APPENDIX

1) *m-sequences:* We use the notation $\{\boldsymbol{a}\}$ to denote the sequence $\{a_1, a_2, \cdots\}$, and $\{\boldsymbol{a}_k\}$ to denote the shifted sequence $\{a_{k+1}, a_{k+2}, \cdots\}$. We consider a binary m-sequence $\{\boldsymbol{a}\}$ with period P. Some properties of m-sequences which will be used in our discussion in Section IV are [10]

- **A1:** One period of an m-sequence contains exactly $(P+1)/2$ ones and $(P-1)/2$ zeros.
- **A2:** *Shift and add:* For any k $(k \neq 0 \bmod P)$, the sum of the m-sequence $\{\boldsymbol{a}\}$ and its k shift $\{\boldsymbol{a}_k\}$ is another shift of the *same* m-sequence; i.e.,

$$(a_n \oplus a_{n+k}) = a_{n+l} \ \forall n$$

where $\oplus$ denotes modulo two addition, and l is some other shift of the same sequence.
- **A3:** *Decimation:* Consider the sequence $\{\boldsymbol{c}\}$ defined by $c_n = a_{Jn}, \ \forall n$. $\{\boldsymbol{c}\}$ is called the *decimation* by J of the sequence $\{\boldsymbol{a}\}$. The period of $\{\boldsymbol{c}\}$ is $P/\gcd(J, P)$. All m-sequences of period P can be constructed by decimations of $\{\boldsymbol{a}\}$.
- **A4:** *Periodic autocorrelation function:* The periodic autocorrelation function of an m-sequence $\{\boldsymbol{a}\}$, is defined as

$$P_{\boldsymbol{aa}}(l) = \sum_{n=1}^{P} (-1)^{[a_n + a_{n+l}]} = \left[P - 2\sum_{n=1}^{P}(a_n \oplus a_{n+l})\right]$$

and is given by

$$P_{\boldsymbol{aa}}(l) = \begin{cases} P, & l = 0 \bmod P \\ -1, & \text{otherwise.} \end{cases}$$

2) *Gold sequences:* Gold's design contains $(P+2)$ sequences constructed as follows:

Given an m-sequence $\{\boldsymbol{a}\}$ of period P, obtain another m-sequence $\{\boldsymbol{a}'\}$ with the same period by decimating $\{\boldsymbol{a}\}$ by a J such that $\gcd(J, P) = 1$. The remaining P sequences, $[\{\boldsymbol{a}^{(0)}\}, \{\boldsymbol{a}^{(1)}\}, \cdots, \{\boldsymbol{a}^{(P-1)}\}]$ are given by

- **A5:** $\{\boldsymbol{a}^{(j)}\} = \{\boldsymbol{a}\} \oplus \{\boldsymbol{a}'_j\}$, i.e., $a_n^{(j)} = [a_n \oplus a'_{n+j}], \ \forall n$.

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Error-Rate Characteristics of Oversampled Analog-to-Digital Conversion

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***Abstract*—Accuracy of simple analog-to-digital conversion depends on both resolution of discretization in amplitude and resolution of discretization in time. For implementation convenience, high conversion accuracy is attained by refining the discretization in time using oversampling. It is commonly believed that oversampling adversely impacts rate-distortion properties of the conversion, since the bit rate, B, increases linearly with oversampling, resulting in a slow error decay in the bit rate, on the order of $O(1/B)$. We demonstrate that the information obtained in the process of oversampled analog-to-digital conversion can easily be encoded in a manner which requires only a logarithmic increase of the bit rate with redundancy, achieving an exponential error decay in the bit rate.**

***Index Terms*—Error-rate characteristics, oversampled A/D conversion.**

I. INTRODUCTION

Digital encoding of an analog signal requires discretization in both time and amplitude. The simplest approach to this discretization is sampling with an interval τ followed by a uniform scalar quantization with a quantization step q. This process is the so-called *simple analog-to-digital (A/D) conversion*. Fundamental properties of analog-to-digital conversion, such as accuracy and its dependence on the bit rate, are not fully understood even in this most elementary form.

