# Strong Converse to the Quantum Channel Coding Theorem 

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#### Abstract

A lower bound on the probability of decoding error of quantum communication channel is presented. The strong converse to the quantum channel coding theorem is shown immediately from the lower bound. It is the same as Arimoto's method except for the difficulty due to noncommutativity.


## Keywords

Quantum channel coding theorem, average error probability, strong converse, operator monotone

## 1 Introduction

Recently, the quantum channel coding theorem was established by Holevo 99 and by Schumacher and Westmoreland [15], after the breakthrough of Hausladen et al. [7]. Furthermore, a upper bound on the probability of decoding error, in case rate below capacity, was derived by Burnashev and Holevo [2]. It is limited in pure signal state. They conjectured on a upper bound in general signal state, which corresponds to Gallager's bound [5] in classical information theory. We will show a lower bound on the probability of decoding error, in case rate above capacity, which corresponds to Arimoto's bound [1]. The strong converse to the quantum channel coding theorem is shown immediately from the lower bound.

Let $\mathcal{H}$ be Hilbert space which represents a physical system of information carrier. We suppose $\operatorname{dim} \mathcal{H}<\infty$ for simplicity. Quantum channel 11 is defined as mapping $i \in \mathcal{X} \mapsto \rho_{i}(i=1, \cdots, a)$, where $\mathcal{X}=\{1, \cdots, a\}$ is the set of input alphabet and $\rho_{i}(i=1, \cdots, a)$ is a density operator in $\mathcal{H}$, i.e., non-negative operator with trace one. For a more general treatment, see Fujiwara and Nagaoka [4].

[^0]To describe asymptotic property, we use $n$-th extension of the channel. The messages $\left\{1, \cdots, M_{n}\right\}$ is encoded to a codebook $\mathcal{C}^{(n)}=\left\{u^{1}, \cdots, u^{M_{n}}\right\}$, where each $u^{k}=i_{1}^{k} \cdots i_{n}^{k} \in \mathcal{X}^{n}\left(k=1, \cdots, M_{n}\right)$ is a codeword, and is mapped to $\rho_{u^{k}}=\rho_{i_{1}^{k}} \otimes \cdots \otimes \rho_{i_{n}^{k}}$, which is a density operator in $\mathcal{H}^{\otimes n}$. Decoding process $X^{(n)}=\left\{X_{0}, X_{1}, \cdots, X_{M_{n}}\right\}$ is a quantum measurement 11], that is a resolution of identity in $\mathcal{H}^{\otimes n}$, i.e., $X_{k} \geq 0\left(k=0, \cdots, M_{n}\right)$ and $\sum_{k=0}^{M_{n}} X_{k}=I$. We think of $X_{0}$ as evasion of decoder. A pair of encoding and decoding process $\left(\mathcal{C}^{(n)}, X^{(n)}\right)$ is called a code with cardinality $M_{n} . R_{n}=\log M_{n} / n$ is called transmission rate for a code $\left(\mathcal{C}^{(n)}, X^{(n)}\right)$. In the sequel, we will omit the subscript $n$ when no confusion is likely to arise.

The conditional probability of output $k$, when message $l$ was sent, is given by $P(k \mid l)=\operatorname{Tr} \rho_{u^{l}} X_{k}$. If all messages arise with uniform probability, the average error probability of code $(\mathcal{C}, X)$ is

$$
\operatorname{Pe}(\mathcal{C}, X)=1-\frac{1}{M} \sum_{k=1}^{M} \operatorname{Tr} \rho_{u^{k}} X_{k}
$$

Let us denote the minimum of the average error probability as

$$
\operatorname{Pe}\left(M_{n}, n\right)=\min _{\mathcal{C}, X} \operatorname{Pe}(\mathcal{C}, X)
$$

The (operational) capacity 10 is defined as the number $C$ such that $\operatorname{Pe}\left(e^{n R}, n\right)$ tends to zero as $n \rightarrow \infty$ for any $0 \leq R<C$ and does not tend to zero if $R>C$.

