as well. This means that every finite subset of the set

$$\{ R(a', x) \} \cup \{ ST(i(\bar{b'}_l), x) \mid i(\bar{w}_l) \in \Gamma \} \cup \{ \neg ST(i(\bar{b'}_l), x) \mid i(\bar{w}_l) \in \Delta \}$$

is satisfiable at $[\alpha, a', \bar{b'}_l]$. (We set $\Delta' = \{ ST(\bot, x) \}$ if the finite set in question has an empty intersection with Δ and $\Gamma' = \{ ST(\bot \to \bot, x) \}$ if it has an empty intersection with Γ.) Therefore, by compactness of first-order logic, this set is consistent with $Th([\alpha, a', \bar{b'}_l])$ and, by ω-saturation of both M and N it must be satisfied in $[\alpha, a', \bar{b'}_l]$ by some $a'' \in D(\alpha)$. So for any such a'' we will have $a' R^\alpha a''$ and, moreover

$$\alpha, a'', \bar{b'}_l \models \{ ST(i(\bar{w}_l), x) \mid i(\bar{w}_l) \in \Gamma \} \cup \{ \neg ST(i(\bar{w}_l), x) \mid i(\bar{w}_l) \in \Delta \}.$$

Thus, by choice of Γ and Δ plus independence of truth at a pointed model from the choice of free variables in a formula we will have both $(\alpha, a'', \bar{b'}_l) \leq_\Theta (\beta, c'', \bar{d'}_l)$ and $(\beta, c'', \bar{d'}_l) \leq_\Theta (\alpha, a'', \bar{b'}_l)$ and condition (77) is verified.

To verify *condition* (78), choose any $\alpha, \beta \in \{ M, N \}$, any $(a'; \bar{b'}_l) \in D(\alpha) \times D(\alpha)^l$, $(c'; \bar{d'}_l) \in D(\beta) \times D(\beta)^l$ such that $(\alpha, a', \bar{b'}_l) \leq_\Theta (\beta, c', \bar{d'}_l)$ and choose any $b'' \in D(\alpha)$ for which we have $E^\alpha(a', b'')$.

Then choose any variables $x, \bar{w}_n, w'$ and consider the following set:

$$\Gamma = \{ i(\bar{w}_l, w') \mid \alpha, a', \bar{b'}_l, b'' \models ST(i(\bar{w}_l, w'), x) \}.$$

We have by the choice of Γ that for every finite $\Gamma' \subseteq \Gamma$ the formula $ST(\exists w' \bigwedge(\Gamma'), x)$ is verified by b'' for $(\alpha, a', \bar{b'}_l)$. So, by our premise that $(\alpha, a', \bar{b'}_l) \leq_\Theta (\beta, c', \bar{d'}_l)$, the standard translation of every such quantified conjunction must be true at $(\beta, c', \bar{d'}_l)$ as well. This means that every finite subset of the set

$$\{ E(c', w') \} \cup \{ ST(i(\bar{d'}_l, w'), c') \mid i(\bar{w}_l, w') \in \Gamma \}$$

is satisfiable at $[\beta, c', \bar{d'}_l]$. Therefore, by compactness of first-order logic, this set is consistent with $Th([\beta, c', \bar{d'}_l])$ and, by ω-saturation of both M and N, it must be

[1] This definition of A makes sense only when $D(M) \cap D(N) = \varnothing$. However, the latter can always be assumed without a loss of generality.

satisfied in $[\beta, c', \bar{d}'_l]$ by some $d'' \in D(\beta)$. So for any such d'' we will have $E^\beta(c', d'')$ and, moreover

$$\beta, c', \bar{d}'_l, d'' \models \{\, ST(i(\bar{w}_l, w'), x) \mid i(\bar{w}_l, w') \in \Gamma \,\}.$$

Thus, by choice of Γ plus independence of truth at a pointed model from the choice of free variables in a formula we will have $(\alpha, a', \bar{b}'_l, b'') \leq_\Theta (\beta, c', \bar{d}'_l, d'')$ and condition (78) is verified.