Accuracy of analog-to-digital conversion is commonly measured as the error of a linear reconstruction algorithm, which amounts to linear interpolation between the quantized samples using a sequence of $\sin(x)/x$ functions (see Fig. 1). If the analog signal f is band-limited, which is a good model for a broad class of signals that appear in engineering applications, and τ is smaller than the Nyquist sampling interval τ_N, then the discretization in time is reversible. However, the amplitude discretization introduces an irreversible loss of information. Consequently, the reconstructed signal f_r is generally different from the original, and the error $e = f - f_r$ is referred to as *quantization error*. A common way to study a quantization error is to model it as white additive noise, independent of the input [1]. Under the assumption that f is band-limited and that $\tau < \tau_N$, the

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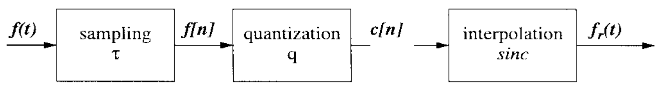

Fig. 1. Block diagram of simple A/D conversion followed by classical reconstruction. The input σ-band-limited signal $f(t)$ is first sampled at a frequency $f_s = 1/\tau$, which is above the Nyquist frequency $f_N = 1/\tau_N$. The sequence of samples $f[n]$ is then discretized in amplitude with a quantization step q. Classical reconstruction amounts to linear interpolation between the quantized samples, using $\mathrm{sinc}\,(\sin(\mathrm{x})/\mathrm{x})$ functions, and gives a signal $f_r(t)$.

white noise model shows that the error average power is given by

$$E(e(t)^2) = \frac{1}{12}\frac{q^2}{r} \tag{1}$$

where r is the oversampling ratio, $r = \tau_N/\tau$. This formula suggests that conversion accuracy can be improved by refining resolution of either discretization in time or discretization in amplitude. For convenience, high accuracy of analog-to-digital conversion is usually attained by refining time discretization. This technique is referred to as *oversampled analog-to-digital conversion*.

Only recently it was observed that (1) is misleading about conversion accuracy. Namely, it can be shown that a band-limited signal can be reconstructed from its digital representation with an error which behaves in the squared norm[1] as $\|e\|^2 = O(1/r^2)$ [2], [3]. This conversion accuracy is attained using nonlinear reconstruction algorithms. Moreover, the white noise model is not asymptotically valid, and experimental results demonstrate that for high oversampling ratios the error decay rate of linear reconstruction is lower than that implied by the white noise model [2].

The purpose of this correspondence is to set forth some new facts about the dependence of accuracy of oversampled analog-to-digital conversion on the bit rate. It is commonly believed that even though oversampling improves accuracy, it has adverse impact on the overall rate-distortion performance of the conversion. That is, since the bit rate B increases linearly with the oversampling ratio, the error (its average power or its squared norm) decays as $O(1/B)$ or $O(1/B^2)$, depending on the reconstruction algorithm. However, if the quantization step is successively reduced for a fixed sampling interval, the error decays exponentially in the bit rate.

This poor performance of oversampling with respect to quantization refinement can be significantly improved with adequate lossless encoding of the sequence of quantized samples. As the sampling interval tends towards zero, quantized samples of a band-limited signal become more and more correlated. Pulse-code modulation (PCM), which is the standard way to binary-encode these samples, does not take advantage of these correlations to reduce the bit rate. In this correspondence we demonstrate that the information obtained in the process of oversampled analog-to-digital conversion can be represented in a simple and efficient manner, and that the required bit rate increases only as a logarithm of the oversampling ratio. This follows from the observation that oversampled analog-to-digital conversion amounts to characterizing a signal by its quantization threshold crossings, that are given in time with precision determined by the sampling interval τ. The error of the conversion can thus be shown to be an exponentially decaying function of the bit rate, with an exponent that is close to the exponent of the error-rate characteristic that is obtained when the quantization step tends towards zero for a fixed sampling interval.

