# Categorical Comprehensions and Recursion 

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#### Abstract

A new categorical setting is defined in order to characterize the subrecursive classes belonging to complexity hierarchies. This is achieved by means of coercion functors over a symmetric monoidal category endowed with certain recursion schemes that imitate the bounded recursion scheme. This gives a categorical counterpart of generalized safe composition and safe recursion.


Keywords: Symmetric Monoidal Category, Safe Recursion, Ramified Recursion.

## 1. Introduction

Various recursive function classes have been characterized in categorical terms. It has been achieved by considering a category with certain structure endowed with a recursion scheme. The class of Primitive Recursive Functions ( $\mathcal{P R}$ in the sequel), for instance, has been chased simply by means of a cartesian category and a Natural Numbers Object with parameters (nno in the sequel, see [11]). In [13] it can be found a generalization of that characterization to a monoidal setting, that is achieved by endowing a monoidal category with a special kind of nno (a left nno) where the tensor product is included. It is also known that other classes containing $\mathcal{P} \mathcal{R}$ can be obtained by adding more structure: considering for instance a topos ([8]), a cartesian closed category ([14]) or a category with finite limits $\left([12){ }^{1}\right.$

Less work has been made, however, on categorical characterizations of subrecursive function classes, that is, those contained in $\mathcal{P R}$ (see 44 and [5). In $\mathcal{P} \mathcal{R}$ there is at least a sequence of functions such that every function in it has a more complex growth than the preceding function in the sequence. Such function scale allows us to define a hierarchy in $\mathcal{P} \mathcal{R}$ with which we can classify the

[^0]primitive recursive functions according to its level of complexity. This is the case of the Grzegorzcyk Hierarchy.

A reason to not have more studies of subrecursive function classes in Category Theory at our disposal is that we lack a recursive diagram with enough expressiveness to characterize the operation of bounded recursion under which most of those classes are closed and looking like

$$
\left\{\begin{array}{l}
f(u, 0)=g(u) \\
f(u, x+1)=h(u, x, f(u, x)) \\
f(u, x) \leq j(u, x)
\end{array}\right.
$$

The problem arises when, given those functions $g, h$ and $j$, we want to know if there exists a function $f$ satisfying the three conditions in the bounded recursion scheme.

The known as safe recursion scheme was introduced by Bellantoni and Cook in [2] as a way to substitute the bounding condition in the above scheme by a syntactical condition. The central idea of S. Bellantoni and S. Cook was to define two different kinds of variables (normal and safe variables) according to the use we make of them in the process of computation (see [2] for more details). In [2] the class of polynomial time functions has been characterized and, subsequently, several other subrecursive classes.

The ramified recursion, in turn, is a way to avoid impredicativity problems. In a ramified system the objects are defined using levels such that the definition of an object in level $i$ depends only on levels below $i$. According to [9], by considering recursion over a word algebra $\mathbf{A}$, we can get a collection of levels $\mathbf{A}_{j}$ of $\mathbf{A}$ seen as types or universes where everyone of them contains a copy of the constructors.

The method we will use consists in considering a collection of copies of $\mathbf{N}$, denoted by $\mathbf{N}_{k}$, such that the functions defined in every (isomorphic) copy are:

- in $\mathbf{N}_{0}$ certain initial functions where zero and successor are always present
- in $\mathbf{N}_{k+1}$ the definable functions using functions defined in $\mathbf{N}_{j}$ with $j \leq k$ and certain operators, among which are recursion operators, and whose recursion has been made over values in $\mathbf{N}_{s}$ with $s \leq k$.

We will call these $\mathbf{N}_{i}$ levels of the natural numbers and they have a close relation with different function classes according to its complexity level.

The thesis [10] uses categories of ordinal numbers ${ }^{2}$ to define coercion functors with the idea of chasing the ramification conditions of [9]. Using this method, and introducing the concept of symmetric monoidal 2- and 3-Comprehensions, J. R. Otto tries to characterize several subrecursive function classes such as linear time, polynomial time, polynomial space and the classes $\mathcal{E}^{2}$ and $\mathcal{E}^{3}$ of Grzegorzcyk Hierarchy.

[^1]The aim of this paper is to give a categorical characterization of subrecursive hierarchies based on the operations of safe recursion and composition.

## 2. Basic structures

Definition 1. For each $n \in \mathbf{N}$ the category $\mathbf{n}$ has as objects the natural numbers lower than $n$ and as arrows

$$
0 \longrightarrow 1 \longrightarrow \cdots \longrightarrow n-1
$$

corresponding to the order of $n$. We denote by $m_{i, j}$ the only arrow from $i$ to $j$ with $0 \leq i \leq j<n$.

Definition 2. Let $M_{n}^{o p}$ be the monoid of endofunctors in $\mathbf{n}$ in which the product $f g$ is the composition $g \circ f b^{3}$

Let's establish a set of elements in $M_{n}^{o p}$ from which one can generate the rest of elements by means of multiplication. This set is used in [10] in the case of $n=2$ and $n=3$.

Let be for every $0 \leq k<n-1$ the functors $i d: \mathbf{n} \longrightarrow \mathbf{n}, T_{k}: \mathbf{n} \longrightarrow \mathbf{n}$ and $G_{k}: \mathbf{n} \longrightarrow \mathbf{n}$ such that for all $j \in \mathbf{n}$ :

$$
i d(j)=j \quad T_{k}(j)=\left\{\begin{array}{ll}
k+1 & \text { if } j=k \\
j & \text { if } j \neq k
\end{array} \quad G_{k}(j)= \begin{cases}k & \text { if } j=k+1 \\
j & \text { if } j \neq k+1\end{cases}\right.
$$

taking the form

$\left[T_{1}\right] 0 \longrightarrow 0 \quad \cdots \quad\left[T_{n-2}\right] \quad 0 \longrightarrow 0$
$2 \longrightarrow 2$

$1 \longrightarrow 1$
$\vdots$

$$
n-1 \longrightarrow n-1
$$

$$
n-1 \longrightarrow n-1
$$


and

[^2]

In the sequel we will refer to different $T$ and $G$ as coercion functors.
Proposition 3. For every $n \in \mathbf{N}$ the monoid $M_{n}^{o p}$ can be generated by the finite set

$$
\left\{G_{0}, \cdots, G_{n-2}, T_{0}, \cdots, T_{n-2}\right\} .
$$

Now we consider some particular natural transformations in $\mathbf{n}$.
Definition 4. Let $\epsilon_{k}: G_{k} \Longrightarrow i d$ and $\eta_{k}: i d \Longrightarrow T_{k}(0 \leq k \leq n-2)$ be such that for $i \in \mathbf{n}$

$$
\epsilon_{k}(i)=\left\{\begin{array}{ll}
m_{i, i} & \text { if } i \neq k+1 \\
m_{i-1, i} & \text { if } i=k+1
\end{array} \quad \eta_{k}(i)=\left\{\begin{array}{ll}
m_{i, i} & \text { if } i \neq k \\
m_{i, i+1} & \text { if } i=k
\end{array} .\right.\right.
$$

Theorem 5. Every non-identity natural transformation in $\mathbf{n}$ can be generated by means of a composition of natural transformations from Definition 4 and right and left multiplication of those natural transformations and functors from Proposition 3 .
$M_{n}^{o p}$ can be seen as a category whose objects are the endofunctors in $\mathbf{n}$ and whose arrows are the natural transformations in $\mathbf{n}$.

Theorem 6. For the definitions given above we have the following chain of adjunctions

$$
T_{k} \dashv G_{k} \dashv T_{k+1} \dashv G_{k+1}
$$

for every $k \in\{0,1, \ldots, n-3\}$.

## 3. SM n-Comprehensions

For the definition of $S M$ n-Comprehension in this Section we need to consider relations among categories allowing the definition of categorical structures
arising from other structures based on certain properties that the former inherits from the latter. A category will then have the same certain bicategorical property of another category if the same commutative diagrams are satisfied for them both. That is, if there exists a bifunctor between them.