To verify *condition* (79), choose any $\alpha, \beta \in \{\, M, N \,\}$, any $(a'; \bar{b}'_l) \in D(\alpha) \times D(\alpha)^l$, $(c'; \bar{d}'_l) \in D(\beta) \times D(\beta)^l$ such that $(\alpha, a', \bar{b}'_l) \leq_\Theta (\beta, c', \bar{d}'_l)$ and choose any $c'', d'' \in D(\beta)$ for which we have $c' R^\beta c''$ and $E^\beta(c'', d'')$.

Then choose any variables $x, \bar{w}_n, w'$ and consider the following set:

$$\Delta = \{\, i(\bar{w}_l, w') \mid \beta, c'', \bar{d}'_l, d'' \models \neg ST(i(\bar{w}_l, w'), x) \,\}.$$

We have by the choice of Δ that for every finite $\Delta' \subseteq \Delta$ the formula $ST(\forall w' \bigvee(\Delta'), x)$ is disproved by c'', d'' for $(\beta, c', \bar{d}'_l)$. So, by our premise that $(a'; \bar{b}'_l) \leq_\Theta (c'; \bar{d}'_l)$, the standard translation of every such quantified disjunction must be false at $(\alpha, a', \bar{b}'_l)$ as well. This means that every finite subset of the set

$$\{\, R(a', x), E(x, w') \,\} \cup \{\, \neg ST(i(\bar{b}'_l, w'), x) \mid i(\bar{w}_l, w') \in \Delta \,\}$$

is satisfiable at $[\alpha, a', \bar{b}'_l]$. Therefore, by compactness of first-order logic, this set is consistent with $Th([\alpha, a', \bar{b}'_l])$ and, by ω-saturation of both M and N, it must be satisfied in $[\alpha, a', \bar{b}'_l]$ by some $a'', b'' \in D(\alpha)$. So for any such a'' and b'' we will have $a' R^\alpha a''$, $E^\alpha(a'', b'')$ and, moreover

$$\alpha, a'', \bar{b}'_l, b'' \models \{\, \neg ST(i(\bar{w}_l, w'), x) \mid i(\bar{w}_l, w') \in \Delta \,\}.$$

Thus, by choice of Δ plus independence of truth at a pointed model from the choice of free variables in a formula we will have $(\alpha, a'', \bar{b}'_l, b'') \leq_\Theta (\beta, c'', \bar{d}'_l, d'')$ and condition (79) is verified. □

We are prepared now to state and prove our main result.

Theorem 3. *Let $\varphi(x, \bar{w}_n)$ be invariant with respect to asimulations. Then $\varphi(x, \bar{w}_n)$ is equivalent to a standard x-translation of an intuitionistic formula.*

Proof. We may assume that $\varphi(x, \bar{w}_n)$ is satisfiable, for $\perp$ is clearly invariant with respect to asimulations and $\perp \leftrightarrow ST(\perp, x)$ is a valid formula. In what follows we will write $IC(\varphi(x, \bar{w}_n))$ for the set of Σ_φ-formulas in variables $x, \bar{w}_n$ that are standard x-translations of intuitionistic formulas following from $\varphi(x, \bar{w}_n))$. For any n-ary evaluation Σ_φ-point $(M, a, \bar{b}_n)$ we will denote the set of Σ_φ-formulas in variables $x, \bar{w}_n$ that are standard x-translations of intuitionistic formulas true at $(M, a, \bar{b}_n)$, or *intuitionistic Σ_φ-theory* of $(M, a, \bar{b}_n)$ by $IT_\varphi(M, a, \bar{b}_n)$. It is obvious that for any n-ary evaluation Σ_φ-points $(M, a, \bar{b}_n)$ and $(N, c, \bar{d}_n)$ we will have $(M, a, \bar{b}_n) \leq_{\Sigma_\varphi} (N, c, \bar{d}_n)$ if and only if $IT_\varphi(M, a, \bar{b}_n) \subseteq IT_\varphi(N, c, \bar{d}_n)$.