Notation: A signal f is said to be σ-band-limited if it is square-integrable, $f \in L^2(\boldsymbol{R})$, and its Fourier transform $\hat{f}$ satisfies $|\hat{f}(\omega)| = 0$, $|\omega| > \sigma$. A space of periodic band-limited signals, with a period T, is a space of trigonometric polynomials

$$f(t) = a_0 + \sum_{i=1}^{N}(a_i \cos(2\pi it/T) + b_i \sin(2\pi it/T)).$$

The dimension of this space of periodic band-limited signals is $2N+1$. In this correspondence, the term *band-limited signals* will, unless otherwise stated, refer to both periodic band-limited signals and σ-band-limited signals. The Nyquist sampling interval τ_N, in the space of periodic band-limited signals with the period T and dimension $2N+1$, is equal to $\tau_N = T/(2N+1)$. In the space of σ-band-limited signals it is given by $\tau_N = \sigma/\pi$.

II. Improved Reconstruction Accuracy

Information contained in the digital representation, which is generated in the process of oversampled analog-to-digital conversion, allows for reconstruction of the corresponding analog signal with an error that behaves in the squared norm as $O(1/r^2)$. This improved performance with respect to linear reconstruction can be achieved by so-called *consistent reconstruction*. The latter denotes an algorithm which always gives a consistent estimate of the original analog signal, that is, a band-limited signal which has the same digital representation as the original. This fact was first pointed out for the case of periodic band-limited signals [2], and later also proven for band-limited signals in $L^2(\boldsymbol{R})$ [3]. The result about accuracy of oversampled A/D conversion in $L^2(\boldsymbol{R})$ is formulated in the following theorem.

Theorem 1: Let $g \in C^1$ be a consistent estimate of a σ-band-limited signal f, with respect to the digital representation of f obtained in the process of simple oversampled analog-to-digital conversion with a sampling interval $\tau < \pi/\sigma$. If the sequence of quantization threshold crossings of f forms a sequence of stable sampling for the space of σ-band-limited signals, there exists a positive constant δ such that for $\tau < \delta$

$$\|f - g\|^2 \le \frac{k}{r^2} \tag{2}$$

where $r = \pi/\sigma\tau$ and k is a constant which does not depend on r or g.

The notion of *sequence of stable sampling* was introduced by Landau [4] to denote a sequence of sampling points which provides a complete and numerically stable description of band-limited signals. If a space of periodic band-limited signals is considered, this condition is satisfied by any set of points with cardinality greater than or equal to the dimension of the space. The constant of proportionality k in (2) depends on the norm of f, and also on the distribution of its quantization threshold crossings [2], [3]. It is important to note that if f is a periodic band-limited signal, its digital representation allows for reconstruction with an error which either tends towards zero as $\|e\|^2 = O(1/r^2)$, or does not approach zero at all. The latter occurs when f does not have a sufficient number of quantization threshold crossings. The situation with band-limited signals in $L^2(\boldsymbol{R})$ is much more subtle [3]. However, if reconstruction error is to converge to zero in a numerically stable manner as $r \to \infty$, then quantization threshold crossings of the signal should constitute a sequence of stable sampling, and in that case $\|e\|^2 = O(1/r^2)$.

Algorithms for consistent reconstruction were proposed in [2] and [5], and are based on alternating projections onto convex constraints which are determined by the digital representation. Note that in the case of conversion with sampling at the Nyquist rate, linear $\sin(x)/x$ interpolation is also a consistent reconstruction algorithm; however, this is no longer true if some oversampling is introduced.

[1] The term *norm* is used throughout the paper to denote the L^2 norm, hence $\|f\| = \left(\int_{-\infty}^{+\infty} |f(t)|^2\, dt\right)^{1/2}$.

