Unifying logics via context-sensitiveness

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Abstract

The goal of this paper is to design a uniform proof-theoretical framework encompassing classical, non monotonic and paraconsistent logic. This framework is obtained by the *control sets* logical device, a syntactical apparatus for controlling derivations. A basic feature of control sets is that of leaving the underlying syntax of a proof system unchanged, while affecting the very combinatorial structure of sequents and proofs. We prove the cut-elimination theorem for a version of *controlled* propositional classical logic, i.e. the sequent calculus for classical propositional logic to which a suitable system of control sets is applied. Finally, we outline the skeleton of a new (positive) account of non-monotonicity and paraconsistency in terms of concurrent processes.

1 Motivation

As is well known, classical consequence relation is both *monotonic* and *explosive*. Monotonicity expresses the fact that whenever a set of premises Γ entails the conclusion A, so does the superset $\Gamma \cup \Delta$ of Γ, that is:

$$\Gamma \vdash A \Rightarrow \Gamma, \Delta \vdash A.$$

Monotonicity of classical logic is represented in sequent calculus by the left weakening rule [21, 33][1]:

$$\frac{\Gamma \vdash \Delta}{\Gamma, B \vdash \Delta}\; weak \vdash.$$

On the other hand, explosiveness corresponds to the principle that contradictions *trivialize* the theory, i.e. any contradiction constitutes the sufficient condition for proving *any* formula. More formally, for any A and B,

$$A \wedge \neg A \vdash B.$$

The explosion principle is proved by means of left-negation and right-weakening rules:

$$\cfrac{\cfrac{\cfrac{\cfrac{}{A \vdash A}\; ax.}{A, \neg A \vdash}\; \neg \vdash}{A \wedge \neg A \vdash}\; \wedge \vdash}{A \wedge \neg A \vdash B}\; \vdash weak.$$

The left-negation rule allows us to clear the right-hand side of the sequent, so that we may introduce, by means of the right-weakening rule, whatever we want. Note that if one retains left-weakening — as it happens, for instance, in Johansson minimal logic [23] — one can 'reduce' explosion by restricting it to negated formulas:

$$\cfrac{\cfrac{\cfrac{\cfrac{}{A \vdash A}\; ax.}{A, \neg A \vdash}\; \neg \vdash}{A \wedge \neg A, B \vdash}\; weak \vdash}{A \wedge \neg A \vdash \neg B}\; \vdash \neg.$$

In the last decades an enormous amount of literature has been devoted to 'non-monotonic' and 'paraconsistent' logics, i.e., logics where monotonicity and explosiveness do no hold, respectively (see, e.g., [27, 30, 26, 4, 32, 9, 14]). These two kinds of logics are typically investigated in a separate fashion (but see [1], [2]), though the question of their relationship emerges quite naturally.

[1]"Weakening" is Curry's translation of Gentzen's "Verdünnung", whereas Kleene employs the term "thinning" which is closer to the original German word [25, 16]. On the admissibility of weakening (and contraction) in linear logic, where these rules are dismissed, see [29] and [15].

Some authors maintain that non-monotonicity is nothing else but paraconsistency in disguise [6, 7], whilst others contend that non-monotonicity is a too wide phenomenon to be judged as parasitic on that of paraconsistency [20]. A discussion of the matter would take us too far afield. For present purposes, however, the remark that needs to be made is that both non-monotonic and paraconsistent logics can be regarded as governing *conflicts* that may arise in expanding our information. On the one hand, paraconsistent logics *circumscribe* the conflict, i.e. the impact of a contradiction on a certain theory; on the other, one might say that non monotonic logics push the contradiction *outside* of logic, inasmuch as such a con tradiction is supposed to be confined to some background knowledge: why $A \vdash B$ *but* $A \wedge C \nvdash B$? Because the information expressed by C is at odds with the (extra-logical) information conveyed by A and B.

Our concern in this paper is to investigate a uniform proof-theoretical framework encompassing non-monotonicity and paraconsistency as two regions of the same logical space. To this end, we appeal to a particular regimentation of *classical* logic. We propose, indeed, a novel conceptualization of the sequent calculus for propositional classical logic able to provide a uniform and *context-sensitive* account of classical, non-monotonic and paraconsistent logic. How are we to achieve this? The technical tool is presented by the idea of *control set*, which was introduced in [11, 17] for handling controlled monotonicity and applied in [12] to logically account for biochemical pathways. Informally, a control set refers to a set of sets of formulas that are supposed to prohibit a certain derivation within a given proof system. In practice a control set $\mathbf{S}$, attached to a sequent $\Gamma \vdash \Delta$, lists a set of informations which overturn the derivability of Δ from Γ. This is indicated by adding a subscript to the turnstile so: $\Gamma \vdash_{\mathbf{S}} \Delta$, whereas the absence of contrary information or prohibitions for deriving Δ from Γ is denoted by $\Gamma \vdash_{\varnothing} \Delta$, where $\varnothing$ refers to the *empty* control set.

From this perspective, the role of control sets may be regarded prima facie as similar to that of the extra-logical or empirical knowledge in Default Logics, originally introduced in [32]. For example, it is known that people suffering from dengue should not take aspirin because it can aggravate bleeding. Default logics augment classical logic by rules like this: *aspirin is recommended for flu-like symptoms, unless they are caused by dengue*. This protocol is represented by the default:

$$\frac{\textsf{flu-like symptoms} : \neg\textsf{dengue}}{\textsf{aspirin}}\,.$$

In our approach, this default can be restated by the following sequent:

$$\Gamma, \textsf{flu-like symptoms} \vdash_{\{\{\textsf{dengue}\}\}} \textsf{aspirin}.$$

The singleton $\{\{dengue\}\}$ is the control set which constraints the soundness of the sequent: from $\mathsf{flu\text{-}like\ symptoms}$ one can derive $\mathsf{aspirin}$, *provided that the context* Γ *does not contain the information* dengue. Thus, when $\mathsf{flu\text{-}like\ symptoms}$ is associated with dengue, the conclusion $\mathsf{aspirin}$ is no longer derivable. Accordingly, the sequent below is unsound:

$$\Gamma, \mathsf{dengue}, \mathsf{flu\text{-}like\ symptoms} \vdash_{\{\{\mathsf{dengue}\}\}} \mathsf{aspirin}.$$

Of course, people who are allergic to aspirin must to avoid it as well. So, if we want to include this constraint, we have to enlarge the control set $\{\{\mathsf{dengue}\}\}$ attached to the sequent. That is:

$$\Gamma, \mathsf{flu\text{-}like\ symptoms} \vdash_{\{\{\mathsf{dengue}\},\{\mathsf{asp\text{-}allergy}\}\}} \mathsf{aspirin}.$$

In this case the inference from $\mathsf{flu\text{-}like\ symptoms}$ to $\mathsf{aspirin}$ is blocked whenever the context Γ contains dengue or $\mathsf{asp\text{-}allergy}$ or both.

However, control sets differ from traditional default constraints in an important regard: instead of the extra-logical information for dealing with exceptions being managed in a *super-classical* calculus, we have the extra-logical information being handled through a control mechanism in a *classical* sequent calculus. The proof-theoretical advantage is twofold. On the one hand, we recover cut-elimination and hence the subformula property. Although there have been other attempts to grasp specific aspects of default reasoning through Gentzen sequent systems [10, 28], our system is, as far as we know, the first with a cut rule and a relative normalization procedure. On the other hand, the modularity of the control sets approach allows us to bridge the gap between non-monotonicity and paraconsistency phenomena, which can be inscribed within the *same* classical sequent calculus.

The paper is organized as follows. In the next section, we set up the notation and present the basic concepts of the control set machinery, which we are going to use. We then introduce, in Section 3, a *controlled* version of classical logic, and we establish that its proofs involving the cut rule can be rewritten as cut-free proofs (the cut-elimination theorem). The key to proving this theorem lies in distinguishing between *proofs* and *paraproofs* (that is, roughly speaking, 'proofs' made by valid inferences in which nevertheless circulate conflicting extra-logical information). Since the normalization procedure for the controlled classical logic does not preserve the property of being a proof, the cut-elimination is shown to depend on the fact that paraproofs in normal form are always proofs. Section 4 delineates a unifying approach to non-monotonicity and paraconsistency. The control sets apparatus allows one to induce both non-monotonic and paraconsistent features in *any* two-sided sequent calculus under consideration. This is the effect of viewing paraconsistency and non-monotonicity through the lens of context-sensitiveness. Moreover, we sketch at the informal level a new (positive) interpretation of non-monotonicity and paraconsistency in terms of concurrent processes. We close, in Section 5, by pointing out some more technical questions which still remains open and themes for further research.

2 Control sets and controlled systems

Let us introduce the basic notions and terminology about control sets. These syntactical objects are indicated with, possibly indexed, boldface capital letters $\mathbf{S}, \mathbf{T}, \ldots$. Standardly, we use capital Greek letters $\Gamma, \Delta, \ldots$ to range over *contexts*, i.e. finite sequences of formulas separated by commas. Moreover, we denote as $\{\Gamma\}$ the set of the formulas occurring in Γ.