Our strategy will be to show that $IC(\varphi(x, \bar{w}_n)) \models \varphi(x, \bar{w}_n)$. Once this is done we will apply compactness of first-order logic and conclude that $\varphi(x, \bar{w}_n)$ is equivalent to a finite conjunction of standard x-translations of intuitionistic formulas and hence to a standard x-translation of the corresponding intuitionistic conjunction.

To show this, take any n-ary evaluation Σ_φ-point $(M, a, \bar{b}_n)$ such that $M, a, \bar{b}_n \models IC(\varphi(x, \bar{w}_n))$. Such a model exists, because $\varphi(x, \bar{w}_n)$ is satisfiable and $IC(\varphi(x, \bar{w}_n))$ will be satisfied in any pointed model satisfying $\varphi(x, \bar{w}_n)$. Then we can also choose an n-ary evaluation Σ_φ-point $(N, c, \bar{d}_n)$ such that $N, c, \bar{d}_n \models \varphi(x, \bar{w}_n)$ and $IT_\varphi(N, c, \bar{d}_n) \subseteq IT_\varphi(M, a, \bar{b}_n)$.

For suppose otherwise. Then for any n-ary evaluation Σ_φ-point $(N, c, \bar{d}_n)$ such that $N, c, \bar{d}_n \models \varphi(x, \bar{w}_n)$ we can choose an intuitionistic formula $i_{(N,c,\bar{d}_n)}(\bar{w}_n)$ such that $ST(i_{(N,c,\bar{d}_n)}(\bar{w}_n), x)$ is a Σ_φ-formula true at $(N, c, \bar{d}_n)$ but not at $(M, a, \bar{b}_n)$. Then consider the set

$$S = \{\, \varphi(x, \bar{w}_n) \,\} \cup \{\, \neg ST(i_{(N,c,\bar{d}_n)}(\bar{w}_n), x) \mid N, c, \bar{d}_n \models \varphi(x, \bar{w}_n) \,\}$$

Let $\{\, \varphi(x, \bar{w}_n), \neg ST(i_1(\bar{w}_n), x) \ldots, \neg ST(i_u(\bar{w}_n), x) \,\}$ be a finite subset of this set. If this set is unsatisfiable, then we must have $\varphi(x) \models ST(i_1(\bar{w}_n), x) \vee \ldots \vee ST(i_u(\bar{w}_n), x)$, but then we will also have $(ST(i_1(\bar{w}_n), x) \vee \ldots \vee ST(i_u(\bar{w}_n), x)) \in IC(\varphi(x, \bar{w}_n)) \subseteq IT_\varphi(M, a, \bar{b}_n)$, and hence $(ST(i_{(N_1,b_1)}, x) \vee \ldots \vee ST(i_{(N_u,b_u)}, x))$ will be true at $(M, a, \bar{b}_n)$. But then at least one of $ST(i_1(\bar{w}_n), x) \ldots, ST(i_u(\bar{w}_n), x)$ must also be true at $(M, a, \bar{b}_n)$, which contradicts the choice of these formulas. Therefore, every finite subset of S is satisfiable, and by compactness S itself is satisfiable as well. But then take any pointed Σ_φ-model $(N', c', \bar{d'}_n)$ of S and this will be a model for which we will have both $N', c', \bar{d'}_n \models ST(i_{(N',c',\bar{d'}_n)}(\bar{w}_n), x)$ by choice of $i_{(N',c',\bar{d'}_n)}$ and $N', c', \bar{d'}_n \models \neg ST(i_{(N',c',\bar{d'}_n)}(\bar{w}_n), x)$ by the satisfaction of S, a contradiction.

