On Quantum Fidelities and Channel Capacities

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Abstract— We show the equivalence of two different notions of quantum channel capacity: that which uses the entanglement fidelity as its criterion for success in transmission, and that which uses the minimum fidelity of pure states in a subspace of the input Hilbert space as its criterion. As a corollary, any source with entropy less than the capacity may be transmitted with high entanglement fidelity. We also show that a restricted class of encodings is sufficient to transmit any quantum source which may be transmitted on a given channel. This enables us to simplify a known upper bound for the channel capacity. It also enables us to show that the availability of an auxiliary classical channel from encoder to decoder does not increase the quantum capacity.

Keywords— Channel capacity, Quantum channels, Quantum information.

I. Introduction

A theory of quantum information is emerging which shows striking parallels with, but also fascinating differences from, classical information theory. One of the principal concerns of such theories is the capacity of a noisy channel for transmitting the state of a system despite some uncertainty about that state; that is, for rendering the state of some other system virtually identical to the initial state of the system at hand. In classical information theory, this is one of a set of mutually exclusive classical states; in quantum mechanics, a quantum state represented by a vector in a Hilbert space, or a density operator on that space. Classically, the input system may retain its original state, while the no-cloning theorem and related results [1], [2], [3], [4], [5], [6], [7], [8], [9], [10], [11], [12] imply that in the quantum case the input system cannot in general remain in its initial state. Both theories allow the use of encoding and decoding operations to increase the fidelity with which states are transmitted. Due partly to the peculiarly quantum fact that a system's state may be entangled with that of other systems, a greater variety of definitions of capacity has arisen in quantum mechanics, depending, for example, on whether the entanglement of a system with some reference system is required to be preserved by the transmission process, or not. Here we concentrate on two notions of quantum capacity, one investigated for example in [13],[14],[15],[16], concerned with the maximum entropy of a density operator whose entanglement with a reference system which does not undergo the noise process can be preserved with high fidelity, and another arising for example in [17],[18], [19],[20] and concerned with the maximum size of a Hilbert space all of whose pure states can be preserved with high fidelity. We show that these two definitions of capacity are in fact equivalent, in the situation in which sources are required to satisfy the quantum analogue of the asymptotic equipartition principle. We also show that any source with entropy less than the capacity may be sent with high entanglement fidelity, so that quantum entropy and capacity parallel classical entropy and capacity in this respect.

We also establish that any source that may be transmitted may be transmitted using only a maximal partial isometry as an encoding. This can be interpreted as meaning that encoding can be a unitary process, except for an initial projection of the source onto a subspace small enough to fit into the channel, if the channel is smaller than the source

This fact, which is in some ways analogous to the sourcechannel coding separation theorem of classical information theory, allows us to simplify a known upper bound on the quantum channel capacity, by removing from the expression a maximization over encodings, confirming an earlier conjecture. The conjecture has also been confirmed by [21], but the result that *any* source that may be transmitted may be transmitted using partially isometric encodings is slightly stronger than that obtained in [21].

In [22] Adami and Cerf express the view that "Whether a capacity can be defined *consistently* that characterizes the "purely" quantum component of a channel is an open question." In our view, the pure-state capacity defined below and in earlier papers is just such a consistently defined capacity, and the result that any source with entropy less than the entanglement capacity of a channel may be transmitted with high entanglement fidelity removes the last possible objection to the capacity for entanglement transmission as another such notion of "purely quantum" capacity.

We note that besides those cited above, many authors have worked on the problem of quantum information transmission through quantum channels; some of this work calculates or or bounds the capacity we study here, for particular channels or classes of channels: an incomplete list that could serve as an entry to the literature includes [17], [23], [20], [18], [19], [7], [9], [24]. Some of the extensive literature on the more algebraic approach to quantum coding also yields information about the quantum capacity.