Definition 1 (control set) A *control set* is a set of sets of logical formulas, set-theoretically completed under conjunction and disjunction as follows:

- $\{\Gamma, A \wedge B\} \in \mathbf{S} \Rightarrow \{\Gamma, A, B\} \in \mathbf{S}$,
- $\{\Gamma, A \vee B\} \in \mathbf{S} \Rightarrow \{\Gamma, A\} \in \mathbf{S}$ and $\{\Gamma, B\} \in \mathbf{S}$.

We say that $\mathbf{C}_\Gamma$ is the smallest control set $\mathbf{S}$ such that $\{\Gamma\} \in \mathbf{S}$. If Γ is the empty context (i.e. $\{\Gamma\} = \varnothing$), then we pose $\mathbf{C}_\Gamma = \varnothing$.

Remark 2 According to Definition 1, if $A \in \{\Lambda\}$ with $\{\Lambda\} \in \mathbf{C}_\Gamma$, then A is a subformula of some formula in Γ. This observation implies the *finiteness* of the control set $\mathbf{C}_\Gamma$, for any context Γ.

Example 3 We give some examples to illustrate the Definition 1.

$$\mathbf{C}_{p\wedge(q\vee p)} = \{\{p \wedge (q \vee p)\}, \{p, q \vee p\}, \{p, q\}, \{p\}\}$$

$$\mathbf{C}_{p\vee q, r\wedge s} = \{\{p\vee q, r\wedge s\}, \{p\vee q, r, s\}, \{p, r\wedge s\}, \{q, r\wedge s\}, \{p, r, s\}, \{q, r, s\}\}$$

$$\mathbf{C}_{p\vee(q\wedge r)} = \{\{p \vee (q \wedge r)\}, \{p\}, \{q \wedge r\}, \{q, r\}\}.$$

Lemma 4 For each $\{\Sigma'\} \in \mathbf{C}_\Delta$, there is a $\{\Sigma\} \in \mathbf{C}_{\Gamma,\Delta}$ such that $\{\Sigma'\} \subseteq \{\Sigma\}$.

Proof It suffices to assume $\{\Sigma\} = \{\Gamma, \Sigma'\}$ and notice that, by Definition 1, $\{\Gamma, \Sigma'\} \in \mathbf{C}_{\Gamma,\Delta}$.

Example 5 Let $\Delta = p \vee q$ and $\Gamma = p \vee q, r \wedge s$. Thus:

$$\mathbf{C}_{p\vee q} = \{\{p \vee q\}, \{p\}, \{q\}\}$$

$$\mathbf{C}_{p\vee q, r\wedge s} = \{\{p\vee q, r\wedge s\}, \{p\vee q, r, s\}, \{p, r\wedge s\}, \{q, r\wedge s\}, \{p, r, s\}, \{q, r, s\}\}.$$

According to Lemma 4 we have:

$$\{p \vee q\} \subseteq \{p \vee q, r \wedge s\}$$

$$\{p\} \subseteq \{p, r \wedge s\}$$

$$\{q\} \subseteq \{q, r \wedge s\}.$$

Definition 6 (compatibility) A context Γ is *compatible* with a control set $\mathbf{S}$, in symbols $\Gamma \parallel \mathbf{S}$, if, for all $\{\Sigma\} \in \mathbf{C}_\Gamma$ and all $\{\Lambda\} \in \mathbf{S}$, $\{\Lambda\} \nsubseteq \{\Sigma\}$.

Example 7 According to Definition 1 and Example 3, we have:

$$p \vee q, r \wedge s \parallel \{\{p, q, r, s\}\}$$

$$p \vee q, r \wedge s \nparallel \{\{q, r, s\}\}$$

$$p \vee q, r \wedge s \nparallel \{\{p, r, s\}\}.$$

Remark 8 For any context Γ: $\Gamma \parallel \varnothing$.

Theorem 9

1. If $\Gamma, \Delta \parallel \mathbf{S}$ and $\mathbf{T} \subseteq \mathbf{S}$, then $\Delta \parallel \mathbf{T}$.
2. $\Gamma, A \parallel \mathbf{S}$ iff $\Gamma, A, A \parallel \mathbf{S}$.
3. $\Gamma, A \wedge B \parallel \mathbf{S}$ iff $\Gamma, A, B \parallel \mathbf{S}$.
4. $\Gamma, A \vee B \parallel \mathbf{S}$ iff $\Gamma, A \parallel \mathbf{S}$ and $\Gamma, B \parallel \mathbf{S}$.

Proof

1. Suppose by absurd that $\Delta \nparallel \mathbf{T}$, i.e. that there exists two sets of formulas $\{\Sigma'\} \in \mathbf{C}_\Delta$ and $\{\Lambda\} \in \mathbf{T}$ such that $\{\Lambda\} \subseteq \{\Sigma'\}$. By hypothesis, $\{\Lambda\} \in \mathbf{S}$. Moreover, by Lemma 4, there exists a $\{\Sigma\} \in \mathbf{C}_{\Gamma,\Delta}$ such that $\{\Sigma'\} \subseteq \{\Sigma\}$. Thus, it would follow $\{\Lambda\} \subseteq \{\Sigma\}$ against our assumption that $\Gamma, \Delta \parallel \mathbf{S}$.
2. It suffices to observe that $\mathbf{C}_{\Gamma,A} = \mathbf{C}_{\Gamma,A,A}$.

3. ($\Rightarrow$) Notice that, by Definition 1, $\mathbf{C}_{\Gamma,A,B} \subseteq \mathbf{C}_{\Gamma,A\wedge B}$.

   ($\Leftarrow$) Suppose by absurd that $\Gamma, A, B \parallel \mathbf{S}$ whereas $\Gamma, A\wedge B \nparallel \mathbf{S}$, i.e., that there are two sets $\{\Sigma\} \in \mathbf{C}_{\Gamma,A\wedge B}$ and $\{\Lambda\} \in \mathbf{S}$ such that $\{\Lambda\} \subseteq \{\Sigma\}$. Notice that $A \wedge B \in \{\Lambda\}$, otherwise there would be a set $\{\Sigma'\} \in \mathbf{C}_{\Gamma,A,B}$ such that $\{\Lambda\} \subseteq \{\Sigma'\}$ and so $\Gamma, A, B \nparallel \mathbf{S}$. Let us pose $\{\Lambda\} = \{\Lambda', A\wedge B\}$ and $\{\Sigma\} = \{\Sigma', A \wedge B\}$. By Definition 1, if $\{\Lambda', A \wedge B\} \in \mathbf{C}_{\Gamma,A\wedge B}$, then $\{\Lambda', A, B\} \in \mathbf{C}_{\Gamma,A\wedge B}$. Now, $\{\Lambda', A, B\} \subseteq \{\Sigma', A, B\}$ with $\{\Sigma', A, B\} \in \mathbf{C}_{\Gamma,A,B}$ and so $\Gamma, A, B \nparallel \mathbf{S}$.

4. ($\Rightarrow$) Again, notice that by Definition 1 we have both $\mathbf{C}_{\Gamma,A} \subseteq \mathbf{C}_{\Gamma,A\vee B}$ and $\mathbf{C}_{\Gamma,B} \subseteq \mathbf{C}_{\Gamma,A\vee B}$.

   ($\Leftarrow$) Similarly to the proof of part (3), suppose by absurd that $\Gamma, A \parallel \mathbf{S}$, but $\Gamma, A\vee B \nparallel \mathbf{S}$, i.e., there are two sets $\{\Sigma\} \in \mathbf{C}_{\Gamma,A\vee B}$ and $\{\Lambda\} \in \mathbf{S}$ such that $\{\Lambda\} \subseteq \{\Sigma\}$. Now, $A \vee B \in \{\Lambda\}$, otherwise there would be a set $\{\Sigma'\} \in \mathbf{C}_{\Gamma,A}$ (or $\{\Sigma'\} \in \mathbf{C}_{\Gamma,B}$) such that $\{\Lambda\} \subseteq \{\Sigma'\}$ and so $\Gamma, A \nparallel \mathbf{S}$ (or $\Gamma, B \nparallel \mathbf{S}$) against our assumptions. Let us put $\{\Lambda\} = \{\Lambda', A \vee B\}$ and $\{\Sigma\} = \{\Sigma', A \vee B\}$. By Definition 1, $\{\Lambda', A\} \in \mathbf{S}$ and $\{\Sigma', A\} \in \mathbf{C}_{\Gamma,A}$. Since $\{\Lambda', A\} \subseteq \{\Sigma', A\}$ we obtain that $\Gamma, A \nparallel \mathbf{S}$.

Definition 10 (controlled sequent, soundness) A *controlled sequent* is a standard sequent $\Gamma \vdash \Delta$ with attached:

- a control set $\mathbf{S}$,
- a context Σ called *repository*.

Controlled sequents will be expressed as follows:

$$\Sigma \,|\, \Gamma \vdash_{\mathbf{S}} \Delta.$$

When the repository stores no formulas, we will simply omit it and write:

$$\cdot \,|\, \Gamma \vdash_{\mathbf{S}} \Delta.$$

The sequent $\Sigma \,|\, \Gamma \vdash_{\mathbf{S}} \Delta$ is said to be *sound* whenever $\Sigma, \Gamma \parallel \mathbf{S}$.