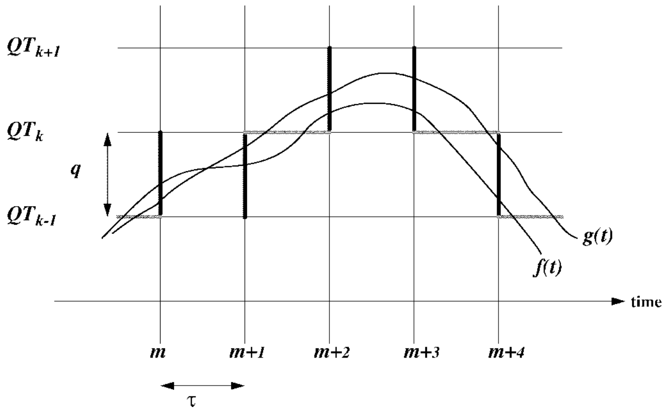

Fig. 2. Quantization threshold crossings. Two band-limited signals f and g, having the same pattern of quantization threshold crossings, also have the same quantized values of corresponding samples.

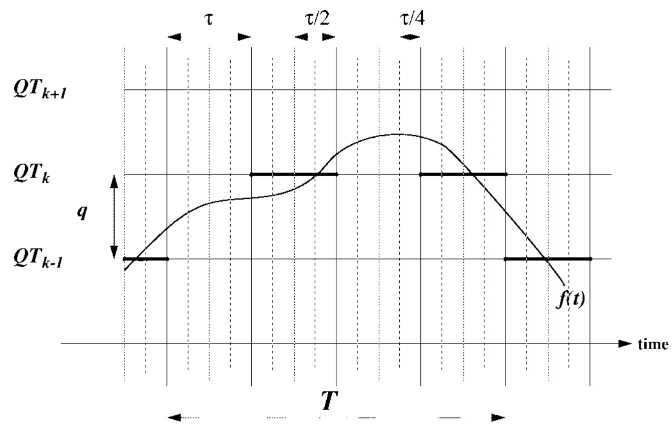

Fig. 3. Quantization threshold crossings encoding. Quantization threshold crossings are grouped on intervals of a length T. Refining the sampling interval by a factor 2^k requires additional k bits per quantization threshold crossing, to specify the position of the crossing within the interval T.

III. An Efficient Encoding Scheme

An efficient scheme for lossless encoding of quantized samples of a band-limited signal, follows from the observation that for sufficiently fine sampling, these can be determined from the corresponding sequence of quantization threshold crossings.

Consider a band-limited signal f and suppose that its quantization threshold crossings are separated; that is, there is an $\epsilon > 0$ such that no two threshold crossings are closer than ϵ. Note that since f is a band-limited signal of finite energy, it has a bounded slope, so that there is always an $\epsilon_1 > 0$ such that f cannot go through more than one quantization threshold in a time interval shorter than ϵ_1. The condition for separated quantization threshold crossings requires in addition that the intervals between consecutive crossings through any given threshold are limited from below by an $\epsilon_2 > 0$. For any sampling interval smaller than $\epsilon \leq \min(\epsilon_1, \epsilon_2)$, all quantization threshold crossings of f occur in distinct sampling intervals. Under this condition, quantized samples of f are completely determined by the corresponding sequence of quantization threshold crossings (see Fig. 2). Another effect of high oversampling is that the quantized values of consecutive samples differ with a small probability. An efficient way to represent such a digital sequence would be to encode incidences of data changes, that is, sampling intervals where quantization threshold crossings occur, rather than the quantized samples themselves.