II. QUANTUM SOURCES AND CHANNEL CAPACITY

A. Mathematical preliminaries and notation

The effect of encoding procedures, decoding procedures, and noisy quantum channels on the state of a system may be described by completely positive linear maps \mathcal{N} , from the space $B(H_c)$ of bounded linear operators on a input

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Hilbert space H_c , to the space $B(H_o)$ of bounded linear operators on an output Hilbert space H_o [25],[26],[27]. In this paper, we consider only discrete channels, which we define as having finite-dimensional input and output Hilbert spaces (the word "bounded" in the specification of the input and output spaces is redundant in the discrete case). We will sometimes use the term quantum operation for a trace-nonincreasing completely positive map. Such maps have representations in terms of linear operators A_i [25],[26],

$$\mathcal{A}(\rho) = \sum_{i} A_{i} \rho A_{i}^{\dagger} , \qquad (1)$$

with

$$\sum_{i} A_i^{\dagger} A_i \le I \; ; \tag{2}$$

equality holds in the latter when the map is tracepreserving. We call the set $\{A_i\}$ an operator decomposition, or simply decomposition, of the operation \mathcal{A} , and sometimes write:

$$\mathcal{A} \sim \{A_i\} \tag{3}$$

to indicate that $\{A_i\}$ is an operator decomposition of \mathcal{A} . Any two decompositions of the same operation, $\{A_i\}$ having r operators and $\{B_i\}$ having $s \leq r$ operators, are related by [27]:

$$A_i = \sum_{j=1}^s m_{ij} B_j \tag{4}$$

where m is the matrix of a maximal partial isometry from the complex vector space \mathcal{C}^s to \mathcal{C}^r . A partial isometry is a generalization of a unitary operator, which must satisfy $VV^{\dagger} = \Pi$ for some projector Π . Such an isometry will then also satisfy $V^{\dagger}V = \Gamma$ for some projector Γ having the same dimensionality as Π . If the range and domain spaces of a linear operator V have different dimensionality, it will not be possible to find a unitary mapping between the two: the best one can do is find a partial isometry V such that one of VV^{\dagger} and $V^{\dagger}V$ is the identity (whichever one operates on the smaller space). We will call such a map a maximal partial isometry between the spaces S_1 and S_2 . A partial isometry with VV^{\dagger} having dimension C may be thought of as projecting onto a C-dimensional subspace of V's domain Hilbert space and then mapping that subspace unitarily to a C-dimensional subspace of the range Hilbert space. Thus if $s \leq r$ in 4, m's columns are s orthonormal vectors in \mathcal{C}^r :

$$\sum_{i} m_{ij}^* m_{kj} = \delta_{ik} \tag{5}$$

or in other words:

$$mm^{\dagger} = I^{(s)} . {(6)}$$

Sometimes an operation \mathcal{A} will have a decomposition consisting of a single operator A; in this case, we will often

use the roman letter A to denote the operation \mathcal{A} as well as the operator A when no confusion will result. We note that care is needed when the operator includes a scalar factor z: thus if $\mathcal{A} \sim \{A\}$ while $\mathcal{B} \sim \{zA\}$, we may also refer to the operation B as either zA or $|z|^2\mathcal{A}$.

We write \mathcal{AE} for the operation of \mathcal{E} followed by \mathcal{A} ; thus $\mathcal{AE}(\rho) \equiv \mathcal{A}(\mathcal{E}(\rho))$.

Any quantum operation on a system Q may be realized [28],[25],[26] by a "unitary representation" in which the Hilbert space Q is extended by adjoining an environment E prepared in a standard state $|0^E\rangle$, and the system and environment undergo a unitary interaction, followed by a projection on the environment system. Any such unitary interaction with a given initial environment state determines a quantum operation. (In the case of a trace-preserving operation, the environment projection is the identity.) That is,

$$\mathcal{A}(\rho) = \operatorname{tr}_{E}(\pi^{E}U^{QE}|0^{E}\rangle\langle 0^{E}| \otimes \rho^{Q}U^{\dagger QE}\pi^{E}) . \tag{7}$$

The operators A_i in the operator decomposition representation discussed above, turn out to be the "operator matrix elements"

$$A_i^Q = \langle i^E | U^{QE} | 0^E \rangle \tag{8}$$

of the unitary interaction, between the initial environment state and orthonormal environment vectors vectors $|i\rangle$ of the basis used for the partial trace over the environment. The freedom (4) to "unitarily remix" the operators A_i , obtaining another valid decomposition, is just the freedom to do the environment partial trace in a different environment basis (related to the first by that same unitary).

