Implicit Kripke Semantics and Ultraproducts in Stratified Institutions

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Abstract

We propose *stratified institutions* (a decade old generalised version of the theory of institutions of Goguen and Burstall) as a fully abstract model theoretic approach to modal logic. This allows for a uniform treatment of model theoretic aspects across the great multiplicity of contemporary modal logic systems. Moreover Kripke semantics (in all its manifold variations) is captured in an implicit manner free from the sometimes bulky aspects of explicit Kripke structures, also accommodating other forms of concrete semantics for modal logic systems. The conceptual power of stratified institutions is illustrated with the development of a modal ultraproducts method that is independent of the concrete details of the actual modal logical systems. Consequently, a wide array of compactness results in concrete modal logics may be derived easily.

1. Introduction

The model theory oriented formalisation by Goguen and Burstall [14] of the notion of a logical system as an *institution* has started a line of important developments of adequately abstract and general approaches to the foundations of software specifications and formal system development (see [20]) as well as a modern version of very abstract model theory (see [8]). One of the main original motivations for introducing institution theory was to respond to the explosion in the population of logics in use in computing about three decades ago, a situation that continues today perhaps at an accelerated pace. Among the logics with relevance in various areas of informatics there is of course the family of modal logics, with its great multiplicity of flavours. The recent works on 'modalizations' of institutions [9–11, 18] (see also [8]), in which only the modalities (and eventually nominals and @) and Kripke semantics are kept explicit, while the other ingredients (e.g. sorts, functions, predicates, constraints, etc.) are abstracted away, has intensified the quest for a fully abstract institution theoretic approach that has the potential to address adequately the specificities of modality and Kripke semantics while leaving none of these explicit.

Our paper proposes stratified institutions of [2] as a general framework for a fully abstract approach to the semantics of modal logic. In particular this means no explicit modalities, no explicit Kripke structures, while still retaining the essence of Kripke semantics. Consequently a very general form of model theory uniformly applicable to a wide range of concrete modal logic systems, either conventional or more eccentric, can be developed. Results can be developed in a top-down manner with hypotheses kept as general as possible and introduced on a by-need basis, the whole development process being guided by structurally clean causality. From the perspective of institution theory, our proposal yields an institution theoretic structure fully capable of addressing modality. The conventional definition of institution [14] may lack enough

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structure to capture various specificities of modal logics, hence our work can be regarded as a minimal but sufficient refinement of the concept of institution towards modal logics.

We illustrate the power of our concepts with the development of very general modal-oriented ultraproducts method. This provides rather automatically Łoś-style theorems [5, 16] for a wide range of concrete modal systems, as a puzzle of preservation results in the style of [7, 8, 11]. In conventional model theory the method of ultraproducts is renowned as extremely powerful and pervading a lot of deep results (see [5], for example), many of these been lifted to the level of abstract institutions (see [8]). Our developments may represent the beginning of a similar journey in the realm of modality and Kripke semantics. From the many consequences of ultraproducts, here we focus only on compactness results. Hence we derive a series of modal compactness results for our benchmark examples, this process having a generic nature.

Summary and Contributions.

1. We recall briefly some category and institution theoretic concepts and notations that are necessary for our paper.
2. We from [2] the concept of stratified institution and slightly upgrade it. Ordinary institutions arise as stratified institutions with a trivial stratification; in this way stratified institutions can be seen as more general than ordinary institutions. The move in the other direction is given by two general interpretations of stratified institutions as ordinary institutions. They represent high abstractions of the concepts of *local* and *global satisfaction* from modal logic, respectively.
3. We provide a series of examples of stratified institutions that include both conventional and eccentric modal logic systems. The former category includes propositional and first order modal logic, possibly with hybrid and polyadic modalities features, while the latter includes the double hybridization of [10, 17] and a first order valuation semantics for first order modal logic that is based upon the 'internal stratification' example introduced in [2]. These are to be used as benchmark examples for the further developments in the paper.
4. We give a straightforward extension of the well known institution theoretic semantics of the Boolean connectives $\wedge$, $\neg$, etc. and of the quantifiers $\forall$, $\exists$ to the more refined level of stratified institutions and establish the relationship with their correspondents from the local and the global institutions associated to the stratified institution.
5. We introduce a semantics for modalities and for hybrid features in abstract stratified institutions. This is one of the crucial contributions of this paper.
6. We extend the institution theoretic method of ultraproducts [7, 8] to stratified institutions. The core contributions here consist of a series of general preservation results across the abstract semantics for Boolean connectives, quantifiers, modalities, nominals, @. These cover related previous developments from [11] (also to be found in [8]), but with significant differences in generality: (1) stratified institutions with their lack of commitment to explicit modalities and Kripke structures are much more general than the 'modalized' institutions of [11]; (2) the results of our paper cover polyadic modalities and hybrid features while [11] considers only the unary $\Box$ and $\Diamond$. The above mentioned differences reflect very much in the way the preservation results are actually obtained.
7. Derivation of compactness properties for the local and the global institutions associated to a stratified institution via ultraproducts.

2. Category and institution theoretic preliminaries

In this section we recall some category and institution theoretic notions that will be used in the paper.

We will use the diagrammatic notation for compositions of arrows in categories, i.e. if $f: A \to B$ and $g: B \to C$ are arrows then $f;g$ denotes their composition. A *concrete category* $(\mathcal{A}, U)$ consists of a category $\mathcal{A}$ and a faithful functor $U: \mathcal{A} \to \mathbf{Set}$.[1] A functor of concrete categories $F: (\mathcal{A}, U) \to (\mathcal{B}, V)$ is just a functor $F: \mathcal{A} \to \mathcal{B}$ such that $U = F;V$. Let **CCAT** denote the category that has the concrete categories as objects and functors of concrete categories as arrows. When it is clear from the context we may omit U and simply refer to $(\mathcal{A}, U)$ as $\mathcal{A}$. This implies also that for $A \in |\mathcal{A}|$ we may write $a \in A$ instead of $a \in U(A)$. We use double arrow $\Rightarrow$ rather than single arrow $\to$ for natural transformations. A functor $\mathcal{U}: C \to C'$ *preserves* a (co-)limit μ of a functor $D: J \to C$ when $\mu\mathcal{U}$ is a (co-)limit of $D;\mathcal{U}$. It *lifts* a (co-)limit μ' of $D;\mathcal{U}$, if there exists a (co-)limit μ of D such that $\mu\mathcal{U} = \mu'$.

The original standard reference for definitions below of institutions and institution morphisms is [14].

Definition 2.1 (Institution). *An* institution $\mathcal{I} = (Sign^{\mathcal{I}}, Sen^{\mathcal{I}}, Mod^{\mathcal{I}}, \models^{\mathcal{I}})$ *consists of*

- *a category $Sign^{\mathcal{I}}$ whose objects are called* signatures,
- *a sentence functor $Sen^{\mathcal{I}}: Sign^{\mathcal{I}} \to \mathbf{Set}$ defining for each signature a set whose elements are called* sentences *over that signature and defining for each signature morphism a* sentence translation *function,*
- *a model functor $Mod^{\mathcal{I}}: (Sign^{\mathcal{I}})^{\mathrm{op}} \to \mathbf{CAT}$ defining for each signature Σ the category $Mod^{\mathcal{I}}(\Sigma)$ of* Σ-models *and Σ-model homomorphisms, and for each signature morphism φ the* reduct *functor $Mod^{\mathcal{I}}(\varphi)$,*
- *for every signature Σ, a binary* Σ-satisfaction relation $\models^{\mathcal{I}}_{\Sigma} \subseteq |Mod^{\mathcal{I}}(\Sigma)| \times Sen^{\mathcal{I}}(\Sigma)$,

such that for each morphism $\varphi: \Sigma \to \Sigma' \in Sign^{\mathcal{I}}$, the Satisfaction Condition

(1) $\quad M' \models^{\mathcal{I}}_{\Sigma'} Sen^{\mathcal{I}}(\varphi)(\rho)$ *if and only if* $Mod^{\mathcal{I}}(\varphi)(M') \models^{\mathcal{I}}_{\Sigma} \rho$

holds for each $M' \in |Mod^{\mathcal{I}}(\Sigma')|$ and $\rho \in Sen^{\mathcal{I}}(\Sigma)$.

$$\begin{tikzcd}
\Sigma \arrow[d, "\varphi"'] & \left|Mod^{\mathcal{I}}(\Sigma)\right| \arrow[r, dash, "\models^{\mathcal{I}}_{\Sigma}"] & Sen^{\mathcal{I}}(\Sigma) \arrow[d, "Sen^{\mathcal{I}}(\varphi)"] \\
\Sigma' & \left|Mod^{\mathcal{I}}(\Sigma')\right| \arrow[u, "Mod^{\mathcal{I}}(\varphi)"] \arrow[r, dash, "\models^{\mathcal{I}}_{\Sigma'}"'] & Sen^{\mathcal{I}}(\Sigma')
\end{tikzcd}$$

*We may omit the superscripts or subscripts from the notations of the components of institutions when there is no risk of ambiguity. For example, if the considered institution and signature are clear, we may denote $\models^{\mathcal{I}}_{\Sigma}$ just by $\models$. For $M = Mod(\varphi)(M')$, we say that M is the φ-*reduct *of M' and that M' is a φ-*expansion *of M.*

Notation 2.1. *In any institution as above we use the following notations:*

– *for any $E \subseteq Sen(\Sigma)$, E^* denotes $\{M \in |Mod(\Sigma)| \mid M \models_{\Sigma} \rho$ for each $\rho \in E\}$.*
– *for any $E, E' \subseteq Sen(\Sigma)$, $E \models E'$ denotes $E^* \subseteq E'^*$.*

Definition 2.2 (Compactness [8]). *An institution $\mathcal{I}$ is*

[1] This is most commonly accepted definition for concrete categories, although in [1] this is called ‘concrete over **Set**’ or ‘construct’.

– m-compact *when for each set E of Σ-sentences, $E^* \neq \emptyset$ if and only if for each $E_0 \subseteq E$ finite, $E_0^* \neq \emptyset$;*
– compact *when for each set E of Σ-sentences and each Σ-sentence ρ, if $E \models_\Sigma \rho$ then there exists a finite $E_0 \subseteq E$ such that $E_0 \models_\Sigma \rho$.*

Definition 2.3 (Morphism of institutions)**.** *Given two institutions $\mathcal{I}_i = (Sign_i, Sen_i, Mod_i, \models_i)$, with $i \in \{1, 2\}$, an institution morphism $(\Phi, \alpha, \beta) : \mathcal{I}_2 \to \mathcal{I}_1$ consists of*

- *a signature functor $\Phi : Sign_2 \to Sign_1$,*
- *a natural transformation $\alpha : Sen_1 \Rightarrow \Phi; Sen_2$, and*
- *a natural transformation $\beta : Mod_2 \Rightarrow \Phi^{\mathrm{op}}; Mod_1$*

such that the following satisfaction condition holds for any $\mathcal{I}_2$-signature Σ_2, Σ_2-model M_2 and $\Phi(\Sigma_2)$-sentence ρ:

$$M_2 \models_2 \alpha_{\Sigma_2}(\rho) \text{ if and only if } \beta_{\Sigma_2}(M_2) \models_1 \rho.$$

The literature (e.g. [8, 20]) shows myriads of logical systems from computing or from mathematical logic captured as institutions. In fact, an informal thesis underlying institution theory is that any 'logic' may be captured by the above definition. While this should be taken with a grain of salt, it certainly applies to any logical system based on satisfaction between sentences and models of any kind. The institutions introduced in the following couple of examples will be used intensively in the paper in various ways.

Example 2.1 (Propositional logic ($\mathcal{PL}$))**.** This is defined as follows. $Sign^{\mathcal{PL}} = \mathbf{Set}$, for any set P, $Sen(P)$ is generated by the grammar

$$S ::= P \mid S \wedge S \mid \neg S$$

and $Mod^{\mathcal{PL}}(P) = (2^P, \subseteq)$. For any function $\varphi : P \to P'$, $Sen^{\mathcal{PL}}(\varphi)$ replaces the each element $p \in P$ that occur in a sentence ρ by $\varphi(p)$, and $Mod^{\mathcal{PL}}(\varphi)(M') = \varphi; M$ for each $M' \in 2^{P'}$. For any P-model $M \subseteq P$ and $\rho \in Sen^{\mathcal{PL}}(P)$, $M \models \rho$ is defined by induction on the structure of ρ by $(M \models p) = (p \in M)$, $(M \models \rho_1 \wedge \rho_2) = (M \models \rho_1) \wedge (M \models \rho_2)$ and $(M \models \neg\rho) = \neg(M \models \rho)$.

Example 2.2 (First order logic ($\mathcal{FOL}$))**.** For reasons of simplicity of notation, our presentation of first order logic considers only its single sorted, without equality, variant. A detailed presentation of full many sorted first order logic with equality as institution may be found in numerous works in the literature (e.g. [8], etc.).

The $\mathcal{FOL}$ signatures are pairs $(F = (F_n)_{n\in\omega}, P = (P_n)_{n\in\omega})$ where F_n and P_n are sets of function symbols and predicate symbols, respectively, of arity n. Signature morphisms $\varphi : (F, P) \to (F', P')$ are tuples $(\varphi^f = (\varphi^f_n)_{n\in\omega}, \varphi^p = (\varphi^p_n)_{n\in\omega})$ such that $\varphi^f_n : F_n \to F'_n$ and $\varphi^p_n : P_n \to P'_n$. Thus $Sign^{\mathcal{FOL}} = \mathbf{Set}^\omega \times \mathbf{Set}^\omega$.

For any $\mathcal{FOL}$-signature (F, P), the set S of the (F, P)-sentences is generated by the grammar:

$$S ::= \pi(t_1, \ldots, t_n) \mid S \wedge S \mid \neg S \mid (\exists x)S' \tag{2}$$

where $\pi(t_1, \ldots, t_n)$ are the atoms with $\pi \in P_n$ and $t_1, \ldots, t_n$ being terms formed with function symbols from F, and where S' denotes the set of $(F + x, P)$-sentences with $F + x$ denoting the family of function symbols obtained by adding the single variable x to F_0.

An (F, P)-model M is a tuple

$$M = (|M|, \{M_\sigma : |M|^n \to |M| \mid \sigma \in F_n, n \in \omega\}, \{M_\pi \subseteq |M|^n \mid \pi \in P_n, n \in \omega\}).$$

where $|M|$ is a set called the *carrier of* M. An (F, P)-model homomorphism $h : M \to N$ is a function $|M| \to |N|$ such that $h(M_\sigma(x_1, \ldots, x_n)) = N_\sigma(h(x_1), \ldots, h(x_n))$ for any $\sigma \in F_n$ and $h(M_\pi) \subseteq N_\pi$ for each $\pi \in P_n$.

The satisfaction relation $M \models^{\mathcal{FOL}}_{(F,P)} \rho$ is the usual Tarskian style satisfaction defined on induction on the structure of the sentence ρ.

Given a signature morphism $\varphi : (F, P) \to (F', P')$, the induced sentence translation $Sen^{\mathcal{FOL}}(\varphi)$ just replaces the symbols of any (F, P)-sentence with symbols from (F', P') according φ, and the induced model reduct $Mod^{\mathcal{FOL}}(\varphi)(M')$ leaves the carrier set as it is and for any x function or predicate symbol of (F, P), it interprets x as $M'_{\varphi(x)}$.