Quantization threshold crossings can be grouped on consecutive time intervals of a given length, for instance T. In the case of periodic band-limited signals we can take T to be equal to the signal period. For each of the crossings, at most $1 + \log_2(T/\tau)$ bits are needed to specify its position inside an interval of length T. Each threshold level can be specified with respect to the level of the preceding threshold crossing; for this information, only one additional bit is needed, to denote the direction of the crossing (upwards or downwards). Hence, in order to encode the information on quantization threshold crossings on an interval where Q of them occur, at most $Q(2+\log_2(T/\tau))+C_0$ bits are needed (see Fig. 3). Here, C_0 denotes the number of bits used to specify the level of the first crossing on the interval. The bit rate B is then bounded as

$$B \leq \frac{Q_m}{T}\left(2 + \log_2\left(\frac{T}{\tau}\right)\right) + \frac{C_0}{T} \tag{3}$$

where Q_m denotes the maximal number of crossings on an interval of length T. Recall that if the samples are themselves encoded using pulse-code modulation, the bit rate increases linearly with the oversampling ratio; therefore, the quantization threshold crossings encoding is substantially more efficient. If this efficient representation is used together with consistent reconstruction, it follows from (2) and (3) that the error of oversampled analog-to-digital conversion can be bounded as

$$\|e\|^2 \leq K 2^{-2\beta B} \tag{4}$$

where $K = 16kT^2 2^{2C_0/Q_m}$ and $\beta = T/Q_m$. The C_0 factor can be made arbitrarily small if instead of specifying the level of the first threshold crossing in each of the length T intervals, we specify it once per a number of consecutive intervals, and specify levels of other crossings with respect to that level.

The result in (4) can be established in a nonconstructive, but more general manner. That is, if for a given sampling interval τ_0 at most B_m bits are needed for encoding quantization threshold crossings on an interval of length T, then as the sampling interval is successively refined, the bit rate can be bounded as

$$B \leq \frac{B_m}{T}\left(2 + \log_2 \frac{\tau_0}{\tau}\right).$$

This also immediately gives an exponentially decaying error-rate characteristic.

IV. An Estimate of the β Exponent

Considerations in this subsection pertain to band-limited signals in $L^2(\boldsymbol{R})$. Analogous results for the case of periodic band-limited signals are straightforward. In order to estimate the exponent $\beta = T/Q_m$ in (4), we consider the two types of quantization threshold crossings and denote them as d-crossings and s-crossings. A quantization threshold crossing is said to be a d-crossing if it is preceded by a crossing of a different quantization threshold, and an s-crossing if it is preceded by a crossing of the same threshold (see Fig. 4). The total number of quantization threshold crossings of a σ-band-limited signal f on an interval T is the sum of these two types of crossings. The count of d-crossings, Q_d, depends on the slope of f as well as on the quantization step size q. The slope of f can be bounded as $|f'(t)| \leq \sigma^{3/2}\|f\|$, which gives

$$\frac{Q_d}{T} \leq \frac{\sigma^{3/2}}{q}\|f\|. \tag{5}$$

For the count of s-crossings, Q_s, we can investigate some average density. A sequence of points $\{\lambda_n\}$ on the real axis is said to have uniform density d if there exist two numbers $\delta > 0$ and $L < \infty$ such that the following is satisfied:

$$|\lambda_n - n/d| \leq L, \qquad n = 0, \pm 1, \pm 2, \cdots$$
$$|\lambda_n - \lambda_m| \geq \delta, \qquad n \neq m.$$

TABLE I

ERROR-RATE CHARACTERISTICS OF OVERSAMPLED A/D CONVERSION AS THE SAMPLING INTERVAL TENDS TO ZERO, FOR THE FOUR DIFFERENT COMBINATIONS OF RECONSTRUCTION AND ENCODING. THE QUANTIZATION ERROR e IS EXPRESSES AS A FUNCTION OF THE BIT RATE B

	Linear Reconstruction	Consistent Reconstruction
PCM	$E(e(t)^2) = O\left(\frac{1}{B}\right)$	$\|e\|^2 = O\left(\frac{1}{B^2}\right)$
Threshold Crossings Encoding	$E(e(t)^2) = O\left(2^{-\beta B}\right)$	$\|e\|^2 = O\left(2^{-2\beta B}\right)$