B. Transmission and capacity

We now review the problem of entanglement transmission, as discussed more fully in [13], [14],[16]. A fuller discussion of the problem may be found in those articles. Here the goal is to use block coding to send the density operator of a source in a manner which preserves its entanglement with whatever reference system it may be entangled with. We imagine the density operator ρ^Q of our quantum system to arise from a pure state on a larger composite system RQ, by tracing out the "reference" system R. That is,

$$\rho^{Q} = \operatorname{tr}_{R}(|\psi^{RQ}\rangle\langle\psi^{RQ}|). \tag{9}$$

For R with dimension at least as great as that of ρ^Q 's support, such purifications always exist; different purifications of the same ρ^Q are related by unitary transformations on R. We define the entanglement fidelity as

$$F_e(\rho^Q, \mathcal{A}) \equiv \langle \psi^{RQ} | \mathcal{I} \otimes \mathcal{A}(|\psi^{RQ}\rangle \langle \psi^{RQ}|) | \psi^{RQ} \rangle ,$$
 (10)

the matrix element of the final, noise-affected state of the system RQ, with the initial state $|\psi^{RQ}\rangle$. This is easily shown to be independent of which purification $|\psi^{RQ}\rangle$ is used, and to have the form:

$$F_e(\rho, \mathcal{A}) = \sum_i |\operatorname{tr} A_i \rho|^2.$$
 (11)

Note that while [16] defined F_e as the renormalized entanglement fidelity $\sum_i |\text{tr } A_i \rho|^2 / \text{tr } \mathcal{A}(\rho)$, we have omitted the normalization, since the unrenormalized version is most useful in the present context. When we need the renormalized entanglement fidelity, just defined, we will use the symbol \hat{F}_e .

We define a quantum source $\Sigma = (H_s, \Upsilon)$ to consist of a Hilbert space H_s and a sequence $\Upsilon = \{\rho_s^{(1)}, \rho_s^{(2)}, ..., \rho_s^{(n)}, ...\}$ where $\rho_s^{(1)}$ is a density operator on H_s , $\rho_s^{(2)}$ a density operator on $H_s \otimes H_s$, and $\rho_s^{(n)}$ a density operator on $H_s^{\otimes n}$, etcetera. We define the entropy rate of a source Σ as

$$S(\Sigma) \equiv \limsup_{n \to \infty} \frac{S(\rho_s^{(n)})}{n}.$$
 (12)

(Sometimes we use the term "entropy of a source" to mean its entropy rate.) A quantum channel will be a trace-preserving map

$$\mathcal{N}: B(H_c) \to B(H_o) \tag{13}$$

from operators over a channel input space H_c to operators over a channel output space H_o . A coding scheme for a given source into a given channel consists of a sequence $(\mathcal{E}^{(n)}, \mathcal{D}^{(n)})$ of trace-preserving encoding maps and decoding maps

$$\mathcal{E}^{(n)} : B(H_s^{\otimes n}) \to B(H_c^{\otimes n})$$

$$\mathcal{D}^{(n)} : B(H_o^{\otimes n}) \to B(H_s^{\otimes n}). \tag{14}$$

We say that a source Σ may be sent reliably over a quantum channel $\mathcal N$ if there exists a coding scheme such that

$$\lim_{n \to \infty} F_e(\rho^{(n)}, \mathcal{D}^{(n)} \mathcal{N}^{\otimes n} \mathcal{E}^{(n)}) = 1.$$
 (15)

We say that rate R is achievable with a quantum channel \mathcal{N} if there is a source Σ with entropy R which may be sent reliably over the channel. We define the quantum capacity of the channel for transmission of entanglement, $Q_e(\mathcal{N})$, as the supremum of rates achievable with the channel \mathcal{N} . This definition of channel capacity leaves open the possibility that although some sources with entropy close to the capacity can be sent reliably, not all such sources can. Classically, it turns out that this is not the case: any source with entropy less than the classical capacity may be sent reliably. In what follows, we will establish that this is also the case for the quantum capacity. We will also establish the equality of the capacity for entanglement transmission Q_e , with the capacity for transmission of pure states in a subspace, Q_s , used for example in [19], [18]. We define the minimum pure-state fidelity, or simply pure-state fidelity, of a subspace H of the channel input Hilbert space as

$$F_p(H, \mathcal{A}) \equiv \min_{|\psi\rangle \in H} \langle \psi | \mathcal{A}(|\psi\rangle \langle \psi|) | \psi \rangle$$
 (16)

We say the rate R of transmission of subspace dimensions is achievable with channel \mathcal{N} if there exists a sequence of subspaces $H^{(n)}$ of $H_c^{\otimes n}$ such that

$$\limsup_{n \to \infty} \frac{\log \dim(H^{(n)})}{n} = R \tag{17}$$

and there is a coding scheme which sends it reliably in the sense that

$$\lim_{n \to \infty} F_p(H^{(n)}, \mathcal{D}^{(n)} \mathcal{N}^{\otimes n} \mathcal{E}^{(n)}) = 1.$$
 (18)

We define the capacity of the channel \mathcal{N} for transmission of subspaces, Q_s , as the supremum of achievable rates of transmission of subspace dimensions with channel \mathcal{N} .