Definition 7. A $S M$ n-Comprehension $\left(\mathcal{C},\left\langle T_{k}^{\mathcal{C}}\right\rangle,\left\langle G_{k}^{\mathcal{C}}\right\rangle,\left\langle\eta_{k}^{\mathcal{C}}\right\rangle,\left\langle\epsilon_{k}^{\mathcal{C}}\right\rangle\right)$ consists of

- A SM category $\mathcal{C}=(\otimes, \top, l, a, \sigma){ }^{4}$
- for every $k$ such that $0 \leq k<n-1$ the $S M$ functors $T_{k}^{\mathcal{C}}, G_{k}^{\mathcal{C}}: \mathcal{C} \longrightarrow \mathcal{C} \square^{5}$
- for every $k$ such that $0 \leq k<n-1$ the $S M$ transformations $\eta_{k}^{\mathcal{C}}$ : id $\Longrightarrow T_{k}^{\mathcal{C}}$ and $\epsilon_{k}^{\mathcal{C}}: G_{k}^{\mathcal{C}} \Longrightarrow i d{ }^{6}$
and the existence of a bifunctor $\Im: M_{n}^{o p} \rightarrow(\mathcal{C}, \mathcal{C})$ such that $\Im\left(T_{k}\right)=T_{k}^{\mathcal{C}}$, $\Im\left(G_{k}\right)=G_{k}^{\mathcal{C}}, \Im\left(\eta_{k}\right)=\eta_{k}^{\mathcal{C}}$ and $\Im\left(\epsilon_{k}\right)=\epsilon_{k}^{\mathcal{C}} \square^{7}$

We will denote $\left(\mathcal{C},\left\langle T_{k}\right\rangle,\left\langle G_{k}\right\rangle,\left\langle\eta_{k}\right\rangle,\left\langle\epsilon_{k}\right\rangle\right)$ for $\left(\mathcal{C},\left\langle T_{k}^{\mathcal{C}}\right\rangle,\left\langle G_{k}^{\mathcal{C}}\right\rangle,\left\langle\eta_{k}^{\mathcal{C}}\right\rangle,\left\langle\epsilon_{k}^{\mathcal{C}}\right\rangle\right)$ when there is no ambiguity.

We will now see that an analogous structure can be defined for the exponential of a category by considering two different starting cases: a given $S M$ n-Comprehension (Exemple 8) or simply a $S M$ category (Exemple 9). We will see for both structures how a sort of exponential SM n-Comprehension can be constructed in quite a different way. This is achieved by using the cotensor product of two $\mathcal{V}$-categories, a concept we recall in Appendix 1, specialized to the case of $\mathbf{n} \multimap \mathcal{C}$.

[^3]Example 8. An exemple of an exponential $S M$ n-Comprehension from a given a $S M n$-Comprehension $\left(\mathcal{C},\left\langle T_{k}^{\mathcal{C}}\right\rangle,\left\langle G_{k}^{\mathcal{C}}\right\rangle,\left\langle\eta_{k}^{\mathcal{C}}\right\rangle,\left\langle\epsilon_{k}^{\mathcal{C}}\right\rangle\right)$ is constructed in the following. We denote for $\chi_{k}^{\mathcal{C}}=\eta_{k}^{\mathcal{C}} \circ \epsilon_{k}^{\mathcal{C}}$ a natural transformation from $G_{k}^{\mathcal{C}}$ to $T_{k}^{\mathcal{C}}$ for each $k=0, \ldots, n-2$ and for which we have the obvious equalities

$$
T_{k}^{\mathcal{C}} \epsilon_{k}^{\mathcal{C}}=G_{k}^{\mathcal{C}} \eta_{k}^{\mathcal{C}}=\chi_{k}^{\mathcal{C}}
$$

We define a functor between $\mathcal{C}$ and $\mathbf{n} \multimap \mathcal{C}$ by taking the following endofunctors in $d^{8}$

$$
G_{n-2}^{\mathcal{C}} \ldots G_{k}^{\mathcal{C}} T_{0}^{\mathcal{C}} \ldots T_{k-1}^{\mathcal{C}}
$$

whose corresponding functors in $\mathbf{n}$ give constant values for $0 \leq k \leq n-1$. That is

$$
G_{n-2} \ldots G_{k} T_{0} \ldots T_{k-1}(j)=k
$$

for every $j=0, \ldots, n-1$.
We denote $\bar{k}$ for $G_{n-2}^{\mathcal{C}} \ldots G_{k}^{\mathcal{C}} T_{0}^{\mathcal{C}} \ldots T_{k-1}^{\mathcal{C}}$ where $0 \leq k \leq n-1$. Let $\chi$ be then the assignation

$$
\begin{gathered}
\chi(k)=\bar{k} \\
\chi\left(m_{k, k+1}\right)=\bar{k} \chi_{k}^{\mathcal{C}}
\end{gathered}
$$

and

$$
\chi(f \circ g)=\chi(f) \circ \chi(g),
$$

for all $k=0,1, \ldots, n-2$ and for every pair of morphisms $f$ and $g$ in $\mathbf{n}$. We then have the following assignations:

[^4]$$
\mathrm{n} \xrightarrow{\chi} \mathcal{S M}(\mathcal{C}, \mathcal{C})
$$


We stress here that $\chi_{k}^{\mathcal{C}}: G_{k}^{\mathcal{C}} \Longrightarrow T_{k}^{\mathcal{C}}$ are natural transformations for endofunctors in $\mathcal{C}$ while $\chi$ can be seen as a bifunctor with domain $\mathbf{n}$, seen as a bicategory, and $\mathcal{S} \mathcal{M}(\mathcal{C}, \mathcal{C})$ as codomain.

For $\mathbf{n} \multimap \mathcal{C}$ functors are chains of natural transformations in the form

$$
\bar{k} \chi_{k}
$$

with $0 \leq k \leq n-2$. That is, starting from the unique $(n-1)$-tuple of natural transformations

$$
\left[\overline{0} \chi_{0}, \overline{1} \chi_{1}, \ldots, \overline{(n-2)} \chi_{n-2}\right]
$$

in $\mathcal{S} \mathcal{M}(\mathcal{C}, \mathcal{C})$, that can be seen as the assignation of

$$
\chi: \mathbf{n} \rightarrow \mathcal{S M}(\mathcal{C}, \mathcal{C})
$$

for constant values in Cat, it can be generated another assignation ${ }^{9}$

$$
\bar{\chi}: \mathcal{C} \rightarrow \mathbf{n} \multimap \mathcal{C}
$$

in $\mathcal{S M}$. This new assignation has in the case of a $n$-Comprehension, among others, the form we have introduced above.
With this construction we can assert that whenever $\left(\mathcal{C},\left\langle T_{k}\right\rangle,\left\langle G_{k}\right\rangle,\left\langle\eta_{k}\right\rangle,\left\langle\epsilon_{k}\right\rangle\right)$

[^5]is a $S M$-Comprehension we can construct a new tuple
$$
\left(\mathbf{n} \multimap \mathcal{C},\left\langle T_{k}^{\mathbf{n}}\right\rangle,\left\langle G_{k}^{\mathbf{n}}\right\rangle,\left\langle\eta_{k}^{\mathbf{n}}\right\rangle,\left\langle\epsilon_{k}^{\mathbf{n}}\right\rangle\right)
$$
being itself a $S M$-Comprehension.
Example 9. An exemple of an exponential $S M$ n-Comprehension from a given $S M$ category $\mathcal{C}$ is constructed by considering again $\mathbf{n} \multimap \mathcal{C}$.

We now define some endofunctors $T^{e}$ and $G^{e}$ acting in such a way that for every

$$
X_{0} \xrightarrow{h_{0}} \ldots \xrightarrow{h_{n-2}} X_{n-1}
$$

we obtain

$$
T_{k}^{e}\left(X_{0} \rightarrow \ldots \rightarrow X_{n-1}\right)=X_{0} \rightarrow \ldots \rightarrow X_{k-1} \xrightarrow{t} X_{k+1} \xrightarrow{i d} X_{k+1} \rightarrow \ldots \rightarrow X_{n-1}
$$

and

$$
G_{k}^{e}\left(X_{0} \rightarrow \ldots \rightarrow X_{n-1}\right)=X_{0} \rightarrow \ldots \rightarrow X_{k} \xrightarrow{i d} X_{k} \xrightarrow{g} X_{k+2} \rightarrow \ldots \rightarrow X_{n-1}
$$

where $t=h_{k-1} \circ h_{k}$ and $g=h_{k} \circ h_{k+1}$ and for every chain of vertical arrows $\left(f_{0}, \ldots, f_{n-1}\right)$ we obtain ${ }^{10}$

$$
T_{k}^{e}\left(f_{0}, \ldots, f_{n-1}\right)=\left(f_{0}, \ldots, f_{k-1}, f_{k+1}, f_{k+1}, \ldots, f_{n-1}\right)
$$

and

$$
G_{k}^{e}\left(f_{0}, \ldots, f_{n-1}\right)=\left(f_{0}, \ldots, f_{k}, f_{k}, f_{k+2}, \ldots, f_{n-1}\right)
$$

Making of $\mathbf{n} \multimap \mathcal{C}$ a SM n-Comprehension.
Fixing a single object $X$ there are some special objects in the form

where the chains are formed by $n$ objects and $n-1$ arrows. We call these objects the levels of $X$. This levels of an object $X$ can also be generated by applications