Therefore, we will assume in the following that $(M, a, \bar{b}_n)$, $(N, c, \bar{d}_n)$ are n-ary evaluation Σ_φ-points, $M, a, \bar{b}_n \models IC(\varphi(x, \bar{w}_n))$, $N, c, \bar{d}_n \models \varphi(x, \bar{w}_n)$, and $IT_\varphi(N, c, \bar{d}_n) \subseteq IT_\varphi(M, a, \bar{b}_n)$. Then, according to Lemma 6, consider ω-saturated elementary extensions M', N' of M and N, respectively. We have:

$$M, a, \bar{b}_n \models \varphi(x, \bar{w}_n) \Leftrightarrow M', a, \bar{b}_n \models \varphi(x, \bar{w}_n) \tag{80}$$

$$N', c, \bar{d}_n \models \varphi(x, \bar{w}_n) \tag{81}$$

Also since M', N' are elementarily equivalent to M, N we have

$$IT_\varphi(N', c, \bar{d}_n) = IT_\varphi(N, c, \bar{d}_n) \subseteq IT_\varphi(M, a, \bar{b}_n) = IT_\varphi(M', a, \bar{b}_n).$$

But then we have $(N', c, \bar{d}_n) \leq_{\Sigma_\varphi} (M', a, \bar{b}_n)$, and, by ω-saturation of M', N', relation A as defined in Lemma 7 is an $\langle (N', c, \bar{d}_n), (M', a, \bar{b}_n) \rangle$-asimulation. But then by (81) and asimulation invariance of $\varphi(x, \bar{w}_n)$ we get $M', a, \bar{b}_n \models \varphi(x, \bar{w}_n)$, and further, by (80) we conclude that $M, a, \bar{b}_n \models \varphi(x, \bar{w}_n)$. Therefore, $\varphi(x, \bar{w}_n)$ in fact follows from $IC(\varphi(x, \bar{w}_n))$. □

The following theorem is an immediate consequence of Corollary 2 and Theorem 3:

Theorem 4. *A formula $\varphi(x, \bar{w}_n)$ is invariant with respect to asimulations iff it is equivalent to a standard x-translation of an intuitionistic formula.*

4 Criteria for first-order definable classes

Theorem 4 stated above establishes a criterion for the equivalence of first-order formula to a standard translation of intuitionistic formula on arbitrary first-order models.

But one may have a special interest in a proper subclass K of the class of first-order models viewing the models which are not in this subclass as irrelevant, non-intended etc. In this case one may be interested in the criterion for equivalence of a given first-order formula to a standard translation of an intuitionistic predicate formula *over* this particular subclass. It turns out that if some parts of this subclass are first-order axiomatizable then only a slight modification of our general criterion is necessary to solve this problem.

To tighten up on terminology, we introduce the following definitions:

Definition 7. *Let K be a class of models. Then:*

1. $K(\Theta) = \{ M \in K \mid K$ *is a* Θ*-model*$\}$*;*
2. $K(\Theta)$ *is first-order axiomatizable iff there is a set Ax of Θ-sentences, such that a Θ-model M is in K iff $M \models Ax$;*
3. *A set Γ of Θ-formulas is K-satisfiable iff Γ is satisfied by some model in K;*
4. *A Θ-formula φ K-follows from Γ ($\Gamma \models_K \varphi$) iff $\Gamma \cup \{ \varphi \}$ is K-unsatisfiable;*
5. *Θ-formulas φ and ψ are K-equivalent iff $\varphi \models_K \psi$ and $\psi \models_K \varphi$.*

It is clear that for any class K, such that Ax first-order axiomatizes $K(\Theta)$, any set Γ of Θ-formulas and any Θ-formula φ, Γ is K-satisfiable iff $\Gamma \cup Ax$ is satisfiable, and $\Gamma \models_K \varphi$ iff $\Gamma \cup Ax \models \varphi$.

Definition 8. *A formula $\varphi(x, \bar{w}_n)$ is K-invariant with respect to asimulations iff for any Θ such that $\Sigma_\varphi \subseteq \Theta$, any n-ary evaluation Θ-points $(M, a, \bar{b}_n)$ and $(N, c, \bar{d}_n)$, if $M, N \in K$, there exists an $\langle (M, a, \bar{b}_n), (N, c, \bar{d}_n) \rangle$-asimulation A, and $M, a, \bar{b}_n \models \varphi(x, \bar{w}_n)$, then $N, c, \bar{d}_n \models \varphi(x, \bar{w}_n)$.*

Now for the criterion of K-equivalence:

