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Ghulchak, Andrey

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Duality in Robust Control: Uncertainty vs. Controller

Andrey Ghulchak
Department of Automatic Control
Lund Institute of Technology, Box 118, 221 00 Lund, Sweden
Phone: (+46)-46-2228796, Fax: (+46)-46-138118,
Email: ghulchak@control.lth.se

Abstract

To find a controller that provides the maximal stability margin to an LTI system under rank-one perturbations is a quasiconvex problem. In this paper, the dual quasiconvex problem is obtained, using the convex duality arguments in the Hardy space $\mathbf{H}^\infty$. It is shown that the dual problem can be viewed as minimization of a "length" of uncertainties that destabilize the system. Several examples establishing a connection with such classical results as the corona theorem and the Adamyan-Arov-Krein theorem are considered.

Keywords: robust stabilization, stability radius, rank-one problem, quasiconvex optimization, duality.

1 Introduction

Most control designs are based on the use of a design model. A good model should be simple enough to facilitate design, yet complex enough to capture important properties of the true plant. One way to bridge the gap between model and reality is to incorporate uncertainties that reflect both our knowledge of the physical mechanism of the plant and our ability to solve control problems with such uncertainties.

During the last two decades, much research efforts have been devoted to the robust control of uncertain models. However, design problems appear to be very hard, and only a few methods have proved to be efficient in the synthesis of robust controllers. Among them, $\mathbf{H}^\infty$ optimization and μ synthesis should be mentioned. $\mathbf{H}^\infty$ optimization gives a convenient and very efficient tool for robust synthesis. However, it is limited to the class of *unstructured* uncertainties and becomes particularly complicated in the case of nonrational plants. In contrast, μ synthesis allows for a much more flexible uncertainty structure, but design procedures become very involved, and one has to use conservative simplifications and upper bounds.

Halfway between the unstructured uncertainties and μ synthesis is a rank-one uncertainty model where the structure of uncertainty is limited to a vector (or a rank one matrix). This class of models turns out to be quite rich to describe, for instance, many physical plants with real parametric uncertainties, yet simple enough to be dealt with by rigour mathematical analysis. In particular, the set of all robustly stabilizing controllers can be parameterized as a convex specification [5], and a linear programming approach can be used to design the optimal controller in case of "diamond" and "square" uncertainty sets [4].

In this paper, we continue the study of the rank-one problem where the uncertainty set is relaxed to an arbitrary convex compact set in a finite-dimensional vector space. The primal problem of robust stabilization, which is the maximization of stability radius over all stabilizing controllers, can be stated as a quasiconvex optimization. The paper presents the precise analytical form of the dual quasiconvex problem for this optimization. It is shown that the dual problem can be stated as a minimization of the "length" of uncertainties that destabilize the plant. We also trace a relationship between the dual problem and such classical results as the corona and Adamyan-Arov-Krein theorems.

The paper is organized as follows. All main notations used throughout the paper are collected in Section 2. Section 3 introduces the robust stabilization problem as the primal quasiconvex optimization. The dual problem is obtained in Section 4. Section 5 clarifies the primal and the dual problems through considering two particular cases. Further analysis of the dual problem in Section 6 makes it clear that the dual problem is the "norm" minimization of destabilizing uncertainties. Section 7 provides a detailed consideration of the primal-dual pair in case of scalar $\mathbf{H}^\infty$ optimization and traces the relation between the dual problem and Adamyan-Arov-Krein's result. Section 8 offers a numerical example where the primal-dual method is applied to a nonconvex $\mathbf{H}^\infty$ optimization. The conclusion is found in Section 9.

2 Notation

By $\mathbb{R}$ (or $\mathbb{C}$) we denote the field of real (or complex) numbers. The subset of $\mathbb{R}$ of nonnegative numbers is denoted by $\mathbb{R}_+$. The unit circle and the open unit disc in $\mathbb{C}$ are denoted by $\mathbb{T}$ and $\mathbb{D}$ respectively. For any measurable $Y \subset \mathbb{C}^n$, the notation $\mathbf{L}^p(Y)$ stands for the standard Lebesgue space of functions $f\colon \mathbb{T} \to Y$ equipped with the norm

$$\|f\|_p = \begin{cases} \left(\int_{\mathbb{T}} |f(z)|^p \, dm(z)\right)^{1/p}, & 1 \le p < +\infty, \\ \operatorname{ess\,sup}_{z\in\mathbb{T}} |f(z)|, & p = +\infty. \end{cases}$$