C. The Quantum Asymptotic Equipartition Property

The ϵ -typical subspace for an n-block of material $\rho^{(n)}$ produced by a quantum source Σ on a Hilbert space H is defined to be the subspace $T_{\epsilon}^{(n)}$ of $H^{\otimes n}$ spanned by the eigenvectors $|\lambda\rangle$ of $\rho^{(n)}$ whose eigenvalues λ satisfy:

$$2^{-n(S(\Sigma)+\epsilon)} \le \lambda \le 2^{-n(S(\Sigma)-\epsilon)} . \tag{19}$$

An equivalent requirement is:

$$\left| -\frac{1}{n}\log\lambda - S(\Sigma) \right| \le \epsilon. \tag{20}$$

The definition derives its interest from the fact that for some interesting sources—for example, the i.i.d. source with $\rho^{(n)} = \rho^{\otimes n}$ [29]—all but a negligible portion of the source becomes concentrated in an ϵ -typical subspace as n goes to infinity, no matter how small ϵ is chosen to be. More formally, the i.i.d. source satisfies the Quantum Asymptotic Equipartition Property (QAEP). (Here and elsewhere, we will sometimes use the phrase "for large enough n, P(n) is true" to mean "there exists an n_0 such that for all $n > n_0$, P(n) is true".)

Definition 1: A source $\Sigma = \{\rho^{(1)}, ..., \rho^{(n)}, ...\}$ is said to satisfy the Quantum Asymptotic Equipartition Property if for any positive ϵ and δ , for large enough n the ϵ -typical subspace of $\rho^{(n)}$ satisfies:

$$\operatorname{tr} \Lambda^{(n)} \rho^{(n)} \Lambda^{(n)} > 1 - \delta , \qquad (21)$$

where $\Lambda^{(n)}$ is the projector onto $T_{\epsilon}^{(n)}$.

An immediate consequence of satisfaction of the QAEP is the following bound on the the dimension of the typical subspace, which holds for n large enough that the trace bound in the QAEP is satisfied:

$$(1 - \delta)2^{n(S(\Sigma) - \epsilon)} \le \dim(T_{\epsilon}^{(n)}) \le 2^{n(S(\Sigma) + \epsilon)}, \qquad (22)$$

A slightly more involved consequence is that for large enough n no subspace of dimension smaller than the lower bound $(1-\delta)2^{n(S(\Sigma)-\epsilon)}$ on the size of the typical subspace, has probability greater than δ . That is, if Π is the projector onto such a space,

$$\operatorname{tr} \Pi \rho < \delta$$
. (23)

See [30].

The classical Shannon-McMillan-Breiman theorem states that all stationary ergodic classical sources satisfy the (classical) AEP; however, these are not necessarily all the sources which satisfy it. There is as yet no known quantum analogue of the Shannon-McMillan-Breiman theorem, providing a broad and natural class of sources satisfying the QAEP, although there has been work in this direction [31].

III. USEFUL FACTS ABOUT FIDELITIES

A. Convexity of Entanglement Fidelity in the Input Density Operator

Lemma 1: The entanglement fidelity is convex in the input density operator,

$$F_e(\lambda \rho_1 + (1-\lambda)\rho_2), \mathcal{E}) \leq \lambda F_e(\rho_1, \mathcal{E}) + (1-\lambda)F_e(\rho_2, \mathcal{E}).$$
(24)

Proof: Note that the entanglement fidelity may be viewed as the squared norm $||\mathbf{a}||^2 \equiv \sum_i |a_i|^2$ of a complex vector \mathbf{a} whose components are:

$$a_i \equiv \operatorname{tr} A_i \rho_1 \ . \tag{25}$$

as

Then, letting also

$$b_i \equiv \operatorname{tr} A_i \rho_2 \,, \tag{26}$$

the entanglement fidelity of the convex combination of ρ_1 and ρ_2 may be written

$$F_e(\lambda \rho_1 + (1 - \lambda)\rho_2, \mathcal{E}) = ||\lambda \mathbf{a} + (1 - \lambda)\mathbf{b}||^2.$$
 (27)

Any norm is easily shown to be convex (see e.g. [32] for real vector spaces), and since a norm is positive its square is also convex and the lemma follows.