Let $\pi=\left\{\pi_{i}\right\}_{i=1}^{a}$ be a probability distribution on $\mathcal{X}$, and define (formal) quantum mutual information (11 as

$$
I(\pi)=H\left(\bar{\rho}_{\pi}\right)-\sum_{i=1}^{a} \pi_{i} H\left(\rho_{i}\right)
$$

where $\bar{\rho}_{\pi}=\sum_{i=1}^{a} \pi_{i} \rho_{i}$ and $H(\rho)=-\operatorname{Tr} \rho \log \rho$, which is Von Neumann entropy. The quantum coding theorem states that $\max _{\pi} I(\pi)$ is equal to the operational capacity $C$. The aim of this correspondence is to show the strong converse to the quantum channel coding theorem, i.e., $\mathrm{Pe}\left(e^{n R}, n\right)$ tends to one exponentially as $n \rightarrow \infty$ if $R>C$.

## 2 lower bound on the average error probability

To begin with, we will show the following Lemma.
Lemma 1 For an arbitrary measurement $X=\left\{X_{k}\right\}_{k=0}^{M}$

$$
\begin{equation*}
\operatorname{Pe}(\mathcal{C}, X) \geq 1-\frac{1}{M} \operatorname{Tr}\left(\sum_{k=1}^{M} \rho_{u^{k}}^{\frac{1}{\beta}}\right)^{\beta} \quad(0<\beta \leq 1) \tag{1}
\end{equation*}
$$

holds.

Proof: $\quad \rho_{u^{l}}^{\frac{1}{\beta}} \leq \sum_{k=1}^{M} \rho_{u^{k}}^{\frac{1}{\beta}}(l=1, \cdots, M)$ is obvious. Since $x^{\beta}(0<\beta \leq 1)$ is a operator monotone function (see ex. [6]),

$$
\begin{equation*}
\rho_{u^{l}}=\left(\rho_{u^{l}}^{\frac{1}{\beta}}\right)^{\beta} \leq\left(\sum_{k=1}^{M} \rho_{u^{k}}^{\frac{1}{\beta}}\right)^{\beta} \quad(l=1, \cdots, M) \tag{2}
\end{equation*}
$$

holds. Hence,

$$
\begin{aligned}
\operatorname{Pe}(\mathcal{C}, X) & =1-\frac{1}{M} \sum_{l=1}^{M} \operatorname{Tr} \rho_{u^{l}} X_{l} \\
& \geq 1-\frac{1}{M} \sum_{l=1}^{M} \operatorname{Tr}\left(\sum_{k=1}^{M} \rho_{u^{k}}^{\frac{1}{\beta}}\right)^{\beta} X_{l} \\
& \geq 1-\frac{1}{M} \operatorname{Tr}\left(\sum_{k=1}^{M} \rho_{u^{k}}^{\frac{1}{\beta}}\right)^{\beta}
\end{aligned}
$$

where we used $\sum_{l=1}^{M} X_{l} \leq I$ in the last inequality.
Following Arimoto [1] , let us apply random coding technic to Lemma 11 with a probability distribution

$$
P\left(u^{1}, \cdots, u^{M}\right)=\operatorname{Prob}\left\{\mathcal{C}=\left(u^{1}, \cdots, u^{M}\right)\right\}
$$

For this purpose, we shall need next two conditions.

1. $\mathrm{E}_{P}\left[\min _{X} \operatorname{Pe}(\mathcal{C}, X)\right]=\min _{\mathcal{C}} \min _{X} \operatorname{Pe}(\mathcal{C}, X)$
2. $P\left(u^{1}, \cdots, u^{M}\right)$ is invariant under a permutation of $u^{1}, \cdots, u^{M}$.