Let us now provide an informal account of what control sets and controlled calculi are meant to be.

As we said, control sets are concerned with gathering all contexts which are supposed to block a certain derivation. According to Definition 1, 'forbidden contexts' have to be set-theoretically completed in order to constitute a control set and to be thus amenable to a logical treatment. Completion under conjunction is obvious: if $\Gamma, A \wedge B$ represents a 'forbidden context', then Γ, A, B should be considered to be 'forbidden' as well. In other words, the conjunction operator fails to return a new resource from A and B, but it pairs them, so that the formula $A \wedge B$ records this pairing operation. Thus:

$$\{\{\Gamma, A \wedge B\}, \{\Gamma, A, B\}\} \subseteq \mathbf{C}_{\Gamma, A\wedge B}.$$

The disjunction connective comes with an exclusive meaning. If $\Gamma, A \vee B$ is on a blacklist of contexts, then it seems quite reasonable to put both Γ, A and Γ, B on the same blacklist. Thus:

$$\{\{\Gamma, A \vee B\}, \{\Gamma, A\}, \{\Gamma, B\}\} \subseteq \mathbf{C}_{\Gamma, A\vee B}.$$

The exclusive feature of the disjunction emerges from the observation that the context Γ, A, B does not necessarily appear among the 'forbidden contexts'. This absence is unproblematic since any context containing $\{\Gamma, A, B\}$ will be blocked as it contains both the subsets $\{\Gamma, A\}$ and $\{\Gamma, B\}$. Anyway, the effects of dealing with an inclusive disjunction can be easily recovered by taking the union of the two control sets induced by $\Gamma, A \wedge B$ and $\Gamma, A \vee B$. In this way, we get:

$$\{\{\Gamma, A, B\}, \{\Gamma, A\}, \{\Gamma, B\}\} \subseteq \mathbf{C}_{\Gamma, A\vee B} \cup \mathbf{C}_{\Gamma, A\wedge B}.$$

Although here we are focused on classical logic, a crucial point to be aware of is that a system of control sets $\mathcal{S}$ can be associated with any *two-sided* sequent calculus, which provide sequents in a form suitable for our techniques (i.e., the compatibility between the assumptions Γ on the left with a control set may be verified). A calculus for a given logic obtained by means of a system of control sets will be referred to as *controlled calculus.*

Informally speaking, $\mathcal{S}$ grows out of the assignment of a control set $\mathbf{S}$ to each atom of p, so that the corresponding axiom is:

$$\frac{}{\cdot \,|\, p \vdash_{\mathbf{S}} p}\, ax.$$

Note that uniquely atomic axioms, i.e. axioms introducing atomic propositions, are authorized. Moreover, the general task of $\mathcal{S}$ is to indicate how to combine and transform control sets and repositories along derivations. Let $\mathscr{L}$ be a logic in a two-sided sequent calculus formulation. The essential idea is that *each single application of the rules of $\mathscr{L}$ along derivations has to preserve, besides validity, the soundness of the proved sequent.* As an example, let us consider the following controlled version of the standard weakening rule:

$$\frac{\Sigma \,|\, \Gamma \vdash_{\mathbf{S}} A}{\Sigma \,|\, \Gamma, B \vdash_{\mathbf{S}} A} \, weak \vdash.$$

When considered in a classical framework, this rule is clearly sound. However, this does not enable us to draw the conclusion $\Sigma \,|\, \Gamma, B \vdash_{\mathbf{S}} A$ from the premise $\Sigma \,|\, \Gamma \vdash_{\mathbf{S}} A$: the compatibility between the wider context Σ, Γ, B and the control set $\mathbf{S}$ needs indeed to be verified.

Now, let $\mathscr{L}^{\mathcal{S}}$ be the logical system obtained from $\mathscr{L}$ according to a given system of control sets $\mathcal{S}$. $\mathscr{L}^{\mathcal{S}}$ is a subsystem of $\mathscr{L}$ since the set of the theorems of $\mathscr{L}^{\mathcal{S}}$ is included into the set of theorems of $\mathscr{L}$. This means that, for any system $\mathcal{S}$, if the basis $\mathscr{L}$ is non-trivial, then $\mathscr{L}^{\mathcal{S}}$ will be non-trivial as well. Furthermore, let $\mathscr{L}^{\varnothing}$ be the controlled calculus obtained from $\mathscr{L}$ by attaching everywhere the empty control set $\varnothing$, that is, for any axiom p:

$$\frac{}{\cdot \,|\, p \vdash_{\varnothing} p} \, ax.$$

In $\mathscr{L}^{\varnothing}$, the inclusion mechanism in Definition 6 is blocked, so that one reaches a deductive collapse: $\mathscr{L}^{\varnothing} = \mathscr{L}$.

We close this section by formally elucidating the notion of system of control sets on a given logic.

Notation 11 Let $\mathscr{L}$ be a logic in a two-sided sequent calculus formulation (for the sake of simplicity, we assume that $\mathscr{L}$ has only unary and binary rules). We denote with:

- $\mathtt{At}$ the set of its atoms,
- $\mathtt{For}$ the set of its formulas,
- $\mathtt{Con}$ the set of all its possible contexts,
- $\mathtt{CoSet}$ the set of all possible control sets definable upon its language,
- $\mathtt{R}^1$ the set of its unary rules,
- $\mathtt{R}^2$ the set of its binary rules.

Definition 12 (system of control sets) A system of control sets $\mathcal{S}$ on a logic $\mathscr{L}$ is a 5-uple $(\mathtt{S}^1, \mathscr{F}, \mathscr{G}^1, \mathscr{G}^2, \mathscr{S}^1)$ such that $\mathtt{S}^1$ is a set of, finitely many, new (unary) structural rules and $\mathscr{F}, \mathscr{G}^1, \mathscr{G}^2, \mathscr{S}^1$ are functions as follows:

- $\mathscr{F} : \texttt{At} \mapsto \texttt{Con} \times \texttt{CoSet}$
- $\mathscr{G}^1 : \texttt{R}^1 \times (\texttt{For} \times \texttt{Con} \times \texttt{CoSet}) \mapsto \texttt{Con} \times \texttt{CoSet}$
- $\mathscr{G}^2 : \texttt{R}^2 \times (\texttt{For} \times \texttt{Con} \times \texttt{CoSet}) \times (\texttt{For} \times \texttt{Con} \times \texttt{CoSet}) \mapsto \texttt{Con} \times \texttt{CoSet}$
- $\mathscr{S}^1 : \texttt{S}^1 \times (\texttt{For} \times \texttt{Con} \times \texttt{CoSet}) \mapsto \texttt{Con} \times \texttt{CoSet}$.

Definition 13 (spectrum) A *spectrum* $\mathfrak{S}$ is set of systems of control sets defined by the 4-uple $(\texttt{S}^1, \mathscr{G}^1, \mathscr{G}^2, \mathscr{S}^1)$.

A spectrum can be viewed as the logical substratum of different systems of control sets, obtained by abstracting away from the extra-logical information encoded in the systems by the function $\mathscr{F}$. In other words, if $\mathcal{S}, \mathcal{S}' \in \mathfrak{S}$ and $\mathcal{S} \neq \mathcal{S}'$, then they respectively are of the form $(\texttt{S}^1, \mathscr{F}, \mathscr{G}^1, \mathscr{G}^2, \mathscr{S}^1)$ and $(\texttt{S}^1, \mathscr{F}', \mathscr{G}^1, \mathscr{G}^2, \mathscr{S}^1)$ with $\mathscr{F} \neq \mathscr{F}'$. The functions in common set the way in which the extra-logical information attached to atomic axioms is transmitted in and manipulated by the logical calculus.

Notation 14 Let $\mathcal{S} = (\texttt{S}^1, \mathscr{F}, \mathscr{G}^1, \mathscr{G}^2, \mathscr{S}^1)$ be a system of control sets. For convenience we write $\mathcal{S}(p)$ for referring to the control set assigned to the atom p by the function $\mathscr{F}$.

3 Controlling LK

3.1 The minimal spectrum and the controlled calculus $\mathsf{LK}^{\mathcal{S}}$

We proceed to consider the spectrum $\mathfrak{S}_{\mathsf{LK}}$ devised to control classical logic. The way in which $\mathfrak{S}_{\mathsf{LK}}$ transmits control sets and repositories along derivations is described in Table 1. Once that a system of control sets $\mathcal{S} \in \mathfrak{S}_{\mathsf{LK}}$ has been fixed, the resulting controlled sequent calculus will be referred to as $\mathsf{LK}^{\mathcal{S}}$.

Minimality. With regards to control sets, $\mathfrak{S}_{\mathsf{LK}}$ is a *minimal* spectrum. This means that:

1. $\mathfrak{S}_{\mathsf{LK}}$ transmits the *same* control set from the upper controlled sequent to the lower one in all unary inference rules, with the exception of the structural rule σ (which arbitrarily 'expands' the control set).
2. $\mathfrak{S}_{\mathsf{LK}}$ attaches to the lower sequent of the binary rules the *union* of the control sets assigned to the upper sequents.