The notation $\mathbf{H}^p(Y)$ denotes the Hardy space of functions in $L^p(Y)$ that have an analytical continuation inside the unit disc, and $\mathbf{H}_0^p(Y)$ denotes the shifted $\mathbf{H}^p(Y)$, that is

$$\mathbf{H}_0^p(Y) = z\mathbf{H}^p(Y) = \{f \in \mathbf{H}^p(Y) \mid f(0) = 0\}.$$

We use the notation $\mathbf{RH}^\infty(Y)$ for all real-rational functions from $\mathbf{H}^\infty(Y)$. The space of all continuous functions $f\colon \mathbb{T} \to Y$ is denoted by $C(Y)$. The notation $\mathbf{A}(Y)$ stands for the disc algebra $\mathbf{H}^\infty(Y) \cap C(Y)$.

The short notations $\mathbf{L}^p$, $\mathbf{H}^p$ etc will be used if $Y = \mathbb{C}^n$ and the dimension n is clear from context or makes no difference for presentation.

Given $\Theta \in \mathbf{L}^\infty$, the subspace $\mathbf{H}^2 \ominus \Theta\mathbf{H}^2$ contains all functions $f \in \mathbf{H}^2$ such that $f^*\Theta \in \mathbf{H}_0^2$.

The prefix B denotes the unit ball in the corresponding space, and S the unit sphere. The superscript ^{T} stands for transposition, † for pseudoinverse. Re is the real part of a complex number. A bar over a function denotes the complex conjugate. For two sets A and B

$$A \setminus B = \{a \in A \mid a \notin B\}.$$

3 Preliminaries

Since the Hardy classes $\mathbf{H}^\infty$ in the unit disc and in the right half plane are equivalent modulo the conformal bilinear transformation $s = (1-z)/(1+z)$, only the former will be considered.

Given a nominal LTI plant P and an uncertainty set $\Delta \ni 0$, the general robust controller design problem is to find a controller K that robustly stabilizes the whole family of perturbed plants

$$\begin{aligned} \begin{pmatrix} y \\ z \end{pmatrix} &= P\begin{pmatrix} w \\ u \end{pmatrix}, \\ w &= \delta^T z, \quad \delta \in \nu\Delta \end{aligned}$$

for as large a ν as possible. The rank-one problem is a particular case when w is a scalar or, equivalently, the uncertainty δ is of rank one (that is just a vector). This particular case is appealing since in this case, a *convex* parameterization of all robustly stabilizing controllers is available [5] provided that the uncertainty set Δ is convex. Briefly speaking, for the rank-one problem the Youla parameterization of all admissible closed-loop transfer functions from w to z has the form $T_{zw} = T_1 + T_2 Q$, $Q \in \mathbf{RH}^\infty$, with the given functions $T_1 \in \mathbf{A}(\mathbb{C}^{N_z})$, $T_2 \in \mathbf{A}(\mathbb{C}^{N_z \times N_u})$. Then, the robustly stabilizing controllers are those whose Q parameter satisfies

$$1 + \delta^T(T_1 + T_2 Q) \neq 0, \quad \forall z \in \mathbb{T}, \forall \delta \in \nu\Delta. \tag{1}$$

Hyperplane separation of convex sets and a little bit of simple analysis (see details in [5]) yield the equivalent condition to (1) in terms of the function $h \in \mathbf{RH}^\infty$ as

$$\operatorname{Re}(F + \delta^T G(z))h(z) > 0, \quad \forall z \in \mathbb{T}, \forall \delta \in \nu\Delta \tag{2}$$

where $F = \begin{pmatrix} 1 & 0 \end{pmatrix}$ and $G = \begin{pmatrix} T_1 & T_2 \end{pmatrix}$. The function $Q \in \mathbf{RH}^\infty$ can be reconstructed from h by the simple formula $Q = \begin{pmatrix} h_2 & \dots & h_{N_u+1} \end{pmatrix}^T / h_1$.

Let $\Delta \ni 0$ be a convex compact set in $\mathbb{C}^n$. Denote

$$p_\Delta(x) = \sup_{\delta \in \Delta} \operatorname{Re} \delta^T x. \tag{3}$$

This is a gauge of the polar set of Δ (i.e. almost a norm), so the problem (2) is a quasiconvex optimization

$$\nu_{opt} = \sup_{h \in \mathbf{RH}^\infty} \inf_{z \in \mathbb{T}} \frac{\operatorname{Re} Fh(z)}{p_\Delta(G(z)h(z))}. \tag{4}$$

The purpose of this paper is to provide a dual form of (4).