In what follows we shall also consider the following parts (or 'sub-institutions') of $\mathcal{FOL}$ that are determined by restricting the $\mathcal{FOL}$ signatures as follows:

- $\mathcal{REL}$: no function symbols (hence $Sign^{\mathcal{REL}} \cong \mathbf{Set}^\omega$);
- $\mathcal{BREL}$: no function symbols and only one binary predicate symbol λ (hence $Sign^{\mathcal{BREL}} \cong \{\lambda\}$);
- $\mathcal{SETC}$: no predicate symbols and no function symbols of arity greater than 0 (hence $Sign^{\mathcal{SETC}} \cong \mathbf{Set}$);
- $\mathcal{BRELC}$: one binary predicate symbol and no function symbols of arity greater than 0 (hence $Sign^{\mathcal{BRELC}} \cong \mathbf{Set}$);

3. Stratified institutions

The structure and contents of this section is as follows:

1. We recall the definition of stratified institution of [2] and slightly upgrade it;
2. We provide two canonical extractions of ordinary institutions out of stratified institutions, corresponding to the local and global satisfaction in modal logic, respectively;
3. We present a series of examples of modal logical systems captured as stratified institutions.

3.1. Stratified institutions: the concept

Informally, the main idea behind the concept of stratified institution as introduced in [2] is to enhance the concept of institution with 'states' for the models. Thus each model M comes equipped with a *set* $[\![M]\!]$. A typical example is given by the Kripke models, where $[\![M]\!]$ is the set of the possible worlds in the Kripke structure M.

Definition 3.1 (Stratified institution)**.** *A* stratified institution $\mathcal{I} = (Sign^{\mathcal{I}}, Sen^{\mathcal{I}}, Mod^{\mathcal{I}}, [\![\_]\!]^{\mathcal{I}}, \models^{\mathcal{I}})$ *consists of:*

– *a category $Sign^{\mathcal{I}}$ of signatures,*
– *a sentence functor $Sen^{\mathcal{I}} : Sign^{\mathcal{I}} \to \mathbf{Set}$;*
– *a model functor $Mod^{\mathcal{I}} : (Sign^{\mathcal{I}})^{\mathrm{op}} \to \mathbf{CAT}$;*
– *a "stratification" lax natural transformation $[\![\_]\!]^{\mathcal{I}} : Mod^{\mathcal{I}} \Rightarrow SET$, where $SET : Sign^{\mathcal{I}} \to \mathbf{CAT}$ is a functor mapping each signature to* **Set***; and*
– *a satisfaction relation between models and sentences which is parameterized by model states, $M\ (\models^{\mathcal{I}})^w_\Sigma\ \rho$ where $w \in [\![M]\!]^{\mathcal{I}}_\Sigma$ such that*

$$(3) \qquad Mod^{\mathcal{I}}(\varphi)(M)\ (\models^{\mathcal{I}})^{[\![M]\!]_\varphi(w)}_\Sigma\ \rho \ \text{ if and only if } \ M\ (\models^{\mathcal{I}})^w_{\Sigma'}\ Sen^{\mathcal{I}}(\varphi)(\rho)$$

holds for any signature morphism $\varphi : \Sigma \to \Sigma'$, Σ'-model M, $w \in [\![M]\!]^{\mathcal{I}}_{\Sigma'}$, and Σ-sentence ρ.

Like for ordinary institutions, when appropriate we shall also use simplified notations without superscripts or subscripts that are clear from the context.

The lax natural transformation property of $[\![\_]\!]$ is depicted in the diagram below

$$
\begin{tikzcd}
\Sigma'' & & Mod(\Sigma'') \arrow[r, "{[\![\_]\!]_{\Sigma''}}"] \arrow[d, "Mod(\varphi')"'] \arrow[dr, bend left] \arrow[dr, phantom, "\Leftarrow {[\![\_]\!]_{\varphi'}}"] & \mathbf{Set} \arrow[d, "="] \\
\Sigma' \arrow[u, "\varphi'"'] & & Mod(\Sigma') \arrow[r, "{[\![\_]\!]_{\Sigma'}}"] \arrow[d, "Mod(\varphi)"'] \arrow[dr, bend left] \arrow[dr, phantom, "\Leftarrow {[\![\_]\!]_{\varphi}}"] & \mathbf{Set} \arrow[d, "="] \\
\Sigma \arrow[u, "\varphi"'] & & Mod(\Sigma) \arrow[r, "{[\![\_]\!]_{\Sigma}}"'] & \mathbf{Set}
\end{tikzcd}
$$

with the following compositionality property for each Σ''-model M'':

$$[\![M'']\!]_{(\varphi';\varphi)} = [\![M'']\!]_{\varphi'}; [\![Mod(\varphi')(M'')]\!]_{\varphi}.$$

Moreover the natural transformation property of each $[\![\_]\!]_{\varphi}$ is given by the commutativity of the following diagram:

$$
\begin{tikzcd}
M' \arrow[d, "h'"'] & & {[\![M']\!]_{\Sigma'}} \arrow[r, "{[\![M']\!]_{\varphi}}"] \arrow[d, "{[\![h']\!]_{\Sigma'}}"'] & {[\![Mod(\varphi)(M')]\!]_{\Sigma}} \arrow[d, "{[\![Mod(\varphi)(h')]\!]_{\Sigma}}"] \\
N' & & {[\![N']\!]_{\Sigma'}} \arrow[r, "{[\![N']\!]_{\varphi}}"'] & {[\![Mod(\varphi)(N')]\!]_{\Sigma}}
\end{tikzcd} \tag{4}
$$

The satisfaction relation can be presented as a natural transformation $\models : Sen \Rightarrow [\![Mod(\_) \to \mathbf{Set}]\!]$ where the functor $[\![Mod(\_) \to \mathbf{Set}]\!] : Sign \to \mathbf{Set}$ is defined by

- for each signature $\Sigma \in |Sign|$, $[\![Mod(\Sigma) \to \mathbf{Set}]\!]$ denotes the set of all the mappings $f : |Mod(\Sigma)| \to \mathbf{Set}$ such that $f(M) \subseteq [\![M]\!]_{\Sigma}$; and
- for each signature morphism $\varphi : \Sigma \to \Sigma'$, $[\![Mod(\varphi) \to \mathbf{Set}]\!](f)(M') = [\![M']\!]_{\varphi}^{-1}(f(Mod(\varphi)(M')))$.

A straightforward check reveals that the Satisfaction Condition (3) appears exactly as the naturality property of $\models$:

$$
\begin{tikzcd}
\Sigma \arrow[d, "\varphi"'] & & Sen(\Sigma) \arrow[r, "\models_{\Sigma}"] \arrow[d, "Sen(\varphi)"'] & {[\![Mod(\Sigma) \to \mathbf{Set}]\!]} \arrow[d, "{[\![Mod(\varphi)\to\mathbf{Set}]\!]}"] \\
\Sigma' & & Sen(\Sigma') \arrow[r, "\models_{\Sigma'}"'] & {[\![Mod(\Sigma') \to \mathbf{Set}]\!]}
\end{tikzcd}
$$

Ordinary institutions are the stratified institutions for which $[\![M]\!]_{\Sigma}$ is always a singleton set. In Dfn. 3.1 we have removed the surjectivity condition on $[\![M']\!]_{\varphi}$ from the definition of the stratified institutions of [2] and will rather make it explicit when necessary. This is motivated by the fact that most of the results developed do not depend upon this condition which however holds in all examples known by us. In fact in most of the examples $[\![M']\!]_{\varphi}$ are even identities, which makes $[\![\_]\!]$ a strict rather than lax natural transformation. A notable exception, when $[\![\_]\!]$ is a proper lax natural transformation is given by Ex. 3.6. Also the definition of stratified institution of [2] did not introduce $[\![\_]\!]$ as a lax natural transformation, but rather as an indexed family of mappings without much compositionality properties, which was enough for the developments in [2].

The following very expected property does not follow from the axioms of Dfn. 3.1, hence we impose it explicitly. It holds in all the examples discussed in this paper.

Assumption: In all considered stratified institutions the satisfaction is preserved by model isomorphisms, i.e. for each Σ-model isomorphism $h : M \to N$, each $w \in [\![M]\!]_\Sigma$, and each Σ-sentence ρ,

$$M \models^w \rho \text{ if and only if } N \models^{[\![h]\!](w)} \rho.$$

3.2. Reducing stratified institutions to ordinary institutions

The following construction will be used systematically in what follows for reducing stratified institution theoretic concepts to ordinary institution theoretic concepts, and consequently for reusing results from the latter to the former realm.

Fact 3.1. *Each stratified institution $\mathcal{I} = (Sign, Sen, Mod, [\![\_]\!], \models)$ determines the following ordinary institution $\mathcal{I}^\sharp = (Sign, Sen, Mod^\sharp, \models^\sharp)$ (called the* local institution of $\mathcal{I}$*) where*

- *the objects of $Mod^\sharp(\Sigma)$ are the pairs (M, w) such that $M \in |Mod(\Sigma)|$ and $w \in [\![M]\!]_\Sigma$;*
- *the Σ-homomorphisms $(M, w) \to (N, v)$ are the pairs (h, w) such that $h : M \to N$ and $[\![h]\!]_\Sigma(w) = v$;*
- *for any signature morphism $\varphi : \Sigma \to \Sigma'$ and any Σ'-model (M', w')*

$$Mod^\sharp(\varphi)(M', w') = (Mod(\varphi)(M'), [\![M']\!]_\varphi(w'));$$

- *for each Σ-model M, each $w \in [\![M]\!]_\Sigma$, and each $\rho \in Sen(\Sigma)$*

$$((M, w) \models^\sharp_\Sigma \rho) = (M \models^w_\Sigma \rho).$$

The preservation of $\models$ under model isomorphisms imply the preservation of $\models^\sharp$ under model isomorphisms. This follows immediately by noting that (h, w) is a model isomorphism in $\mathcal{I}^\sharp$ if and only if h is a model isomorphism in $\mathcal{I}$.

The following second interpretation of stratified institutions as ordinary institutions has been given in [2]. Note that unlike $\mathcal{I}^\sharp$ above, $\mathcal{I}^*$ below shares with $\mathcal{I}$ the model functor.

Definition 3.2. *For any stratified institution $\mathcal{I} = (Sign, Sen, Mod, [\![\_]\!], \models)$ we say that $[\![\_]\!]$* is surjective *when for each signature morphism $\varphi : \Sigma \to \Sigma'$ and each Σ'-model M', $[\![M']\!]_\varphi : [\![M']\!]_{\Sigma'} \to [\![Mod(\varphi)(M')]\!]_\Sigma$ is surjective.*

Fact 3.2. *Each stratified institution $\mathcal{I} = (Sign, Sen, Mod, [\![\_]\!], \models)$ with $[\![\_]\!]$ surjective determines an (ordinary) institution $\mathcal{I}^* = (Sign, Sen, Mod, \models^*)$ (called the* global institution of $\mathcal{I}$*) by defining*

$$(M \models^*_\Sigma \rho) = \bigwedge\{M \models^w_\Sigma \rho \mid w \in [\![M]\!]_\Sigma\}.$$

Fact 3.3. *Let $\mathcal{I}$ be a stratified institution $\mathcal{I}$ with $[\![\_]\!]$ surjective. For each $E \subseteq Sen(\Sigma)$ and each $\rho \in Sen(\Sigma)$, we have that*

$$E \models^\sharp \rho \text{ implies } E \models^* \rho.$$

The institutions $\mathcal{I}^\sharp$ and $\mathcal{I}^*$ represent generalizations of the concepts of local and global satisfaction, respectively, from modal logic (e.g. [4]).

3.3. Examples of stratified institutions

Example 3.1 (Modal propositional logic ($\mathcal{MPL}$))**.** This is the most common form of modal logic (e.g. [4], etc.).

Let $Sign^{\mathcal{MPL}} = \mathbf{Set}$. For any signature P, commonly referred to as 'set of propositional variables', the set of its sentences $Sen^{\mathcal{MPL}}(P)$ is the set S defined by the following grammar

(5) $\quad S \ ::= \ P \mid S \wedge S \mid \neg S \mid \Diamond S$

A P-model is Kripke structure (W, M) where

- $W = (|W|, W_\lambda)$ consists of set (of 'possible worlds') $|W|$ and an 'accesibility' relation $W_\lambda \subseteq |W| \times |W|$; and
- $M : \ |W| \to 2^P$.

A homomorphism $h : \ (W, M) \to (V, N)$ between Kripke structures is a homomorphism of binary relations $h : \ W \to V$ (i.e. $h : \ |W| \to |V|$ such that $h(W_\lambda) \subseteq V_\lambda$) and such that for each $w \in |W|$, $M^w \subseteq N^{h(w)}$.

The satisfaction of any P-sentence ρ in a Kripke structure (W, M) at $w \in |W|$ is defined by recursion on the structure of ρ:

- $((W, M) \models^w_P \pi) = (\pi \in M^w)$;
- $((W, M) \models^w_P \rho_1 \wedge \rho_2) = ((W, M) \models^w_P \rho_1) \wedge ((W, M) \models^w_P \rho_2)$;
- $((W, M) \models^w_P \neg\rho) = \neg((W, M) \models^w_P \rho)$; and
- $((W, M) \models^w_P \Diamond\rho) = \bigvee_{(w,w')\in W_\lambda}((W, M) \models^{w'}_P \rho)$.

For any function $\varphi : \ P \to P'$ the φ-translation of a P-sentence just replaces each $\pi \in P$ by $\varphi(\pi)$ and the φ-reduct of a P'-structure (W, M') is the P-structure (W, M) where for each $w \in |W|$, $M^w = \varphi; M'^w$.

The stratification is defined by $[\![(W, M)]\!]_P = |W|$.

Various 'sub-institutions' of $\mathcal{MPL}$ are obtained by restricting the semantics to particular classes of frames. Important examples are $\mathcal{MPL}t$, $\mathcal{MPL}s4$, and $\mathcal{MPL}s5$ which are obtained by restricting the frames W to those which are respectively, reflexive, preorder, or equivalence (see e.g. [4]).

Example 3.2 (First order modal logic ($\mathcal{MFOL}$))**.** First order modal logic [12] extends classical first order logic with modalities in the same way propositional modal logic extends classical propositional logic. However there are several variants that differ slightly in the approach of the quantifications. Here we present a capture of one of the most common variants of first order modal logic as a stratified institution.

$\mathcal{MFOL}$ has the category of signatures of $\mathcal{FOL}$ but for the sentences adds $S \ ::= \Diamond S$ to the $\mathcal{FOL}$ grammar (2). The $\mathcal{MFOL}$ (F, P)-models upgrade the $\mathcal{MPL}$ Kripke structures (W, M) to the first order situation by letting $M : \ |W| \to |Mod^{\mathcal{FOL}}(F, P)|$ such that the following sharing conditions hold: for any $i, j \in |W|$, $|M^i| = |M^j|$ and also $M^i_x = M^j_x$ for each constant $x \in F_0$. The concept of $\mathcal{MFOL}$-model homomorphism is also an upgrading of the concept of $\mathcal{FOL}$-model homomorphism as follows: $h : \ (W, M) \to (V, N)$ is pair (h_0, h_1) where $h_0 : \ W \to V$ is a homomorphism of binary relations (like in $\mathcal{MPL}$) and $h_1 : \ M^w \to N^{h_0(w)}$ is an (F, P)-homomorphism of $\mathcal{FOL}$-models for each $w \in |W|$.