Furthermore, concerning atomic axioms, we require that $\mathcal{S} \in \mathfrak{S}_{\mathsf{LK}}$ is such that:

1. For all atoms p, $p \notin \bigcup \mathcal{S}(p)$.
2. The attached repository is always the empty context.

The rationale for the first condition will become clear later (see Remark 17). Henceforth, when we mention a system of control sets $\mathcal{S}$, we always refer to a system taken from this minimal spectrum.

The minimality of the spectrum substantiates basic intuitions about context-sensitiveness. For instance, it appears quite natural to formulate the right conjunction rule as follows:

$$\frac{\Sigma \,|\, \Gamma \vdash_{\mathbf{S}} A, \Delta \qquad \Sigma' \,|\, \Gamma' \vdash_{\mathbf{T}} B, \Delta'}{\Sigma, \Sigma' \,|\, \Gamma, \Gamma' \vdash_{\mathbf{S} \cup \mathbf{T}} A \wedge B, \Delta, \Delta'} \vdash \wedge.$$

The rule is soundly applied whenever the context $\Sigma, \Sigma', \Gamma, \Gamma'$ is compatible with $\mathbf{S} \cup \mathbf{T}$. The idea here is that if the contexts in $\mathbf{S}$ halt the derivation of A, Δ from Γ and the contexts in $\mathbf{T}$ the derivation of B, Δ' from Γ', then the contexts in $\mathbf{S} \cup \mathbf{T}$ will block the derivation of $A \wedge B, \Delta, \Delta'$ from Γ, Γ'. On the other hand, the left conjunction rule comes as follows:

$$\frac{\Sigma \,|\, \Gamma, A, B \vdash_{\mathbf{S}} \Delta}{\Sigma \,|\, \Gamma, A \wedge B \vdash_{\mathbf{S}} \Delta} \wedge \vdash.$$

Notice that this rule is 'innocent' with regard to compatibility, insofar as, for any system $\mathcal{S}$ and context Σ, Γ, it is always soundly applied (see Theorem 9(3)).

The $\vdash\to$, $\to\vdash$ and $\vdash \neg$-rules deserve a separate comment, since here repositories perform their specific task. For the sake of explanation, let us consider the following instance of the $\vdash\to$ rule.

$$\frac{\cdot \,|\, \Gamma, A \vdash_{\mathbf{S}} B}{A \,|\, \Gamma \vdash_{\mathbf{S}} A \to B} \vdash\to$$

Axiom:

$$\frac{}{\cdot \,|\, p \vdash_{\mathcal{S}(p)} p}\, ax. \qquad \text{with } p \text{ atomic}$$

Cut rule:

$$\frac{\Sigma \,|\, \Gamma \vdash_{\mathbf{S}} A, \Delta \qquad \Sigma' \,|\, \Gamma', A \vdash_{\mathbf{T}} \Delta'}{\Sigma, \Sigma' \,|\, \Gamma, \Gamma' \vdash_{\mathbf{S} \cup \mathbf{T}} \Delta, \Delta'}\, cut$$

Structural rules:

$$\frac{\Sigma \,|\, \Gamma \vdash_{\mathbf{S}} \Delta}{\Sigma \,|\, \Gamma, A \vdash_{\mathbf{S}} \Delta}\, weak \vdash \qquad \frac{\Sigma \,|\, \Gamma \vdash_{\mathbf{S}} \Delta}{\Sigma \,|\, \Gamma \vdash_{\mathbf{S}} \Delta, A} \vdash weak$$

$$\frac{\Sigma \,|\, \Gamma, A, A \vdash_{\mathbf{S}} \Delta}{\Sigma \,|\, \Gamma, A \vdash_{\mathbf{S}} \Delta}\, cont \vdash \qquad \frac{\Sigma \,|\, \Gamma \vdash_{\mathbf{S}} \Delta, A, A}{\Sigma \,|\, \Gamma \vdash_{\mathbf{S}} \Delta, A} \vdash cont$$

$$\frac{\Sigma \,|\, \Gamma, A, B \vdash_{\mathbf{S}} \Delta}{\Sigma \,|\, \Gamma, B, A \vdash_{\mathbf{S}} \Delta}\, exch \vdash \qquad \frac{\Sigma \,|\, \Gamma \vdash_{\mathbf{S}} \Delta, A, B}{\Sigma \,|\, \Gamma \vdash_{\mathbf{S}} \Delta, B, A} \vdash exch$$

$$\frac{\Sigma \,|\, \Gamma \vdash_{\mathbf{S}} \Delta}{\Sigma \,|\, \Gamma \vdash_{\mathbf{S} \cup \mathbf{T}} \Delta}\, \sigma \qquad \frac{\Sigma \,|\, \Gamma \vdash_{\mathbf{S}} \Delta}{\Sigma, A \,|\, \Gamma \vdash_{\mathbf{S}} \Delta}\, \rho$$

Logical rules:

$$\frac{\Sigma \,|\, \Gamma, A, B \vdash_{\mathbf{S}} \Delta}{\Sigma \,|\, \Gamma, A \wedge B \vdash_{\mathbf{S}} \Delta} \wedge \vdash \qquad \frac{\Sigma \,|\, \Gamma \vdash_{\mathbf{S}} \Delta, A \qquad \Sigma' \,|\, \Gamma' \vdash_{\mathbf{T}} \Delta', B}{\Sigma, \Sigma' \,|\, \Gamma, \Gamma' \vdash_{\mathbf{S} \cup \mathbf{T}} \Delta, \Delta', A \wedge B} \vdash \wedge$$

$$\frac{\Sigma \,|\, \Gamma, A \vdash_{\mathbf{S}} \Delta \qquad \Sigma' \,|\, \Gamma', B \vdash_{\mathbf{T}} \Delta'}{\Sigma, \Sigma' \,|\, \Gamma, \Gamma', A \vee B \vdash_{\mathbf{S} \cup \mathbf{T}} \Delta, \Delta'} \vee \vdash \qquad \frac{\Sigma \,|\, \Gamma \vdash_{\mathbf{S}} \Delta, A, B}{\Sigma \,|\, \Gamma \vdash_{\mathbf{S}} \Delta, A \vee B} \vdash \vee$$

$$\frac{\Sigma \,|\, \Gamma \vdash_{\mathbf{S}} A, \Delta \qquad \Sigma' \,|\, \Gamma', B \vdash_{\mathbf{T}} \Delta'}{\Sigma, \Sigma', B \,|\, \Gamma, \Gamma', A \to B \vdash_{\mathbf{S} \cup \mathbf{T}} \Delta, \Delta'} \to\vdash \qquad \frac{\Sigma \,|\, \Gamma, A \vdash_{\mathbf{S}} B, \Delta}{\Sigma, A \,|\, \Gamma \vdash_{\mathbf{S}} A \to B, \Delta} \vdash\to$$

$$\frac{\Sigma \,|\, \Gamma \vdash_{\mathbf{S}} A, \Delta}{\Sigma \,|\, \Gamma, \neg A \vdash_{\mathbf{S}} \Delta} \neg \vdash \qquad \frac{\Sigma \,|\, \Gamma, A \vdash_{\mathbf{S}} \Delta}{\Sigma, A \,|\, \Gamma \vdash_{\mathbf{S}} \neg A, \Delta} \vdash \neg$$

Table 1: The controlled sequent calculus $\mathsf{LK}^{\mathcal{S}}$

The premise says that Γ and A produce B, provided that no 'forbidden context' among those listed in $\mathbf{S}$ is included in Γ, A. Due to the ambiguous status of the turnstile symbol in sequent calculus (it comes, indeed, as a sort of metatheoretical rendition of implication), the conclusion says quite the same: once that A is added to Γ, the formula B is produced. Thus, it is natural to consider A as it were actually present on the left-hand side of the sequent. Likewise, when a $\rightarrow\vdash$-rule introduces $A \rightarrow B$ as principal formula, the consequent B should be considered as still lying on the left-hand side of the turnstile sign.

Similar observations can be afforded in case of the $\vdash\neg$-rule being $A \rightarrow B$ and $\neg A \vee B$ classically equivalent:

$$\dfrac{\dfrac{\cdot \,|\, \Gamma, A \vdash_{\mathbf{S}} B}{A \,|\, \Gamma \vdash_{\mathbf{S}} \neg A, B}\vdash\neg}{A \,|\, \Gamma \vdash_{\mathbf{S}} \neg A \vee B}\vdash\vee.$$

More in general, the contribution of the repository consists in keeping trace of formulas shifted by the rules from the left-hand side of the sequent to the right-hand side. As we shall see in the next section, repositories allow us to prove cut-elimination by implementing a Gentzen-style normalization procedure (cf. Lemma 18).