4 Primal and Dual Problems

Let $F \in \mathbf{A}(\mathbb{C}^{1\times n})$, $G \in \mathbf{A}(\mathbb{C}^{n\times n})$ and $\Delta \subset \mathbb{C}^n$ be a convex compact set containing the origin. For any $h \in \mathbf{H}^\infty$ define

$$\Gamma_\nu(h) = \operatorname*{ess\,inf}_{z \in \mathbb{T}} (\operatorname{Re} F(z)h(z) - \nu p_\Delta(G(z)h(z)))$$

where p_Δ is defined in (3). The problem of finding $h \in \mathbf{RH}^\infty$ such that $\Gamma_\nu(h) > 0$ will be referred to as *primal problem.* Obviously it has a solution if and only if $\nu < \nu_{opt}$

$$\nu < \nu_{opt} \quad \Leftrightarrow \quad \gamma_{opt} := \sup_{h \in B\mathbf{A}} \Gamma_\nu(h) > 0. \tag{5}$$

Thus by solving the convex problem (5) and checking if $\gamma_{opt} > 0$, one can find the lower bound on ν_{opt} along with the suboptimal solution h. Numerically it can be implemented as a convex optimization over a n-dimensional subspace $H(n) \subset \mathbf{RH}^\infty$ since

$$\gamma_n := \sup_{h \in BH(n)} \Gamma_\nu(h) \nearrow \gamma_{opt}.$$

Once $\gamma_n > 0$, we conclude that $\gamma_{opt} > 0$ and, hence, $\nu < \nu_{opt}$. However, the case $\gamma_n = 0$ cannot indicate that $\gamma_{opt} = 0$ and provide an upper bound on ν_{opt}. The upper bound can be found from a dual representation of γ_{opt} which gives us a condition to determine whether $\gamma_{opt} = 0$.

Before the dual representation is considered, let us slightly modify the problem (5). The following lemma states that γ_{opt} can be calculated by the optimization over the larger set $B\mathbf{H}^\infty$.

Lemma 1

$$\gamma_{opt} = \sup_{h \in B\mathbf{H}^\infty} \Gamma_\nu(h). \tag{6}$$

Now we are in a position to give the dual representation of γ_{opt}. The following theorem is a generalization of [4, Theorem 2] to the case of *any* convex compact sets Δ.

Theorem 1 *Let $F \in \mathbf{H}^\infty(\mathbb{C}^{1\times n})$, $G \in \mathbf{H}^\infty(\mathbb{C}^{m\times n})$ and $\Delta \subset \mathbb{C}^m$ be a convex compact set. Then the following equality holds*

$$\gamma_{opt} = \inf_{\delta\in\mathbf{L}^\infty(\nu\Delta)} \inf_{w\in S\mathbf{L}^1(\mathbb{R}_+)} \inf_{p\in\mathbf{H}_0^1(\mathbb{C}^{1\times n})} \|(F+\delta^T G)w-p\|_1.$$

Theorem 1 provides an upper bound on γ_{opt} by which we can determine the case when $\gamma_{opt} = 0$.

Corollary 1

1. A number ν is an upper bound on ν_{opt} in (4) if and only if there exists a sequence of functions $\{(w_i, \delta_i, p_i)\}_{i=0}^{+\infty}$ such that $w_i \in S\mathbf{L}^1(\mathbb{R}_+)$, $\delta_i \in \mathbf{L}^\infty(\nu\Delta)$, $p_i \in \mathbf{H}_0^1(\mathbb{C}^{1\times n})$ and

$$\|(F+\delta_i^T G)w_i - p_i\|_1 \to 0, \quad i \to +\infty. \tag{7}$$

2. If there exist $w \in \mathbf{L}^1(\mathbb{R}_+) \setminus 0$ and $\delta \in \mathbf{L}^\infty(\nu\Delta)$ such that $(F+\delta^T G)w \in \mathbf{H}_0^1$ then $\nu \geq \nu_{opt}$.

To find the optimizing sequence of functions in (7) is not an easy problem in general. However in our case, the functions F and G belong to the disc algebra $\mathbf{A}$, and the problem can be essentially simplified. The optimal function w is proved to be either a regular function in $SL^1(\mathbb{R}_+)$ or Dirac's δ-function, and by this, the dual problem can be naturally split into two parts — regular and singular. The regular part has been already covered by Corollary 1, and the dual problem is completed by the singular part in the next theorem.