The satisfaction $(W, M) \models^{\mathcal{MFOL}}_{(F,P)} \rho$ is defined by recursion on the structure of ρ, like in $\mathcal{MPL}$ for $\wedge$, $\neg$, and $\Diamond$, for the atoms the $\mathcal{FOL}$ satisfaction relation is used, and for the quantifier case $(W, M) \models_{(F,P)} (\exists x)\rho$ if and only if there is a valuation of x into $|M|$ such that $(W, M') \models_{(F+x,P)} \rho$ for the corresponding expansion (W, M') of (W, M) to $(F+x, P)$. (This makes sense because in any $\mathcal{MFOL}$ Kripke structure the interpretations of the carriers and of the constants are shared.)

The translation of sentences and the model reducts corresponding to an $\mathcal{MFOL}$ signature morphism are obtained by the obvious blend of the corresponding translations and reducts, respectively, in $\mathcal{MPL}$ and $\mathcal{FOL}$.

The stratification is like in $\mathcal{MPL}$, with $[\![(W,M)]\!]_{(F,P)} = |W|$.

In the institution theory literature (e.g. [8, 9, 11, 18]) first order modal logic is often considered in a more general form in which the symbols that have shared interpretations are 'user defined' rather than being 'predefined' like here. In short this means that the signatures exhibit designated symbols (sorts, function, or predicate) that are 'rigid' in the sense that in a given Kripke structure they share the same interpretations across the possible worlds. For the single reason of making the reading easier we stick here with a simpler variant that has constants and the single sort being predefined as rigid.

Example 3.3 (Hybrid logics ($\mathcal{HPL}$, $\mathcal{HFOL}$))**.** Hybrid logics [3, 19] refine modal logics by adding explicit syntax for the possible worlds. Our presentation of hybrid logics as stratified institutions is related to the recent institution theoretic works on hybrid logics [9, 18].

The refinement of modal logics to hybrid ones is achieved by adding a set component (Nom) to the signatures for the so-called 'nominals' and by adding to the respective grammars

(6) $\quad S ::= i\text{-sen} \mid @_i S \mid (\exists i)S'$

where $i \in \mathrm{Nom}$ and S' is the set of the sentences of the signature that extends Nom with the nominal variable i. The models upgrade the respective concepts of Kripke structures to (W, M) by adding to W interpretations of the nominals, i.e. $W = (|W|, \{W_i \in |W| \mid i \in \mathrm{Nom}\}, W_\lambda)$. The satisfaction relations between models (i.e. Kripke structures) and sentences extend the satisfaction relations of the corresponding non-hybrid modal institutions with

- $((W,M) \models^w i\text{-sen}) = (W_i = w)$;
- $((W,M) \models^w @_i\rho) = ((W,M) \models^{W_i} \rho)$; and
- $((W,M) \models^w (\exists i)\rho) = \bigvee\{(W',M) \models^w \rho \mid W' \text{ expansion of } W \text{ to } \mathrm{Nom}{+}i\}$.

Note that quantifiers over nominals allow us to simulate the binder operator $(\downarrow\rho)$ of [15] by $(\forall i)i \Rightarrow \rho$.

The translation of sentences and model reducts corresponding to signature morphisms are canonical extensions of the corresponding concepts from $\mathcal{MPL}$ and $\mathcal{MFOL}$.

The stratifications of $\mathcal{HPL}$ and $\mathcal{HFOL}$ are like for $\mathcal{MPL}$ and $\mathcal{MFOL}$, i.e. $[\![(W,M)]\!]_{(\mathrm{Nom},\Sigma)} = |W|$.

Example 3.4 (Polyadic modalities ($\mathcal{MMPL}$, $\mathcal{MHPL}$, $\mathcal{MMFOL}$, $\mathcal{MHFOL}$))**.** Multi-modal logics (e.g. [13]) exhibit several modalities instead of only the traditional $\Diamond$ and $\Box$ and moreover these may have various arities. If one considers the sets of modalities to be variable then they have to be considered as part of the signatures. We may extend each of $\mathcal{MPL}$, $\mathcal{HPL}$, $\mathcal{MFOL}$ and $\mathcal{HFOL}$ to the multi-modal case,

- by adding an '$\mathcal{M}$' in front of each of these names;
- by adding a component $\Lambda = (\Lambda_n)_{n\in\omega}$ to the respective signature concept (with Λ_n standing for the modalities symbols of arity n), e.g. an $\mathcal{MHFOL}$ signature would be a tuple of the form $(\mathrm{Nom}, \Lambda, (F,P))$;
- by replacing in the respective grammars the rule $S ::= \Diamond S$ by the set of rules

$$\{S ::= \langle\lambda\rangle S^n \mid \lambda \in \Lambda_{n+1}, n \in \omega\};$$

- by replacing the binary relation W_λ from the models (W,M) with a set of interpretations $\{W_\lambda \subseteq |W|^n \mid \lambda \in \Lambda_n, n \in \omega\}$.

Consequently the definition of the satisfaction relation gets upgraded with

$$\text{for each } \lambda \in \Lambda_{n+1},\ ((W,M) \models^{w} \langle\lambda\rangle(\rho_1,\dots,\rho_n)) = (\bigvee_{(w,w_1,\dots,w_n)\in W_\lambda} \bigwedge_{1\leq i\leq n} (W,M) \models^{w_i} \rho_i).$$

The stratification is the same like in the previous examples, i.e. $[\![(W,M)]\!]_{(\mathrm{Nom},\Lambda,\Sigma)} = |W|$.

Example 3.5 (Modalizations of institutions; $\mathcal{HHPL}$)**.** In a series of works [9, 11, 18] modal logic and Kripke semantics are developed by abstracting away details that do not belong to modality, such as sorts, functions, predicates, etc. This is achieved by extensions of abstract institutions (in the standard situations meant in principle to encapsulate the atomic part of the logics) with the essential ingredients of modal logic and Kripke semantics. The result of this process, when instantiated to various concrete logics (or to their atomic parts only) generate uniformly a wide range of hierarchical combinations between various flavours of modal logic and various other logics. Concrete examples discussed in [9, 11, 18] include various modal logics over non-conventional structures of relevance in computing science, such as partial algebra, preordered algebra, etc. Various constraints on the respective Kripke models, many of them having to do with the underlying non-modal structures, have also been considered. All these arise as examples of stratified institutions like the examples presented above in the paper. This great multiplicity of non-conventional modal logics constitute an important range of applications for this work.

An interesting class of examples that has emerged quite smoothly out of the general works on hybridization[2] of institutions is that of multi-layered hybrid logics that provide a logical base for specifying hierarchical transition systems (see [17]). As a single simple example let us present here the double layered hybridization of propositional logic, denoted $\mathcal{HHPL}$.[3] This amounts to a hybridization of $\mathcal{HPL}$, its models thus being “Kripke structures of Kripke structures”.

The $\mathcal{HHPL}$ signatures are triples $(\mathrm{Nom}^0, \mathrm{Nom}^1, P)$ with Nom^0 and Nom^1 denoting the nominals of the first and second layer of hybridization, respectively. The $(\mathrm{Nom}^0, \mathrm{Nom}^1, P)$-sentences are built over the two hybridization layers by taking the (Nom^0, P)-sentences as atoms in the grammar for the $\mathcal{HPL}$ sentences with nominals from Nom^1. In order to prevent potential ambiguities, in general we tag the symbols of the respective layers of hybridization by the superscripts 0 (for the first layer) and 1 (for the second layer). This convention should include nominals and connectives ($\Diamond$, $\wedge$, etc.) as well as quantifiers. For instance, the expression $@_{j^1} k^0 \wedge^1 \Box^1 \rho$ is a sentence of $\mathcal{HHPL}$ where the symbols k and j represent nominals of the first and second level of hybridization and ρ a $\mathcal{PL}$ sentence. On the other hand, according to this tagging convention the expression $@_{j^0} k^1 \wedge^1 \Box^1 \rho$ would not parse.

Our tagging convention extends also to $\mathcal{HHPL}$ models. A $(\mathrm{Nom}^0, \mathrm{Nom}^1, P)$-model is a pair (W^1, M^1) with W^1 being a $Mod^{\mathcal{BREL}}(\lambda)$ model and $M^1 = ((M^1)^w)_{w\in|W^1|}$ where $(M^1)^w$ is a (Nom^0, P)-model in $\mathcal{HPL}$, denoted $((W^0)^w, (M^0)^w)$. We also require that for all $w, w' \in |W^1|$, we have that $|(W^0)^w| = |(W^0)^{w'}|$ and $(W^0)^w_i = (W^0)^{w'}_i$ for each $i \in \mathrm{Nom}^0$.

These definitions extend in the obvious way to signature morphisms, sentence translations, model reducts and satisfaction relation. We leave these details as exercise for the reader. Then $\mathcal{HHPL}$ has the same stratified structure like $\mathcal{HPL}$ and $\mathcal{HFOL}$, namely $[\![(W^1, M^1)]\!]_{(\mathrm{Nom}^0,\mathrm{Nom}^1,P)} = |W^1|$.

It is easy to see that in $\mathcal{HHPL}$ the semantics of the Boolean connectors and of the quantifications with nominals of the lower layer is invariant with respect to the hybridization layer; this means that in these cases the tagging is not necessary. For example if ρ is an $\mathcal{HPL}$ sentence then $(\forall^1 i^0)\rho$ and $(\forall^0 i^0)\rho$ are semantically

[2] I.e. Modalization including also hybrid logic features.

[3] Other interesting examples that may be obtained by double or multiple hybridizations of logics would be $\mathcal{HHFOL}$, $\mathcal{HHHPL}$, etc., and also their polyadic multi-modalities extensions.

equivalent, while if ρ is not an $\mathcal{HPL}$ sentence (which means it has some ingredients from the second layer of hybridization) then $(\forall^0 i^0)\rho$ would not parse. In both cases just using the notation $(\forall i^0)$ would not carry any ambiguities.

The next series of examples include multi-modal first order logics whose semantics are given by ordinary first order rather than Kripke structures.

Example 3.6 (Multi-modal open first order logic ($\mathcal{OFOL}$, $\mathcal{MOFOL}$, $\mathcal{HOFOL}$, $\mathcal{HMOFOL}$)). The stratified institution $\mathcal{OFOL}$ is a the $\mathcal{FOL}$ instance of $St(\mathcal{I})$, the 'internal stratification' abstract example developed in [2]. An $\mathcal{OFOL}$ signature is a pair (Σ, X) consisting of $\mathcal{FOL}$ signature Σ and a finite block of variables. An $\mathcal{OFOL}$ signature morphism $\varphi : (\Sigma, X) \to (\Sigma', X')$ is just a $\mathcal{FOL}$ signature morphism $\varphi : \Sigma \to \Sigma'$ such that $X \subseteq X'$.

We let $Sen^{\mathcal{OFOL}}((F, P), X) = Sen^{\mathcal{FOL}}(F + X, P)$ and $Mod^{\mathcal{OFOL}}((F, P), X) = Mod^{\mathcal{FOL}}(F, P)$.

For each $((F, P), X)$-model M, each $w \in |M|^X$, and each $((F, P), X)$-sentence ρ we define

$$(M(\models^{\mathcal{OFOL}}_{(F,P),X})^w \rho) = (M^w \models^{\mathcal{FOL}}_{(F+X,P)} \rho)$$

where M^w is the expansion of M to $(F + X, P)$ such that $M^w_X = w$. This is a stratified institution with $[\![M]\!]_{\Sigma,X} = |M|^X$ for each (Σ, X)-model M. For any signature morphism $\varphi : (\Sigma, X) \to (\Sigma', X')$ and any (Σ', X')-model M', $[\![M']\!]_\varphi : |M'|^{X'} \to |M'|^X$ is defined by $[\![M']\!]_\varphi(a) = a|_X$ (i.e. the restriction of a to X). Note that $[\![M']\!]_\varphi$ is surjective and that this provides an example when $[\![\_]\!]$ is a proper lax natural transformation.

We may refine $\mathcal{OFOL}$ to a multi-modal logic ($\mathcal{MOFOL}$) by adding

$$\{S ::= \langle\pi\rangle S^n \mid \pi \in P_{n+1}, n \in \omega\}$$

to the grammar defining each $Sen^{\mathcal{OFOL}}((F, P), X)$ and consequently by extending the definition of the satisfaction relation with

- $(M \models^w \langle\pi\rangle(\rho_1, \ldots, \rho_n)) = \bigvee_{(w,w_1,\ldots,w_n)\in(M^X)_\pi} \bigwedge_{1\le i\le n}(M \models^{w_i} \rho_i)$ for each $\pi \in P_{n+1}$, $n \in \omega$.

(Here and elsewhere M^X denotes the X-power of M in the category of $\mathcal{FOL}$ (F, P)-models.)

Or else we may refine $\mathcal{OFOL}$ with nominals ($\mathcal{HOFOL}$) by adding the grammar for nominals (6), for each constant $i \in F_0$, to the grammar defining each $Sen^{\mathcal{OFOL}}((F, P), X)$ and consequently extending the definition of the satisfaction relation with

- $(M \models^w_{(F,P),X} i\text{-sen}) = ((M^X)_i = w)$;
- $M \models^w_{(F,P),X} @_i\rho) = (M \models^{(M^X)_i}_{(F,P),X} \rho)$;
- $(M \models^w_{(F,P),X} (\exists i)\rho) = \bigvee\{M' \models^w_{(F+i,P),X} \rho \mid M' \text{ expansion of } M \text{ to } (F+i, P)\}$.

We can also have $\mathcal{HMOFOL}$ as the blend between $\mathcal{HOFOL}$ and $\mathcal{MOFOL}$.

4. The logic of stratified institutions

We start the section by extending the definition of the semantics of Boolean connectives and quantifiers from ordinary institutions (see [7, 8, 22] etc.) to stratified institutions. After this, based on the stratified structure of stratified institutions, we define the semantics of modalities, nominals, @ at the level of abstract stratified institutions. In each of these cases a minimally sufficient additional structure is employed.