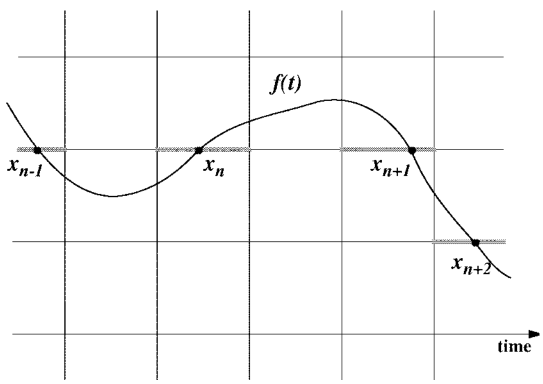

Fig. 4. Quantization threshold crossing types. A quantization threshold crossing can be immediately preceded by a crossing of the same quantization threshold, as illustrated by the crossings at points x_n and x_{n+1}. These are denoted as s-crossings, and each of them is preceded by a point where the considered signal assumes an extremum. The other type of quantization threshold crossings, d-crossings, are those which occur after a crossing of a different quantization threshold. The threshold crossing at x_{n+2} is of this type.

Each of the s-crossings of f is preceded by a point where f assumes a local extremum. If s-crossings constitute a sequence of a uniform density d, then a subset of zeros of the first derivative of f also constitutes a sequence of uniform density d. According to results on nonharmonic Fourier expansions, d should be bounded as $d \leq \sigma/\pi$ [6], except in a degenerate case, where the first derivative of f is identically equal to zero. If we assume that the sequence of quantization threshold crossings of f is a realization of an ergodic process, then when the interval T grows

$$\frac{Q_s}{T} \to c \tag{6}$$

where $c \leq \sigma/\pi$. Hence, the exponent β in the error-rate characteristic in (4)

$$\|e\|^2 = O(2^{-2\beta B})$$

has the form $\beta = 1/(\alpha_1 + \alpha_2)$ where $\alpha_1 \leq \sigma^{3/2}\|f\|/q$ and $\alpha_2 \leq \sigma/\pi$. Note that the average error power of analog-to-digital conversion with a fixed sampling frequency $f_s \geq \sigma/\pi$, when the quantization step tends towards zero, behaves as

$$E(e(t)^2) = O(2^{-2\alpha B})$$

where $\alpha = \pi/\sigma$. Thus with consistent reconstruction and efficient encoding the error-rate characteristic of oversampled analog-to-digital conversion is drastically improved, approaching the characteristic of conversion with a fixed sampling interval and quantization refinement.

V. SUMMARY OF ERROR-RATE CHARACTERISTICS

It is interesting to assess the accuracy of oversampled analog-to-digital conversion as a function of the bit rate, for the four combinations of reconstruction and encoding, i.e., linear versus consistent reconstruction, and pulse-code modulation (PCM) versus quantization threshold crossings encoding.

Recall that the accuracy of linear reconstruction is characterized by average error power and that it is given by $E(e(t)^2) = O(1/r)$, where r is the oversampling ratio. On the other hand, results on the accuracy of consistent reconstruction are obtained using deterministic analysis, which asserts that the error squared norm behaves as $\|e\|^2 = O(1/r^2)$.

Results concerning the dependence of the bit rate on the oversampling factor are established without reference to a particular kind of reconstruction. Thus $B = O(r)$ for the PCM, and $B = O(\log r)$ for the quantization threshold crossings encoding, regardless of whether linear or consistent reconstruction is used. The overall error-rate characteristics are summarized in Table I.

VI. CONCLUSION

In this correspondence we discuss error-rate characteristics of oversampled analog-to-digital conversion. We describe a simple and efficient scheme for lossless encoding of digital sequences produced in the conversion process. With this scheme we demonstrate that the information on analog signals which is retained in the oversampled analog-to-digital conversion can be efficiently represented, and that exponential decay of quantization error in the bit rate can easily be attained.

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