[^6]of the endofunctors $G_{k}^{e}$ starting from $X^{0}$ :
$$
X^{k}=G_{k-1}^{e} X^{k-1}
$$
or else, starting from $X^{n-2}$ and excluding $X^{n-1}$, by
$$
X^{k}=T_{k+1}^{e} X^{k+1}
$$
when $k=0, \ldots, n-2$. It gives the following table for the levels of the object $X$

|  | $X^{0}$ | $X^{1}$ | ... | $X^{n-3}$ | $X^{n-2}$ | $X^{n-1}$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $T_{0}^{e}$ | $1^{n}$ | $X^{1}$ | $\ldots$ | $X^{n-3}$ | $X^{n-2}$ | $X^{n-1}$ |
| $G_{0}^{e}$ | $X^{1}$ | $X^{1}$ | $\ldots$ | $X^{n-3}$ | $X^{n-2}$ | $X^{n-1}$ |
| $T_{1}^{e}$ | $X^{0}$ | $X^{0}$ | ... | $X^{n-3}$ | $X^{n-2}$ | $X^{n-1}$ |
| $G_{1}^{e}$ | $X^{0}$ | $X^{2}$ | ... | $X^{n-3}$ | $X^{n-2}$ | $X^{n-1}$ |
| $\vdots$ |  |  |  |  |  |  |
| $G_{n-3}^{e}$ | $X^{0}$ | $X^{1}$ | .. | $X^{n-2}$ | $X^{n-2}$ | $X^{n-1}$ |
| $T_{n-2}^{e}$ | $X^{0}$ | $X^{1}$ | $\ldots$ | $X^{n-3}$ | $X^{n-3}$ | $X^{n-1}$ |
| $G_{n-2}^{e}$ | $X^{0}$ | $X^{1}$ | ... | $X^{n-3}$ | $X^{n-1}$ | $X^{n-1}$ |

## 4. SM n-Comprehensions with Recursion

Following [13], where some categorical structures giving rise to primitive recursive functions in the initial monoidal category with a left natural numbers object were introduced, we can establish for some objects in the free $S M \mathrm{n}$ Comprehension with Recursion analogous results. We will see in fact, in the following Section, that the morphisms generated in the free $S M$ n-Comprehension with Recursion are morphisms between cocommutative comonoids in a $S M$ category (see Appendix 2 for a description of these concepts). This is done to justify the introduction of the safe dependent recursion schemes in the class of SM n-Comprehensions with Recursion for the so-called cartesian objects below.

We have the following Theorem related to this point (taken from [1]).
Theorem 10. Let $\mathcal{C}$ be a SM category, $\triangle: \mathcal{C} \longrightarrow \mathcal{C}$ a functor such that $\triangle(C)=$ $C \otimes C$ and $t: \mathcal{C} \longrightarrow \mathcal{C}$ a functor such that $t(C)=\top$ for every object $C$ in $\mathcal{C}$ with monoidal natural transformations $\delta: i d \longrightarrow \triangle$ and $\tau: i d \longrightarrow t$ such that for every object $C$ in $\mathcal{C}$ the diagrams

commute. Then $\mathcal{C}$ is cartesian SM .
Proof. See Appendix 3.

This Theorem says essentially that every $S M$ category is a cartesian $S M$ category if we can duplicate and delete data and, roughly speaking, duplicate and delete the same datum is the same thing than doing nothing ${ }^{11}$

We now define the basic categorical structure from which we'll develop recursion in $n$-Comprehensions. That is done by taking a class of SM nComprehensions endowed with more structure, that is, some recursive diagrams. We then proceed to modify and enrich the structure with initial diagrams and recursive operators. For that we denote by $\mathcal{C} \mathcal{R}^{n}$ a new class named $S M$ nComprehension with Recursion obtained from a SM n-Comprehension in the form of the following Definition.

Definition 11. We define the class of $S M$ n-Comprehensions with Recursion, denoted by $\mathcal{C} \mathcal{R}^{n}$, as the class of $S M{ }_{n}$-Comprehensions in the form $\left(\mathcal{C},\left\langle T_{k}\right\rangle,\left\langle G_{k}\right\rangle,\left\langle\eta_{k}\right\rangle,\left\langle\epsilon_{k}\right\rangle\right)$

- containing an object $N_{0}$ and two arrows $0_{0}$ and $s_{0}$ whose diagram (named initial diagram) is

$$
\top \xrightarrow{0_{0}} N_{0} \xrightarrow{s_{0}} N_{0} .
$$

We define recursively for each $i=1, \ldots, n-2$ the objects $N_{i}$ by the rules ${ }^{12}$

$$
\begin{gathered}
N_{1}=G_{0} N_{0} \\
N_{i+1}=G_{i} N_{i}
\end{gathered}
$$

and morphisms $0_{j}$ and $s_{j}{ }^{13}$ In $\mathcal{C}$ we have for each $i=0,1, \ldots, n-2$ and $j=0,1, \ldots, n-1$

$$
T_{i} N_{j}=\left\{\begin{array}{ll}
\top & \text { if } i=j=0 \\
N_{i-1} & \text { if } i=j \neq 0 \\
N_{j} & \text { otherwise }
\end{array} \quad G_{i} N_{j}= \begin{cases}N_{i+1} & \text { if } i=j \\
N_{j} & \text { otherwise }\end{cases}\right.
$$

With these definitions we can generate all initial diagrams in the form

$$
\top \xrightarrow{0_{j}} N_{j} \xrightarrow{s_{j}} N_{j}
$$

for $0<j<n-1$ as well as

$$
T_{0}\left(\top \xrightarrow{0} N_{0} \xrightarrow{s} N_{0}\right)=\top \xrightarrow{1} \top \xrightarrow{1} \top ;
$$

- closed under flat recursion (FR):

[^7]for all morphisms
$$
g: X \longrightarrow Y \text { and } h: N_{0} \otimes X \longrightarrow Y
$$
where $X$ and $Y$ are in the form $N_{0}^{\alpha}$ there exist a unique
$$
f: N_{0} \otimes X \longrightarrow Y
$$
in $\mathcal{C}$, which we will denote by $F R(g, h)$, such that the following diagram commute ${ }^{14}$


- closed under safe ramified recursion diagrams on each level $k\left(S R R_{k}\right)$ :
for all $k=0,1, \ldots, n-2$ and for all morphisms

$$
g: X \longrightarrow Y \text { and } h: Y \longrightarrow Y
$$

where $T_{k} \ldots T_{0} Y$ is isomorphic to $\top$ there exist a unique

$$
f: N_{k+1} \otimes X \longrightarrow Y
$$

in $\mathcal{C}$, which we will denote by $S R R_{k}(g, h)$, such that the following diagram commutes


- naming cartesian objects in $\mathcal{C} \mathcal{R}^{n}$ the objects in the form $\bigotimes_{i=0}^{n-1} N_{i}^{\alpha_{i}}$, we have that for every cartesian object $\mathcal{C} \mathcal{R}^{n}$ is also closed under safe dependent recursion in each level $k\left(S D R_{k}\right)$ :
for all $k=0, \ldots, n-2$ and for all morphisms

$$
g: X \longrightarrow Y \text { and } h:\left(N_{k+1} \otimes X\right) \otimes Y \longrightarrow Y
$$

[^8]where $T_{k} \ldots T_{0} Y$ is isomorphic to $\top$ and $X$ and $Y$ are cartesian objects there exist a unique
$$
f: N_{k+1} \otimes X \longrightarrow Y
$$
in $\mathcal{C}$, which we will denote by $S D R_{k}(g, h)$, such that the following diagram commutes


Elements of $\mathcal{C} \mathcal{R}^{n}$ are then $S M$-Comprehensions with four different shaped diagrams and certain bounding conditions on the objects over which those diagrams are acting. Note at this point also that the number of nested recursions made in every step is exactly the recursion level in every scheme (see [3]).

Example 12. Our exemple of SM n-Comprehension with Recursion consists of defining a cotensor in the form of a presheaf. Consider the category Set $t^{{ }^{o p}}$ which we denote by $\widehat{S e t}$. Its objects are chains of sets indexed by $\mathbf{n}^{o p}$ :

$$
X_{n-1} \xrightarrow{f_{n-2}} \ldots \xrightarrow{f_{0}} X_{0}
$$

and its arrows squares built out of them.
By fixing a single set $X$ we have some special objects $X^{k}$ for $k=0, \ldots, n-1$ in the same form than those given in the Exemple 9