Definition 4.1. *In any stratified institution* $\mathcal{I} = (Sign, Sen, Mod, [\![\_]\!], \models)$

- *a* Σ*-sentence* $\rho_1 \wedge \rho_2$ *is an* external conjunction *of* Σ*-sentences* ρ_1 *and* ρ_2 *when for each* Σ*-model* M *and each* $w \in [\![M]\!]_\Sigma$,

  $$(M \models^w \rho_1 \wedge \rho_2) = (M \models^w \rho_1) \wedge (M \models^w \rho_2);$$

- *a* Σ*-sentence* $\rho_1 \Rightarrow \rho_2$ *is an* external implication *of* Σ*-sentences* ρ_1 *and* ρ_2 *when for each* Σ*-model* M *and each* $w \in [\![M]\!]_\Sigma$,

  $$(M \models^w \rho_1 \Rightarrow \rho_2) = (M \models^w \rho_1) \Rightarrow (M \models^w \rho_2);$$

- *a* Σ*-sentence* $\rho_1 \vee \rho_2$ *is an* external disjunction *of* Σ*-sentences* ρ_1 *and* ρ_2 *when for each* Σ*-model* M *and each* $w \in [\![M]\!]_\Sigma$,

  $$(M \models^w \rho_1 \vee \rho_2) = (M \models^w \rho_1) \vee (M \models^w \rho_2);$$

- *a* Σ*-sentence* $\neg\rho$ *is the* external negation *of a* Σ*-sentence* ρ *when for each* Σ*-model* M *and each* $w \in [\![M]\!]_\Sigma$,

  $$(M \models^w \neg\rho) = \neg(M \models^w \rho)$$

- *a* Σ*-sentence* $(\forall\chi)\rho'$ *is an* external universal χ-quantification *of a* Σ'*-sentence* ρ' *for* $\chi : \Sigma \to \Sigma'$ *signature morphism when for any* Σ*-model* M *and each* $w \in [\![M]\!]_\Sigma$

  $$(M \models^w_\Sigma (\forall\chi)\rho') = \bigwedge_{Mod(\chi)(M')=M} \Big(\bigwedge_{w' \in [\![M']\!]_\chi^{-1}(w)} (M' \models^{w'}_{\Sigma'} \rho') \Big)$$

- *a* Σ*-sentence* $(\exists\chi)\rho'$ *is an* external existential χ-quantification *of a* Σ'*-sentence* ρ' *for* $\chi : \Sigma \to \Sigma'$ *signature morphism when for any* Σ*-model* M *and each* $w \in [\![M]\!]_\Sigma$

  $$(M \models^w_\Sigma (\exists\chi)\rho') = \bigvee_{Mod(\chi)(M')=M} \Big(\bigvee_{w' \in [\![M']\!]_\chi^{-1}(w)} (M' \models^{w'}_{\Sigma'} \rho') \Big)$$

Remark 4.1. In Dfn. 4.1 the notations $\rho_1 \wedge \rho_2$, $\neg\rho$, etc. are meta-notations in the sense that they may not correspond to how the actual sentences appear in *Sen*. For example in $Sen^{\mathcal{MPL}}(\{\pi, \pi'\})$ (see Ex. 3.1), according to the respective grammar, there is no actual sentence such as $\pi \Rightarrow \pi'$, however $\mathcal{MPL}$ has implications, in the realm of the meta notations $\pi \Rightarrow \pi'$ corresponding to the actual sentence $\neg(\pi \wedge \neg\pi')$. So, these meta-notations of Dfn. 4.1 rather denote semantical equivalence classes of sentences[4], which goes well with our work since here we never need to distinguish between semantically equivalent sentences. We will keep employing such meta-notations also below in the paper when introducing the semantics for modalities (Dfn. 4.3) or for the hybrid features (Dfn 4.5).

On the one hand, the concepts of Boolean connectives and quantifications in ordinary institutions (e.g. from [7, 8, 21] etc.) arise as an instance of Dfn. 4.1 when the underlying set of each $[\![M]\!]_\Sigma$ is a singleton set. On the other hand, Fact 4.1 below shows that Dfn. 4.1 is not a proper generalization of the corresponding ordinary institution theoretic concepts since the stratified institution theoretic concepts of Boolean connectives and quantifications may also be regarded as corresponding instances of the respective ordinary institution theoretic concepts. The importance of Dfn. 4.1 resides thus in the fact that it gives an explicit account of how Boolean connectors and quantifications reflect in a stratified setup.

[4]Classes of sentences that hold exactly in the same models.

Fact 4.1. *When they exist, the conjunctions, disjunctions, implications, negations, universal/existential χ-quantifications coincide in $\mathcal{I}$ and $\mathcal{I}^\sharp$.*

Corollary 4.1. *In any stratified institution we have the following:*

1. $\neg(\neg\rho_1 \wedge \neg\rho_2)$ *is an external disjunction* $\rho_1 \vee \rho_2$*;*
2. $\neg\rho_1 \vee \rho_2$ *is an external implication* $\rho_1 \Rightarrow \rho_2$*;*
3. $\neg(\exists\chi)\neg\rho$ *is an external universal quantification* $(\forall\chi)\rho$*.*

Proposition 4.1. *In any stratified institution $\mathcal{I}$ with $[\![\_]\!]$ surjective*

1. *any external conjunctions in $\mathcal{I}$ is an external conjunction in $\mathcal{I}^*$ too; and*
2. *for any signature morphism χ, any external universal χ-quantifications in $\mathcal{I}$ is an external universal χ-quantifications in $\mathcal{I}^*$ too.*

Proof. 1. For each Σ-model M and any conjunction $\rho_1 \wedge \rho_2$ in $\mathcal{I}$ we have that

$$\begin{array}{rcll} M \models^* \rho_1 \wedge \rho_2 & = & \bigwedge_{w\in[\![M]\!]}(M \models^w \rho_1 \wedge \rho_2) & \text{(by definition of } \models^*) \\ & = & \bigwedge_{w\in[\![M]\!]}((M \models^w \rho_1) \wedge (M \models^w \rho_2)) & \text{(since } \rho_1 \wedge \rho_2 \text{ is conjunction in } \mathcal{I}) \\ & = & (\bigwedge_{w\in[\![M]\!]}(M \models^w \rho_1)) \wedge (\bigwedge_{w\in[\![M]\!]}(M \models^w \rho_2)) & \\ & = & (M \models^* \rho_1) \wedge (M \models^* \rho_2) & \text{(by definition of } \models^*). \end{array}$$

2. Let M be a Σ-model and $(\forall\chi)\rho$ a universally quantified Σ-sentence in $\mathcal{I}$ for $\chi\colon\ \Sigma \to \Sigma'$ signature morphism. We have that

$$\begin{array}{rcl} M \models^*_\Sigma (\forall\chi)\rho & = & \bigwedge_{w\in[\![M]\!]}(M \models^w_\Sigma (\forall\chi)\rho) \\ & = & \bigwedge\{(M' \models^{w'}_{\Sigma'} \rho \mid w \in [\![M]\!], Mod(\chi)(M') = M, w' \in [\![M']\!]^{-1}_\chi(w)\}. \end{array} \tag{7}$$

On the other hand we have that

$$\bigwedge_{Mod(\chi)(N')=M} (N' \models^*_{\Sigma'} \rho') = \bigwedge_{Mod(\chi)(N')=M} \Big(\bigwedge_{v'\in[\![N']\!]_{\Sigma'}} (N' \models^{v'}_{\Sigma'} \rho)\Big) \tag{8}$$

In order to show that $(\forall\chi)\rho$ is an external universal quantification in $\mathcal{I}^*$ we have to prove that the values in the equations (7) and (8) are equal.

$\boxed{(7) \leq (8)}$ For each $Mod(\chi)(N') = M$ and $w' \in [\![N']\!]_{\Sigma'}$ like in (8) we consider $M' = N'$, $w' = v'$ and $w = [\![M']\!]_\chi(w')$ in (7).

$\boxed{(8) \leq (7)}$ For each $w \in [\![M]\!]$, $Mod(\chi)(M') = M$ and $w' \in [\![M']\!]^{-1}_\chi(w)$ like in (7) we take $N' = M'$ and $v' = w'$ in (8). □

In general, $\mathcal{I}^*$ may lack other connectives besides conjunction and also the existential quantifications that $\mathcal{I}$ does have.

Definition 4.2 (Frame extraction)**.** *Given a stratified institution $\mathcal{I}$, a* frame extraction *is a pair L, Fr consisting of a functor $L\colon\ Sign^{\mathcal{I}} \to Sign^{\mathcal{REL}}$ and a lax natural transformation $Fr\colon\ Mod^{\mathcal{I}} \Rightarrow L; Mod^{\mathcal{REL}}$ such that $[\![\_]\!] = Fr; L(Mod^{\mathcal{REL}} \Rightarrow SET)$.*

$$\begin{tikzcd} Mod(\Sigma) \arrow[rr, "{[\![\_]\!]_\Sigma}"] \arrow[dr, "Fr_\Sigma"'] & & \mathbf{Set} \\ & Mod^{\mathcal{REL}}(L(\Sigma)) \arrow[ur, "forgetful"'] & \end{tikzcd}$$

Example 4.1. The following table shows some frame extractions for the stratified institutions introduced above.

stratified institution	L	Fr
$\mathcal{MPL}, \mathcal{MFOL}, \mathcal{HPL}, \mathcal{HFOL}, \mathcal{HHPL}$	$L(\Sigma) = \{\lambda : 2\}$	$Fr_\Sigma(W, M) = (\lvert W\rvert, W_\lambda)$
$\mathcal{MMPL}, \mathcal{MMFOL}, \mathcal{MHPL}, \mathcal{MHFOL}$	$L(\Sigma, \Lambda) = \Lambda$	$Fr_\Sigma(W, M) = (\lvert W\rvert, (W_\lambda)_{\lambda \in \Lambda_{n+1}, n\in\omega})$
$\mathcal{MOFOL}, \mathcal{HMOFOL}$	$L((F, P), X) = P$	$Fr_\Sigma(M) = (\lvert M\rvert^X, ((M^X)_\pi)_{\pi \in P_{n+1}, n\in\omega})$.

Definition 4.3. *Let $\mathcal{I}$ be a stratified institution endowed with a frame extraction L, Fr. For any $\lambda \in L(\Sigma)_{n+1}$ and any Σ-sentences $\rho_1, \ldots, \rho_n$*

- *a Σ-sentence $\langle\lambda\rangle(\rho_1, \ldots \rho_n)$ is an* external λ-possibility of *$\rho_1, \ldots, \rho_n$ when*

$$(M \models^w \langle\lambda\rangle(\rho_1, \ldots \rho_n)) = \bigvee_{(w, w_1, \ldots, w_n) \in (Fr_\Sigma(M))_\lambda} (\bigwedge_{1 \le i \le n} M \models^{w_i} \rho_i);$$

- *a Σ-sentence $[\lambda](\rho_1, \ldots \rho_n)$ is an* external λ-necessity of *$\rho_1, \ldots, \rho_n$ when*

$$(M \models^w [\lambda](\rho_1, \ldots \rho_n)) = \bigwedge_{(w, w_1, \ldots, w_n) \in (Fr_\Sigma(M))_\lambda} (\bigvee_{1 \le i \le n} M \models^{w_i} \rho_i);$$

for each Σ-model M and for each $w \in [\![M]\!]_\Sigma$.

Fact 4.2. *In any stratified institution like in Dfn. 4.3, $\neg\langle\lambda\rangle(\neg\rho, \ldots, \neg\rho_n)$ is a λ-necessity of $\rho_1, \ldots, \rho_n$.*

Definition 4.4 (Nominals extraction)**.** *Given a stratified institution $\mathcal{I}$, a* nominals extraction *is a pair N, Nm consisting of a functor $N : Sign^{\mathcal{I}} \to Sign^{\mathcal{SETC}}$ and a lax natural transformation $Nm : Mod^{\mathcal{I}} \Rightarrow N; Mod^{\mathcal{SETC}}$ such that $[\![\_]\!] = Nm; N(Mod^{\mathcal{SETC}} \Rightarrow SET)$.*

$$\begin{tikzcd} Mod(\Sigma) \arrow[rr, "{[\![\_]\!]_\Sigma}"] \arrow[dr, "Nm_\Sigma"'] & & \mathbf{Set} \\ & Mod^{\mathcal{SETC}}(N(\Sigma)) \arrow[ur, "forgetful"'] & \end{tikzcd}$$

Example 4.2. The following table shows some nominals extractions for the stratified institutions introduced above. Note that $\mathcal{HHPL}$ admits two such nominals extractions.

stratified institution	N	Nm
$\mathcal{HPL}, \mathcal{HFOL}, \mathcal{MHPL}, \mathcal{MHFOL}$	$N(\mathrm{Nom}, \Sigma) = \mathrm{Nom}$	$Nm_{(\mathrm{Nom},\Sigma)}(W, M) = (\lvert W\rvert, (W_i)_{i\in\mathrm{Nom}})$
$\mathcal{HHPL}$	$N(\mathrm{Nom}^0, \mathrm{Nom}^1, P) = \mathrm{Nom}^0$	$Nm(W^1, M^1) = (\lvert (W^0)^w\rvert, ((W^0)^w_i)_{i\in\mathrm{Nom}^0})$
	$N(\mathrm{Nom}^0, \mathrm{Nom}^1, P) = \mathrm{Nom}^1$	$Nm(W^1, M^1) = (\lvert W^1\rvert, (W^1_i)_{i\in\mathrm{Nom}^1})$
$\mathcal{HOFOL}, \mathcal{HMOFOL}$	$N((F, P), X) = F_0$	$Nm(M) = (\lvert M\rvert^X, ((M^X)_i)_{i\in F_0})$

Definition 4.5. *Let $\mathcal{I}$ be a stratified institution endowed with a nominals extraction N, Nm. For any $i \in Nom(\Sigma)$*

- *a Σ-sentence i*-sen *is an i*-sentence *when*

$$(M \models^w i\text{-sen}) = ((Nm_\Sigma(M))_i = w);$$

- *for any* Σ*-sentence* ρ*, a* Σ*-sentence* $@_i\rho$ *is the* satisfaction of ρ at i *when*

$$(M \models^w @_i\rho) = (M \models^{(Nm_\Sigma(M))_i} \rho);$$

for each Σ*-model* M *and for each* $w \in [\![M]\!]_\Sigma$*.*

Example 4.3. The following table shows what of the properties of Dfn. 4.1, 4.3 and 4.5 are satisfied by the examples of stratified institutions given above in the paper.

	$\wedge$	$\vee$	$\neg$	$\Rightarrow$	$(\forall\chi)$	$(\exists\chi)$	$\langle\lambda\rangle$	$[\lambda]$	i-sen	$@_i$
$\mathcal{MPL}$	✓	✓	✓	✓			$\Diamond$	$\Box$		
$\mathcal{MFOL}$	✓	✓	✓	✓	$(\forall x)$	$(\exists x)$	$\Diamond$	$\Box$		
$\mathcal{HPL}$	✓	✓	✓	✓	$(\forall i)$	$(\exists i)$	$\Diamond$	$\Box$	✓	✓
$\mathcal{HFOL}$	✓	✓	✓	✓	$(\forall x)$, $(\forall i)$	$(\exists x)$, $(\exists i)$	$\Diamond$	$\Box$	✓	✓
$\mathcal{MMPL}$	✓	✓	✓	✓			✓	✓		
$\mathcal{MHPL}$	✓	✓	✓	✓	$(\forall i)$	$(\exists i)$	✓	✓	✓	✓
$\mathcal{MMFOL}$	✓	✓	✓	✓	$(\forall x)$	$(\exists x)$	✓	✓		
$\mathcal{MHFOL}$	✓	✓	✓	✓	$(\forall x)$, $(\forall i)$	$(\exists x)$, $(\exists i)$	✓	✓	✓	✓
$\mathcal{HHPL}$	✓	✓	✓	✓	$(\forall i^0)$, $(\forall i^1)$	$(\exists i^0)$, $(\exists i^1)$	$\Diamond$	$\Box$	i^0-sen, i^1-sen	$@_{i^0}$, $@_{i^1}$
$\mathcal{OFOL}$	✓	✓	✓	✓	$(\forall x)$	$(\exists x)$				
$\mathcal{MOFOL}$	✓	✓	✓	✓	$(\forall x)$	$(\exists x)$	✓	✓		
$\mathcal{HOFOL}$	✓	✓	✓	✓	$(\forall x)$, $(\forall i)$	$(\exists x)$, $(\exists i)$			✓	✓
$\mathcal{HMOFOL}$	✓	✓	✓	✓	$(\forall x)$, $(\forall i)$	$(\exists x)$, $(\exists i)$	✓	✓	✓	✓

In the table $(\forall x)$, $(\forall i)$ stand for $(\forall\chi)$ where χ is an extension of the signature with a first order variable, or a nominal variable, respectively, and similarly for the existential quantifiers. The case of the quantifiers reminds us once more that in spite of the abstract simplicity of the institution theoretic approach to quantifiers, just based upon model reducts, they are a very powerful concept supporting a wide range of quantifications within a single uniform definition. Basically, one may quantify over any syntactic entity that is supported by the respective concept of signature morphisms. In our examples this means first order variables and nominals alike. An particularly interesting situation is given by $\mathcal{HHPL}$, where the concept of signature supports quantification over two kinds of nominals, corresponding to the two layers of hybridization.