Actually, such a probability distribution on the set of all codebook exists. Suppose that $\hat{\mathcal{C}}=\left(\hat{u}^{1}, \cdots, \hat{u}^{M}\right)$ attains the minimum of condition 1 , then from symmetry of average error probability, a permutation of $\left(\hat{u}^{1}, \cdots, \hat{u}^{M}\right)$ also attains the minimum. Therefore

$$
\hat{P}\left(u^{1}, \cdots, u^{M}\right)= \begin{cases}\frac{1}{M!} & \left(u^{1}, \cdots, u^{M}\right) \text { is } \\ 0 & \text { a permutation of }\left(\hat{u}^{1}, \cdots, \hat{u}^{M}\right) \\ \text { otherwise }\end{cases}
$$

is the probability distribution which satisfies above two conditions. Furthermore, the marginal probability distributions of $\hat{P}\left(u^{1}, \cdots, u^{M}\right)$ does not depend on $u^{1}, \cdots, u^{M}$ by condition 2 , i.e.,

$$
\begin{aligned}
& \hat{P}\left(u^{1}\right)=\cdots=\hat{P}\left(u^{M}\right) \\
& =\sum_{u^{2} \in \mathcal{X}^{n}} \cdots \sum_{u^{M} \in \mathcal{X}^{n}} \hat{P}\left(u^{1}, u^{2}, \cdots, u^{M}\right)
\end{aligned}
$$

By taking average of (1) with $\hat{P}$, we obtain

$$
\begin{align*}
& \min _{\mathcal{C}, X} \operatorname{Pe}(\mathcal{C}, X)=\mathrm{E}_{\hat{P}}\left[\min _{X} \operatorname{Pe}(\mathcal{C}, X)\right] \\
\geq & 1-\frac{1}{M} \mathrm{E}_{\hat{P}}\left[\operatorname{Tr}\left(\sum_{k=1}^{M} \rho_{u^{k}}^{\frac{1}{\beta}}\right)^{\beta}\right] \tag{3}
\end{align*}
$$

Using Jensen type inequality, which is derived from operator concavity of $x^{\beta}(0<$ $\beta \leq 1)($ see $[6]),(3)$ is bounded as

$$
\begin{align*}
& \geq 1-\frac{1}{M} \operatorname{Tr}\left(\sum_{k=1}^{M} \mathrm{E}_{\hat{P}}\left[\rho_{u^{k}}^{\frac{1}{\beta}}\right]\right)^{\beta}  \tag{4}\\
& =1-M^{\beta-1} \operatorname{Tr}\left(\sum_{u \in \mathcal{X}^{n}} \hat{P}(u) \rho_{u}^{\frac{1}{\beta}}\right)^{\beta}  \tag{5}\\
& \geq 1-M^{\beta-1} \max _{P \in \mathcal{P}_{\mathcal{X}^{n}}} \operatorname{Tr}\left(\sum_{u \in \mathcal{X}^{n}} P(u) \rho_{u}^{\frac{1}{\beta}}\right)^{\beta} \tag{6}
\end{align*}
$$

where we used the notation $\mathcal{P}_{\mathcal{X}^{n}}$, which represents the set of all the probability distributions on $\mathcal{X}^{n}$.

The following Lemma is the same as classical information theory (see [5] [1]) except for one point that derivative of a function is not easy due to noncommutativity. We will give the proof for convenience.
Lemma 2 Let $\mathcal{P}_{\mathcal{X}}$ be the set of all the probability distributions on $\mathcal{X}$. Then

$$
\max _{P \in \mathcal{P}_{\mathcal{X}^{n}}} \operatorname{Tr}\left(\sum_{u \in \mathcal{X}^{n}} P(u) \rho_{u}^{\frac{1}{\beta}}\right)^{\beta}=\left[\max _{\pi \in \mathcal{P}_{\mathcal{X}}} \operatorname{Tr}\left(\sum_{i=1}^{a} \pi_{i} \rho_{i}^{\frac{1}{\beta}}\right)^{\beta}\right]^{n}
$$