- $\widehat{S e t}$ is a SM category
- It has as terminal object chains $1 \rightarrow \ldots \rightarrow 1$ denoted by $1^{n}$ where 1 is whatever set with a single object
- For $k \in\{0,1, \ldots, n-1\}$ and taking 0 (zero) and $s$ (successor) from the usual diagram $1 \xrightarrow{0} \mathbf{N} \xrightarrow{s} \mathbf{N}$ in Set we have the chains of functions
$-0_{k}: 1^{n} \rightarrow \mathbf{N}^{k}$ in the form

with $k$ zero arrows and $n-k-1$ arrows with no name which are identities
$-s_{k}: \mathbf{N}^{k} \rightarrow \mathbf{N}^{k}$ in the form

with $k$ successor arrows and $n-k-1$ arrows with no name which are identities.
- We define the endofunctors $T_{k}^{\widehat{e}}$ and $G_{k}^{\widehat{e}}$ in $\widehat{S e t}$ in the same way than Exemple 9 but reversing the subindexes:
for every $X_{n-1} \xrightarrow{h_{n-2}} \ldots \xrightarrow{h_{0}} X_{0}$ we obtain

$$
\begin{gathered}
T_{k}^{\widehat{e}}\left(X_{n-1} \rightarrow \ldots \rightarrow X_{0}\right)= \\
=X_{n-1} \rightarrow \ldots \rightarrow X_{n-1-k} \xrightarrow{i d} X_{n-1-k} \xrightarrow{t} X_{n-3-k} \rightarrow \ldots \rightarrow X_{0}
\end{gathered}
$$

and

$$
\begin{gathered}
G_{k}^{\widehat{e}}\left(X_{n-1} \rightarrow \ldots \rightarrow X_{0}\right)= \\
X_{n-1} \rightarrow \ldots \rightarrow X_{n-k} \xrightarrow{g} X_{n-2-k} \xrightarrow{i d} X_{n-2-k} \rightarrow \ldots \rightarrow X_{0}
\end{gathered}
$$

where $t=h_{n-3-k} \circ h_{n-2-k}$ and $g=h_{n-1-k} \circ h_{n-2-k}$ and for every chain of vertical arrows $\left(f_{n-1}, \ldots, f_{0}\right)$ we obtain ${ }^{15}$

$$
T_{k}^{\widehat{e}}\left(f_{n-1}, \ldots, f_{0}\right)=\left(f_{n-1}, \ldots, f_{n-1-k}, f_{n-1-k}, f_{n-3-k}, \ldots, f_{0}\right)
$$

and

$$
G_{k}^{\widehat{e}}\left(f_{n-1}, \ldots, f_{0}\right)=\left(f_{n-1}, \ldots, f_{n-k}, f_{n-k}, f_{n-2-k}, \ldots, f_{0}\right)
$$

- We can define a bifunctor $\Im: M_{n}^{o p} \rightarrow(\widehat{\text { Set }}, \widehat{\text { Set }})$ sending every $T_{k}, G_{k}, \eta_{k}, \epsilon_{k}$ to $T_{k}^{\widehat{e}}, G_{k}^{\widehat{e}}, \eta_{k}^{\widehat{e}}, \epsilon_{k}^{\widehat{e}}$ respectively.
Proposition 13. Every cartesian object in $\mathcal{C} \in \mathcal{C} \mathcal{R}^{n}$ is endowed with diagonal and eraser morphisms satisfying the hypothesis of Theorem 10.

Proof. Eraser and duplication morphisms can be both defined on every cartesian object in $\mathcal{C} \mathcal{R}^{n}$. Let then be $X$ a cartesian object belonging to $\mathcal{C}$ in $\mathcal{C} \mathcal{R}^{n}$ :

1. Eraser morphisms $\tau_{X}: X \longrightarrow \top$ in $\mathcal{C}$ can be defined recursively by considering:

- if $X=\top$ we take $\tau_{\top}=1_{\top}$

[^9]- if $X=N_{k+1}$ with $k=0, \ldots, n-1$ we can form the following instance of safe ramified recursion

and the composition $\tau_{N_{k+1}}=f \circ r^{-1}{ }^{16}$
- if $X=Y \otimes Z$ with $Y$ and $Z$ in any of the former cases then we also have the eraser morphism by recalling that $\tau_{X} X=\tau_{Y} Y \otimes \tau_{Z} Z$.

2. Duplication morphisms $\delta_{N_{k}}$ can be obtained by the following diagrams

for each $k=1, \ldots, n-2$ and the composition $\delta_{N_{k}}=G_{k}\left(f \circ r^{-1}\right)$.
Squares of Theorem 10 involving eraser and duplication are also commutative in $\mathcal{C} \in \mathcal{C R}^{n}$ because of their uniqueness.

Remark 14. $\delta_{N_{0}}$ has the problem that we have not at our disposal neither a diagram giving it nor coercion functors allowing us, when $n=2$, to lower the level of the object over which it is acting. We'll consider therefore in the sequel $n>2$.

Example 15. Exemple 12 can be extended to get cartesian objects. They exist obviously in $\widehat{\operatorname{Set}}$ as those chains of sets $X_{n-1} \rightarrow \ldots \rightarrow X_{0}$ where each $X_{k}$ is of


With the last result we point that every cartesian object in $\mathcal{C} \in \mathcal{C R}^{n}$ behave as we expect, that is, they are really cartesian in the sense of Theorem 10. That concept of cartesian object was devoted in the Definition of $\mathcal{\mathcal { C }}{ }^{n}$ to introduce the so-called safe dependent recursion and is inspired on the results in [13, where it was proven that all the objects in the initial monoidal category with a left natural numbers object are powers of it.

[^10]
## 5. The free $S M$ n-Comprehension with Recursion

By endowing the initial $S M$ category with all initial diagrams and all required recursion schemes, we consider the free SM n-Comprehension with Recursion, which we denote $\mathcal{F} \mathcal{R}^{n}$. Now, regarding some concepts of the previous section and some results of [13], we see that $\mathcal{F} \mathcal{R}^{n}$ is actually a cartesian $S M$ category, which allows us to consider $S D R$ diagrams in it.

Theorem 16. $\mathcal{F R}^{n}$ is cartesian.
Proof. It is a consequence of Proposition 13.
Now we have the following results related to the concept of cocommutative comonoid, given in Appendix 2, that were first stated in 13. We won't mention the subscripts of $\delta$ and $\tau$ when they are obvious and $n$ will be greater than 2 for the following.

Proposition 17. $\left(N_{i}, \delta, \tau\right)$ are cocommutative comonoids in $\mathcal{F R}^{n}$ for all $i=$ $0,1, \ldots, n-1$.

Corollary 18. $\left(N_{i}^{k}, \delta, \tau\right)$ are cocommutative comonoids in $\mathcal{F R}^{n}$ for all $k \in \mathbf{N}$ and for all $i=0,1, \ldots, n-1$.

Theorem 19. The tensor product of two cartesian objects in $\mathcal{F}^{n}$ is a cartesian product.

Proof. All cartesian objects in $\mathcal{F} \mathcal{R}^{n}$ are cocommutative comonoids.
It's important here to note that this Theorem allowed us to introduce $S D R$ diagrams in $\mathcal{F} \mathcal{R}^{n}$ as it was seen in Proposition 13 .

## 6. The standard model

The Freyd Cover, technique that we will use to prove some properties of the syntactical structures defined up to now, is a particular case of the following Definition.

Definition 20. Given a functor $\Gamma: \mathcal{C} \longrightarrow$ Set we call Artin Glueing the comma category $S e t / \Gamma$ generated from $\Gamma$ :

- whose objects are groups of three $(X, f, U)$ where
$-X$ is a set
- $U$ is an an object of $\mathcal{C}$
$-f$ is a function $X \longrightarrow \Gamma U$
- whose morphisms between the objects $\left(X, f_{1}, U\right)$ and $\left(Y, f_{2}, V\right)$ are commutative squares

that is, ordered pairs $\left(h_{1}, h_{2}\right)$ where $X \xrightarrow{h_{1}} Y$ and $U \xrightarrow{h_{2}} V$.
Definition 21. If $\mathcal{C}$ is a category with a terminal object 1 its Freyd Cover is the Artin Glueing for the functor $\Gamma=\mathcal{C}(1,-)$.

Morphisms in $\mathcal{F} \mathcal{R}^{n}$ that we will call formal, because of their resemblance with the terms in the formal languages, can be identified with programs generated in that category.

Definition 22. The standard model of formal morphisms is the functor $\Gamma_{n}$ given by the diagram

that is $\Gamma_{n}=(n \multimap \Gamma) \circ \bar{\chi}$ where $\Gamma: \mathcal{F R}^{n} \longrightarrow$ Set is defined by $\Gamma X=\mathcal{F}^{n}(\top, X)$ and $\Gamma f=f \circ-{ }^{17}$

Taking into account that the functor $\Gamma_{n}$ acts over the objects $\top$ and $N_{n-1}$ in $\mathcal{F} \mathcal{R}^{n}$ as

$$
\Gamma_{n} \top=1 \longrightarrow 1 \longrightarrow \ldots \longrightarrow 1
$$

and

$$
\Gamma_{n} N_{n-1}=\mathbf{N}_{n-1} \longrightarrow \mathbf{N}_{n-1} \longrightarrow \ldots \longrightarrow \mathbf{N}_{n-1}
$$

where the arrows are identities, its expressions over the elements in $\mathcal{F} \mathcal{R}^{n}$ are:

- over the objects $N_{j}$ in $\mathcal{F R}^{n}$ for $0 \leq j \leq n-2$ we have

$$
\Gamma_{n} N_{j}=[(n \multimap \Gamma) \circ \bar{\chi}]\left(N_{j}\right)
$$

[^11]being equal to $n \multimap \Gamma$ applied $t q^{18}$
$$
\overline{0} N_{j} \longrightarrow \overline{1} N_{j} \longrightarrow \quad \cdots \quad \longrightarrow \overline{n-2} N_{j} \longrightarrow \overline{n-1} N_{j}
$$
and giving
$$
\overline{0} \mathbf{N}_{j} \longrightarrow \overline{1} \mathbf{N}_{j} \longrightarrow \quad \cdots \quad \longrightarrow \overline{n-2} \mathbf{N}_{j} \longrightarrow \overline{n-1} \mathbf{N}_{j}
$$