5. Model ultraproducts in stratified institutions

The structure of the section is as follows:

1. We start with a recollection of the concept of filtered product in abstract categories.
2. Then we discuss filtered products of models in stratified institutions and develop some technical results about the representation of filtered products of models in $\mathcal{I}^\sharp$, the local institution associated to a stratified institution $\mathcal{I}$.
3. The last part of this section is concerned with the development of a Łoś styled theorem for abstract stratified institutions that carry some implicit modal structure. This means a gathering of relevant preservation properties for the connectives commonly used in sentences in various modal logic systems; the connectives are considered by their semantic definitions given in Sect. 4. Here also the compactness consequence of Łoś theorem is studied both at the level of abstract structured institutions and at the level of concrete examples.

5.1. *A reminder of categorical filtered products*

For each non-empty set I we denote the set of all subsets of I by $\mathcal{P}(I)$. A *filter F over I* is defined to be a set $F \subseteq \mathcal{P}(I)$ such that

- $I \in F$,
- $X \cap Y \in F$ if $X \in F$ and $Y \in F$, and
- $Y \in F$ if $X \subseteq Y$ and $X \in F$.

A filter F is *proper* when F is not $\mathcal{P}(I)$ and it is an *ultrafilter* when $X \in F$ if and only if $(I \setminus X) \notin F$ for each $X \in \mathcal{P}(I)$. Notice that ultrafilters are proper filters. We will always assume that all our filters are proper.

Let F be a filter over I and $I' \subseteq I$. The *reduction of F to I'* is denoted by $F|_{I'}$ and defined as $\{I' \cap X \mid X \in F\}$.

Fact 5.1. *The reduction of any filter is still a filter.*

Definition 5.1. *A class $\mathcal{F}$ of filters is* closed under reductions *if and only if $F|_J \in \mathcal{F}$ for each $F \in \mathcal{F}$ and $J \in F$.*

Examples of classes of filters closed under reductions include the class of all filters, the class of all ultrafilters, the class $\{\{I\} \mid I \text{ set}\}$, etc.

Definition 5.2 (Categorical filtered products)**.** *Let F be a filter over I and $(M_i)_{i\in I}$ a family of objects in a category with small direct products. Then an F-*filtered product *of $(M_i)_{i\in I}$ (or F-*product, *for short) is a co-limit $\{\mu_J : M_J \to M_F \mid J \in F\}$ of the directed diagram of canonical projections $\{p_{J\supseteq J'} : M_J \to M_{J'} \mid J' \subseteq J \in F\}$, where for each $J \in F$, $\{p_{J,i} : M_J \to M_i \mid i \in J\}$ is a direct product of $(M_j)_{j\in J}$.*

$$\begin{tikzcd}
 & M_J \arrow[dl, "p_{J,i}"'] \arrow[d, "p_{J\supseteq J'}"] \arrow[dr, bend left, "\mu_J"] & \\
M_i & M_{J'} \arrow[l, "p_{J',i}"] \arrow[r, "\mu_{J'}"'] & M_F
\end{tikzcd}$$

If F is an ultrafilter then F-products are called ultraproducts.

Note that a direct product $\prod_{i\in I} A_i$ is the same as an $\{I\}$-product of $(A_i)_{i\in I}$. Obviously, as co-limits of diagrams of products, filtered products are unique up to isomorphisms. Since the co-limits defining filtered products are directed, a sufficient condition for the existence of filtered products, which applies to many situations, is the existence of small products and of directed co-limits of models. Note however that this is not a necessary condition because only co-limits over diagrams of projections are involved. For example models of higher order logic [6, 8] in general are known to have only direct products and ultraproducts.

Definition 5.3 (Preservation/lifting of filtered products [7, 8])**.** *Consider a functor $G : \mathcal{C}' \to \mathcal{C}$ and F a filter over a set I.*

- *G* preserves F-products *when for each F-product μ' of a family $(M'_i)_{i\in I}$ in $|\mathcal{C}'|$, $G(\mu')$ is an F-product (in $\mathcal{C}$) of $(G(M'_i))_{i\in I}$.*
- *G* lifts F-products *when for each family $(M'_i)_{i\in I}$ in $|\mathcal{C}'|$ and each F-product μ in $\mathcal{C}$ of $(G(M'_i))_{i\in I}$, there exists an F-product μ' of $(M'_i)_{i\in I}$ in $\mathcal{C}'$ such that $G(\mu') = \mu$.*

For any class $\mathcal{F}$ of filters, we say that a functor preserves/lifts $\mathcal{F}$-products *if it preserves/lifts all F-products for each filter $F \in \mathcal{F}$.*

Fact 5.2. *If G lifts F-products then it also preserves them.*

In many situations the following applies.

Fact 5.3. *A functor G preserves/lifts F-products if it preserves/lifts direct products and directed co-limits.*

The concept has been introduced first time in [7] under a different terminology and in a slightly different form, and has been subsequently used in several works most notably in [8, 11].

Definition 5.4 (Inventing of filtered products)**.** *Let $\mathcal{F}$ be a class of filters closed under reductions. A functor $G : C' \to C$* invents $\mathcal{F}$-products *when for each $F \in \mathcal{F}$, for each F-product $\{\mu_J : M_J \to M_F \mid J \in F\}$ of a family $(M_i)_{i\in I}$ in $|C|$, and for each $B \in |C'|$ such that $G(B) = M_F$,*

- *there exists $J \in F$ and $(M'_i)_{i\in J}$ a family in $|C'|$ such that $G(M'_i) = M_i$ for each $i \in J$ and such that*
- *there exists an $F|_J$-product $\{\mu'_{J'} : M'_{J'} \to B \mid J' \in F|_J\}$ of $(M'_i)_{i\in J}$ such that $G(\mu'_{J'}) = \mu_{J'}$ for each $J' \in F|_J$.*

When $J = I$ we say that G lifts completely *the respective F-product. (Note that in this case the closure of $\mathcal{F}$ under reductions is redundant.)*

In essence, the inventing property of Dfn. 5.4 means that each $\mathcal{F}$-product construction of $G(B)$ can be established as the image by G of an $\mathcal{F}$-product construction of B by means of a filter reduction.

5.2. Filtered products in stratified institutions

Definition 5.5. *Let $\mathcal{F}$ be any class of filters. A stratified institution* has (concrete) $\mathcal{F}$-products *when for each signature Σ, $Mod(\Sigma)$ has $\mathcal{F}$-products (and $[\![\_]\!]_\Sigma : Mod(\Sigma) \to$* **Set** *preserves $\mathcal{F}$-products).*

As the following examples show, in practice it is common that the $\mathcal{F}$-products are concrete.

Example 5.1. In all examples of Sect. 3.3 the respective stratified institutions have all F-products, which are concrete, as follows.

1. The F-products in $\mathcal{MPL}$, $\mathcal{MFOL}$, $\mathcal{HPL}$, $\mathcal{HFOL}$, $\mathcal{HHPL}$ are obtained as direct instances of the general result on existence of F-products developed in [11]. In the case of $\mathcal{HHPL}$ this has to be applied twice, first for getting F-products in $\mathcal{HPL}$ from the F-products in $\mathcal{PL}$, and then for getting the F-products in $\mathcal{HHPL}$ from the F-products in $\mathcal{HPL}$.
2. In the case of $\mathcal{MMPL}$, $\mathcal{MHPL}$, $\mathcal{MMFOL}$, $\mathcal{MHFOL}$ we may apply a straightforward extension of the above mentioned result of [11] to the multi-modal situation.
3. In the case of $\mathcal{OFOL}$, $\mathcal{MOFOL}$, $\mathcal{HOFOL}$, $\mathcal{HMOFOL}$ the F-products are much simpler than in the previous cases because the models in all these institutions are just $\mathcal{FOL}$ models.

In the case of $\mathcal{MPL}$, $\mathcal{MFOL}$, $\mathcal{HPL}$, $\mathcal{HFOL}$, $\mathcal{MMPL}$, $\mathcal{MHPL}$, $\mathcal{MMFOL}$, $\mathcal{MHFOL}$, $\mathcal{HHPL}$, according to [11] the construction of filtered products is done in two steps, first at the level of the Kripke frames and next lifted to the level of the Kripke models in $Mod(\Sigma)$; this shows that $[\![\_]\!]_\Sigma$ creates filtered products. For example, in $\mathcal{MFOL}$ an F-product of a family $(W_i, M_i)_{i\in I}$ is $\{\mu_J : (W_J, M_J) \to (W_F, M_F) \mid J \in F\}$ where

- $\{(\mu_J)_0 : W_J \to W_F \mid J \in F\}$ is an F-product of the family of $\mathcal{BREL}$ models $(W_i)_{i\in I}$ where W_J is the cartezian product of $(W_i)_{i\in J}$; and
- for each $(w_i)_{i\in I} \in |W_I|$ and each $J \in F$ we let $M_J^{(w_j)_{j\in J}}$ denote the cartezian product of $(M_j^{w_j})_{j\in J}$; note that both $|M_J^{(w_j)_{j\in J}}|$ and $(M_J^{(w_j)_{j\in J}})_x$ for x constant are invariant with respect to $(w_i)_{i\in I}$;

- let $\{(\mu_J)_1 : \ |M_J^{(w_j)_{j\in J}}| \to |M_F| \mid J \in F\}$ be a directed co-limit in **Set**;
- since the underlying carrier functor $|\_| : \ Mod^{\mathcal{FOL}}(\Sigma) \to$ **Set** creates directed co-limits, for each $(w_i)_{i\in I} \in |W_I|$ we lift the directed co-limit of the previous item to a directed co-limit $\{(\mu_J)_1 : \ M_J^{(w_j)_{j\in J}} \to M_F^{(\mu_I)_0((w_i)_{i\in I})} \mid J \in F\}$ of $Mod^{\mathcal{FOL}}(\Sigma)$-models; it is not difficult to check that the definition of M_F is correct in the sense that $(\mu_I)_0((w_i)_{i\in I}) = (\mu_I)_0((v_i)_{i\in I})$ implies that $M_F^{(\mu_I)_0((w_i)_{i\in I})} = M_F^{(\mu_I)_0((v_i)_{i\in I})}$.

In the case of $\mathcal{OFOL}$, $\mathcal{MOFOL}$, $\mathcal{HOFOL}$, $\mathcal{HMOFOL}$, $[\![\_]\!]_\Sigma$ is just the composition between a $\mathcal{FOL}$ underlying carrier functor $M \mapsto |M|$, and a power functor $|M| \mapsto |M|^X$, which are known (e.g. [8], etc.) to create direct products and directed co-limits, and thus filtered products.

The following result gives a representation of F-products in the local institution $\mathcal{I}^\sharp$ from the F-products in the stratified institution $\mathcal{I}$.

Proposition 5.1. *If a stratified institution $\mathcal{I}$ has concrete F-products, then $\mathcal{I}^\sharp$ has F-products, which for any family $\{(M_i, w_i) \mid M_i \in |Mod(\Sigma)|, w_i \in [\![M_i]\!]_\Sigma, i \in I\}$ may be defined by*

(9) $\{(\mu_J, w_J) : \ (M_J, w_J) \to (M_F, [\![\mu_I]\!](w_I)) \mid J \in F\}$,

where $\{\mu_J : \ M_J \to M_F \mid J \in F\}$ is an F-product in $Mod(\Sigma)$ and w_J is the unique element of $[\![M_J]\!]$ such that for each $i \in J$, $[\![p_{J,i}]\!](w_J) = w_i$.

Proof. Let $(M_i)_{i\in I}$ be a family in $|Mod(\Sigma)|$ and F be a filter over I. We first show that for each $J \in F$,

(10) $\{(p_{J,i}, w_J) : \ (M_J, w_J) \to (M_i, w_i) \mid i \in J\}$

is a direct product in $Mod^\sharp(\Sigma)$. By the definition of w_J, each $(p_{J,i}, w_J)$ is well defined, i.e. $[\![p_{J,i}]\!](w_J) = w_i$.

For any family $\{(f_i, v) : \ (N, v) \to (M_i, w_i) \mid i \in J\}$, by the universal property of the direct products in $Mod(\Sigma)$ there exists an unique $f : \ N \to M_J$ such that for each $i \in J$, $f; p_{J,i} = f_i$.

$$\begin{tikzcd}
(M_J, w_J) \arrow[d, "{(p_{J,i}, w_J)}"'] & (N, v) \arrow[l, "{(f, v)}"'] \arrow[dl, "{(f_i, v)}"] \\
(M_i, w_i) &
\end{tikzcd}$$

Hence, for each $i \in J$, $[\![p_{J,i}]\!]([\![f]\!](v)) = [\![f_i]\!](v) = w_i$. Since $[\![p_{J,i}]\!]$ are cartezian projections, it follows that $[\![f]\!](v) = w_J$. This completes the proof of the universal property of the direct product (10).