Proof: Let us define a function of $\pi \in \mathcal{P}_{\mathcal{X}}$ as

$$
f(\pi)=\operatorname{Tr}\left(\sum_{i=1}^{a} \pi_{i} \rho_{i}^{\frac{1}{\beta}}\right)^{\beta}
$$

We note that $f$ is a concave function. First, we show that necessary and sufficient condition on the probability distribution $\pi^{*} \in \mathcal{P}_{\mathcal{X}}$, which attains the maximum of $f(\pi)$, is

$$
\begin{align*}
& \left.\operatorname{Tr} S^{\beta-1} \rho_{i}^{\frac{1}{\beta}} \leq \operatorname{Tr} S^{\beta} \quad \text { (equality if } \pi_{i}^{*}>0\right)  \tag{7}\\
& \text { where } \quad S=\sum_{i=1}^{a} \pi_{i}^{*} \rho_{i}^{\frac{1}{\beta}} \tag{8}
\end{align*}
$$

To show this, introduce Lagrange multiplier $s_{i}\left(s_{i} \geq 0, i=1, \cdots, a\right)$ and $\lambda$, define another function of $\pi$ as

$$
g(\pi)=-f(\pi)-\sum_{i=1}^{a} \pi_{i} s_{i}-\lambda\left(\sum_{i=1}^{a} \pi_{i}-1\right)
$$

differentiate $g(\pi)$ by $\pi_{i}$ and make it to 0 . From general theory of Lagrange multiplier method (see ex. [13]), we assert that necessary and sufficient condition on $\pi^{*} \in \mathcal{P}_{\mathcal{X}}$, which attains maximum of $f(\pi)$ (i.e. minimum of $g(\pi)$ ), is that there exist $s_{i} \geq 0(i=0, \cdots, a)$ and $\lambda$ which satisfies next conditions.

$$
\begin{align*}
-\beta \operatorname{Tr} S^{\beta-1} \rho_{i}^{\frac{1}{\beta}}-s_{i}-\lambda & =0  \tag{9}\\
\sum_{i=1}^{a} \pi_{i}^{*} s_{i} & =0 \tag{10}
\end{align*}
$$

where we used derivative of $f(\pi)$ (see Appendix Lemma 5)

$$
\frac{\partial f}{\partial \pi_{i}}=\operatorname{Tr}\left(\sum_{i=1}^{a} \pi_{i} \rho_{i}^{\frac{1}{\beta}}\right)^{\beta-1} \rho_{i}^{\frac{1}{\beta}}
$$

By multiplying $\pi_{i}^{*}$ to the both sides of (9) and summing over, we obtain

$$
-\lambda=\beta \operatorname{Tr} S^{\beta}
$$

Meanwhile, $s_{i}=0$ if $\pi_{i}^{*}>0$ by (10). Hence (9) (10) is equivalent to

$$
\begin{aligned}
\exists s_{i} \geq 0 \quad \text { s.t. } \quad \operatorname{Tr} S^{\beta-1} \rho_{i}^{\frac{1}{\beta}} & =-\frac{s_{i}}{\beta}+\operatorname{Tr} S^{\beta} \\
\left(s_{i}\right. & \left.=0 \text { if } \pi_{i}^{*}>0\right)
\end{aligned}
$$

Moreover, this is equivalent to (7) (8). Now, Suppose $\pi^{*}$ satisfies (7)(8), and put

$$
P^{*}(u)=P^{*}\left(i_{1}, \cdots, i_{n}\right)=\pi_{i_{1}}^{*} \cdots \pi_{i_{n}}^{*}
$$

which is i.i.d. extension of $\pi^{*}$. Then it is clear $P^{*}$ satisfies

$$
\begin{aligned}
& \operatorname{Tr} \tilde{S}^{\beta-1} \rho_{u}^{\frac{1}{\beta}} \leq \operatorname{Tr} \tilde{S}^{\beta} \quad\left(\text { equality if } P^{*}(u)>0\right) \\
& \text { where } \quad \tilde{S}=\sum_{u \in \mathcal{X}^{n}} P^{*}(u) \rho_{u}^{\frac{1}{\beta}}
\end{aligned}
$$