This is both an object in $n \multimap$ Set and a function composition in Set ${ }^{19}$

- over morphisms $f: N_{k} \longrightarrow N_{j}$ in $\mathcal{F} \mathcal{R}^{n}$ for $1 \leq k, j \leq n-2$ it is represented by commutative squares in the form

$$
\Gamma_{n} f=[(n \multimap \Gamma) \circ \bar{\chi}](f)=(n \multimap \Gamma)\left(\bar{\chi} N_{k} \longrightarrow \bar{\chi} N_{j}\right)
$$

[^12]${ }^{19}$ In terms of sequences out of 1 and $\mathbf{N}$ we had $n-1$ chains of commutative squares.
giving


This is both an arrow in $n \multimap$ Set and a chain of commutative squares in Set ${ }^{20}$

In a more general case we could consider objects in the form $N_{j}^{\alpha_{j}}$. In that case, given that the endofunctors $T$ and $G$ preserve the tensor product, we had chains in the form

$$
\overline{0} \mathbf{N}_{j} \otimes \stackrel{\alpha_{j}}{\cdots} \otimes \overline{0} \mathbf{N}_{j} \rightarrow \ldots \rightarrow \overline{n-1} \mathbf{N}_{j} \otimes \stackrel{\alpha_{j}}{\cdots} \otimes \overline{n-1} \mathbf{N}_{j}
$$

and an analogous expression for the morphisms. We will work modulo tensor powers due to the enormous length of those expressions.

Definition 23. In the case of the $n$-Comprehension $n \multimap S e t$ the Freyd Cover of $\mathcal{F} \mathcal{R}^{n}$ is given by the comma category $(n \multimap S e t) / \Gamma_{n}$ with the functor $\Gamma_{n}: \mathcal{F} \mathcal{R}^{n} \longrightarrow$ $n \multimap$ Set whose

[^13]- objects are triples $(X, f, U)$ where
$-X$ is an object of $n \multimap S e t$, that is, a chain in the form

$$
X_{0} \rightarrow X_{1} \rightarrow \ldots \rightarrow X_{n-1}
$$

- $U$ is an object of $\mathcal{F} \mathcal{R}^{n}$, that is, the tensor product of distinct tensor powers of objects $N_{k}$ in the form

$$
\bigotimes_{j=0}^{n-1} N_{j}^{\alpha_{j}}
$$

- $f$ is a function $X \longrightarrow \Gamma_{n} U$ in $n \multimap S e t$, that is, a chain of squares
- morphisms between objects $\left(X, f_{1}, U\right)$ and $\left(Y, f_{2}, V\right)$ are commutative squares

that is, pairs $\left(h_{1}, h_{2}\right)$ where $X \xrightarrow{h_{1}} Y$ belongs to $n \multimap$ Set and $U \xrightarrow{h_{2}} V$ to $\mathcal{F} \mathcal{R}^{n}$ and therefore $\Gamma_{n} U \xrightarrow{\Gamma_{n} h_{2}} \Gamma_{n} V$ also belongs to $n \multimap$ Set.
Those squares can be seen as chains of commutative cubes in $n \multimap$ Set.
To complete this Section we give two results connecting the syntactical structure here described with the semantics of numerical functions.

Proposition 24. The image of the objects $N_{k}$ by the functor $\Gamma$ are sets whose elements have the form $\Gamma N_{k}=\left\{s^{2} d_{k} n / n \in \mathbf{N}\right\}$ where $s t d_{k}: \mathbf{N} \longrightarrow \Gamma N_{k}$ is defined by the scheme

$$
\left\{\begin{array}{l}
s t d_{k} 0=0_{k} \\
s t d_{k}(s n)=s_{k}\left(s t d_{k} n\right)
\end{array}\right.
$$

with $k=0,1, \ldots, n-1$.
Corollary 25. $\Gamma N_{k}=\mathbf{N}_{k}$ for all $k=0,1, \ldots, n-1$.
This Proposition and its Corollary indicate that the sets generated by the functor $\Gamma$ applied to the levels of the natural numbers in $\mathcal{F} \mathcal{R}^{n}$ behave as the natural numbers themselves. This fact is a consequence of the use of the Freyd Cover, where every arrow $\top \longrightarrow N_{k}$ has the form $s_{k}^{n} \circ 0_{k}$ for some $n \in \mathbf{N}$.

## 7. Recursive functions in $\mathcal{F} \mathcal{R}^{n}$

To show how hierarchies of subrecursive functions can be defined in $\mathcal{F} \mathcal{R}^{n}$ we introduce a language containing $n$ different species of variables, separated by semicolons, which we will denote by the numbers $0,1, \ldots, n-1$. We assign at the same time a level to every function as is explained in the following:

- we say that a function $f$ is of the type $\left(a_{k}, a_{k-1}, \ldots, a_{0} ; a_{m}\right)$ if its arguments belong to the species $a_{k}, a_{k-1}, \ldots, a_{0}$ in its domain and its codomain belong to the species $a_{m}$. We express this fact by

$$
\underset{a_{k} a_{k-1} \ldots a_{0} ; a_{m}}{f}
$$

- we define the level of a function as the species of its codomain.

If a variable belongs to the $n$-th species then it also belongs to the $(n+1)$-th species.

Now we need to make use of a new recursion scheme in $\mathcal{F} \mathcal{R}^{n}$ with $n>2$ that will turn out to be a particular instance of $S D R$ scheme.

Definition 26. We say that a morphism $f: N_{k+1} \otimes X \longrightarrow Y$ in $\mathcal{F R}^{n}$ with $n>2$ is defined by the parameterized safe ramified recursion scheme on the level $k$ if it is the unique such that for all $g: X \longrightarrow Y$ and $h: X \otimes Y \longrightarrow Y$ with $T_{k} \ldots T_{0} Y$ isomorphic to $\top$ the following diagram

commutes. We denote $f$ by $P S R R_{k}(g, h)$.
Theorem 27. Every function defined using a PSRR scheme can also be defined using a $S D R$ scheme.

With this result we can argue that a doctrine which is closed under the $S D R$ scheme is also closed under the $P S R R$ scheme.

We now define some functions ${ }^{21}$

[^14]- addition in $\mathcal{F} \mathcal{R}^{1}$ denoted by $\underset{10 ; 0}{\bigoplus}: N_{1} \otimes N_{0} \longrightarrow N_{0}$ is defined by $S R R$ :

such that

$$
\left\{\begin{array}{l}
\bigoplus(0, n)=n \\
\bigoplus(s x, n)=s(\bigoplus(x, n))
\end{array}\right.
$$

- multiplication in $\mathcal{F} \mathcal{R}^{2}$ denoted by $\underset{11 ; 0}{\bigotimes}: N_{1} \otimes N_{1} \longrightarrow N_{0}$ is defined by PSRR:

such that

$$
\left\{\begin{array}{l}
\bigotimes(0, y)=0 \\
\bigotimes(s x, y)=\bigoplus(y, \bigotimes(x, y))
\end{array}\right.
$$

- exponentiation in $\mathcal{F} \mathcal{R}^{3}$ denoted by $\underset{21 ; 1}{\uparrow}: N_{2} \otimes N_{1} \longrightarrow N_{1}$ is defined by PSRR ${ }^{22}$

such that

$$
\left\{\begin{array}{l}
\uparrow(0, y)=c_{1} \\
\uparrow(s x, y)=G_{0} \otimes(y, \uparrow(x, y))
\end{array}\right.
$$

[^15]- tetration in $\mathcal{F R}{ }^{4}$ denoted by $\uparrow: N_{3} \otimes N_{2} \longrightarrow N_{1}$ is defined by PSRR: $32 ; 1$

such that

$$
\left\{\begin{array}{l}
\Uparrow(0, y)=y \\
\Uparrow(s x, y)=\uparrow(y, \Uparrow(x, y))
\end{array}\right.
$$

## 8. Safe composition

Safe composition, as defined in the following Definition, has a representation in $\mathcal{F R}^{n}$ by means of diagrams associated to natural transformations in the form $T_{0} \ldots T_{k-1} \eta_{k}$.
Definition 28. We say that a function $f$ is defined by safe composition from functions $\overline{r_{0}}, \ldots, \overline{r_{n}}$ and $h$ if

$$
f\left(\overline{x_{n}} ; \ldots ; \overline{x_{0}}\right)=h\left(\overline{r_{n}}\left(\overline{x_{n}} ;\right) ; \overline{r_{n-1}}\left(\overline{x_{n}} ; \overline{x_{n-1}}\right) ; \ldots ; \overline{r_{0}}\left(\overline{x_{n}} ; \ldots ; \overline{x_{0}}\right)\right)
$$

where the level of $f$ is the level of $h$ while the level of $r_{n}$ is less or equal than $n$ and that of $r_{0}$ is 0 .