It follows immediately that for each $J' \subset J \in F$, $(p_{J \supseteq J'}, w_J) : \ (M_J, w_J) \to (M_{J'}, w_{J'})$ is a corresponding canonical projection in $Mod^\sharp(\Sigma)$. Let us show that (9) is a co-limit in $Mod^\sharp(\Sigma)$.

$$\begin{tikzcd}
& (M_J, w_J) \arrow[dl, "{(p_{J \supseteq J'}, w_J)}"'] \arrow[d, "{(\mu_J, w_J)}"] \arrow[ddr, bend left, "{(\nu_J, w_J)}"] & \\
(M_{J'}, w_{J'}) \arrow[r, "{(\mu_{J'}, w_{J'})}"'] \arrow[drr, bend right, "{(\nu_{J'}, w_{J'})}"'] & (M_F, [\![\mu_I]\!](w_I)) \arrow[dr, "{(f, [\![\mu_I]\!](w_I))}"'] & \\
& & (N, v)
\end{tikzcd}$$

First, note that each (μ_J, w_J) is well defined, i.e. that $[\![\mu_J]\!](w_J) = [\![\mu_I]\!](w_I)$, which is given by the following calculation:

$$[\![\mu_I]\!](w_I) = [\![p_{I \supset J}; \mu_J]\!](w_I) = [\![\mu_J]\!]([\![p_{I \supset J}]\!](w_I)) = [\![\mu_J]\!](w_J).$$

For establishing the universal property of the co-cone $(\mu_J, w_J)_{J\in F}$ let us consider another co-cone $(\nu_J, w_J)_{J\in F}$ over $(p_{J\supset J'}, w_J)_{J\supset J'\in F}$. Let (N, v) denote it vertex. By the universal property of $(\mu_J)_{J\in F}$ in $Mod(\Sigma)$ there exists an unique $f : M_F \to N$ such that for each $J \in F$, $\mu_J; f = \nu_J$. The argument is completed if we showed that $[\![f]\!]([\![\mu_I]\!](w_I)) = v$. This holds by the following calculation:

$$[\![f]\!]([\![\mu_I]\!](w_I)) = [\![\mu_I; f]\!](w_I) = [\![\nu_I]\!](w_I) = v.$$

□

Corollary 5.1. *For any signature morphism χ in any stratified institution $\mathcal{I}$ with concrete F-products, if $Mod(\chi)$ preserves F-products in $\mathcal{I}$ then $Mod^\sharp(\chi)$ preserves F-products in $\mathcal{I}^\sharp$.*

Proof. Let $\chi : \Sigma \to \Sigma'$ be signature morphism such that $Mod(\chi)$ preserves F-products and let

$$\{(\mu'_J, w_J) : (M'_J, w_J) \to (M'_F, [\![\mu'_I]\!](w_I)) \mid J \in F\}$$

be an F-product in $Mod^\sharp(\Sigma')$ like in Prop. 5.1. We denote $Mod(\chi)(M'_i) = M_i$, $Mod(\chi)(M'_J) = M_J$, $Mod(\chi)(M'_F) = M_F$, and $Mod(\chi)(\mu'_J) = \mu_J$. We have to show that

$$\{(\mu_J, [\![M'_J]\!]_\chi(w_J)) : (M_J, [\![M'_J]\!]_\chi(w_J)) \to (M_F, [\![M'_F]\!]_\chi([\![\mu'_I]\!](w_I))) \mid J \in F\}$$

is an F-product in $Mod^\sharp(\Sigma)$. First we should establish that for each $J \in F$

$$\{(Mod(\chi)(p_{J,i}), [\![M'_J]\!]_\chi(w_J)) : (M_J, [\![M'_J]\!]_\chi(w_J)) \to (M_i, [\![M'_i]\!]_\chi(w_i)) \mid i \in J\} \tag{11}$$

is a direct product. Consider

$$\{(f_i, v) : (N, v) \to (M_i, [\![M'_i]\!]_\chi(w_i)) \mid i \in J\}.$$

Since $Mod(\chi)$ preserves products in $Mod(\Sigma)$, we have that the $\mathcal{I}$ part of (11) is a direct product, hence let $f : N \to M_i$ such that $f; Mod(\chi)(p_{J,i}) = f_i$. For showing that (11) is a direct product in $Mod^\sharp(\Sigma)$ it remains to show that $[\![f]\!]_\Sigma(v) = [\![M'_J]\!]_\chi(w_J)$. This holds by the following calculation

$$\begin{array}{rcll}
[\![Mod(\chi)(p_{J,i})]\!]_\Sigma([\![f]\!]_\Sigma(v)) & = & [\![f_i]\!](v) & \\
& = & [\![M'_i]\!]_\chi(w_i) & \text{(by the definition of } f_i) \\
& = & [\![M'_i]\!]_\chi([\![p_{J,i}]\!]_{\Sigma'}(w_J)) & \text{(by the definition of } w_J) \\
& = & [\![Mod(\chi)(p_{J,i})]\!]_\Sigma([\![M'_J]\!]_\chi(w_J)) & \text{(by (4))}
\end{array}$$

and by the fact that $Mod(\chi)$ and $[\![\_]\!]_\Sigma$ preserve direct products, we hav that $[\![Mod(\chi)(p_{J,i})]\!]_\Sigma$ are direct product projections.

Then it follows immediately that $\{(Mod(\chi)(p_{J\supseteq J'}), [\![M'_J]\!]_\chi(w_J)) \mid J' \subseteq J \in F\}$ is a diagram of projections. Now consider any co-cone for the above diagram as follows:

$$\{(\nu_J, [\![M'_J]\!]_\chi(w_J)) : (M_J, [\![M'_J]\!]_\chi(w_J)) \to (N, v) \mid J \in F\}.$$

Since $Mod(\chi)$ preserves F-products it follows that $\{\mu_J : M_J \to M_F \mid J \in F\}$ is an F-product in $Mod(\Sigma)$, hence there exists an unique $f : M_F \to N$ such that for each $J \in F$, $\mu_J; f = \nu_J$. In order to show that $(f, [\![M'_F]\!]_\chi([\![\mu'_I]\!](w_I)))$ is a $Mod^\sharp(\Sigma)$ homomorphism $(M_F, [\![M'_F]\!]_\chi([\![\mu'_I]\!](w_I))) \to (N, v)$ we still have to show that $[\![f]\!]_\Sigma([\![M'_F]\!]_\chi([\![\mu'_I]\!]_{\Sigma'}(w_I))) = v$. This holds by the following calculation:

$$\begin{array}{rcll}
[\![f]\!]_\Sigma([\![M'_F]\!]_\chi([\![\mu'_I]\!]_{\Sigma'})) & = & [\![f]\!]_\Sigma([\![\mu_I]\!]_\Sigma([\![M'_I]\!]_\chi(w_I))) & \text{(by (4))} \\
& = & [\![\nu_I]\!]_\Sigma(([\![M'_I]\!]_\chi(w_I)) & \text{(since } \mu_I; f = \nu_I) \\
& = & v & \text{(by the homomorphism property of } (\nu_I, ([\![M'_I]\!]_\chi(w_I))).
\end{array}$$

□

5.3. Łoš theorem in stratified institutions

The following definition generalizes the corresponding modal preservation concept of [8, 11] to the much more general setup of stratified institutions.

Definition 5.6. *Let $\mathcal{F}$ be a class of filters and let $\mathcal{I}$ be a stratified institution with $\mathcal{F}$-products. A Σ-sentence ρ is*

- preserved by $\mathcal{F}$-products *when for each $w \in [\![M_F]\!]$, "there exists $J \in F$ and $k \in [\![\mu_J]\!]^{-1}(w)$ such that $M_j \models^{k_j} \rho$ for each $j \in J$" implies $M_F \models^w \rho$, and*
- preserved by $\mathcal{F}$-factors *when for each $w \in [\![M_F]\!]$, $M_F \models^w \rho$ implies "there exists $J \in F$ and $k \in [\![\mu_J]\!]^{-1}(w)$ such that $M_j \models^{k_j} \rho$ for each $j \in J$"*

for each filter $F \in \mathcal{F}$ over a set I and for each family $(M_j)_{j\in I}$ of Σ-models, and where $\{\mu_J : M_J \to M_F \mid J \in F\}$ denotes an F-product of $(M_j)_{j\in I}$ and $k_j = [\![p_{J,j}]\!]_\Sigma(k)$.

When all $[\![M]\!]_\Sigma$ have singletons as their underlying sets, Dfn. 5.6 yields the preservation by $\mathcal{F}$-products/factors in ordinary institutions as defined in [7, 8]. On the other hand, the following result shows that stratified preservations by $\mathcal{F}$-products/factors of Dfn. 5.6 may be an instance of their ordinary versions from [7, 8].

Proposition 5.2. *For any stratified institution $\mathcal{I}$ with concrete $\mathcal{F}$-products the following are equivalent for any Σ-sentence ρ:*

1. *ρ is preserved by $\mathcal{F}$-products/factors in $\mathcal{I}$; and*
2. *ρ is preserved by $\mathcal{F}$-products/factors in $\mathcal{I}^\sharp$.*

Proof. In this proof we use the notations of Prop. 5.1. First note that since $\mathcal{I}$ has $\mathcal{F}$-products, by Prop. 5.1 $\mathcal{I}^\sharp$ has $\mathcal{F}$-products too. Moreover, by the assumption of preservation of satisfaction by model isomorphisms, without any loss of generality, we may consider only the F-products given by (9) of Prop. 5.1.

$\boxed{1. \Rightarrow 2.}$ For the preservation by $\mathcal{F}$-products, let $(M_i, w_i)_{i\in I}$ and $F \in \mathcal{F}$ filter over I and assume that there exists $J \in F$ such that for each $j \in J$, $(M_j, w_j) \models^\sharp \rho$. By the definition of $\models^\sharp$ we have that for each $j \in J$, $M_j \models^{w_j} \rho$. By 1. it follows that $M_F \models^{[\![\mu_J]\!](w_J)} \rho$. Since $[\![\mu_I]\!](w_I) = [\![\mu_J]\!]([\![p_{I\supseteq J}]\!](w_I)) = [\![\mu_J]\!](w_J)$ it follows that $(M_F, [\![\mu_I]\!](w_I)) \models \rho$.

For the preservation by $\mathcal{F}$-factors, let $(M_i, w_i)_{i\in I}$ and $F \in \mathcal{F}$ filter over I such that $(M_F, [\![\mu_I]\!](w_I)) \models^\sharp \rho$. Hence $M_F \models^{w'} \rho$ where $w' = [\![\mu_I]\!](w_I)$. By the hypothesis 1. there exists $J \in F$ and $k \in [\![\mu_J]\!]^{-1}(w')$ such that for each $j \in J$, $M_j \models^{k_j} \rho$. Because $[\![\mu_J]\!](k) = [\![\mu_J]\!](w_J)$ we have that there exists $J \supseteq J' \in F$ such that $[\![p_{J,J'}]\!](k) = w_{J'}$. Hence for each $j \in J'$, $(M_j, w_j) \models^\sharp \rho$.

$\boxed{2. \Rightarrow 1.}$ For the preservation by $\mathcal{F}$-products, let $(M_i)_{i\in I}$ and $F \in \mathcal{F}$ filter over I and for any fixed $w \in [\![M_F]\!]$ assume that there exists $J \in F$ and $k \in [\![\mu_J]\!]^{-1}(w)$ such that for each $j \in J$, $M_j \models^{k_j} \rho$. Let us take any $k_i \in [\![M_i]\!]$ for each $i \notin J$ and let k_I be defined by $[\![p_{I,i}]\!](k_I) = k_i$ for each $i \in I$. Since for each $j \in J$, $(M_j, k_j) \models^\sharp \rho$, by 2. it follows that $(M_F, [\![\mu_I]\!](k_I)) \models^\sharp \rho$. Since $[\![\mu_I]\!](k_I) = [\![\mu_J]\!](k) = w$ it means that $M_F \models^w \rho$.

For the preservation by $\mathcal{F}$-factors, let $(M_i)_{i\in I}$ and $F \in \mathcal{F}$ filter over I. Let us assume that $M_F \models^w \rho$. Let $k \in [\![\mu_I]\!]^{-1}(w)$. By (9) of Prop. 5.1 we have that (M_F, w) is the F-product of $((M_i, k_i))_{i\in I}$. By the hypothesis 2. there exists $J \in F$ such that for each $j \in J$, $(M_j, k_j) \models^\sharp \rho$, which means $M_j \models^{k_j} \rho$. □

Proposition 5.3. *For any stratified institution $\mathcal{I}$ with F-products, if a sentence ρ is preserved by F-products in $\mathcal{I}$ then it is preserved by F-products in $\mathcal{I}^*$ too.*

Proof. Let us assume that $J' = \{j \in I \mid M_j \models^* \rho\} \in F$ for $\{\mu_J : M_J \to M_F \mid J \in F\}$ an F-product of a family $(M_j)_{j\in I}$ of Σ-models. Let $w \in [\![M_F]\!]$. For any $k \in [\![\mu_{J'}]\!]^{-1}(w)$ and each $j \in J'$ we have that $M_j \models^{k_j} \rho$ (since $M_j \models^* \rho$). Because ρ is preserved by F-products in $\mathcal{I}$ it follows that $M_F \models^w \rho$. Hence $M_F \models^* \rho$. ☐

According to [7, 8] any institution in which all its sentences are preserved by ultraproducts is m-compact. Hence from Prop. 5.3 and 5.2 we get the following consequence.

Corollary 5.2. *Let $\mathcal{I}$ be a stratified institution with ultraproducts such that each of its sentences are preserved by ultraproducts. Then*

1. *$\mathcal{I}^*$ is m-compact; and*
2. *if in addition the ultraproducts are concrete then $\mathcal{I}^\sharp$ is m-compact too.*

The following consequence of Prop. 5.2 represents a transfer of preservation results from ordinary institutions to stratified institutions.

Corollary 5.3. *In any stratified institution $\mathcal{I}$ with concrete $\mathcal{F}$-products*

1. *both the sentences preserved by $\mathcal{F}$-products and those preserved by $\mathcal{F}$-factors are closed under conjunctions;*
2. *if ρ is preserved by $\mathcal{F}$-products then $\neg\rho$ is preserved by $\mathcal{F}$-factors;*
3. *if ρ is preserved by $\mathcal{F}$-factors and $\mathcal{F}$ contains only ultrafilters then $\neg\rho$ is preserved by $\mathcal{F}$-products; and*
4. *if $\mathcal{F}$ is closed under reductions, $Mod(\chi)$ preserves $\mathcal{F}$-products, and ρ is preserved by $\mathcal{F}$-products then $(\exists\chi)\rho$ is preserved by $\mathcal{F}$-products.*

Proof. 1., 2., 3. By Fact 4.1, the conjunction and negation coincide in $\mathcal{I}$ and $\mathcal{I}^\sharp$. By Prop. 5.2, preservation by $\mathcal{F}$-products/factors also coincides in $\mathcal{I}$ and $\mathcal{I}^\sharp$. The conclusions for 1., 2., 3. follow because by [7, 8] the considered preservation properties hold in general in any ordinary institution and in particular in $\mathcal{I}^\sharp$.

4. By Prop. 5.2 ρ is preserved by $\mathcal{F}$-products in $\mathcal{I}^\sharp$. By Cor. 5.1 it follows that $Mod^\sharp(\chi)$ preserves $\mathcal{F}$-products. From [7, 8] we know that in general, in any (ordinary) institution, from such conditions it follows that $(\exists\chi)\rho$ is preserved by $\mathcal{F}$-products. We apply this conclusion within $\mathcal{I}^\sharp$. By Fact 4.1 (existential quantification coincide in $\mathcal{I}$ and in $\mathcal{I}^\sharp$) and by Prop. 5.2 it now follows that $(\exists\chi)\rho$ is preserved by $\mathcal{F}$-products in $\mathcal{I}$. ☐

The conclusions of Cor. 5.3 may be obtained directly without reliance upon Prop. 5.2. Some of them may be obtained under the slightly milder condition that does not require the F-products to be concrete, however this generality is largely meaningless in the applications because the F-products are usually concrete (in fact we do not know examples of F-products that are not concrete).

Proposition 5.4. *In any stratified institution $\mathcal{I}$ with $\mathcal{F}$-products, if $\mathcal{F}$ is closed under reductions, $Mod(\chi)$ invents $\mathcal{F}$-products, and ρ is preserved by $\mathcal{F}$-factors then $(\exists\chi)\rho$ is preserved by $\mathcal{F}$-factors.*

Proof. Let $\chi : \Sigma \to \Sigma'$ signature morphism, let $F \in \mathcal{F}$, and let $\{\mu_J : M_J \to M_F \mid J \in F\}$ be an F-product of a family $(M_i)_{i\in I}$ of Σ-models. Assume that $M_F \models^w (\exists\chi)\rho$.