Theorem 2 *The optimal value ν_{opt} in (4) has the following dual representation*

$$\nu_{opt} = \min\{\nu_{opt|c}, \nu_{opt|s}\}, \tag{8}$$

where

$$\nu_{opt|c} = \inf\{\nu \mid \exists w \in \mathbf{L}^1(\mathbb{R}_+)\setminus 0,\ \delta \in \mathbf{L}^\infty(\nu\Delta): (F+\delta^T G)w \in \mathbf{H}_0^1\}, \tag{9}$$

$$\nu_{opt|s} = \inf\{\nu \mid \exists z \in \mathbb{T},\ \delta \in \nu\Delta: F(z)+\delta^T G(z) = 0\}. \tag{10}$$

5 Two Particular Cases of Δ

To get better insight to the dual problem, we consider in details two particular cases of the set Δ. These two cases are extremal in the following sense: one is for $\Delta = 0$, i.e. there is no uncertanty at all, and second is for $\Delta = B\mathbb{C}^m$, i.e. all directions in $\mathbb{C}^m$ are allowed for the uncertainty.

Case 1: $\Delta = 0$.

The primal problem is to find a function $h \in \mathbf{RH}^\infty(\mathbb{C}^n)$ such that

$$\operatorname{Re} F(z)h(z) > 0, \quad \forall z \in \mathbb{T}$$

for given $F \in \mathbf{A}(\mathbb{C}^{1\times n})$. The dual representation for γ_{opt} in Theorem 1 is simplified to

$$\gamma_{opt} := \inf_{w\in S\mathbf{L}^1(\mathbb{R}_+)} \inf_{p\in\mathbf{H}_0^1} \|Fw-p\|_1,$$

and the duality result in Theorem 2 claims that the primal problem has no solution if and only if

1. $\exists w \in \mathbf{L}^1(\mathbb{R}_+)\setminus 0$ such that $Fw \in \mathbf{H}_0^1$ or
2. $\exists z \in \mathbb{T}$ such that $|F(z)| = 0$.

The second condition is the absence of zeros of F on the unit circle. Let us show that the first one is related to that in the open unit disc.

Proposition 1 *Let $F \in \mathbf{A}(\mathbb{C}^{1\times n})$. The following conditions are equivalent:*

1. $\exists \lambda \in \mathbb{D}$ such that $|F(\lambda)| = 0$.

2. $\exists w \in \mathbf{L}^1(\mathbb{R}_+)\setminus 0$ such that $Fw \in \mathbf{H}_0^1(\mathbb{C}^{1\times n})$.

Note that the existence of $h \in \mathbf{A}$ such that $\operatorname{Re} F(z)h(z) > 0$, $\forall z \in \mathbb{T}$, is equivalent to the existence of $g \in \mathbf{A}$ such that $Fg = 1$ (if we just set $g = h(Fh)^{-1}$). Thus the duality theorem in the particular case $\Delta = 0$ gives the well-known result [6] concerning Gelfand's theory of maximal ideals in disc algebra $\mathbf{A}$:

$$\exists g \in \mathbf{A}: Fg = 1 \quad \Leftrightarrow \quad \inf_{\lambda\in\mathbb{D}} |F(\lambda)| > 0. \tag{11}$$

A remarkable generalization of this result to functions in $\mathbf{H}^\infty$, called the corona theorem, was proved by Carleson in 1962 [2].

Case 2: $\Delta = B\mathbb{C}^m$.

The following proposition shows that this case is reduced to the standard $\mathbf{H}^\infty$ optimization.

Proposition 2 *Let $F \in \mathbf{A}(\mathbb{C}^{1\times n})$, $G \in \mathbf{A}(\mathbb{C}^{m\times n})$ and $\Delta = BC^m$. Then the following statements are equivalent:*

1. $\exists h \in \mathbf{A}(\mathbb{C}^{n\times 1})$: $\forall z \in \mathbb{T}, \forall \delta \in \nu\Delta$

$$\operatorname{Re}(F(z)+\delta^T G(z))h(z) > 0. \tag{12}$$

2. $\exists g \in \mathbf{A}(\mathbb{C}^{n\times 1})$: $Fg = 1$, $\|Gg\|_\infty < \nu^{-1}$.