The only structural rules of $\mathsf{LK}^{\mathcal{S}}$ which specifically act on control sets and repositories are the following:

$$\dfrac{\Sigma \,|\, \Gamma \vdash_{\mathbf{S}} \Delta}{\Sigma \,|\, \Gamma \vdash_{\mathbf{S}\cup\mathbf{T}} \Delta}\,\sigma \qquad \text{and} \qquad \dfrac{\Sigma \,|\, \Gamma \vdash_{\mathbf{S}} \Delta}{\Sigma, A \,|\, \Gamma \vdash_{\mathbf{S}} \Delta}\,\rho.$$

Both these rules may be paraphrased as saying that one can arbitrarily strengthen the constraint imposed by context-sensitiveness, provided that the soundness of the conclusion is preserved. On the one hand, the σ-rule permits us to introduce a new bunch of 'forbidden contexts' while producing a derivation; on the other, the ρ-rule allows us to further restrict the fan of admissible proofs by extending the context. In sum, both rules make it harder to satisfy the compatibility relation between the context Γ and the control set.

As we will see in a moment, the ρ-rule is required to the cut-elimination procedure. However, it should be clear that they are both 'open' rules being concerned with injecting new extra-logical information in $\mathsf{LK}^{\mathcal{S}}$ derivations. To illustrate the use of the σ-rule, consider the example of Section 1:

$$\Sigma \mid \Gamma, \mathsf{flu\text{-}like\ symptoms} \vdash_{\{\{\mathsf{dengue}\}\}} \mathsf{aspirin}.$$

The σ-rule permits us to 'upgrade' the control by the additional information about aspirin allergy as follows:

$$\frac{\Sigma \mid \Gamma, \mathsf{flu\text{-}like\ symptoms} \vdash_{\{\{\mathsf{dengue}\}\}} \mathsf{aspirin}}{\Sigma \mid \Gamma, \mathsf{flu\text{-}like\ symptoms} \vdash_{\{\{\mathsf{dengue}\},\{\mathsf{asp\text{-}allergy}\}\}} \mathsf{aspirin}}\ \sigma.$$

Now, the compatibility between the context Σ, Γ and the fresh control set $\{\{\mathsf{dengue}\}, \{\mathsf{asp\text{-}allergy}\}\}$ becomes harder to be satisfied, due to the inclusion of asp-allergy in the set of forbidden contexts.

3.2 Cut-elimination for $\mathsf{LK}^{\mathcal{S}}$

Before embarking on the proof of the cut-elimination theorem, we need to introduce an important distinction and then to prove a related lemma.

Definition 15 (proof, paraproof) Consider a rooted, finitely branching tree π whose nodes are sequents of $\mathsf{LK}^{\mathcal{S}}$, and such that it is recursively built up from axioms by means of the rules of $\mathsf{LK}^{\mathcal{S}}$. If each sequent in π is sound, π is said to be a *proof* of $\mathsf{LK}^{\mathcal{S}}$, otherwise π is called a *paraproof.*

Previous definition places upon a proof π of $\mathsf{LK}^{\mathcal{S}}$ two independent requirements. The first, *logical validity*, standardly establishes that each one of the deductive steps performed in π has to be accomplished in accordance with the rules of $\mathsf{LK}^{\mathcal{S}}$. The second requirement, *soundness*, tells that each single application of the rules in π must be *soundness preserving* so that each sequent occurring in π is sound with respect to the control set attached to it. The fulfilment of the latter condition is what turns a paraproof into a $\mathsf{LK}^{\mathcal{S}}$ proof.

Example 16 Let $\mathcal{S}$ be a system of control sets such that $\{p\} \in \mathcal{S}(q)$. Inasmuch as the sequent $q \mid p, p \to q \vdash_{\mathbf{S} \cup \{\{p\},\dots\}} q$ is unsound, the following derivation constitutes a paraproof.

$$\dfrac{\dfrac{\dfrac{}{\cdot \mid p \vdash_{\mathcal{S}(p)} p}\ ax. \qquad \dfrac{}{\cdot \mid q \vdash_{\{\{p\},\dots\}} q}\ ax.}{q \mid p, p \to q \vdash_{\mathcal{S}(p) \cup \{\{p\},\dots\}} q}\ {\to}{\vdash}}{p, q \mid p \to q \vdash_{\mathcal{S}(p) \cup \{\{p\},\dots\}} p \to q}\ {\vdash}{\to}$$

Remark 17 The previous example shows that equalities (i.e. the provable equivalence of any formula with itself) are not necessarily guaranteed in controlled calculi. With a little ingenuity, however, we can at least preserve equalities involving atoms by adopting systems of control sets such that $p \notin \bigcup \mathcal{S}(p)$, for any atom p.

We now prove the lemma.

Lemma 18 Any cut-free paraproof in $\mathsf{LK}^{\mathcal{S}}$ is a proof if and only if its end-sequent is sound.

Proof Let r be every rule of $\mathsf{LK}^{\mathcal{S}}$, except the cut rule. Consider the following configuration:

$$\frac{\Sigma \,|\, \Gamma \vdash_{\mathbf{S}} \Delta \qquad \ldots}{\Sigma' \,|\, \Gamma' \vdash_{\mathbf{S}'} \Delta'}\, r$$

where $\Sigma \,|\, \Gamma \vdash_{\mathbf{S}} \Delta$ is either the only premise of r, or one (not necessarily the first) of the two premises. It suffices to remark that, for each r, $\{\Sigma, \Gamma\} \subseteq \{\Sigma', \Gamma'\}$ and $\mathbf{S} \subseteq \mathbf{S}'$. By Theorem 9(1), one gets the soundness of the premise $\Sigma \,|\, \Gamma \vdash_{\mathbf{S}} \Delta$ from that of the conclusion $\Sigma' \,|\, \Gamma' \vdash_{\mathbf{S}'} \Delta'$. In the case of left-contraction, left-conjuction and left-disjunction, combine Theorem 9(1) with Theorems 9(2), (3) and (4), respectively.

Finally, we are in position to prove the main result of this section.

Theorem 19 *(Cut-elimination).* Any sequent $\Sigma \,|\, \Gamma \vdash_{\mathbf{S}} \Delta$ which is provable in $\mathsf{LK}^{\mathcal{S}}$ has a cut-free proof.

Proof We show, at first, how the standard cut-elimination algorithm for LK can be tailored for controlled calculi. We restrict our attention to the more meaningful cases of reduction, namely the following:

- *Parallel reduction* $\to\vdash$ / $\vdash\to$.

$$\dfrac{\dfrac{\begin{array}{c}\pi_1\\ \vdots\\ \Sigma \,|\, \Gamma, A \vdash_{\mathbf{S}} B, \Delta\end{array}}{\Sigma, A \,|\, \Gamma \vdash_{\mathbf{S}} A \to B, \Delta}\,\vdash\to \qquad \dfrac{\begin{array}{c}\pi_2\\ \vdots\\ \Sigma' \,|\, \Gamma' \vdash_{\mathbf{T}} A, \Delta'\end{array} \qquad \begin{array}{c}\pi_3\\ \vdots\\ \Sigma'' \,|\, \Gamma'', B \vdash_{\mathbf{U}} \Delta''\end{array}}{\Sigma', \Sigma'', B \,|\, \Gamma', \Gamma'', A \to B \vdash_{\mathbf{T} \cup \mathbf{U}} \Delta', \Delta''}\,\to\vdash}{\Sigma, \Sigma', \Sigma'', A, B \,|\, \Gamma, \Gamma', \Gamma'' \vdash_{\mathbf{S} \cup \mathbf{T} \cup \mathbf{U}} \Delta, \Delta', \Delta''}\, cut$$

$$\downarrow$$

$$\dfrac{\dfrac{\begin{array}{c}\pi_1\\ \vdots\\ \Sigma \mid \Gamma, A \vdash_{\mathbf{S}} B, \Delta\end{array} \quad \begin{array}{c}\pi_2\\ \vdots\\ \Sigma' \mid \Gamma' \vdash_{\mathbf{T}} A, \Delta'\end{array}}{\Sigma, \Sigma' \mid \Gamma, \Gamma' \vdash_{\mathbf{S}\cup\mathbf{T}} B, \Delta, \Delta'}\,cut \quad \begin{array}{c}\pi_3\\ \vdots\\ \Sigma'' \mid \Gamma'', B \vdash_{\mathbf{U}} \Delta''\end{array}}{\dfrac{\dfrac{\Sigma, \Sigma', \Sigma'' \mid \Gamma, \Gamma', \Gamma'' \vdash_{\mathbf{S}\cup\mathbf{T}\cup\mathbf{U}} \Delta, \Delta', \Delta''}{\Sigma, \Sigma', \Sigma'', A \mid \Gamma, \Gamma', \Gamma'' \vdash_{\mathbf{S}\cup\mathbf{T}\cup\mathbf{U}} \Delta, \Delta', \Delta''}\,\rho}{\Sigma, \Sigma', \Sigma'', A, B \mid \Gamma, \Gamma', \Gamma'' \vdash_{\mathbf{S}\cup\mathbf{T}\cup\mathbf{U}} \Delta, \Delta', \Delta''}\,\rho}\,cut$$

- *Parrallel reduction* $\vdash \neg / \neg \vdash$.