It follows that there exists M' and w' such that $M_F = Mod(\chi)(M')$, $w' \in [\![M']\!]_\chi^{-1}(w)$, and $M' \models^{w'} \rho$. By the inventing condition there exists $J \in F$ and an $F|_J$-product $\{\mu'_{J'} : M'_{J'} \to M' \mid J' \in F|_J\}$ of a family $(M'_j)_{j\in J}$ of Σ'-models such that $Mod(\chi)(M'_j) = M_j$ for each $j \in J$ and for all $J'' \subseteq J' \in F|_J$ we have that $Mod(\chi)(p'_{J'\supseteq J''}) = p_{J'\supseteq J''}$ and $Mod(\chi)(\mu'_{J'}) = \mu_{J'}$. Since ρ is preserved by $\mathcal{F}$-factors there exists $J' \in F|_J$

and $k' \in [\![\mu'_{J'}]\!]^{-1}_{\Sigma'}(w')$ such that $M'_j \models^{k'_j} \rho$ for each $j \in J'$. Let $k = [\![M'_{J'}]\!]_\chi(k')$. For each $j \in J'$ we have the following:

$$\begin{array}{rll} k_j = & [\![Mod(\chi)(p_{J',j})]\!](k) & \text{(by the definition of } k_j) \\ = & [\![Mod(\chi)(p_{J',j})]\!]([\![M'_{J'}]\!]_\chi(k')) & \text{(by the definition of } k) \\ = & [\![M'_j]\!]_\chi([\![p_{J',j}]\!](k')) & \text{(by (4))} \\ = & [\![M'_j]\!]_\chi(k'_j) & \text{(by the definition of } k'_j). \end{array}$$

Since $Mod(\chi)(M'_j) = M_j$ we get that $M_j \models^{k_j} (\exists\chi)\rho$. It remains to show that $[\![\mu_{J'}]\!]_\Sigma(k) = w$, which holds by the following calculation:

$$\begin{array}{rll} [\![\mu_{J'}]\!]_\Sigma(k) = & [\![\mu_{J'}]\!]_\Sigma([\![M'_{J'}]\!]_\chi(k')) & \text{(by the definition of)k)} \\ = & [\![M']\!]_\chi([\![\mu'_{J'}]\!]_{\Sigma'}(k')) & \text{(by (4))} \\ = & [\![M']\!]_\chi(w') & \text{(since } k' \in [\![\mu'_{J'}]\!]^{-1}_{\Sigma'}(w')) \\ = & w. & \end{array}$$

□

Proposition 5.5. *Let $\mathcal{I}$ be a stratified institution endowed with a frame extraction $L : Sign^{\mathcal{I}} \to Sign^{\mathcal{REL}}$, $Fr : Mod^{\mathcal{I}} \Rightarrow L; Mod^{\mathcal{REL}}$. Assume that $\mathcal{I}$ has F-products for a filter F over a set I.*

1. *If Fr_Σ preserves direct products and $\rho_1, \ldots, \rho_n$ are preserved by F-products then any λ-possibility $\langle\lambda\rangle(\rho_1, \ldots \rho_n)$ is also preserved by F-products.*
2. *If Fr_Σ preserves F-products and $\rho_1, \ldots, \rho_n$ are preserved by F-factors then any λ-possibility $\langle\lambda\rangle(\rho_1, \ldots \rho_n)$ is also preserved by F-factors.*

Proof. 1. We consider an F-product $\{\mu_J : M_J \to M_F \mid J \in F\}$ for a family $(M_i)_{i\in I}$ of Σ-models and assume that there exists $J \in F$ and $k \in [\![\mu_J]\!]^{-1}(w)$ such that for each $j \in J$, $M_j \models^{k_j} \langle\lambda\rangle(\rho_1, \ldots \rho_n)$. We have to prove that $M_F \models^w \langle\lambda\rangle(\rho_1, \ldots, \rho_n)$, i.e. that there exists $(w, w_1, \ldots, w_n) \in (Fr_\Sigma(M))_\lambda$ such that $M_F \models^{w_i} \rho_i$ for each $1 \le i \le n$.

For each $j \in J$, $M_j \models^{k_j} \langle\lambda\rangle(\rho_1, \ldots \rho_n)$ means that there exists $(k_j, k^1_j, \ldots, k^n_j) \in (Fr_\Sigma(M_j))_\lambda$ such that $M_j \models^{k^i_j} \rho_i$ for each $1 \le i \le n$. Since Fr_Σ preserves products we have that $\{Fr_\Sigma(p_{J,j}) : Fr_\Sigma(M_J) \to Fr_\Sigma(M_j) \mid j \in J\}$ is direct product in $Mod^{\mathcal{REL}}(L(\Sigma))$. Hence for each $1 \le i \le n$, there exists $k^i \in [\![M_J]\!]$ such that $[\![p_{J,j}]\!](k^i) = k^i_j$ for each $j \in J$. We define $w_i = [\![\mu_J]\!](k^i)$.

By the direct product property of $Fr_\Sigma(M_J)$ in $Mod^{\mathcal{REL}}(L(\Sigma))$ we have that $(k_j, k^1_j, \ldots, k^n_j) \in (Fr_\Sigma(M_j))_\lambda$ for each $j \in J$ implies that $(k, k^1, \ldots, k^n) \in (Fr_\Sigma(M_J))_\lambda$. Since $Fr_\Sigma(\mu_J)$ is a homomorphism of $Mod^{\mathcal{REL}}(L(\Sigma))$-models it follows that $(w, w_1, \ldots, w_n) = ([\![\mu_J]\!](k), [\![\mu_J]\!](k^1), \ldots, [\![\mu_J]\!](k^n)) \in (Fr_\Sigma(M_F))_\lambda$.

That for each $1 \le i \le n$, $M_F \models^{w_i} \rho_i$, follows from the hypothesis that ρ_i is preserved by F-products and because $k^i \in [\![\mu_J]\!]^{-1}(w_i)$ and $M_j \models^{k^i_j} \rho_i$ for each $j \in J$.

2. We consider an F-product $\{\mu_J : M_J \to M_F \mid J \in F\}$ for a family $(M_i)_{i\in I}$ of Σ-models and assume that $M_F \models^w \langle\lambda\rangle(\rho_1, \ldots, \rho_n)$. We have to prove that there exists $J \in F$ and $k \in [\![\mu_J]\!]^{-1}(w)$ such that for each $j \in J$, $M_j \models^{k_j} \langle\lambda\rangle(\rho_1, \ldots, \rho_n)$, i.e. that there exists $(k_j, k^1_j, \ldots, k^n_j) \in (Fr_\Sigma(M_j))_\lambda$ such that for each $1 \le i \le n$, $M_j \models^{k^i_j} \rho_i$.

From $M_F \models^w \langle\lambda\rangle(\rho_1, \ldots, \rho_n)$ it follows that there exists $(w, w_1, \ldots, w_n) \in (Fr_\Sigma(M_F))_\lambda$ such that $M_F \models^{w_i} \rho_i$ for each $1 \le i \le n$. By the hypothesis that each ρ_i is preserved by F-factors, this means there exists $J_i \in F$ and $l^i \in [\![\mu_{J_i}]\!]^{-1}(w_i)$ such that $M_j \models^{l^i_j} \rho_i$ for each $j \in J_i$.

Since Fr_Σ preserves F-products it follows that $\{Fr_\Sigma(\mu_J): Fr_\Sigma(M_J) \to Fr_\Sigma(M_F) \mid J \in F\}$ is an F-product of $(Fr_\Sigma(M_j))_{j\in I}$ in $Mod^{\mathcal{REL}}(L(\Sigma))$. Hence, $(w, w_1, \dots, w_n) \in (Fr_\Sigma(M_F))_\lambda$ implies that there exists $J' \in F$ and $(v, v_1, \dots, v_n) \in (Fr_\Sigma(M_{J'}))_\lambda$ with $[\![\mu_{J'}]\!](v) = w$ and $[\![\mu_{J'}]\!](v_i) = w_i$ for each $1 \le i \le n$.

Let us take $J'' = J' \cap \bigcap_{1\le i\le n} J_i$. Since filters are closed under intersections, it follows that $J'' \in F$. For each $1 \le i \le n$ we have that

$$[\![\mu_{J''}]\!]([\![p_{J_i\supseteq J''}]\!](l^i)) = [\![\mu_{J_i}]\!](l^i) = w_i = [\![\mu_{J'}]\!](v_i) = [\![\mu_{J''}]\!](p_{J'\supseteq J''}(v_i)).$$

Since $\{Fr_\Sigma(\mu_J): Fr_\Sigma(M_J) \to Fr_\Sigma(M_F) \mid J \in F\}$ is an F-product, which means it is a particular directed co-limit, it follows that there exists $J \subseteq J''$ such that $[\![p_{J_i\supseteq J}]\!](l^i) = [\![p_{J'\supseteq J}]\!](v_i)$ for each $1 \le i \le n$.

For each $1 \le i \le n$ we define $k^i = [\![p_{J_i\supseteq J}]\!](l^i) = [\![p_{J'\supseteq J}]\!](v_i)$. We also let $k = [\![p_{J'\supseteq J}]\!](v)$.

– Since $(v, v_1, \dots, v_n) \in (Fr_\Sigma(M_{J'}))_\lambda$, by the homomorphism property of $Fr_\Sigma(p_{J'\supseteq J})$ it follows that $(k, k^1, \dots, k^n) \in (Fr_\Sigma(M_J))_\lambda$. By the homomorphism property of each $p_{J,j}$ it further follows that $(k_j, k^1_j, \dots, k^n_j) \in (Fr_\Sigma(M_j))_\lambda$ for each $j \in J$.
– Note that for each $1 \le i \le n$ and each $j \in J$

$$l^i_j = [\![p_{J_i,j}]\!](l^i) = [\![p_{J,j}]\!]([\![p_{J_i\supseteq J}]\!](l^i)) = [\![p_{J,j}]\!](k^i) = k^i_j.$$

Since we know that $M_j \models^{l^i_j} \rho_i$ it means that $M_j \models^{k^i_j} \rho_i$ for each $j \in J$.

□

Proposition 5.6. *Let $\mathcal{I}$ be a stratified institution endowed with a nominals extraction $N: Sign^{\mathcal{I}} \to Sign^{\mathcal{SETC}}$, $Nm: Mod^{\mathcal{I}} \Rightarrow N; Mod^{\mathcal{SETC}}$. Assume that $\mathcal{I}$ has F-products for a filter F over a set I. For any signature Σ and any $i \in N(\Sigma)$,*

1. *If Nm_Σ preserves direct products then i-*sen *is preserved by F-products.*
2. *If Nm_Σ preserves F-products then i-*sen *is preserved by F-factors.*
3. *If ρ is preserved by F-products then each sentence $@_i\rho$ is preserved by F-products too.*
4. *If Nm_Σ preserves F-products and ρ is preserved by F-factors then each sentence $@_i\rho$ is preserved by F-products too.*

Proof. We consider $\{\mu_J: M_J \to M_F \mid J \in F\}$ an F-product a family $(M_j)_{j\in I}$ in $Mod(\Sigma)$.

1. Let us assume that there exists $J \in F$ and $k \in [\![\mu_J]\!]^{-1}_\Sigma(w)$ such that $M_j \models^{k_j} i$-sen for each $j \in J$. This means for each $j \in J$

(12) $(Nm_\Sigma(M_j))_i = k_j = Nm_\Sigma(p_{J,j})(k)$.

Also, by the homomorphism property of $Nm_\Sigma(p_{J,j})$ we have that for each $j \in J$

(13) $(Nm_\Sigma(M_j))_i = Nm_\Sigma(p_{J,j})((Nm_\Sigma(M_J))_i)$.

Since Nm_Σ preserves direct products, from (12) and (13) it follows that $(Nm_\Sigma(M_J))_i = k$. We have that

$$\begin{aligned}(Nm_\Sigma(M_F))_i &= Nm_\Sigma(\mu_J)((Nm_\Sigma(M_J))_i) && \text{(by the homomorphism property of } Nm_\Sigma(\mu_J))\\ &= Nm_\Sigma(\mu_J)(k) && = w,\end{aligned}$$

which means $M_F \models^w i$-sen.

2. Let us assume that $M_F \models^w i$-sen, which means $(Nm_\Sigma(M_F))_i = w$. Since Nm_Σ preserves F-products, $\{Nm_\Sigma(\mu_J): Nm_\Sigma(M_J) \to Nm_\Sigma(M_F) \mid J \in F\}$ is a directed co-limit, hence there exists $J \in F$ such that

$Nm_\Sigma(\mu_J)((Nm_\Sigma(M_J))_i) = (Nm_\Sigma(M_F))_i$. Let $k = (Nm_\Sigma(M_J))_i$. For each $j \in J$, by the homomorphism property of $Nm_\Sigma(p_{J,j})$ it follows that $k_j = Nm_\Sigma(p_{J,j})(k) = Nm_\Sigma(p_{J,j})((Nm_\Sigma(M_J))_i) = (Nm_\Sigma(M_j))_i$ which means $M_j \models^{k_j} i$-sen.

3. Let us assume that there exists $J \in F$ and $k \in [\![\mu_J]\!]_\Sigma^{-1}(w)$ such that $M_j \models^{k_j} @_i\rho$ for each $j \in J$, which just means $M_j \models^{(Nm_\Sigma(M_j))_i} \rho$ for each $j \in J$. Since by the homomorphism property of $Nm_\Sigma(\mu_J)$ and of $Nm_\Sigma(p_{J,j})$, for each $j \in J$, we have that $Nm_\Sigma(\mu_J)((Nm_\Sigma(M_J))_i) = (Nm_\Sigma(M_F))_i$ and that $Nm_\Sigma(p_{J,j})((Nm_\Sigma(M_J))_i) = (Nm_\Sigma(M_j))_i$, respectively, and because by hypothesis ρ is preserved by F-products it follows that $M_F \models^{(Nm_\Sigma(M_F))_i} \rho$ which means $M_F \models^w @_i\rho$.

4. Let us assume $M_F \models^w @_i\rho$, which means $M_F \models^{(Nm_\Sigma(M_F))_i} \rho$. It is enough to show that there exists $J \in F$ such that $M_j \models^{(Nm_\Sigma(M_j))_i} \rho$ for each $j \in J$.

– Since Nm_Σ preserves F-products, $\{Nm_\Sigma(\mu_J) : Nm_\Sigma(M_J) \to Nm_\Sigma(M_F) \mid J \in F\}$ is a directed co-limit, hence there exists $J' \in F$ such that $Nm_\Sigma(\mu_{J'})((Nm_\Sigma(M_{J'}))_i) = (Nm_\Sigma(M_F))_i$.
– By the hypothesis that ρ is preserved by F-factors, it follows that there exists $J'' \in F$ and $k'' \in [\![\mu_{J''}]\!]_\Sigma^{-1}((Nm_\Sigma(M_F))_i)$ such that $M_j \models^{k''_j} \rho$ for each $j \in J''$.

Since $[\![\mu_{J'}]\!]_\Sigma((Nm_\Sigma(M_{J'}))_i) = [\![\mu_{J''}]\!]_\Sigma(k'')$ and because $\{[\![\mu_J]\!]_\Sigma : [\![M_J]\!]_\Sigma \to [\![M_F]\!]_\Sigma \mid J \in F\}$ is a directed co-limit, there exists $J \subseteq J' \cap J'' \in F$ such that

(14) $[\![p_{J' \supseteq J}]\!]_\Sigma((Nm_\Sigma(M_{J'}))_i) = [\![p_{J'' \supseteq J}]\!]_\Sigma(k'')$.