By Proposition 2, the primal optimization (4) is reduced to convex optimization

$$\nu_{opt}^{-1} = \inf_{g\in\mathbf{A}}\{\|Gg\|_\infty \mid Fg = 1\}. \tag{13}$$

To obtain the standard $\mathbf{H}^\infty$ setting, we need to perform the parameterization of solutions to $Fg = 1$ (Youla parameterization). If $g_0 \in \mathbf{A}$ is some solution to the equation $Fg = 1$ and $M =\in \mathbf{A}(\mathbb{C}^{n\times r})$ is a basis of the kernel of F (*ie* $FM = 0$), then all solutions can be parameterized as

$$g = g_0 + Mq, \quad q \in \mathbf{A}(\mathbb{C}^{r\times 1}),$$

which gives the standard $\mathbf{H}^\infty$ optimization

$$\nu_{opt}^{-1} = \inf_{q\in\mathbf{A}} \|Gg_0 + GMq\|_\infty.$$

6 Dual Problem as Uncertainty Minimization

Let us take a close look at the bounds $\nu_{opt|s}$ and $\nu_{opt|c}$ in the dual problem (9), (10). Denote

$$\Phi_\delta(z) = F(z) + \delta(z)^T G(z). \tag{14}$$

To calculate $\nu_{opt|s}$ one has to solve the linear equation $\Phi_\delta(z) = 0$ with respect to δ at every $z \in \mathbb{T}$. This is a relatively easy problem. The necessary and sufficient condition for the equation to have a solution is $F(I - G^\dagger G) = 0$, which means that the vector F^T belongs to the range of G^T. Denoting by N a full rank $k \times m$ matrix that annihilates G

$$N(z)G(z) = 0, \tag{15}$$

and $\mathcal{Z} = \{z \in \mathbb{T} \mid F(z)(I - G(z)^\dagger G(z)) = 0\}$, the upper bound $\nu_{opt|s}$ can be calculated as

$$\nu_{opt|s} = \inf_{z \in \mathcal{Z}} \inf_{\lambda \in \mathbb{C}^k} \{\nu \mid \delta(z, \lambda) \in \nu\Delta\} \tag{16}$$

where

$$\delta(z, \lambda) = (\lambda^T N(z) - F(z)G(z)^\dagger)^T. \tag{17}$$

The calculation of $\nu_{opt|c}$ is more complicated since we have a bilinear equation with respect to $\delta \in \mathbf{L}^\infty(\nu\Delta)$ and $w \in \mathbf{L}^1(\mathbb{R}_+)$

$$\Phi_\delta w \in \mathbf{H}_0^1. \tag{18}$$

First of all, we notice that the real positive function $w \in \mathbf{L}^1(\mathbb{R}_+)$ can be factorized as $w(z) = f(z)^* f(z)$ where $f \in \mathbf{H}^2$ is the outer function (that is f has no zeros in $\mathbb{D}$) [6]. Therefore, the condition (18) can be rewritten as

$$\Phi_\delta f^* \in \mathbf{H}_0^2 \quad \Leftrightarrow \quad f \in \mathbf{H}^2 \ominus \Phi_\delta \mathbf{H}^2. \tag{19}$$

Intuitively, it is clear that for (19) to hold, all unstable poles of f^* must be cancelled by zeros of each entry of Φ_δ, so such a function $f \neq 0$ exists only if $|\Phi_\delta|$ has at least one zero in $\mathbb{D}$ (cmp. Proposition 1).

Lemma 2 *Let $\Phi_\delta \in \mathbf{L}^\infty(\mathbb{C}^{1\times n})$, $|\Phi_\delta| \neq 0$ on $\mathbb{T}$, and $f \in \mathbf{H}^2(\mathbb{C})$ be an outer function such that (19) holds. Suppose that Φ_δ has a finite number of unstable zeros $\{a_k\}_{k=1}^N$ counted according to their multiplicities. Then*

1. *$f \in \mathbf{H}^2 \ominus R\mathbf{H}^2$ where $R(z) = \prod_{k=1}^N (z - a_k)$. This means that $f = \widetilde{\pi}/\widetilde{R} \in \mathbf{RH}^\infty$ where $\widetilde{\ }$ denotes the polynomial conjugation ($\widetilde{r}(z) = z^{\deg r}\bar{r}(z^{-1})$) and π is an unstable polynomial of at most $N-1$ degree.*

2. $$\Phi_\delta \in \frac{R}{\pi}\mathbf{H}^\infty.$$

In particular, the lemma gives the following generalization of (11).