For every $\eta_{k}$ and $f: \bigotimes_{j=0}^{n-1} N_{j}^{\alpha_{j}} \longrightarrow N_{m}^{\beta}$ morphism in $\mathcal{F} \mathcal{R}^{n}$ we have commutative diagrams in the following form:

obtained by the action of $T_{0} \ldots T_{k-1} \eta_{k}$ with $k=0,1, \ldots, n-2$ over $f$.
In this diagram we have made use of the identities

$$
T_{0} \ldots T_{k}\left(\bigotimes_{j=0}^{n-1} N_{j}^{\alpha_{j}}\right)=\bigotimes_{j=k+1}^{n-1} N_{j}^{\alpha_{j}}
$$

[^16]and the fact that the arrow
$$
\eta_{k}\left(\bigotimes_{j=k}^{n-1} N_{j}^{\alpha_{j}}\right): \bigotimes_{j=k}^{n-1} N_{j}^{\alpha_{j}} \longrightarrow T_{k}\left(\bigotimes_{j=k}^{n-1} N_{j}^{\alpha_{j}}\right)
$$
is actually an arrow
$$
\bigotimes_{j=k}^{n-1} N_{j}^{\alpha_{j}} \longrightarrow \bigotimes_{j=k+1}^{n-1} N_{j}^{\alpha_{j}}
$$
for every $k=0,1, \ldots, n-3$ with which we have an expression of $f$ in terms of coercions $T_{k}$ due to the fact that they don't change anything over an object in the form $N_{m}^{\beta}$ for $k \leq m-1$.

This grabs the formulation of safe composition from Definition 28 because we obtain an expression of each morphism in $\mathcal{F} \mathcal{R}^{n}$ in terms of other morphisms whose variables belong, as maximum, to the same species of the former. Therefore, the level $n-1$ output does not depend on lower species inputs when we are in $\mathcal{F} \mathcal{R}^{n}$. In general, a $s$ species output does not depend on lower species inputs than $s$.

Theorem 29. For every function

$$
h\left(\bar{x}_{n} ; \ldots ; \overline{x_{k+1}} ; z, \overline{x_{k}} ; \ldots ; \overline{x_{0}}\right)
$$

where $0 \leq k<n$ there exists a function

$$
f\left(\bar{x}_{n} ; \ldots ; \overline{x_{k+1}}, z ; \overline{x_{k}} ; \ldots ; \overline{x_{0}}\right)
$$

obtained by safe composition from $h$ and projections such that

$$
h\left(\bar{x}_{n} ; \ldots ; \overline{x_{k+1}} ; z, \overline{x_{k}} ; \ldots ; \overline{x_{0}}\right)=f\left(\bar{x}_{n} ; \ldots ; \overline{x_{k+1}}, z ; \overline{x_{k}} ; \ldots ; \overline{x_{0}}\right)
$$

Proof. Take projection functions as $\bar{r}$.
That is, every variable being in a species $k$ position can be moved to a species $t>k$ position.

The function classes characterized by this setting will satisfy one of the main features of the subrecursive hierarchies, that is, its growing behaviour: there exist functions not belonging to any previous class in their ordering. Take for exemple those of the Hyperoperation Sequence and its relation with the classes in the Grzegorzcyk Hierarchy denoted by $\mathcal{E}^{n}$ for $n \in \mathbf{N}$. Every ( $n+1$ )-level function in the Hyperoperation Sequence belong to $\mathcal{E}^{n+1}$ but not to $\mathcal{E}^{n}$.

Concurrently, we can give in $\mathcal{E}^{k}$ a copy of each function in $\mathcal{E}^{j}$ for every $k \geq j$ and we forbid in $\mathcal{E}^{j}$ any copy of a function generated in $\mathcal{E}^{k}$. The former is done by the action of a coercion functor $G_{m}$ for $k>m \geq j$ and the latter by avoiding the application of endofunctors $T_{m}$ for $k \geq m>j$ over the arrows generated by means of a recursion scheme in $\mathcal{E}^{k}$. This is done to avoid the structure collapse due to the fact that those coercion functors may reduce subindexes. In these
situations we must consider a subcategory $\mathcal{S F} \mathcal{R}^{n}$ of $\mathcal{F} \mathcal{R}^{n}$ which we describe in Appendix 4.

## 9. Conclusions and future work

Symmetric Monoidal $n$-Comprehensions are proved to be useful for new characterizations of subrecursive function classes, giving a wider point of view of recursion in Category Theory.

This work can be extended, for instance, by considering other (partial) orders as giving rise to a different concept of $n$-Comprehension to chase different function classes (see [10] for this particular). Other investigation line to follow starting from this paper could be a fibrational point of view of the results here given.

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## Appendix 1

We will use some concepts of [7] to get the cotensor of two $\mathcal{V}$-categories. For that $\mathcal{V}$ will be in the sequel a monoidal category.

Definition 30. Let $\mathcal{B}$ be a $\mathcal{V}$-category, $B, C \in \mathcal{B}$ and $X \in \mathcal{V}$. If there exists an object $D$ in $\mathcal{B}$ and a $\mathcal{V}$-natural isomorphism

$$
\mathcal{B}(B, D) \cong[X, \mathcal{B}(B, C)]
$$

we say that $D$ is the cotensor product of $X$ and $C$ in $\mathcal{B}$ and we will denote it by $X \multimap C$.

If it exists for all $X$ and $C$ then we say that the $\mathcal{V}$-category $\mathcal{B}$ is cotensorial. In this case we also have the isomorphism $\mathcal{B}(X \otimes B, C) \cong[X, \mathcal{B}(B, C)]$ which means that we have the isomorphism

$$
\mathcal{B}(X \otimes B, C) \cong \mathcal{B}(B, X \multimap C)
$$

whenever $\mathcal{V}$ is $S M$ closed and its underlying $\mathcal{V}_{0}$ is complete.
Remark 31. Every $S M$ closed category has tensor and cotensor products for every pair of objects and the cotensor product is the hom-object formed by those objects.

Example 32. Let $\mathcal{C}$ be a SM category. We take $\mathcal{B}=\mathcal{S} \mathcal{M}, \mathcal{V}=C a t$ and the 2-functors

$$
G: \mathcal{I} \longrightarrow \mathcal{S M} \text { and } F: \mathcal{I} \longrightarrow C a t
$$

where $\mathcal{I}$ is the unit $\mathcal{V}$-category such that $G$ determines a category $\mathcal{D}$ and $F$ determines $\mathbf{n}$. Then for all $\mathcal{C} \in \mathcal{S} \mathcal{M}$ we have

$$
\mathcal{S M}(\mathbf{n} \otimes \mathcal{D}, \mathcal{C}) \cong \mathcal{S} \mathcal{M}(\mathcal{D}, \mathbf{n} \multimap \mathcal{C}) \cong[\mathbf{n}, \mathcal{S} \mathcal{M}(\mathcal{D}, \mathcal{C})]
$$

and, by taking $\mathcal{D}=\mathcal{C}$,

$$
\mathcal{S M}(\mathbf{n} \otimes \mathcal{C}, \mathcal{C}) \cong \mathcal{S} \mathcal{M}(\mathcal{C}, \mathbf{n} \multimap \mathcal{C}) \cong[\mathbf{n}, \mathcal{S} \mathcal{M}(\mathcal{C}, \mathcal{C})]
$$

where the 2-category at right is isomorphic to $\mathcal{S M}(\mathcal{C}, \mathcal{C})^{n}$.
This construction makes sense due to the fact that $\mathcal{S M}$ admits cotensor objects with the category $\mathbf{n}$ whenever $\mathcal{C} \in \mathcal{S} \mathcal{M}$, a fact that we spell out immediately below. $\mathcal{S M}$ can be seen itself as a $\mathcal{V}$-category with $\mathcal{V}=C$ at a $S M$ closed category, we can then say that the cotensor object is exactly the hom-object. That is

$$
\mathbf{n} \multimap \mathcal{C}=[\mathbf{n}, \mathcal{C}]
$$

It can be defined a symmetric monoidal structure for $\mathbf{n} \multimap \mathcal{C}$ when $\mathcal{C}$ is in $\mathcal{S M}$ given by the following:

- as unit we take the following chain of $n-1$ morphisms

$$
\top \longrightarrow \top \longrightarrow \ldots \longrightarrow \top
$$

- tensor product of the objects

$$
Y_{0} \xrightarrow{y_{0}} \quad \ldots \quad \xrightarrow{y_{n-2}} Y_{n-1}
$$

and

$$
X_{0} \xrightarrow{x_{0}} \quad \cdots \quad \xrightarrow{x_{n-2}} X_{n-1}
$$

is defined by

$$
Y_{0} \otimes X_{0} \xrightarrow{y_{0} \otimes x_{0}} \quad \cdots \quad \stackrel{y_{n-2} \otimes x_{n-2}}{\longrightarrow} Y_{n-1} \otimes X_{n-1}
$$

- tensor product of an object

$$
Y_{0} \xrightarrow{y_{0}} \quad \cdots \quad \xrightarrow{y_{n-2}} Y_{n-1}
$$

and an arrow 24

is defined by

[^17]

- symmetries are

and the rest of diagrams giving the $S M$ structure where commutativity is satisfied in every diagram because of the $S M$ structure in $\mathcal{C}$.