For each $j \in J$ we have that

$(Nm_\Sigma(M_j))_i =$

$$\begin{array}{lll} = & Nm_\Sigma(p_{J,j})((Nm_\Sigma(M_J))_i) & \text{(by the homomorphism property of } Nm_\Sigma(p_{J,j})) \\ = & Nm_\Sigma(p_{J,j})(Nm_\Sigma(p_{J' \supseteq J})((Nm_\Sigma(M_{J'}))_i)) & \text{(by the homomorphism property of } Nm_\Sigma(p_{J' \supseteq J})) \\ = & Nm_\Sigma(p_{J,j})(Nm_\Sigma(p_{J'' \supseteq J})(k'')) & \text{(by (14))} \\ = & [\![p_{J'',j}]\!]_\Sigma(k'') & = k''_j. \end{array}$$

Hence for each $j \in J$, $M_j \models^{(Nm_\Sigma(M_j))_i} \rho$. □

Note that from the six preservation results included in Prop. 5.5 and 5.6, one does not assume anything on the frame/nominals extraction, two assume that the respective extractions preserve direct products, and three that the they preserve F-products.

The preservation results of Cor. 5.3 and of Prop. 5.4–5.6 may be applied for lifting preservation properties from simpler to more complex sentences. They can be used at the induction step when establishing preservation properties by induction on the structure of the sentences. The following result and its corollary constitute a general approach to the base case of such induction proofs, that in general corresponds to the atomic sentences.

Lemma 5.1. *Let* $(\Phi, \alpha, \beta) : \mathcal{B}' \to \mathcal{B}$ *be an institution morphism such that each* β_Σ *preserves* F*-products. Then for any* $\Phi(\Sigma)$*-sentence* ρ *that is preserved by* F*-products/factors, the* Σ*-sentence* $\alpha_\Sigma(\rho)$ *is preserved by* F*-products/factors.*

Proof. Let us assume an F-product $\{\mu'_J : M'_J \to M'_F \mid J \in F\}$ of a family $(M'_i)_{i \in I}$ of Σ-models for a $\mathcal{B}'$-signature Σ. By hypothesis we have that $\{\beta_\Sigma(\mu'_J) : \beta_\Sigma(M'_J) \to \beta_\Sigma(M'_F) \mid J \in F\}$ is an F-product of $(\beta_\Sigma(M'_i))_{i \in I}$ in $Mod^{\mathcal{B}}(\Phi(\Sigma))$.

For the preservation by F-products, let us assume $J \in F$ such that $M'_i \models_\Sigma \alpha_\Sigma(\rho)$ for each $i \in J$. By the satisfaction condition of (Φ, α, β) this means $\beta_\Sigma(M'_i) \models_{\Phi(\Sigma)} \rho$ for each $i \in J$, hence because ρ is preserved by F-products, $\beta_\Sigma(M'_F) \models_{\Phi(\Sigma)} \rho$. By the satisfaction condition of (Φ, α, β) it follows that $M'_F \models_\Sigma \alpha_\Sigma(\rho)$.

For the preservation by F-factors, let us assume that $M'_F \models_\Sigma \alpha_\Sigma(\rho)$. By the satisfaction condition of (Φ, α, β) it follows that $\beta_\Sigma(M'_F) \models_{\Phi(\Sigma)} \rho$. Since ρ is preserved by F-factors, there exists $J \in F$ such that $\beta_\Sigma(M'_i) \models_{\Phi(\Sigma)} \rho$ for each $i \in J$. By the satisfaction condition of (Φ, α, β) we obtain that $M'_i \models_\Sigma \alpha_\Sigma(\rho)$ for each $i \in J$. □

The following is an immediate consequence of Prop. 5.2 and Lemma 5.1, which is applicable in concrete situations.

Corollary 5.4. *Let $\mathcal{I}$ be a stratified institution with concrete $\mathcal{F}$-products. Let $(\Phi, \alpha, \beta) : \mathcal{I}^\sharp \to \mathcal{B}$ be an institution morphism such that each β_Σ preserves $\mathcal{F}$-products. Then for each $\Phi(\Sigma)$-sentence ρ that is preserved by $\mathcal{F}$-products/factors, $\alpha_\Sigma(\rho)$ is preserved by $\mathcal{F}$-products/factors in $\mathcal{I}$.*

Now we can put together the results of this section and apply them to our concrete benchmark examples.

Corollary 5.5. *Let $\mathcal{I} \in \{\mathcal{MPL}, \mathcal{MFOL}, \mathcal{HPL}, \mathcal{HFOL}, \mathcal{MMPL}, \mathcal{MHPL}, \mathcal{MMFOL}, \mathcal{MHFOL}, \mathcal{HHPL}, \mathcal{OFOL}, \mathcal{MOFOL}, \mathcal{HOFOL}, \mathcal{HMOFOL}\}$. Then in $\mathcal{I}$ each sentence is preserved by all ultraproducts and ultrafactors. Consequently $\mathcal{I}^\sharp$ and $\mathcal{I}^*$ are m-compact and in addition $\mathcal{I}^\sharp$ is compact.*

Proof. The first conclusion is proved by induction on the structure of $\mathcal{I}$-sentences through application of the preservation results of Cor. 5.4, 5.3, Prop. 5.4, 5.5, and 5.6 as follows.

From Ex. 5.1 let us note that $\mathcal{I}$ has concrete F-products for any filter F.

The base case of our induction proof on the structure of the $\mathcal{I}$-sentences is represented, with the exception of $\mathcal{HHPL}$, only by atomic sentences. These atomic sentences may be of two kinds, either atomic sentences of $\mathcal{PL}$ or $\mathcal{FOL}$, or else i-sen. In the case of $\mathcal{HHPL}$, besides i^1-sen at the base case we also have the sentences of the $\mathcal{HPL}$ corresponding to the lower layer of hybridization. For the case when the sentence is a not a nominal sentence, we apply Cor. 5.4. Let $\mathcal{APL}$ and $\mathcal{AFOL}$ denote the sub-institutions of $\mathcal{PL}$ (propositional logic) and of $\mathcal{FOL}$ (first order logic), respectively, that have only the atoms as their sentences. Let $\mathcal{B}$ be $\mathcal{HPL}$ when $\mathcal{I} = \mathcal{HHPL}$, $\mathcal{APL}$ when $\mathcal{I} \in \{\mathcal{MPL}, \mathcal{HPL}, \mathcal{MMPL}, \mathcal{MHPL}\}$ and $\mathcal{AFOL}$ otherwise. The institution morphism $(\Phi, \alpha, \beta) : \mathcal{I} \to \mathcal{B}$ is defined as follows:

- Φ forgets the modalities symbols Λ when $\mathcal{I} \in \{\mathcal{MMPL}, \mathcal{MHPL}, \mathcal{MMFOL}, \mathcal{MHFOL}\}$ and the nominals symbols when $\mathcal{I} \in \{\mathcal{HPL}, \mathcal{HFOL}, \mathcal{MHPL}, \mathcal{MHFOL}, \mathcal{HHPL}\}$[5] and is identity otherwise;
- α is just the inclusion of the sentences of $\mathcal{APL}$ or of $\mathcal{AFOL}$ as atomic sentences of $\mathcal{I}$; and
- $\beta_\Sigma(M, w) = M^w$.

The Satisfaction Condition for (Φ, α, β) is an immediate consequence of the satisfaction of atomic sentences in $\mathcal{I}$ (or of the satisfaction of the $\mathcal{HPL}$-sentences in $\mathcal{HHPL}$) and of the definition of $\models^\sharp$ (see Fact 3.1).

Now we establish that each β_Σ preserves all F-products. By Prop. 5.1 we know that F-products in $\mathcal{I}^\sharp$ are of the form

$$\{(\mu_J, w_J) : (M_J, w_J) \to (M_F, [\![\mu_I]\!](w_I)) \mid J \in F\}.$$

According to the definition of β, we have to show that

(15) $\{\mu_J^{w_J} : M_J^{w_J} \to M_F^{[\![\mu_I]\!](w_I)} \mid J \in F\}$

[5] In the $\mathcal{HHPL}$ case we have $\Phi(\mathrm{Nom}^0, \mathrm{Nom}^1, P) = (\mathrm{Nom}^0, P)$.

is an F-product too. Without any loss of generality we may further assume that M_J are cartezian products. Note that $w_J = (w_j)_{j\in J}$ when the $\mathcal{I}$-models are Kripke models and $w_J; p_{J,j} = w_j$ in the other cases. It follows that $M_J^{w_J}$ is the product of $\{M_j^{w_j} \mid j \in J\}$. When the $\mathcal{I}$-models are Kripke models, from the construction of F-products of Kripke models, by Lemma 11.11 of [8] (the same with Lemma 1 of [11]) it follows that (15) is an F-product of $(M_j^{w_j})_{j\in I}$. When $\mathcal{I} \in \{\mathcal{OFOL}, \mathcal{MOFOL}, \mathcal{HOFOL}, \mathcal{HMOFOL}\}$ then the argument that (15) is an F-product is much simpler because $\{\mu_J : M_J \to M_F \mid J \in F\}$ is an F-product of $\mathcal{FOL}$ (F, P)-models and (15) is just an expansion of this to $(F + X, P)$.[6]

When $\mathcal{B} \neq \mathcal{HPL}$ then all $\mathcal{B}$-sentences are atoms, hence according to [7, 8] they are 'finitary basic sentences' and consequently are preserved by all F-products and all F-factors. When $\mathcal{B} = \mathcal{HPL}$ then we have to use the conclusion of this corollary for $\mathcal{I} = \mathcal{HPL}$, that all $\mathcal{HPL}$-sentences are preserved by ultraproducts. This completes the set of conditions for applying Cor. 5.4, which gets us to the conclusion that, apart of the nominal sentences i-sen, all sentences at the base case are preserved by ultraproducts and ultrafactors. For the sentences i-sen we apply the relevant part of Prop. 5.6. For this we have just to note that the condition that Nm_Σ preserves direct products and ultraproducts is covered by the fact that $\mathcal{I}$ has concrete F-products. This covers the base case of our induction proof.

According to the definition of satisfaction in $\mathcal{I}$ all $\mathcal{I}$-sentences are built by iterative application of external Boolean connectives, quantifiers, modalities, $@_i$, from atoms when $\mathcal{I} \neq \mathcal{HHPL}$ and from $\mathcal{HPL}$-sentences plus i^1-sen when $\mathcal{I} = \mathcal{HHPL}$. Hence for the induction step part of the proof, we have to check the conditions of Cor. 5.3, Prop. 5.4, 5.5, and 5.6. The preservation of direct products and of ultraproducts by Fr_Σ, Nm_Σ is a direct consequence of the construction of filtered products of Kripke models. Because the class of all ultrafilters is closed under reductions, it remains only to show that, when applicable, for each signature extension χ with first order variables or with nominals variables, $Mod(\chi)$ preserves and invents ultraproducts.

The preservation property holds for all F-products as follows. First we have to notice it for the direct products. When $\mathcal{I} \in \{\mathcal{OFOL}, \mathcal{MOFOL}, \mathcal{HOFOL}, \mathcal{HMOFOL}\}$ this is just a matter of preservation of direct products of $\mathcal{FOL}$ models by reducts forgetting interpretations of constants, which is obvious. When the $\mathcal{I}$-models are Kripke models, this is a consequence of the fact that whenever we expand a direct product (W, M) of a family $(W_i, M_i)_{i\in I}$ of reducts of Kripke models $(W'_i, M'_i)_{i\in I}$ with an interpretation of a new constant x in (W, M) by $W'_x = ((W'_i)_x)_{i\in I}$ when x is nominal or by $M'_x = ((M'_i)_x)_{i\in I}$ when x is a first order constant, this yields a direct product of $(W'_i, M'_i)_{i\in I}$.[7] The argument is completed by noting that the directed co-limit component of any F-product is preserved by reducts corresponding to signature expansions χ with nominal or first order variables as a consequence of the fact that any model homomorphism $Mod(\chi)(M') \to N$ may be expanded uniquely to a model homomorphism $M' \to N'$.[8] This property holds both in the simpler case when the $\mathcal{I}$-models are $\mathcal{FOL}$-models but also in the case when they are Kripke models; in the latter situation, in the case of the first order variables the uniqueness of N' relies upon the fact that interpretations of the underlying carriers and of the first order constants are shared across the possible worlds.

Now we show that the inventing property holds in the complete form for all F-products. Let $\chi : \Sigma \to \Sigma'$ be a signature extension with nominal or first order variables and let $\{\mu_J : M_J \to M_F \mid J \in F\}$ be an F-product of a family $(M_i)_{i\in I}$ of Σ-models. Let N' be any χ-expansion of M_F. Since $\mu_I : M_I \to M_F$ is

[6] Note that in this argument F is overloaded, it means both the filter and the family of function symbols of the signature.

[7] Note that here, in order to simplify the discussion, we implicitly assumed cartezian products, which is no loss of generality, and that since in all situations for $\mathcal{I}$ the interpretation of first order constants are shared in all possible worlds we may have a notation such as M_x instead of M_x^w.

[8] At the level of abstract institutions, in [8] this property is called 'quasi-representability'; moreover [8] gives a general result that quasi-representable signature morphisms always preserve directed co-limits.

surjective[9] there exists M'_I a χ-expansion of M_I such that μ_I is a Σ'-model homomorphism $M'_I \to N'$. For each $i \in I$ we let M'_i be the χ-expansion of M_i such that $p_{I,i}: M'_I \to M'_i$ is Σ'-homomorphism. This yields a lifting of $\{\mu_J: M_J \to M_F \mid J \in F\}$ to a co-cone $\{\mu_J: M'_J \to N' \mid J \in F\}$ over a directed diagram of projections in $Mod(\Sigma')$. For any other co-cone $\{\nu_J: M'_J \to N'' \mid J \in F\}$ we let $h: M_F \to Mod(\chi)(N'')$ be the unique mediating homomorphism given by the co-limit property of $\{\mu_J: M_J \to M_F \mid J \in F\}$. It remains to show that $h: N' \to N''$ is a homomorphism of Σ'-models. This follows by virtue of the fact that $\mu_I; h = \nu_I$ and because ν_I is a homomorphism of Σ'-models.

The m-compactness properties of $\mathcal{I}^*$ and $\mathcal{I}^\sharp$ follow immediately from the first part of this corollary via Cor. 5.2. The compactness property of $\mathcal{I}^\sharp$ follows from the general result that compactness and m-compactness are equivalent properties in institutions that have external negations and conjunctions (see [8]), which by Fact 4.1 is the case for all institutions $\mathcal{I}^\sharp$ considered here. $\square$

6. Conclusions

In this paper we have showed that the stratified institutions of [2] may serve as a general fully abstract model theoretic framework for modal logical systems. We have shown that stratified institutions allow for an abstract semantics for modalities, nominals, and satisfaction operator (@); in each of these cases we had been able to employ the minimal structures supporting the corresponding semantics. Within this context we have developed a general ultraproducts method, including a general Łoś theorem, applicable to a wide variety of modal logical systems. Compactness results have have been derived from this ultraproducts method. The concepts introduced and the results developed have been applied to a series of concrete benchmark examples that include both well known and quite unconventional modal logical systems from logic and computing. Due to the very high level of generality of our developments, without commitment to explicit forms of Kripke semantics, our work may be easily applicable to a multitude of new unconventional logical systems. Moreover it may constitute a starting point for a deep institution theoretic approach to a dedicated model theory for modal logical systems in the style of [8].

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