$$\dfrac{\dfrac{\begin{array}{c}\pi_1\\ \vdots\\ \Sigma \mid \Gamma, A \vdash_{\mathbf{S}} \Delta\end{array}}{\Sigma, A \mid \Gamma \vdash_{\mathbf{S}} \Delta, \neg A}\vdash\neg \quad \dfrac{\begin{array}{c}\pi_2\\ \vdots\\ \Sigma' \mid \Gamma' \vdash_{\mathbf{T}} \Delta', A\end{array}}{\Sigma' \mid \Gamma', \neg A \vdash_{\mathbf{T}} \Delta'}\neg\vdash}{\Sigma, \Sigma', A \mid \Gamma, \Gamma' \vdash_{\mathbf{S}\cup\mathbf{T}} \Delta, \Delta'}\,cut$$

$$\downarrow$$

$$\dfrac{\dfrac{\begin{array}{c}\pi_1\\ \vdots\\ \Sigma \mid \Gamma, A \vdash_{\mathbf{S}} \Delta\end{array} \quad \begin{array}{c}\pi_2\\ \vdots\\ \Sigma' \mid \Gamma' \vdash_{\mathbf{T}} \Delta', A\end{array}}{\Sigma, \Sigma' \mid \Gamma, \Gamma' \vdash_{\mathbf{S}\cup\mathbf{T}} \Delta, \Delta'}\,cut}{\Sigma, \Sigma', A \mid \Gamma, \Gamma' \vdash_{\mathbf{S}\cup\mathbf{T}} \Delta, \Delta'}\,\rho$$

- *Commutation* $\vdash\rightarrow$ /cut.

$$\dfrac{\dfrac{\begin{array}{c}\pi_1\\ \vdots\\ \Sigma \mid \Gamma, A, C \vdash_{\mathbf{S}} B, \Delta\end{array}}{\Sigma, A \mid \Gamma, C \vdash_{\mathbf{S}} A \rightarrow B, \Delta}\vdash\rightarrow \quad \begin{array}{c}\pi_2\\ \vdots\\ \Sigma' \mid \Gamma' \vdash_{\mathbf{T}} C, \Delta'\end{array}}{\Sigma, \Sigma', A \mid \Gamma, \Gamma' \vdash_{\mathbf{S}\cup\mathbf{T}} A \rightarrow B, \Delta, \Delta'}\,cut$$

$$\downarrow$$

$$
\dfrac{\dfrac{\begin{array}{c}\pi_1\\ \vdots\\ \Sigma \,|\, \Gamma, A, C \vdash_{\mathbf{S}} B, \Delta\end{array} \qquad \begin{array}{c}\pi_2\\ \vdots\\ \Sigma' \,|\, \Gamma' \vdash_{\mathbf{T}} C, \Delta'\end{array}}{\Sigma, \Sigma' \,|\, \Gamma, \Gamma', A \vdash_{\mathbf{S}\cup\mathbf{T}} B, \Delta, \Delta'}\, cut}{\Sigma, \Sigma', A \,|\, \Gamma, \Gamma' \vdash_{\mathbf{S}\cup\mathbf{T}} A \to B, \Delta, \Delta'} \vdash\to
$$

- *Commutation* $\vdash \neg$/cut.

$$
\dfrac{\dfrac{\begin{array}{c}\pi_1\\ \vdots\\ \Sigma \,|\, \Gamma, A, C \vdash_{\mathbf{S}} \Delta\end{array}}{\Sigma, A \,|\, \Gamma, C \vdash_{\mathbf{S}} \Delta, \neg A} \vdash \neg \qquad \begin{array}{c}\pi_2\\ \vdots\\ \Sigma' \,|\, \Gamma' \vdash_{\mathbf{T}} \Delta', C\end{array}}{\Sigma, \Sigma', A \,|\, \Gamma, \Gamma' \vdash_{\mathbf{S}\cup\mathbf{T}} \Delta, \Delta'}\, cut
$$

$$\downarrow$$

$$
\dfrac{\dfrac{\begin{array}{c}\pi_1\\ \vdots\\ \Sigma \,|\, \Gamma, A, C \vdash_{\mathbf{S}} \Delta\end{array} \qquad \begin{array}{c}\pi_2\\ \vdots\\ \Sigma' \,|\, \Gamma' \vdash_{\mathbf{T}} \Delta', C\end{array}}{\Sigma, \Sigma' \,|\, \Gamma, \Gamma', A \vdash_{\mathbf{S}\cup\mathbf{T}} \Delta, \Delta'}\, cut}{\Sigma, \Sigma', A \,|\, \Gamma, \Gamma' \vdash_{\mathbf{S}\cup\mathbf{T}} \Delta, \Delta', \neg A} \vdash \neg
$$

- *Commutation* σ/cut.

$$
\dfrac{\dfrac{\begin{array}{c}\pi_1\\ \vdots\\ \Sigma \,|\, \Gamma, C \vdash_{\mathbf{S}} \Delta\end{array}}{\Sigma \,|\, \Gamma, C \vdash_{\mathbf{S}\cup\mathbf{T}} \Delta}\, \sigma \qquad \begin{array}{c}\pi_2\\ \vdots\\ \Sigma' \,|\, \Gamma' \vdash_{\mathbf{U}} C, \Delta'\end{array}}{\Sigma, \Sigma' \,|\, \Gamma, \Gamma' \vdash_{\mathbf{S}\cup\mathbf{T}\cup\mathbf{U}} \Delta, \Delta'}\, cut
$$

$$\downarrow$$

$$
\dfrac{\dfrac{\begin{array}{c}\pi_1\\ \vdots\\ \Sigma \,|\, \Gamma, C \vdash_{\mathbf{S}} \Delta\end{array} \qquad \begin{array}{c}\pi_2\\ \vdots\\ \Sigma' \,|\, \Gamma' \vdash_{\mathbf{U}} C, \Delta'\end{array}}{\Sigma, \Sigma' \,|\, \Gamma, \Gamma' \vdash_{\mathbf{S}\cup\mathbf{U}} \Delta, \Delta'}\, cut}{\Sigma, \Sigma' \,|\, \Gamma, \Gamma' \vdash_{\mathbf{S}\cup\mathbf{T}\cup\mathbf{U}} \Delta, \Delta'}\, \sigma
$$

- *Commutation* ρ/cut.

$$\dfrac{\dfrac{\begin{matrix}\pi_1\\ \vdots\\ \Sigma \mid \Gamma, C \vdash_{\mathbf{S}} \Delta\end{matrix}}{\Sigma, A \mid \Gamma, C \vdash_{\mathbf{S}} \Delta}\,\rho \qquad \begin{matrix}\pi_2\\ \vdots\\ \Sigma' \mid \Gamma' \vdash_{\mathbf{T}} \Delta', C\end{matrix}}{\Sigma, \Sigma', A \mid \Gamma, \Gamma' \vdash_{\mathbf{S}\cup\mathbf{T}} \Delta, \Delta'}\,\mathit{cut}$$

$$\downarrow$$

$$\dfrac{\dfrac{\begin{matrix}\pi_1\\ \vdots\\ \Sigma \mid \Gamma, C \vdash_{\mathbf{S}} \Delta\end{matrix} \qquad \begin{matrix}\pi_2\\ \vdots\\ \Sigma' \mid \Gamma' \vdash_{\mathbf{T}} \Delta', C\end{matrix}}{\Sigma, \Sigma' \mid \Gamma, \Gamma' \vdash_{\mathbf{S}\cup\mathbf{T}} \Delta, \Delta'}\,\mathit{cut}}{\Sigma, \Sigma', A \mid \Gamma, \Gamma' \vdash_{\mathbf{S}\cup\mathbf{T}} \Delta, \Delta'}\,\rho$$

- *Commutation* $\rightarrow\vdash$ /cut.

$$\dfrac{\dfrac{\begin{matrix}\pi_1\\ \vdots\\ \Sigma \mid \Gamma \vdash_{\mathbf{S}} A, \Delta\end{matrix} \qquad \begin{matrix}\pi_2\\ \vdots\\ \Sigma' \mid \Gamma', B, C \vdash_{\mathbf{T}} \Delta'\end{matrix}}{\Sigma, \Sigma', B \mid \Gamma, \Gamma', A \to B, C \vdash_{\mathbf{S}\cup\mathbf{T}} \Delta, \Delta'}\,\rightarrow\vdash \qquad \begin{matrix}\pi_3\\ \vdots\\ \Sigma'' \mid \Gamma'' \vdash_{\mathbf{U}} C, \Delta''\end{matrix}}{\Sigma, \Sigma', \Sigma'', B \mid \Gamma, \Gamma', \Gamma'', A \to B \vdash_{\mathbf{S}\cup\mathbf{T}\cup\mathbf{U}} \Delta, \Delta', \Delta''}\,\mathit{cut}$$

$$\downarrow$$

$$\dfrac{\begin{matrix}\pi_1\\ \vdots\\ \Sigma \mid \Gamma \vdash_{\mathbf{S}} A, \Delta\end{matrix} \qquad \dfrac{\begin{matrix}\pi_2\\ \vdots\\ \Sigma' \mid \Gamma', B, C \vdash_{\mathbf{T}} \Delta'\end{matrix} \qquad \begin{matrix}\pi_3\\ \vdots\\ \Sigma'' \mid \Gamma'' \vdash_{\mathbf{U}} C, \Delta''\end{matrix}}{\Sigma', \Sigma'' \mid \Gamma', \Gamma'', B \vdash_{\mathbf{T}\cup\mathbf{U}} \Delta', \Delta''}\,\mathit{cut}}{\Sigma, \Sigma', \Sigma'', B \mid \Gamma, \Gamma', \Gamma'', A \to B \vdash_{\mathbf{S}\cup\mathbf{T}\cup\mathbf{U}} \Delta, \Delta', \Delta''}\,\rightarrow\vdash$$

The tricky point about cut-elimination in controlled calculi is that the steps of reduction do not necessarily preserve the soundness of sequents, that is, the normalization procedure may turn proofs into paraproofs.[2]

Therefore, in order to conclude our demonstration we need to show that the normal derivation π', obtained from an $\mathsf{LK}^{\mathcal{S}}$ proof π by means of the cut-elimination algorithm just outlined, is indeed a proof, namely each one of its sequents is sound. By hypothesis, π is a proof, so its endsequent is sound. By Lemma 18, π' is a proof as well.