## Appendix 2

Definition 33. Let $\mathcal{C}$ be a $S M$ category. We denote by $C C(\mathcal{C})$ the category whose

- objects are cocommutative comonoids in $\mathcal{C}$ in the form $\left(A, \delta_{A}, \tau_{A}\right)$
- morphisms between cocommutative comonoids $\left(A, \delta_{A}, \tau_{A}\right)$ and $\left(B, \delta_{B}, \tau_{B}\right)$ are morphisms $f: A \longrightarrow B$ in $\mathcal{C}$ such that the following diagrams commute


Remark 34. $C C(\mathcal{C})$ is cartesian (see [6]). That cartesian product in $C C(\mathcal{C})$ is given by the comonoid $\left(A \otimes B, \delta_{A \otimes B}, \tau_{A \otimes B}\right)$ for $\left(A, \delta_{A}, \tau_{A}\right)$ and $\left(B, \delta_{B}, \tau_{B}\right)$ in $C C(\mathcal{C})$ due to the fact that the following diagram commutes for all $f: C \longrightarrow A$ and $g: C \longrightarrow B$ in $C C(\mathcal{C})$

where the definition of projections in a cartesian SM category (see the following) are given implicitly.

Definition 35. A cartesian symmetric monoidal category (a cartesian SM category in the sequel) is a symmetric monoidal category whose monoidal structure is given by a cartesian product.

Remark 36. From this Definition we can argue that the unit of the tensor in the case of a cartesian $S M$ category is a terminal object in the category.

Every cartesian $S M$ category is endowed with morphisms diagonal in the form $\delta_{C}: C \rightarrow C \otimes C$ and eraser in the form $\tau_{C}: C \rightarrow \top$ for every object $C$. We can think on the interpretation of morphisms $\delta$ and $\tau$ in terms of Computer Science as the one that duplicates a datum and the one that deletes a datum respectively. Those morphisms carry the structure of a cocommutative comonoid over an object in the category. In fact, every object in a cartesian $S M$ category can be seen uniquely as a comonoid as seen in the following.

Theorem 37. Given a cartesian symmetric monoidal category $\mathcal{C}$ every object is endowed with a cocommutative comonoid structure uniquely defined.

Proof. By being a cartesian symmetric monoidal category we know that the unit $\top$ is a terminal object and therefore there exists for every object $C$ in $\mathcal{C}$ a unique arrow $C \longrightarrow \top$ that has to be $\tau_{C}$ for the comonoid structure.

On the other hand, by being cartesian we can construct a commutative diagram in the form

where the unique $h$, denoted by $\langle i d, i d\rangle$, has to be the duplication arrow $\delta_{C}$.

## Appendix 3

Proof. [of Theorem 10 For an object $D$ in $\mathcal{C}$ and arrows $f_{1}: D \longrightarrow C_{1}$ and $f_{2}: D \longrightarrow C_{2}$ we can construct a diagram ${ }^{25}$

where (1) commutes for being $\delta$ a natural transformation and (3) by hypothesis while the commutativity of diagram numbered (2) can be proved by considering

[^18]it as

where (4) commutes for bifunctoriality. Diagram (5) commutes by taking a monoidal natural transformation $C_{1} \otimes \tau$ giving, for $f_{1}$ and $f_{2}$ like above:

giving

both commuting for naturality.
Then the (1), (2), (3)-diagram (together with its analogous for $C_{2}$ ) is a cartesian product diagram where projections are $r \circ\left(C_{1} \otimes \tau_{C_{2}}\right)$ and $l \circ\left(\tau_{C_{1}} \otimes C_{2}\right)$ and the uniqueness is obvious given $f_{1}$ and $f_{2}$.

## Appendix 4

Let $\mathcal{S F} \mathcal{R}^{n}$ be a subcategory of $\mathcal{F R}^{n}$ in which we avoid any application of $T$ over the objects and morphisms of $\mathcal{F} \mathcal{R}^{n}$. We define some of the objects in $\mathcal{S F} \mathcal{R}^{n}$ by means of $G$ as happens in the case of $\mathcal{F} \mathcal{R}^{n}$ but will make use of endofunctors $T$ only for the introduction of a bounding condition in the recursion schemes used in $\mathcal{S F} \mathcal{R}^{n}$.

We introduce in the squares below the description of $\mathcal{S F} \mathcal{R}^{n}$ in the form of a language for its objects and morphisms of $\mathcal{S F} \mathcal{R}^{n}$. The rules into the squares are subject to the following conventions:

- we have omitted defining symmetric monoidal category rules (identity, associativity as well as coherence diagrams)
- $X, Y, Z$ and $W$ denote whatever object
- $f$ and $g$ denote whatever morphism
- $a, l, \sigma$ denote the natural isomorphisms of the $S M$ structure
- subindex $k$ will range between 0 and $n-2$ into the squares when no other indication is given ${ }^{26}$

1. Objects
(a) initial objects

$$
\bar{\top} T-\text { object } \quad \overline{N_{k}} N_{k}-\text { object }
$$

(b) generation of objects

$$
\frac{X \quad Y}{X \otimes Y} \otimes-\text { object }
$$

2. Arrows
(a) initial arrows

$$
\begin{gathered}
\frac{X}{i d_{X}: X \rightarrow X} \text { identity } \frac{}{0_{k}: \top \rightarrow N_{k}} \text { zero } \frac{X}{s_{k}: N_{k} \rightarrow N_{k}} \text { successor } \\
\frac{X}{\tau_{X}: X \rightarrow \top} \text { eraser } \frac{X}{\delta_{X}: X \rightarrow X \otimes X} \text { duplication } \\
\frac{d_{k}: N_{k+1} \rightarrow N_{k}}{\text { drop }}
\end{gathered}
$$

(b) generation of arrows from arrows

$$
\frac{f: X \rightarrow Y \quad g: Y \rightarrow Z}{g \circ f: X \rightarrow Z} \text { composition } \quad \frac{f: X \rightarrow Y \quad g: Z \rightarrow W}{g \otimes f: X \otimes Z \rightarrow Y \otimes W} \otimes-\text { arrow }
$$

(c) generation of arrows from objects and natural isomorphisms

$$
\begin{gathered}
\frac{X}{l: \top \otimes X \rightarrow X} \text { left } \frac{X \quad Y}{\sigma: X \otimes Y \rightarrow Y \otimes X} \text { symmetry } \\
\frac{X \quad Y \quad Z}{a:(X \otimes Y) \otimes Z \rightarrow X \otimes(Y \otimes Z)} \text { associativity }
\end{gathered}
$$

## 3. Flat Recursion

$$
\frac{g: X \rightarrow Y \quad h: N_{k} \otimes X \rightarrow Y}{F R_{k}(g, h): N_{k} \otimes X \rightarrow Y} \mathbf{F R}_{k}
$$

where $X$ and $Y$ are in the form $N_{k}^{\alpha}$
We now define some assignations which turn out to be, respectively, $S M$ endofunctors and $S M$ natural transformations.

[^19]1. Let be

$$
T_{k} X=\left\{\begin{array}{ll}
\top & \text { if } X=N_{0} \text { and } k=0 \\
N_{k-1} & \text { if } X=N_{k} \text { and } k \neq 0 \\
T_{k} Y \otimes T_{k} Z & \text { if } X=Y \otimes Z \\
X & \text { otherwise }
\end{array} \quad G_{k} X= \begin{cases}N_{k+1} & \text { if } X=N_{k} \\
G_{k} Y \otimes G_{k} Z & \text { if } X=Y \otimes Z \\
X & \text { otherwise }\end{cases}\right.
$$

and

$$
T_{k} f=\left\{\begin{array}{ll}
i d_{\top} & \text { if } f=0_{0} \text { or } s_{0} \\
0_{k-1} & \text { if } f=0_{k} \text { and } k \neq 0 \\
s_{k-1} & \text { if } f=s_{k} \text { and } k \neq 0 \\
\tau_{T_{k} X} & \text { if } f=\tau_{X} \\
i d_{N_{k-1}} & \text { if } f=d_{k-1} \text { and } k \neq 0 \\
\tau_{N_{1}} & \text { if } f=d_{0} \text { and } k=0 \\
\delta_{T_{k} X} & \text { if } f=\delta_{X} \\
d_{k-1} \circ d_{k} & \text { if } f=d_{k} \text { and } k \neq 0 \\
T_{k} g \circ T_{k} h & \text { if } f=g \circ h \\
T_{k} g \otimes T_{k} h & \text { if } f=g \otimes h \\
f & \text { otherwise }
\end{array} \quad G_{k} f= \begin{cases}0_{k+1} \\
s_{k+1} & \text { if } f=0_{k} \\
\tau_{G_{k} X} & \text { if } f=s_{X} \\
\delta_{G_{k} X} & \text { if } f=\delta_{X} \\
i d_{N_{k+1}} \\
d_{k-1} \circ d_{k} & \text { if } f=d_{k} \\
G_{k} g \circ G_{k} h & \text { if } f=d_{k-1} \\
T_{k} g \otimes T_{k} h & \text { if } f=g \otimes h \\
f & \text { otherwise }\end{cases}\right.
$$