Theorem 5. *Let K be a class of first-order models such that $K(\Theta)$ is first-order axiomatizablefor any finite Θ, and let $\varphi(x, \bar{w}_n)$ be K-invariant with respect to asimulations. Then $\varphi(x, \bar{w}_n)$ is K-equivalent to a standard x-translation of an intuitionistic formula.*

Proof. Let Ax_φ be the set of first-order sentences that axiomatizes $K(\Sigma_\varphi)$. We may assume that $\varphi(x, \bar{w}_n)$ is $K(\Sigma_\varphi)$-satisfiable, otherwise $\varphi(x, \bar{w}_n)$ is K-equivalent to $ST(\perp, x)$ and we are done. In what follows we will write $KC(\varphi(x, \bar{w}_n))$ for the set of Σ_φ-formulas in variables $x, \bar{w}_n$ that are standard x-translations of intuitionistic formulas K-following from $\varphi(x, \bar{w}_n)$.

Our strategy will be to show that $KC(\varphi(x, \bar{w}_n)) \models_K \varphi(x, \bar{w}_n)$. Once this is done we will conclude that

$$Ax_\varphi \cup KC(\varphi(x, \bar{w}_n)) \models \varphi(x, \bar{w}_n).$$

Then we apply compactness of first-order logic and conclude that $\varphi(x, \bar{w}_n)$ is equivalent to a finite conjunction $\psi_1(x, \bar{w}_n) \wedge \ldots \wedge \psi_m(x, \bar{w}_n)$ of formulas from this set. But it follows then that $\varphi(x, \bar{w}_n)$ is K-equivalent to the conjunction of the set $KC(\varphi(x)) \cap \{ \psi_1(x, \bar{w}_n) \ldots, \psi_m(x, \bar{w}_n) \}$. In fact, by our choice of $KC(\varphi(x, \bar{w}_n))$ we have

$$\varphi(x, \bar{w}_n) \models_K \bigwedge (KC(\varphi(x, \bar{w}_n)) \cap \{ \psi_1(x, \bar{w}_n) \ldots, \psi_m(x, \bar{w}_n) \}),$$

And by our choice of $\psi_1(x, \bar{w}_n) \ldots, \psi_m(x, , \bar{w}_n)$ we have

$$Ax_\varphi \cup (KC(\varphi(x, \bar{w}_n)) \cap \{ \psi_1(x, \bar{w}_n) \ldots, \psi_m(x, \bar{w}_n) \}) \models \varphi(x, \bar{w}_n)$$

and hence

$$KC(\varphi(x, \bar{w}_n)) \cap \{ \psi_1(x, \bar{w}_n) \ldots, \psi_m(x, , \bar{w}_n) \} \models_K \varphi(x, \bar{w}_n).$$

To show that $KC(\varphi(x, \bar{w}_n)) \models_K \varphi(x, \bar{w}_n)$, take any n-ary evaluation Σ_φ-point $(M, a, \bar{b}_n)$ such that $M \in K$ and $M, a, \bar{b}_n \models KC(\varphi(x, \bar{w}_n))$. Such a model exists, because $\varphi(x, \bar{w}_n)$ is $K(\Sigma_\varphi)$-satisfiable and $KC(\varphi(x, \bar{w}_n))$ will be K-satisfied in any n-ary evaluation Σ_φ-point satisfying $\varphi(x, \bar{w}_n)$. Then we can also choose an n-ary evaluation Σ_φ-point $(N, c, \bar{d}_n)$ such that $N \in K$ and $N, c, \bar{d}_n \models \varphi(x, \bar{w}_n)$ and $IT_\varphi(N, c, \bar{d}_n) \subseteq IT_\varphi(M, a, \bar{b}_n)$.

