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# Performance Analysis of a Decoding Algorithm for Algebraic Geometry Codes

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**Abstract** — We analyse the known decoding algorithms for algebraic geometry codes in the case where the number of errors is greater than or equal to  $\lfloor (d_{FR} - 1)/2 \rfloor + 1$ , where  $d_{FR}$  is the Feng-Rao distance.

## I. INTRODUCTION

The fast decoding algorithm for one-point algebraic geometry codes of Sakata, Elbrønd Jensen, and Høholdt [1] decodes any error pattern of weight up to  $\lfloor (d_{FR} - 1)/2 \rfloor$  where  $d_{FR}$  is the so-called Feng-Rao distance of the code. In this paper we analyse the performance of the decoding algorithm, when the number of errors is greater than or equal to  $\lfloor (d_{FR} - 1)/2 \rfloor + 1$ .

## II. THE CODES AND THE DECODING ALGORITHM

Let  $P_1, P_2, \dots, P_n, Q$  be  $F_q$ -rational points on a nonsingular absolutely irreducible curve  $\chi$  of genus  $g$  defined over  $F_q$ . We consider an algebraic geometry code  $C_m$  of type  $C_L(D, G)^\perp = C_\Omega(D, G)$ , where  $D = P_1 + P_2 + \dots + P_n$  and  $G = mQ$ .

If  $f \in R$  and  $\underline{y} \in F_q^n$  we define the syndrome  $S_{\underline{y}}(f)$  to be

$$S_{\underline{y}}(f) = \sum_{i=1}^n y_i f(P_i)$$

so we have  $\underline{y} \in C \iff S_{\underline{y}}(f) = 0$  for all  $f$  such that  $\rho(f) \leq m$ .

In the decoding situation we receive a vector  $\underline{y}$  which is the sum of a codeword  $\underline{c}$  and an error vector  $\underline{e}$ . We have  $S_{\underline{y}}(f) = S_{\underline{e}}(f)$  if  $\rho(f) \leq m$ , so the syndromes  $S_{\underline{e}}(f)$  can be calculated directly from the received word if  $\rho(f) \leq m$ .

If  $\tau$  is the Hamming weight of  $\underline{e}$  then it is well known e.g. [1] or [2] that if one knows the syndromes  $S_{\underline{e}}(f)$  where  $\rho(f) \leq 2(\tau + 2g) - 1$  then the error vector can be easily found. The objective of the decoder is therefore to determine the syndromes  $S_{\underline{e}}(f)$  where  $m < \rho(f) \leq 2(\tau + 2g) - 1$ .

The decoding algorithm is a version of Sakata's generalization of the Berlekamp-Massey algorithm.

This algorithm indeed solves the decoding problem when  $\tau \leq \lfloor (d_{FR} - 1)/2 \rfloor$  (with  $\tau$  being the number of errors). See [2] or [1].

## III. THE RESULTS

Let  $P_1, \dots, P_\tau$  be the error points. We call these *independent*, if they give independent conditions on a function passing through these points, or equivalently that

$$L(\rho Q - (P_1 + \dots + P_\tau)) = 0 \text{ for } \rho \leq \rho_\tau$$

**Theorem 1** If  $m \geq 4g - 2$ ,  $\tau > \lfloor (d_{FR} - 1)/2 \rfloor$ , and the error points are independent then the algorithm fails.

The algorithm can fail by either giving no answer or a wrong answer, and indeed both cases can occur.

When  $m < 4g - 2$  the situation is different. We have developed a fairly simple procedure to determine the performance of the decoding algorithm in this case also. We mention that for the Hermitian curve over  $F_{r^2}$  given by the equation

$$x^{r+1} + y^r + y = 0$$

which has genus  $g = \frac{r(r-1)}{2}$  and  $r^3$   $F_{r^2}$ -rational points we can often do much better than predicted by the Feng-Rao bound.

If  $r = 4$  we can get a  $(64, 57, 4)$ -code over  $F_{16}$ , but two independent errors are always decoded correctly.

If  $r = 8$  we get a  $(512, 476, 9)$ -code over  $F_{64}$ , but here one can always decode 10 independent errors correctly. By similar considerations we can explain the results presented by O'Sullivan in [3].

The error points can fail to be independent in different ways. If we look at the case where  $\tau = \lfloor (d_{FR} - 1)/2 \rfloor + 1$  and

$$L(\rho Q - (P_1 + \dots + P_\tau)) = 0 \text{ for } \rho < \rho_\tau$$

but  $L(\rho_\tau Q - (P_1 + \dots + P_\tau)) \neq 0$ , we have the following two theorems:

**Theorem 2** The function in  $F_M$  with lowest poleorder  $\rho$  at  $Q$  is an element of  $L(\rho Q - (P_1 + \dots + P_\tau))$  for at least  $(q-1)^{\tau-1}$  of the  $(q-1)^\tau$  possible choices of the error values.

**Theorem 3** The algorithm corrects  $\tau = \lfloor (d_{FR} - 1)/2 \rfloor + 1$  dependent errors correctly in almost all cases.

The question whether a random selected set of points on a curve are independent or not seems difficult. We have some numerical evidence for conjecturing that (at least on a Hermitian curve) that the probability of getting independent points is  $1 - \frac{1}{q}$ .

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