Corollary 2 *Given $\nu \in \mathbb{R}_+$, the dual problem has a solution if and only if there exists a function $\delta \in \mathbf{L}^\infty(\nu\Delta)$ such that for the function Φ_δ the number of common unstable zeros is greater than the number of unstable poles.*

Lemma 2 yields the equation for δ

$$F + \delta^T G = \frac{R}{\pi}h, \quad h \in \mathbf{H}^\infty(\mathbb{C}^{1\times n})$$

and, similarly to the singular case, the necessary and sufficient condition for such a δ to exist is

$$(F - Rh/\pi)(I - G^\dagger G) = 0. \tag{20}$$

Denote by $\mathcal{H}_{N-1} \subset \mathbf{L}^\infty$ the set of all meromorphic functions that have not more than $N-1$ poles in $\mathbb{D}$ and

$$\mathcal{Q} = \{q \in \mathcal{H}_{N-1}(\mathbb{C}^{1\times n}) \mid (F - Rq)(I - G^\dagger G) = 0\}.$$

Then the upper bound $\nu_{opt|c}$ can be found from the optimization problem

$$\nu_{opt|c} = \inf_{q \in \mathcal{Q}} \inf_{\lambda \in \mathbf{L}^\infty(\mathbb{C}^k)} \{\nu \mid \delta(\lambda, q, z) \in \nu\Delta, \ \forall z \in \mathbb{T}\} \tag{21}$$

where

$$\delta(\lambda, q, z) = (\lambda(z)^T N(z) - (F(z) - R(z)q(z))G(z)^\dagger)^T. \tag{22}$$

Thus the dual problem can be thought of as the minimization of the "length" of uncertainty (16), (21) that destabilize our system.

7 Scalar $\mathbf{H}^\infty$ Optimization

In this section, the standard $\mathbf{H}^\infty$ optimization is considered:

$$\epsilon_{opt} = \inf_{Q \in \mathbf{H}^\infty(\mathbb{C})} \|T_1 + T_2 Q\|_\infty, \quad T_1, T_2 \in \mathbf{A}(\mathbb{C}). \tag{23}$$

It has been shown in Section 5 (see Proposition 2) that this gives the primal problem with $\Delta = \mathbb{D}$ (this also follows from Small Gain Theorem)

$$\frac{1}{\epsilon_{opt}} = \sup\{\nu \mid \operatorname{Re}[(1\ 0) + \delta(T_1(z)\ T_2(z))]h(z) > 0, \ \forall z \in \mathbb{T}, |\delta| \leq \nu\}.$$

The dual bound (10) is

$$\nu_{opt|s} = \inf\{|\delta| \mid \exists z \in \mathbb{T}\colon \begin{pmatrix}1 + \delta T_1(z) & \delta T_2(z)\end{pmatrix} = 0\}.$$

To obtain the representation (16), we notice that the linear equation $\begin{pmatrix}1 + \delta T_1(z) & \delta T_2(z)\end{pmatrix} = 0$ has a solution if and only if

$$T_2(z) = 0, \quad T_1(z) \neq 0. \tag{24}$$

Thus (16) becomes

$$\nu_{opt|s} = \inf_{z \in \mathbb{T}} \left\{ \frac{1}{|T_1(z)|} \,\middle|\, T_2(z) = 0 \right\}.$$

The interpretation of this bound is very simple. Obviously

$$\frac{1}{\nu_{opt|s}} = \sup_{z \in \mathbb{T}} \{|T_1(z)| \mid T_2(z) = 0\} \leq \epsilon_{opt}.$$

Thus the singular part of the dual problem handles the invariant zeros of the plant on the border of stability region. Under the assumption that T_2 has no zeros on $\mathbb{T}$, the singular part gives only the trivial infinity bound.

Let us have a look at the second bound $\nu_{opt|c}$ (9). In order to obtain the representation (21) we should find all possible zeros of the function $\begin{pmatrix}1+\delta(z)T_1(z) & \delta(z)T_2(z)\end{pmatrix}$ in $\mathbb{D}$. This is a simple job — similarly to (24):

$$T_2(z) = 0, \quad T_1(z) \neq 0. \tag{25}$$

Let us factorize the function $T_2 = RT_{2o}$ so that the polynomial R absorbs all those zeros from (25) and denote $N = \deg R$. Then by Lemma 2 we have

$$\begin{pmatrix}1+\delta T_1 & \delta T_2\end{pmatrix} = \frac{R}{\pi}\begin{pmatrix}h_1 & h_2\end{pmatrix},$$