For suppose otherwise. Then for any Σ_φ-model $N \in K$ and any n-ary evaluation Σ_φ-point $(N, c, \bar{d}_n)$ such that $N, c, \bar{d}_n \models \varphi(x, \bar{w}_n)$ we can choose an intuitionistic formula $i_{(N,c,\bar{d}_n)}(\bar{w}_n)$ such that $ST(i_{(N,c,\bar{d}_n)}(\bar{w}_n), x)$ is a Σ_φ-formula true at $(N, c, \bar{d}_n)$ but not at $(M, a, \bar{b}_n)$. Then consider the set

$$S = \{ \varphi(x, \bar{w}_n) \} \cup \{ \neg ST(i_{(N,c,\bar{d}_n)}(\bar{w}_n), x) \mid N \in K \wedge N, c, \bar{d}_n \models \varphi(x, \bar{w}_n) \}$$

Let $\{ \varphi(x, \bar{w}_n), \neg ST(i_1(\bar{w}_n), x) \ldots, \neg ST(i_u(\bar{w}_n), x) \}$ be a finite subset of this set. If this set is K-unsatisfiable, then we must have

$$\varphi(x, \bar{w}_n) \models_K ST(i_1(\bar{w}_n), x) \vee \ldots \vee ST(i_u(\bar{w}_n), x),$$

but then we will also have

$$(ST(i_1(\bar{w}_n), x) \vee \ldots \vee ST(i_u(\bar{w}_n), x)) \in KC(\varphi(x, \bar{w}_n)) \subseteq IT_\varphi(M, a, \bar{b}_n),$$

and hence $(ST(i_1(\bar{w}_n), x) \vee \ldots \vee ST(i_u(\bar{w}_n), x))$ will be true at $(M, a, \bar{b}_n)$. But then at least one of $ST(i_1(\bar{w}_n), x) \ldots, ST(i_u(\bar{w}_n), x)$ must also be true at $(M, a, \bar{b}_n)$, which contradicts the choice of these formulas. Therefore, every finite subset of S is K-satisfiable. But then every finite subset of the set $S \cup Ax_\varphi$ is satisfiable as well. By compactness of first-order logic $S \cup Ax_\varphi$ is satisfiable, hence S is satisfiable over K.

But then take any n-ary evaluation Σ_φ-point $(N', c', \bar{d}'_n)$ satisfying S such that $N' \in K$ and this will be an evaluation point for which we will have both $N', c', \bar{d}'_n \models ST(i_{(N',c',\bar{d}'_n)}(\bar{w}_n), x)$ by choice of $i_{(N',c',\bar{d}'_n)}$ and $N', c', \bar{d}'_n \models \neg ST(i_{(N',c',\bar{d}'_n)}(\bar{w}_n), x)$ by the satisfaction of S, a contradiction.

Therefore, for any given n-ary evaluation Σ_φ-point $(M, a, \bar{b}_n)$ satisfying $KC(\varphi(x, \bar{w}_n))$ such that $M \in K$ we can choose an n-ary evaluation Σ_φ-point $(N, c, \bar{d}_n)$ such that $N \in K$, $N, c, \bar{d}_n \models \varphi(x, \bar{w}_n)$ and $IT_\varphi(N, c, \bar{d}_n) \subseteq IT_\varphi(M, a, \bar{b}_n)$. Then, reasoning exactly as in the proof of Theorem 3, we conclude that $M, a, \bar{b}_n \models \varphi(x, \bar{w}_n)$. Therefore, $\varphi(x, \bar{w}_n)$ in fact K-follows from $KC(\varphi(x, \bar{w}_n))$. $\square$

Theorem 6. *Let K be a class of first-order models such that for any finite Θ the class $K(\Theta)$ is first-order axiomatizable. Then a formula $\varphi(x, \bar{w}_n)$ is K-invariant with respect to asimulations iff it is K-equivalent to a standard x-translation of an intuitionistic formula.*