Hence

$$
\max _{P \in \mathcal{P}_{\mathcal{X}^{n}}} \operatorname{Tr}\left(\sum_{u \in \mathcal{X}^{n}} P(u) \rho_{u}^{\frac{1}{\beta}}\right)^{\beta}
$$

$$
\begin{aligned}
& =\operatorname{Tr}\left(\sum_{i_{1}} \cdots \sum_{i_{n}} \pi_{i_{1}}^{*} \cdots \pi_{i_{n}}^{*}\left(\rho_{i_{1}} \otimes \cdots \otimes \rho_{i_{n}}\right)^{\frac{1}{\beta}}\right)^{\beta} \\
& =\left[\operatorname{Tr}\left(\sum_{i=1}^{a} \pi_{i}^{*} \rho_{i}^{\frac{1}{\beta}}\right)^{\beta}\right]^{n} \\
& =\left[\max _{\pi \in \mathcal{P} X} \operatorname{Tr}\left(\sum_{i=1}^{a} \pi_{i} \rho_{i}^{\frac{1}{\beta}}\right)^{\beta}\right]^{n}
\end{aligned}
$$

Now, from (8) and Lemma 2 we obtain

$$
\begin{aligned}
& \min _{\mathcal{C}, X} \operatorname{Pe}(\mathcal{C}, X) \\
\geq & 1-M^{\beta-1}\left[\max _{\pi \in \mathcal{P}_{\mathcal{X}}} \operatorname{Tr}\left(\sum_{i=1}^{a} \pi_{i} \rho_{i}^{\frac{1}{\beta}}\right)^{\beta}\right]^{n}
\end{aligned}
$$

Let us put $s=\beta-1$, recall $R=\log M / n$ and define

$$
\begin{equation*}
E_{0}(s, \pi)=-\log \left(\operatorname{Tr}\left(\sum_{i=1}^{a} \pi_{i} \rho_{i}^{\frac{1}{s+1}}\right)^{s+1}\right) \tag{11}
\end{equation*}
$$

then we have proved the following theorem.
Theorem 1 For all code $(\mathcal{C}, X)$

$$
\begin{array}{ll} 
& \operatorname{Pe}(\mathcal{C}, X) \\
\geq & 1-\exp \left[-n\left[-s R+\min _{\pi \in \mathcal{P}_{X}} E_{0}(s, \pi)\right]\right]  \tag{12}\\
& (-1<s \leq 0)
\end{array}
$$

Remark 1 (11) has appeared in [7] as a conjecture on the upper bound on the average error probability, which forms dual with [12). They proved it in case that all $\rho_{i}(i=1, \cdots, a)$ are pure.

## 3 Strong converse to the quantum channel coding theorem

To understand the graph of $E_{0}(s, \pi)$, we show the following lemma.

## Lemma 3

$$
\begin{align*}
& E_{0}(0, \pi)=0  \tag{13}\\
& E_{0}(s, \pi) \leq 0, \quad(-1<s \leq 0)  \tag{14}\\
& \frac{\partial E_{0}(s, \pi)}{\partial s} \geq 0, \quad(-1<s \leq 0)  \tag{15}\\
& \left.\frac{\partial E_{0}(s, \pi)}{\partial s}\right|_{s=0}=I(\pi) \tag{16}
\end{align*}
$$