Note that with this representation of the entanglement fidelity, the freedom to choose an environment basis (equivalently, the freedom to move to a different operator decomposition of a given operation) corresponds to performing a maximal partial isometry V from the complex vector space containing \mathbf{a} to another complex vector space (with dimension equal to the number of operators in the new decomposition). (Since the transformation is length-preserving, it preserves (as it had better!) the entanglement fidelity.) We may use this "unitary" freedom to transform the vector \mathbf{a} into one of the same length with only a particular component, say the first, nonzero. Then the entanglement fidelity will just be the modulus $|\operatorname{tr} A_1 \rho|^2$ of that component. This gives us a useful lemma:

Lemma 2: There exists an operator sum decomposition $\{A_i\}$ of \mathcal{A} such that $F_e(\rho, \mathcal{A}) = F_e(\rho, A_1)$.

It may be instructive to see how this result arises in the RQE or unitary view of operations. The entanglement fidelity is the fidelity of $\rho^{RQ'}$ and the initial state of RQ; this is equal to the squared inner product of $|0^E\rangle|\psi^{RQ}\rangle$ with some purification of $\rho^{RQ'}$. The final pure state of RQE is such a purification, so it is related to the one whose inner product with the initial state gives the fidelity by a unitary on the environment; view this inner product as one between the final state of RQE and some other tensor product state $U^{\dagger E}|0^E\rangle|\psi^{RQ}\rangle$, and the result follows (since the individual terms in the entanglement fidelity correspond to particular states in an orthonormal basis used for the trace over the environment).

A slight variant of this interpretation is useful in the proof of the next lemma. Write the entanglement fidelity

$$\operatorname{tr}_{RQ}(\rho^{RQ'}|\psi^{RQ}\rangle\langle\psi^{RQ}|) = \operatorname{tr}_{RQE}(U^{QE}|0^{E}\rangle|\psi^{RQ}\rangle\langle\psi^{RQ}|\langle0^{E}|U^{QE\dagger}\rangle\langle\psi^{RQ}|\otimes I^{E}))$$

 $= ||(|\psi^{RQ}\rangle\langle\psi^{RQ}|\otimes I^E)U^{QE}|\psi^{RQ}\rangle|0^E\rangle||^2.$

That is, the entanglement fidelity is just squared length of the projection $|\pi^{RQE'}\rangle$ of the evolved pure state of RQEonto the tensor product of the environment and the one dimensional subspace of RQ spanned by the initial state of RQ. The vector **a** above is in fact just this projection; the components of a are the individual terms in the entanglement fidelity in a particular operator decomposition, i.e. the components of the vector a in a particular orthonormal basis. These correspond to the components of the projection $|\pi^{RQE'}\rangle$ in a particular orthonormal basis $|\chi_i^E\rangle|\psi^{RQ}\rangle$ for the subspace onto which we have projected, which corresponds to a choice of orthonormal basis $|\chi_i^E\rangle$ for the environment. So the lemma above is nothing but the observation that if we do the trace (in the definition of the entanglement fidelity) in an environment basis the first vector of which is a normalized version of $|\pi^{RQE'}\rangle$, we only get one term, which is of course the length of this projection.

We use this point of view to derive a lemma which concerns applying operations in sequence: if an operation has high fidelity, then the fidelity of the operation consisting of that operation followed by a second operation, is close to the fidelity of the second operation alone.