Proof. From left to right our theorem follows from Theorem 5. In the other direction, assume that $\varphi(x, \bar{w}_n)$ is K-equivalent to $ST(i(\bar{w}_n), x)$ and assume that for some Θ such that $\Sigma_\varphi \subseteq \Theta$, some n-ary evaluation Θ-points $(M, a, \bar{b}_n)$ and $(N, c, \bar{d}_n)$ such that $M, N \in K$, and some $\langle (M, a, \bar{b}_n), (N, c, \bar{d}_n) \rangle$-asimulation A we have $M, a, \bar{b}_n \models \varphi(x, \bar{w}_n)$. Then, by Corollary 2 we have $N, c, \bar{d}_n \models ST(i(\bar{w}_n), x)$, but since $ST(i(\bar{w}_n), x)$ is K-equivalent to $\varphi(x, \bar{w}_n)$ and N is in K, we also have $N, c, \bar{d}_n \models \varphi(x, \bar{w}_n)$. Therefore, $\varphi(x, \bar{w}_n)$ is K-invariant with respect to asimulations. $\square$

One obvious instantiation for K would be the class of all *intuitionistic* models which are normally viewed as intended models for intuitionistic predicate logic within the framework of Kripke semantics. A first-order axiomatization for $K(\Theta)$ would be $RT \cup Mon \cup ER \cup Type$, where:

$$RT = \{ \forall y R(y, y), \forall yzw((R(y, z) \wedge R(z, w)) \to R(y, w)) \};$$
$$Mon = \{ \forall yz\bar{w}_n((P(y, \bar{w}_n) \wedge R(y, z)) \to P(z, \bar{w}_n)) \mid P \in \Theta \setminus \{ R \} \};$$
$$ER = \{ \forall x(\exists y E(x, y) \leftrightarrow \neg\exists y E(y, x)), \forall xy(R(x, y) \to \exists zw(E(x, z) \wedge E(y, w))) \};$$
$$Type = \{ \forall y\bar{z}_n(P(y, \bar{z}_n) \to \bigwedge_{i=1}^{n}(E(y, z_i)) \mid P \in \Theta \setminus \{ R \} \}.$$

Another instantiation for K might be, e.g. the class of *intuitionistic models with constant domains*. In this case, if $R^2, E^2 \in \Theta$, a first-order axiomatization for $K(\Theta)$ is given by $RT \cup Mon \cup ER \cup Type \cup \{ CD \}$, where

$$CD = \forall x(\exists y E(y, x) \to \forall y E(y, x)).$$

Thus our Theorem 6 yields, among others, a simple equivalence criterion for these two particular classes of models.

5 Conclusion and further research

Theorems 2, 4, and 6 proved above show that the general idea of asimulation for intuitionistic propositional logic is a faithful analogue of the idea of world-object bisimulation for modal predicate logic in many important respects. However, in the predicate case differences from the corresponding notion of bisimulation are much more conspicuous than in the propositional case. Thus, if we introduced 'asimulation games' corresponding to the propositional version of asimulation defined in [Olkhovikov 2011] (the main difference from propositional case being the absence of conditions (78) and (79)) then, given the strength of condition (77) we would have these games indistinguishable from bisimulation games on the segment beginning from the first move of Duplicator. Every link between worlds established by this player would have to be symmetrical and the asymmetry of asimulation would be important only for the intial pair of worlds.[2] This does not hold in the predicate case. Here, depending on the strategy chosen by Spoiler, the whole game might be played with the asymmetrical links between sequences of world and objects; also asymmetry can be reinstated after the

[2] This asymmetry would also possibly lead to exclusion of some successors of the left world of the link from the domain of the bisimulation game to follow.

players reach the first symmetrical link in the game, and the direction of asymmetry can be switched by moves of the players. All these features show that specific features of intuitionistic logic can be actualized within the setting of quantifiers and predicates only, while on the propositional level one can find but mere rudiments and traces of them.

One interesting further question lying beyond the scope of the present paper is the status of the proofs presented above from the viewpoint of intuitionistic philosophy. It is well-known that ω-saturated models whose existence is guaranteed by Lemma 6 might turn out to be uncountable. Hence our proof might be viewed by a hardcore intuitionist as having no sense at all. As it happens, there is a way to give another proof of our main result that looks more favorable to an intuitionistic eye. This proof uses countable models only and employs the notion of recursive saturation instead of saturation *simpliciter*. However, this variant of proof is also a little bit less clear and more indirect, so we postpone its publication to another occasion.

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