with π being any unstable polynomial of at most $N-1$ degree. The functions $(h_1 \; h_2)$ cannot be chosen independently — they must satisfy (20). If we find the function δ from the second equation $\delta = h_2/(T_{2o}\pi)$ and substitute it to the first one, we get

$$Rh_1 - \frac{T_1}{T_{2o}}h_2 = \pi. \tag{26}$$

This is the condition (20) for our example. Note that the functions R and T_1/T_{2o} are both analytical in $\mathbb{D}$ and coprime (since they do not have common unstable zeros). Hence there exists a solution to Diophantine equation

$$RH_1 - \frac{T_1}{T_{2o}}H_2 = 1,$$

and all $(h_1 \; h_2) \in \mathbf{H}^\infty$ satisfying (26) can be parameterized as

$$\begin{pmatrix}h_1 & h_2\end{pmatrix} = \pi\begin{pmatrix}H_1 & H_2\end{pmatrix} + \begin{pmatrix}T_1/T_{2o} & R\end{pmatrix}Q$$

where $Q \in \mathbf{H}^\infty$. Finally

$$\delta = \frac{h_2}{T_{2o}\pi} = \frac{\pi H_2 + RQ}{T_{2o}\pi}$$

and the representation (21) becomes

$$\nu_{opt|c} = \inf_{q\in\mathcal{H}_{N-1}} \|T_{2o}^{-1}(H_2 + Rq)\|_\infty$$

where $\mathcal{H}_{N-1}$ is the set of all meromorphic functions that have not more than $N-1$ unstable poles in $\mathbb{D}$. In particular, if T_1 and T_2 are coprime, the function T_{2o} is outer, and the representation becomes even simpler

$$\nu_{opt|c} = \inf_{q\in\mathcal{H}_{N-1}} \|B^*L_2 + q\|_\infty \tag{27}$$

where $(L_1, L_2) \in \mathbf{H}^\infty$ is a solution to $T_2L_1 - T_1L_2 = 1$ and $B = R(z)/(z^N R(z^{-1}))$ is the Blaschke product. The function L_2 can be found, for example, from $1 + T_1L_2 \in B\mathbf{H}^\infty$ as a Lagrange interpolation polynomial.

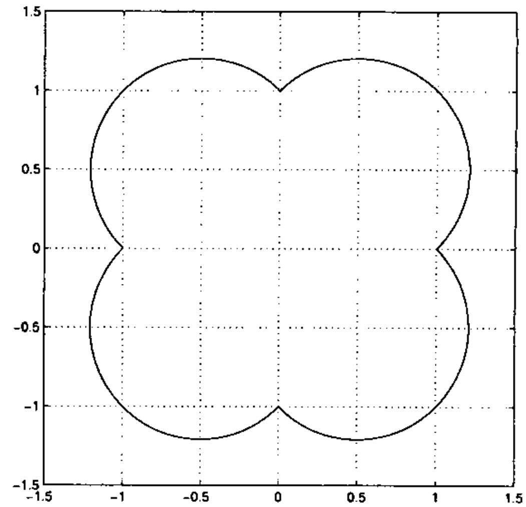

Figure 1: The set Ω from (29).

Recall the definition of Hankel operator with symbol $\phi \in \mathbf{L}^\infty$

$$H_\phi\colon \mathbf{H}^2 \to (\mathbf{H}^2)^\perp, \quad H_\phi(f) = P_-(\phi f).$$

According to Adamyan-Arov-Krein theorem [1], the infimum (27) is equal to the $(N-1)$-th singular value of Hankel operator with symbol B^*L_2

$$\nu_{opt|c} = \sigma_{N-1}(H_{B^*L_2}).$$

It remains to note that the solution to the original $\mathbf{H}^\infty$ optimization (23) is well-known [3]. Nehari theorem claims that ϵ_{opt} is equal to the norm of Hankel operator with symbol B^*T_1

$$\epsilon_{opt} = \|H_{B^*T_1}\| = \sigma_1(H_{B^*T_1}).$$

Thus the duality for coprime T_1 and T_2 with $T_2 \neq 0$ on $\mathbb{T}$ is the relation between singular values of two Hankel operators

$$\sigma_1(H_{B^*T_1})\sigma_{N-1}(H_{B^*L_2}) = 1.$$

8 Numerical example: a nonconvex $\mathbf{H}^\infty$ optimization problem

In this section, a nonconvex $\mathbf{H}^\infty$ optimization problem is solved numerically by the primal-dual method derived above

$$\inf_{Q\in\mathbf{RH}^\infty}\{\gamma \mid T_1(z) + T_2(z)Q(z) \in \gamma\Omega, \; \forall z \in \mathbb{D}\} \tag{28}$$

where Ω is the neighbourhood of origin in $\mathbb{C}$, and $T_1, T_2 \in \mathbf{A}$. If $\Omega = \mathbb{D}$, we have the standard $\mathbf{H}^\infty$ optimization. If Ω is a convex set, the problem (28) is convex.