Corollary 20 *(Subformula Property).* If a sequent is provable in $\mathsf{LK}^{\mathcal{S}}$, then it is provable analytically, namely by means of a derivation in which all formulas are subformulas of those occurring in the end sequent.

Proof By induction on the length of cut-free proofs.

4 Inducing non-monotonic and paraconsistent features

In this section, we explain how it is possible to generate non-monotonic and/or paraconsistent subsystems of LK by selecting suitable systems of control sets from $\mathfrak{S}_{\mathsf{LK}}$. To this aim, it is first convenient to appeal to an extension of the notion of provability as shown in the definition below.

[2]Consider, for instance, the standard implementation of the parallel reduction involving conjunction rules:

$$\dfrac{\dfrac{\begin{array}{c}\pi_1\\ \vdots\\ \Sigma \mid \Gamma \vdash_{\mathbf{S}} A,\Delta\end{array} \qquad \begin{array}{c}\pi_2\\ \vdots\\ \Sigma' \mid \Gamma' \vdash_{\mathbf{T}} B,\Delta'\end{array}}{\Sigma,\Sigma' \mid \Gamma,\Gamma' \vdash_{\mathbf{S}\cup\mathbf{T}} A\wedge B,\Delta,\Delta'}\vdash\wedge \qquad \dfrac{\begin{array}{c}\pi_3\\ \vdots\\ \Sigma'' \mid \Gamma'',A,B \vdash_{\mathbf{U}} \Delta''\end{array}}{\Sigma'' \mid \Gamma'',A\wedge B \vdash_{\mathbf{U}} \Delta''}\wedge\vdash}{\Sigma,\Sigma',\Sigma'' \mid \Gamma,\Gamma',\Gamma'' \vdash_{\mathbf{S}\cup\mathbf{T}\cup\mathbf{U}} \Delta,\Delta',\Delta''}\mathit{cut}$$

$$\downarrow$$

$$\dfrac{\dfrac{\begin{array}{c}\pi_1\\ \vdots\\ \Sigma \mid \Gamma \vdash_{\mathbf{S}} A,\Delta\end{array} \qquad \begin{array}{c}\pi_3\\ \vdots\\ \Sigma'' \mid \Gamma'',A,B \vdash_{\mathbf{U}} \Delta''\end{array}}{\Sigma,\Sigma'' \mid \Gamma,\Gamma'',B \vdash_{\mathbf{S}\cup\mathbf{U}} \Delta,\Delta''}\mathit{cut} \qquad \begin{array}{c}\pi_2\\ \vdots\\ \Sigma' \mid \Gamma' \vdash_{\mathbf{T}} B,\Delta'\end{array}}{\Sigma,\Sigma',\Sigma'' \mid \Gamma,\Gamma',\Gamma'' \vdash_{\mathbf{S}\cup\mathbf{T}\cup\mathbf{U}} \Delta,\Delta',\Delta''}\mathit{cut}$$

It may be the case here that $\Sigma,\Sigma'',\Gamma,\Gamma'',B \nparallel \mathbf{S}\cup\mathbf{U}$ due to some 'forbidden contexts' built up when Σ,Γ and B are put together.

Definition 21 An LK sequent $\Gamma \vdash \Delta$ is said to be provable in $\mathsf{LK}^{\mathcal{S}}$ if there exists a control set $\mathbf{S}$ and a repository Σ such that $\Sigma \,|\, \Gamma \vdash_{\mathbf{S}} \Delta$ is provable in $\mathsf{LK}^{\mathcal{S}}$.

Remark 22 Any $\mathsf{LK}^{\mathcal{S}}$ proof can be turned into an LK proof simply by forgetting any reference to control sets and repositories.

4.1 Non-monotonicity of derivations

Non-monotonic logics are designed mainly to deal with situations where conclusions need to be retracted and inferences blocked given some increments of information. As we said at the beginning, monotonicity in Gentzen-style calculi is usually guaranteed by the left-weakening rule. A system $\mathcal{S}$ assigned to the logic $\mathscr{L}$ in a suitable way allows us to constraint the use of the left-weakening rule making $\mathscr{L}^{\mathcal{S}}$ able to deductively express non-monotonicity phenomena.

Theorem 23 *If $\{A\} \in \mathcal{S}(p)$ and A is a non contradictory statement, then $p, A \nvdash p$.*

Proof Suppose by absurd that the sequent $p, A \vdash p$ is provable in $\mathsf{LK}^{\mathcal{S}}$. According to Definition 21, this means that there exists a control set $\mathbf{S}$ and a repository Σ such that the sequent $\Sigma \,|\, p, A \vdash_{\mathbf{S}} p$ is provable by means of a proof π in $\mathsf{LK}^{\mathcal{S}}$.

Suppose now that the axiom $\dfrac{}{\cdot \,|\, p \vdash_{\mathcal{S}(p)} p}\,ax.$ does not occur among the leaves of π. Then, each occurrence of p is introduced by an application of the weakening rule. Now, these applications of the rule can be permuted down along the derivation until a proof π' is obtained:

$$\dfrac{\dfrac{\begin{array}{c}\vdots\\ \Sigma \,|\, A \vdash_{\mathbf{S}}\end{array}}{\Sigma \,|\, p, A \vdash_{\mathbf{S}}}\,weak \vdash}{\Sigma \,|\, p, A \vdash_{\mathbf{S}} p}\vdash weak.$$

But by hypothesis, A is not a contradiction and so the axiom $\dfrac{}{\cdot \,|\, p \vdash_{\mathcal{S}(p)} p}\,ax.$ *must* occur in the leaves of π, i.e. $\{A\} \subseteq \mathbf{S}$ and so the sequent $\Sigma \,|\, p, A \vdash_{\mathbf{S}} p$ turns out to be unsound whatever repository is attached.

4.2 Paraconsistency

Systems of control sets have also the capacity to induce paraconsistent features in logical calculi. As we said at the beginning, by 'paraconsistent' logic, one standardly means a logic $\mathscr{L}$ in which there exists a pair of formulas A and B and a context Γ such that:

$$\Gamma, A, \neg A \nvdash_{\mathscr{L}} B. \tag{1}$$

We aim at blocking the provability of the sequent:

$$p, \neg p \vdash B \tag{2}$$

for a given atom p and formula B. The next theorem ensures that this effect can be achieved by choosing a system $\mathcal{S}$ such that $\{\neg p\} \in \mathcal{S}(p)$.

Theorem 24 If $\{\neg p\} \in \mathcal{S}(p)$ and B is not a classically valid proposition, then $p, \neg p \nvdash B$.

Proof The argument is similar to that in the proof of Theorem 23. Suppose by absurd that, for a certain control set $\mathbf{S}$ and repository Σ, there exists in $\mathsf{LK}^{\mathcal{S}}$ a proof π of the sequent $\Sigma \,|\, p, \neg p \vdash_{\mathbf{S}} B$. Furthermore, assume that the axiom $\dfrac{}{\cdot \,|\, p \vdash_{\mathcal{S}(p)} p}\,ax.$ does not occur among the leaves of π. This means that both occurrences of p are introduced by the weakening rule (and, possibly, the right-negation rule). Such applications of the weakening can be permuted down along π until one of the following two proofs is obtained:

$$\dfrac{\dfrac{\begin{array}{c}\vdots\\ \Sigma \,|\, \cdot \vdash_{\mathbf{S}} B\end{array}}{\Sigma \,|\, p \vdash_{\mathbf{S}} B}\,weak \vdash}{\Sigma \,|\, p, \neg p \vdash_{\mathbf{S}} B}\,weak \vdash \qquad \dfrac{\dfrac{\dfrac{\begin{array}{c}\vdots\\ \Sigma \,|\, \cdot \vdash_{\mathbf{S}} B\end{array}}{\Sigma \,|\, p \vdash_{\mathbf{S}} B}\,weak \vdash}{\Sigma \,|\, p \vdash_{\mathbf{S}} B, p}\vdash weak}{\Sigma \,|\, p, \neg p \vdash_{\mathbf{S}} B}\,\neg \vdash.$$

By hypothesis, B is not a valid proposition and so we can conclude that the axiom $\dfrac{}{\cdot \,|\, p \vdash_{\mathcal{S}(p)} p}\,ax.$ *must* occur in the leaves of π. It then follows automatically that any control set $\mathbf{S}$ assigned to (2) *must* contain $\{\neg p\}$. In this way, for any repository Σ, we get the unsoundness of $\Sigma \,|\, p, \neg p \vdash_{\mathbf{S}} B$.