Proof: (13) is obvious. From the following Lemma, $E_{0}(s, \pi)$ is shown to be non-decreasing in $(-1,0]$. That is why (14) (15) holds. Using Appendix Lemma 5, we can calculate directly the derivative of $E_{0}(s, \pi)$ to obtain (16), or using

$$
\begin{aligned}
& \left.\frac{\partial}{\partial s} \operatorname{Tr}\left(\sum_{i=1}^{a} \pi_{i} \rho_{i}^{\frac{1}{s+1}}\right)^{s+1}\right|_{s=0} \\
= & \left.\frac{\partial}{\partial s} f(s, t)\right|_{\substack{s=0 \\
t=0}}+\left.\frac{\partial}{\partial t} f(s, t)\right|_{\substack{s=0 \\
t=0}} \\
= & \sum_{i=1}^{a} \pi_{i} H\left(\rho_{i}\right)-H\left(\bar{\rho}_{\pi}\right)
\end{aligned}
$$

where we put $f(s, t)=\operatorname{Tr}\left(\sum_{i=1}^{a} \pi_{i} \rho_{i}^{\frac{1}{s+1}}\right)^{t+1}$, we obtain

$$
\begin{aligned}
\left.\frac{\partial E_{0}(s, \pi)}{\partial s}\right|_{s=0} & =-\left.\frac{\frac{\partial}{\partial s} \operatorname{Tr}\left(\sum_{i=1}^{a} \pi_{i} \rho_{i}^{\frac{1}{s+1}}\right)^{s+1}}{\operatorname{Tr}\left(\sum_{i=1}^{a} \pi_{i} \rho_{i}^{\frac{1}{s+1}}\right)^{s+1}}\right|_{s=0} \\
& =I(\pi)
\end{aligned}
$$

Remark 2 Burnashev and Holevo showed, in case that all $\rho_{i}(i=1, \cdots, a)$ are pure and $0 \leq s \leq 1$, (13)-(10) and

$$
\frac{\partial^{2} E_{0}(s, \pi)}{\partial s^{2}} \leq 0, \quad(0 \leq s \leq 1)
$$

Lemma 4 Let $A_{i}(i=1, \cdots, a)$ be non-negative bounded operators in $\mathcal{H}$. If $0<\alpha \leq \beta \leq 1$ then

$$
\begin{equation*}
\left(\sum_{i=1}^{a} \pi_{i} A_{i}^{\frac{1}{\alpha}}\right)^{\alpha} \geq\left(\sum_{i=1}^{a} \pi_{i} A_{i}^{\frac{1}{\beta}}\right)^{\beta} \tag{17}
\end{equation*}
$$

[^1]Proof: First, we will show

$$
\begin{equation*}
\sum_{i=1}^{a} \pi_{i} A_{i} \leq\left(\sum_{i=1}^{a} \pi_{i} A_{i}^{\frac{1}{\gamma}}\right)^{\gamma} \tag{18}
\end{equation*}
$$

for $0<\gamma \leq 1$ and $A_{i} \geq 0(i=1, \cdots, a)$. Let us put unitary operator $U$ in $\mathcal{H} \oplus \cdots \oplus \mathcal{H}$ as

$$
U=\left(\begin{array}{cc}
\sqrt{\pi_{1}} I &  \tag{19}\\
\vdots & * \\
\sqrt{\pi_{a}} I &
\end{array}\right)
$$

and projection $P$ as

$$
\begin{equation*}
P=\operatorname{diag}[I, 0, \cdots, 0] \tag{20}
\end{equation*}
$$

Generally, for $0<\gamma \leq 1$ and operator $A, C(\|C\| \leq 1), C^{*} A^{\gamma} C \leq\left(C^{*} A C\right)^{\gamma}$ holds. (see [6], or 14], p.18) Using this property, we obtain