Lemma 3: If $F_e(\rho, \mathcal{E}) \geq 1 - \eta$ then for trace-nonincreasing \mathcal{A} ,

$$|F_e(\rho, \mathcal{A}\mathcal{E}) - F_e(\rho, \mathcal{A})| \le 2\eta \ . \tag{29}$$

Proof: Let E1 and E2 be environments inducing the operations \mathcal{E} and \mathcal{A} through unitary interactions U^{QE1} and V^{QE2} respectively. Then:

$$1 - \eta \leq F_{e}(\rho, \mathcal{E})$$

$$= |||\psi^{RQ}\rangle\langle\psi^{RQ}| \otimes I^{E1}U^{QE1}|\psi^{RQ}\rangle|0^{E1}\rangle||^{2}$$

$$= \langle\psi^{RQ}|\langle0^{E2}|\langle\chi^{E1}|U^{QE1}|0^{E1}\rangle|0^{E2}\rangle|\psi^{RQ}\rangle$$
(30)

for some $|\chi^{E1}\rangle$. That is, the two vectors $U^{QE1}|0^{E1}\rangle|0^{E2}\rangle|\psi^{RQ}\rangle$ and $|\chi^{E1}\rangle|0^{E2}\rangle|\psi^{RQ}\rangle$ are close. Now consider the two fidelities the magnitude of whose difference we wish to bound; these may be written as the squared lengths of projections of the two close vectors just considered. That is, define

$$P \equiv |\psi^{RQ}\rangle\langle\psi^{RQ}| \otimes I^{E1E2} \ . \tag{31}$$

Then

$$F_{e}(\rho, \mathcal{A}\mathcal{E}) = ||PV^{QE2}U^{QE1}|\psi^{RQ}\rangle|0^{E1}\rangle|0^{E2}\rangle||^{2}$$

$$= ||(V^{\dagger QE2}PV^{QE2})U^{QE1}|\psi^{RQ}\rangle|0^{E1}\rangle|0^{E2}\rangle||^{2} (32)$$

and

$$F_{e}(\rho, \mathcal{A}) = ||PV^{QE2}U^{QE1}|\psi^{RQ}\rangle|0^{E1}\rangle|0^{E2}\rangle||^{2}$$

$$= ||(V^{\dagger QE2}PV^{QE2})|\psi^{RQ}\rangle|\chi^{E1}\rangle|0^{E2}\rangle||^{2}.$$
(33)

From elementary geometry, if for normalized $|1\rangle$ and $|2\rangle$, $|\langle 1|2\rangle|^2=1-\eta$ then for any projector P, $|\langle 1|P|1\rangle-\langle 2|P|2\rangle|\leq 2\eta$. This may be applied directly to obtain the lemma.

A very simple but useful lemma implies that if two operations have high entanglement fidelity on the same density operator, the final density operators have high fidelity with each other. The notion of fidelity used here is treated in [33], [34], and [35]. It may be defined by

$$F(\rho_1, \rho_2) \equiv \max_{|\psi_1\rangle, \psi_2\rangle} |\langle \psi_1 | \psi_2 \rangle|^2, \tag{34}$$

where $|\psi_i\rangle$ are purifications of ρ_i .

In terms of this fidelity, the lemma is:

Lemma 4: If \mathcal{A},\mathcal{B} are trace-preserving and $F_e(\rho,\mathcal{A}) \geq 1-\epsilon_1$ and $F_e(\rho,\mathcal{B}) \geq 1-\epsilon_2$ then $F(\mathcal{A}(\rho),\mathcal{B}(\rho)) \geq 1-\epsilon_1-\epsilon_2$. Proof: Note that if $\langle 1|1\rangle = \langle 2|2\rangle = 1$, $|\langle 1|2\rangle|^2 > 1-\epsilon_1$ and $|\langle 1|3\rangle|^2 > 1-\epsilon_2$, then $|\langle 2|3\rangle|^2 > 1-\epsilon_1-\epsilon_2$. Apply this with $|2\rangle$ and $|3\rangle$ being the purifications of $\mathcal{A}(\rho)$ and $\mathcal{B}(\rho)$ whose squared inner products with a purification $|1\rangle$ of ρ give the entanglement fidelities, obtaining

$$|\langle 2|3\rangle|^2 \ge 1 - \epsilon_1 - \epsilon_2 \ . \tag{35}$$

Since the fidelity is the maximum squared inner product of purifications, $F(\mathcal{A}(\rho), \mathcal{B}(\rho)) \geq |\langle 2|3\rangle|^2 \geq 1 - \epsilon_1 - \epsilon_2$, as claimed.

B. Continuity of entanglement fidelity in the input operator

We will also need the continuity lemma for entanglement fidelity, trivially extended to the case of unnormalized F_e from [16].