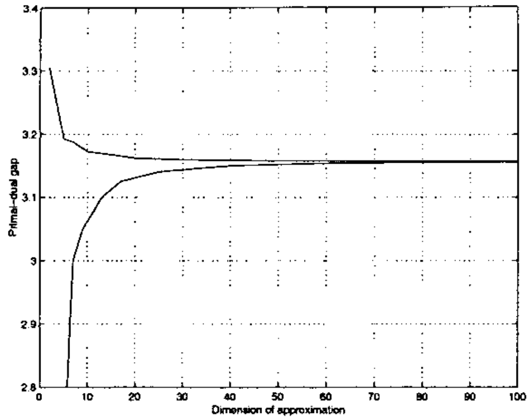

Figure 2: The primal-dual approximations of growing dimensions produce monotonic lower and upper bounds for ν_{opt}.

In this section, we consider the $\mathbf{H}^\infty$ optimization problem (28) with the *nonconvex* set

$$\Omega = \{re^{i\phi} \in \mathbb{C} \mid 0 \leq r < |\cos(\phi)| + |\sin(\phi)|\}. \quad (29)$$

The shape of Ω is shown on Figure 1. This set appears in the optimization of stability radius for a linear system with the "diamond" type of uncertainty in feedback

$$\Delta = \{x + iy \in \mathbb{C} \mid |x| + |y| \leq 1\}.$$

Take the following functions

$$T_1 = \frac{z^5 + 3z^4 + 2z^3 + 4z^2 + 5z + 3}{z^3 - z^2 - 4z + 12}, \quad T_2 = z^2 + \frac{1}{2}.$$

(Note that with the unit disc being *instability* region, the functions can be improper.) Since $T_1/T_2 \notin \mathbf{H}^\infty$, the solution to the optimization problem is not trivial.

Each level set of the primal quasiconvex problem (4) is a linear polytope. Taking into account only the finite grid $\{z_k\}_{k=1}^K$, representing the upper half of $\mathbb{T}$, and the first M coefficients of the function h, the inequality (2) becomes the finite-dimensional linear programming. Hence the primal search can be approximated by the linear programming of growing dimensions [4].

We run the primal LP approximations for different dimensions M and stop the optimization when M has reached 100. It becomes hard for the linear solver to find an approximation of higher degree because the dimension of the linear program at this step has already reached the size 1200×200. The lower bound obtained for $M = 100$ is $\nu_{low} = 3.1560$.

To estimate how far is ν_{low} from the optimal value we use the dual problem in the form (21) which is approximated precisely in the same manner to get the finite-dimensional linear programming.

In our case $R = T_2$ and π is an unstable polynomial of at most first degree. Thus $\pi(z) = z - a$, $|a| < 1$, and the upper bound can be calculated from (21) by the linear programming followed by the line search for a. Taking the same dimension of approximation as in the primal problem $M = 100$, the upper bound $\nu_{upp} = 3.15736$ is obtained. The corresponding value of a is 0.1530. Hence, the primal solution for ν_{low} has a good level of suboptimality (about 0.05%).

The lower and upper bounds produced respectively by the primal and the dual approximations of different dimensions are shown on Figure 2.

9 Conclusion

The robust design problem in case of rank-one uncertainty can be stated as the quasiconvex optimization (4). Given a level of suboptimality ν, the corresponding convex problem can be solved by finite-dimensional approximations. However, these approximations give only lower bounds on the stability radius. In this paper, the dual quasiconvex problem (8) has been derived using the convex duality arguments in the Banach space. The dual problem gives the upper bounds (9), (10) for the stability radius. It has been shown that the corresponding equations can be solved explicitly, and the dual bounds can be viewed as the minimization of the "length" of destabilizing uncertainties in (16), (21). Thus the corresponding quasiconvex optimization can be thought of as a functional quasiconvex game between the stabilizing controller and the destabilizing uncertainty that has a saddle point. The result can be used for numerical optimization of the stability radius by primal-dual methods based on finite-dimensional approximations similar to [4].

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