for $f: X \rightarrow Y$ and for each $k=0,1, \ldots, n-2$.
2. We denote by $\epsilon_{k}: G_{k} \Longrightarrow i d$ and $\eta_{k}: i d \Longrightarrow T_{k}$ some assignations ${ }^{27}$

$$
\epsilon_{k} X=\left\{\begin{array}{lll}
\top & \text { if } X=\top \\
d_{k} & \text { if } X=N_{k} \\
\epsilon_{k} Y \otimes \epsilon_{k} Z & \text { if } X=Y \otimes Z \\
i d_{X} & \text { otherwise }
\end{array} \quad \eta_{k} X= \begin{cases}\top & \text { if } X=\top \\
d_{k-1} & \text { if } X=N_{k} \text { and } k \neq 0 \\
\tau_{N_{0}} & \text { if } X=N_{0} \text { and } k=0 \\
\eta_{k} Y \otimes \eta_{k} Z & \text { if } X=Y \otimes Z \\
i d_{X} & \text { otherwise }\end{cases}\right.
$$

It easy to see that $T_{k}$ and $G_{k}$ are endofunctors and $\epsilon_{k}$ and $\eta_{k}$ are natural transformations in the free $S M$ category defined by the rules above for each $k=0,1, \ldots, n-2$.

## 5. Raising arrows

$$
\frac{f: X \rightarrow Y}{G_{k} f: G_{k} X \rightarrow G_{k} Y} \mathbf{G}_{\mathbf{k}}-\text { arrow }
$$

6. Safe Recursion

$$
\frac{g: X \rightarrow Y \quad h: Y \rightarrow Y}{S R R_{k}(g, h): N_{k+1} \otimes X \rightarrow Y} \mathbf{S R R}_{\mathbf{k}}
$$

where $T_{k} \ldots T_{0} Y$ is isomorphic to $T$
7. Safe Dependent Recursion

$$
\frac{g: X \rightarrow Y \quad h:\left(N_{k+1} \otimes X\right) \otimes Y \longrightarrow Y}{S_{k}(g, h): N_{k+1} \otimes X \rightarrow Y} \mathbf{S D R}_{\mathbf{k}}
$$

where $T_{k} \ldots T_{0} Y$ is isomorphic to $\top$

[^20]
[^0]:    * corresponding author at: boils@uji.es
    ${ }^{1}$ See 15 for a summary of those results.

[^1]:    ${ }^{2}$ Hereafter we will only consider finite ordinals.

[^2]:    ${ }^{3} M_{n}^{o p}$ is exactly the set of monotone functions from $(n, \leq)$ to $(n, \leq)$ with $n=\{0, \ldots, n-1\}$.

[^3]:    ${ }^{4}$ We omit the introduction of the right identity $r: C \otimes \top \rightarrow C$ defined for every object $C$ in $\mathcal{C}$ for being definable in terms of $\sigma$ and $l$ as $C \otimes \top \stackrel{\sigma}{\rightarrow} \top \otimes C \xrightarrow{l} C$. It will be used elsewhere in the sequel, however. We also express the objects modulo associativity and symmetry in the sequel.
    ${ }^{5}$ By SM functors we understand that $T_{k}^{\mathcal{C}}, G_{k}^{\mathcal{C}}: \mathcal{C} \longrightarrow \mathcal{C}$ satisfy:

    \[

    \]

    ${ }^{6}$ By SM functors we understand that $\eta_{k}^{\mathcal{C}}: i d \Longrightarrow T_{k}^{\mathcal{C}}$ and $\epsilon_{k}^{\mathcal{C}}: G_{k}^{\mathcal{C}} \Longrightarrow i d$ satisfy

    $$
    \eta_{k}^{\mathcal{C}} \top=\epsilon_{k}^{\mathcal{C}} \top=1_{\top}, \eta_{k}^{\mathcal{C}}(X \otimes Y)=\eta_{k}^{\mathcal{C}} X \otimes \eta_{k}^{\mathcal{C}} Y \text { and } \epsilon_{k}(X \otimes Y)=\epsilon_{k}^{\mathcal{C}} X \otimes \epsilon_{k}^{\mathcal{C}} Y
    $$

    ${ }^{7}$ That is, what we ask is to commute the same diagrams for $T_{k}^{\mathcal{C}}, G_{k}^{\mathcal{C}}, \eta_{k}^{\mathcal{C}}$ and $\epsilon_{k}^{\mathcal{C}}$ than $T_{k}$, $G_{k}, \eta_{k}$ and $\epsilon_{k}$. For $\Im$ exist we are looking at $M_{n}^{o p}$ as a bicategory with a unique 0 -cell $\mathbf{n}$.

[^4]:    ${ }^{8}$ Whenever $k=0$ this expression takes the form $G_{n-2}^{\mathcal{C}} \ldots G_{0}^{\mathcal{C}}$ and in the case of $k=n-1$ the form $T_{0}^{\mathcal{C}} \ldots T_{n-2}^{\mathcal{C}}$.

[^5]:    ${ }^{9}$ By considering the isomorphism $\mathcal{S} \mathcal{M}(\mathcal{C}, \mathbf{n} \multimap \mathcal{C}) \cong \operatorname{Cat}(\mathbf{n}, \mathcal{S} \mathcal{M}(\mathcal{C}, \mathcal{C}))$ given above.

[^6]:    ${ }^{10}$ With the notation established in the description of $\mathbf{n} \multimap \mathcal{C}$ in Appendix 1.

[^7]:    ${ }^{11}$ In the original in [1] that condition was argued to be actually necessary and sufficient. We state just a direction for being enough for the purpose of this paper.
    ${ }^{12} N_{i}$ will be the levels of $N$.
    ${ }^{13}$ Defined by the following schemes: $\left\{\begin{array}{l}0_{1}=G_{0}\left(0_{0}\right) \\ 0_{j+1}=G_{j}\left(0_{j}\right)\end{array}\right.$ and $\left\{\begin{array}{l}s_{1}=G_{0}\left(s_{0}\right) \\ s_{j+1}=G_{j}\left(s_{j}\right)\end{array}\right.$.

[^8]:    ${ }^{14}$ This is actually a coproduct diagram. By applying $G$ to this diagram we obtain flat recursion for successive levels of $N$, we denote them by $F R_{k}$ for $1 \leq k \leq n-2$. $F R_{k}$ diagrams give to the initial diagrams appropriate properties such as the injectivity of the successor function $s$.

[^9]:    ${ }^{15}$ With the notation established in the description of $\mathbf{n} \multimap \mathcal{C}$ in Appendix 1.

[^10]:    ${ }^{16}$ To obtain the arrow $\tau_{N_{0}}: N_{0} \longrightarrow \top$ we take $\eta N_{0}$.

[^11]:    ${ }^{17}$ This is a special case of the global sections functor.

[^12]:    ${ }^{18}$ We point here that we have the following identities:

    $$
    \bar{k} N_{j}=\left\{\begin{array}{lll}
    \top & \text { if } 0 \leq j \leq k-1 & \bar{k} \mathbf{N}_{j}= \begin{cases}1 & \text { if } 0 \leq j \leq k-1 \\
    N_{n-1} & \text { other }\end{cases} \\
    \mathbf{N}_{n-1} & \text { other }
    \end{array}\right.
    $$

[^13]:    ${ }^{20}$ In terms of sequences out of 1 and $\mathbf{N}$ we had $n-1$ chains of commutative cubes.

[^14]:    ${ }^{21}$ They are functions belonging to the so-called Hyperoperation Sequence, which gives an easy way to classify the functions into the Grzegorzcyk Hierarchy by its complexity.

[^15]:    ${ }^{22} c_{1}$ is the constant function 1.

[^16]:    ${ }^{23}$ Working up to isomorphisms $l$ and $r$.

[^17]:    ${ }^{24} \mathrm{We}$ will express squares like this simply as $\left(f_{0}, \ldots, f_{n-1}\right)$.

[^18]:    ${ }^{25}$ And an analogous for $C_{2}$.

[^19]:    ${ }^{26}$ Due to the syntactical behaviour of the definitions above, we should mention that $d_{k}$ applied to $0_{k+1}$ gives $0_{k}$ and, analogously, for every other natural number in a level $k+1$ it assigns the same number in level $k$.

[^20]:    ${ }^{27}$ They are useful to get morphisms between objects in different levels.