$$
\begin{aligned}
& \operatorname{diag}\left[\sum_{i=1}^{a} \pi_{i} A_{i}, 0, \cdots, 0\right] \\
= & P U^{*} \operatorname{diag}\left[A_{1}, \cdots, A_{a}\right] U P \\
= & P U^{*}\left(\operatorname{diag}\left[A_{1}^{\frac{1}{\gamma}}, \cdots, A_{a}^{\frac{1}{\gamma}}\right]\right)^{\gamma} U P \\
= & P\left(U^{*} \operatorname{diag}\left[A_{1}^{\frac{1}{\gamma}}, \cdots, A_{1}^{\frac{1}{\gamma}}\right] U\right)^{\gamma} P \\
\leq & \left(P U^{*} \operatorname{diag}\left[A_{1}^{\frac{1}{\gamma}}, \cdots, A_{1}^{\frac{1}{\gamma}}\right] U P\right)^{\gamma} \\
= & \operatorname{diag}\left[\left(\sum_{i=1}^{a} \pi_{i} A_{i}^{\frac{1}{\gamma}}\right)^{\gamma}, 0, \cdots, 0\right]
\end{aligned}
$$

which shows (18). Now, change $\gamma$ into $\frac{\alpha}{\beta}$ and $A_{i}$ into $A_{i}^{\frac{1}{\beta}}$ in (18), then

$$
\sum_{i=1}^{a} \pi_{i} A_{i}^{\frac{1}{\beta}} \leq\left(\sum_{i=1}^{a} \pi_{i} A_{i}^{\frac{1}{\alpha}}\right)^{\frac{\alpha}{\beta}}
$$

Since $x^{\beta}$ is a operator monotone function,

$$
\left(\sum_{i=1}^{a} \pi_{i} A_{i}^{\frac{1}{\beta}}\right)^{\beta} \leq\left(\sum_{i=1}^{a} \pi_{i} A_{i}^{\frac{1}{\alpha}}\right)^{\alpha}
$$

Now, from Lemma 3 if $R>C$ there exist $-1<t<0$ such that for all $s \in(t, 0),-s R+\min _{\pi} E_{0}(s, \pi)>0$. Thus, we state the following strong converse theorem.

Theorem 2 If $R>C$, then for all code $(\mathcal{C}, X), \operatorname{Pe}(\mathcal{C}, X)$ goes to 1 exponentially as $n \rightarrow \infty$.

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## Appendix

Lemma 5 Let $f:(a, b) \rightarrow \mathbf{R}$ be an analytic function and $X(t)$ a Hermitian with real parameter $t$, the spectrum of which is in $(a, b)$. Then,

$$
\frac{\partial}{\partial t} \operatorname{Tr} f(X(t))=\operatorname{Tr} f^{\prime}(X(t)) \frac{\partial X(t)}{\partial t}
$$

where $f^{\prime}$ is the derivative of $f$.
Proof: Leu us expand $f$ around $x_{0} \in(a, b)$ as $f(x)=\sum_{n=0}^{\infty} a_{n}\left(x-x_{0}\right)^{n}$ and put $\hat{X}(t)=X(t)-x_{0} I$. We can calculate as follows.

$$
\begin{aligned}
& \frac{\partial}{\partial t} \operatorname{Tr} f(X(t))=\sum_{n=0}^{\infty} a_{n} \operatorname{Tr} \frac{\partial}{\partial t} \hat{X}(t)^{n} \\
= & \sum_{n=1}^{\infty} a_{n} \sum_{i=1}^{n} \operatorname{Tr}(\hat{X}(t) \cdots \hat{X}(t) \underbrace{\frac{\partial \hat{X}(t)}{\partial t}}_{i \text {-th }} \hat{X}(t) \cdots \hat{X}(t)) \\
= & \sum_{n=1}^{\infty} n a_{n} \operatorname{Tr}\left(\hat{X}(t)^{n-1} \frac{\partial}{\partial t} \hat{X}(t)\right) \\
= & \operatorname{Tr} f^{\prime}(X(t)) \frac{\partial X(t)}{\partial t}
\end{aligned}
$$

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[^1]:    ${ }^{1}$ We don't know where to refer about this lemma, but it is in 8 as a exercise.