Remark 25 Theorem 23 and Theorem 24 display in concert a duality, which admits of an intuitive explanation.

- Theorem 23 explains how to block the derivability of the sequent $p, A \vdash p$, *provided that* A *is not a contradiction*. The essential idea is that a contradiction conveys some sort of empty information, which falls short of conflicting with the information expressed by p. This is why the contradiction A cannot block the derivability of $p, A \vdash p$.

- Theorem 24 allows us to control the explosion of a certain contradiction $p \wedge \neg p$, i.e. $p, \neg p \nvdash B$, *provided that* B *is not a classical tautology.* This condition represents the fact that tautologies, being provable without assumptions, should be also provable from a cluster of contradictory assumptions which annihilate each other.

4.3 A concurrent view: a sketch

Another issue in the overall picture is the tantalizing possibility of accounting for non-monotonicity and paraconsistency in terms of *concurrency.* A nice example of concurrent processes is afforded by biochemistry. In the concurrent enzyme inhibition, the enzyme E binds with the substratum S provided that the inhibitor I is not present at the time of the reaction, otherwise E binds with I with S playing the role of the residual element [8]. More formally (with $\star$ denoting the product of the interaction between two elements), we can distinguish two processes:

- $\pi : \Gamma, E, S \rightsquigarrow \Gamma, E \star S$

- $\lambda : \Gamma, E, I \rightsquigarrow \Gamma, E \star I$

such that λ has priority over π. Accordingly, when we have in input Γ, E, S, I, the process λ is implemented so as to produce $\Gamma, E \star I, S$, whilst the process π is blocked ([11, 17]). Another well-known problem involving concurrency between processes is that of the 'dining philosophers' [18].

This is an informal sketch of how a concurrent account of non-monotonicity and paraconsistency might run:

1. *Non-monotonicity.* Take a valid sequent $\Gamma \vdash A$ which becomes invalidated by the introduction of a new element B. $\Gamma \vdash A$ amounts to a process $\pi_1 : \Gamma \rightsquigarrow A$ (the nature of which depending upon the context) that produces A from Γ. Now, the new element B triggers an alternative process π_2: $\Delta, B \rightsquigarrow C$, with $\{\Delta\} \subseteq \{\Gamma\}$ and $C \neq A$, which has priority over π_1. Because of this priority, we can conclude that there exists no (effectively executable) process $\Gamma, B \rightsquigarrow A$ and so the sequent $\Gamma, B \vdash A$ is invalidated.

2. *Paraconsistency.* A contradiction $A \wedge \neg A$ lacks the power to invalidate the whole deductive mechanism of the calculus. Indeed, the explosion principle $\Gamma, A, \neg A \vdash B$ can be blocked by assuming two distinct concurrent processes $\pi_1 : \Delta, A \rightsquigarrow B$ and $\pi_2 : \Delta', \neg A \rightsquigarrow B'$, with $\{\Delta\}, \{\Delta'\} \subseteq \{\Gamma\}$ and $B \neq B'$. When a priority relation between π_1 and π_2 is fixed, for example suppose that π_1 has priority over π_2, the explosive character of $A \wedge \neg A$ is neutralized, inasmuch as $\Gamma, A, \neg A \nvdash B'$. Namely, the opposition between A and $\neg A$ is pushed to the higher level opposition between the two processes π_1 and π_2. This is to say, a contradiction $A \wedge \neg A$ explodes only when no priority is established. To use a catch-phrase: *contradiction is the absence of a fixed priority between two opposite processes.*

On the other hand, control sets can be naturally interpreted in terms of concurrent processes as follows:

- *Provability*: $\Gamma \vdash_{\{\{\Lambda_1\},\ldots,\{\Lambda_n\}\}} \Delta$ means that there exists a process π that 'transforms' Γ into (the disjunction of the formulas listed in) Δ.

- *Monotonicity*: any process triggered by contexts not among $\Lambda_1, \ldots, \Lambda_n$ do not interfere with the original process π.

- *Concurrency*: any process π' triggered by each one of $\Lambda_1, \ldots, \Lambda_n$ is a *competitor* of π to the extent that π' *has priority over* π.

In particular, it may be desirable to apply the intuition outlined above in the context of controlled versions of linear logic and their possible connection with Petri-Nets with inhibition arcs [22, 31, 19, 24].

5 Conclusions and further work

This paper contributes to an ongoing research programme whose overall aim is to investigate the extent to which it is possible to accommodate distinctive features of classical and non-classical logics within a disciplined proof-theoretical framework [11, 17]. The expected achievements might cast light upon the precise relationship between classical and non-classical reasoning as well as upon the very connections among different non-classical proof systems. Specifically, the main goal of this paper has been to illustrate how cut-free controlled calculi derived from classical logic have the capacity to display both non-monotonic and paraconsistent behaviours. This opens the door to a modular treatment of non-monotonicity and paraconsistency which turns out to be conceptually uniform.

One may regard our perspective as similar in spirit to that of Makinson, who is concerned with 'bridging the gap' between classical and non-monotonic logic by means of a logical 'continuous' [26]. However, Makinson's work is essentially a *semantical* one, being focused on the notion of consequence relation. We produce instead fragments, or better, fibrations of classical logic able to preserve cut-elimination in spite of the specific constraints imposed by context-sensitiveness. (This point also marks the fundamental difference between our approach and that of the adaptive logics research program [5, 3, 4]). Moreover, the role of the 'pivotal' information in some of the approaches mentioned by Makinson is *positive*, i.e. the calculus is authorized to enlarge the set of its theorems. Namely, for any fixed cluster of hypotheses Σ, the consequence relation $\vdash_\Sigma$ extends the classical one as follows: $\Gamma \vdash_\Sigma \Delta$ if, and only if, $\Sigma, \Gamma \vdash \Delta$. By contrast, in our setting the role of the 'pivotal' information is played by the notion of control set which expresses a sort of negative information, that is an array of prohibitions.

In closing, we would like to briefly present in a bullet-point fashion some technical questions which still remain open and themes for further research.

- It is not still completely clear whether or not repositories are needed to ensure cut-elimination. However, we conjecture the answer to be negative. Perhaps, then, a possible way to do without such additional device might be that of considering one-side sequents and then impose the control on negative formulas. However, the price to pay would be a decrease in expressiveness due to the the fact that implication and negation disappear as explicit logical connectives.

- The definition of control set here proposed should be calibrated against some unpleasant phenomena of deductive diffraction like the following. Take a system $\mathcal{S}$ so that $\mathcal{S}(p) = \{\{q\}\}$ and $\mathcal{S}(q) = \varnothing$. The first of the two derivations displayed below is a paraproof, the second is a proof.

$$\dfrac{\dfrac{\dfrac{}{\cdot \,|\, p \vdash_{\{\{q\}\}} p}\,ax.}{\cdot \,|\, q, p \vdash_{\{\{q\}\}} p}\,weak \vdash}{\cdot \,|\, q, p \vdash_{\{\{q\}\}} p, q}\vdash weak \qquad\qquad \dfrac{\dfrac{\dfrac{}{\cdot \,|\, q \vdash_{\varnothing} q}\,ax.}{\cdot \,|\, q, p \vdash_{\varnothing} q}\,weak \vdash}{\cdot \,|\, q, p \vdash_{\varnothing} p, q}\vdash weak$$

  A desirable refinement of the notion of control set should be able to block the provability of the sequent $p, q \vdash p, q$ once that p introduces the singleton $\{q\}$ as a 'forbidden context'.

- Another interesting problem concerns the possibility of *unblocking* derivations that have been previously blocked. Given the derivability of A from Γ, we have seen how to block the derivability of A from Γ, B by introducing the control set $\{\{B\}\}$. But similarly, given some new incoming information C *denying* B, we should also be able to unblock the derivability of A from Γ, B, C. The notion of spectrum we have introduced in this paper is unfit for further purpose, since control sets increase their size along derivations.

- As already observed, controlled calculi are vulnerable to the criticism that they do not in general guarantee equalities (see Example 16 and Remark 17). Put in terms of the consequence relation, one could be tempted to overcome this difficulty by saying that controlled consequence relations are not necessarily reflexive. Indeed, the situation is not so clear-cut insofar as controlled calculi do not express a well-defined consequence relation, but rather they seem to be based on the combination of infinitely many different consequent relations, one for each control set. For this reason, a fruitful line of research could be that of approaching controlled calculi as a way to combine different *slices* of classical logic [13].

- Being defined as a set of sets, the notion of control set appears to involve a sort of higher order conceptualization in disguise. For this reason, it would be interesting to evaluate the possibility of reproducing the control sets device by resorting to *second-order* calculi, so as to avoid the resort to "external" decorations of sequents. This would put us in the position to raise the question of whether the natural logical level for dealing with non-monotonicity and paraconsistency is the second-order one.

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