Lemma 5:

$$|F_e(B+\Delta,\mathcal{A}) - F_e(B,\mathcal{A})| \le (\operatorname{tr}(|\Delta|))^2 + 2\operatorname{tr}(|\Delta|), \quad (36)$$

where $|\Delta| \equiv \sqrt{\Delta^{\dagger} \Delta}$.

C. Continuity of entropies in fidelities

Here we will establish a quantitative statement of the continuity of the entropy as a function of the density operator, in terms of the fidelity of neighboring density operators

Lemma 6: For any density operators ρ_1 , ρ_2 , acting on a d-dimensional Hilbert space,

$$|S(\rho_1) - S(\rho_2)| \le 2\sqrt{1 - F(\rho_1, \rho_2)} \log d + 1$$
 (37)

when

$$2\sqrt{1 - F(\rho_1, \rho_2)} < \frac{1}{3} . {38}$$

Proof: The proof begins with an inequality due to Fannes [36], involving an "error" quantity different from $1 - F(\rho_1, \rho_2)$. Defining the L_1 norm of an operator A as

$$||A|| \equiv \operatorname{tr} |A| \equiv \operatorname{tr} \sqrt{A^{\dagger} A},$$
 (39)

and the function $\eta(\cdot)$ by $\eta(x) = -x \log x$, we have (when $||\rho_1 - \rho_2|| < \frac{1}{3}$)

$$|S(\rho_1) - S(\rho_2)| \le ||\rho_1 - \rho_2|| \log d + \eta(||\rho_1 - \rho_2||).$$
 (40)

For our purposes, we may note that for $x < \frac{1}{3}$, $\eta(x) < \frac{\log 3}{3} < 1$, and use the weaker inequality

$$|S(\rho_1) - S(\rho_2)| \le ||\rho_1 - \rho_2|| \log d + 1$$
. (41)

Defining $p^{(1)}, p^{(2)}$ to be probability distributions given by the eigenvalues of ρ_1 and ρ_2 respectively, we note that if the two density matrices commute, then $||\rho_1 - \rho_2|| = 2d_K(p^{(1)}, p^{(2)})$, where d_K is the Kolmogorov distance or total variation distance between two probability distributions,

$$d_K(p^{(1)}, p^{(2)}) \equiv \frac{1}{2} \sum_i |p_i^{(1)} - p_i^{(2)}|.$$
 (42)

Since the entropy difference is invariant under independent unitary rotations of each density matrix,

$$|S(\rho_1) - S(\rho_2)| \le 2d_K(p^{(1)}, p^{(2)}) \log d + 1,$$
 (43)

where we may take the eigenvalues to be arranged in order of size in both probability distributions. An inequality of C. H. Kraft [37] [38] implies

$$d_K(p^{(1)}, p^{(2)}) \le \sqrt{1 - B(p^{(1)}, p^{(2)})},$$
 (44)

where B is the Bhattacharyya-Wootters overlap

$$B(p^{(1)}, p^{(2)}) \equiv \sum_{i} \sqrt{p_i^{(1)} p_i^{(2)}}$$
 (45)

Moreover,

$$B(p^{(1)}, p^{(2)}) \ge F(\rho_1, \rho_2),$$
 (46)

since, given the eigenvalues of both density operators, the fidelity is maximized by choosing their eigenvectors to be the same, assigned to eigenvalues in order of size. This follows easily from [39],[40] and the representation of the square root of the fidelity as

$$\max_{\text{unitary } U} \text{tr } \rho_1^{1/2} \rho_2^{1/2} U . \tag{47}$$

This completes the proof of the lemma.

Now consider the situation where d is the dimension of each of two spaces Q and R, and ρ^{RQ} a density operator on the d^2 -dimensional space $R \otimes Q$. We use the notation $\rho^Q \equiv \operatorname{tr}_R \rho^{RQ}$. Using Lemma 6 and the fact that $F(\rho_1^Q, \rho_2^Q) \geq$

 $F(\rho_1^{RQ}, \rho_2^{RQ})$, one easily obtains a continuity relation for the entropy of Q conditional on R, defined as

$$S(Q|R) \equiv S(\rho^{RQ}) - S(\rho^{Q}). \tag{48}$$

Lemma 7: Continuity of conditional entropy.

$$|S(Q_1|R_1) - S(Q_2|R_2)| \le 6\sqrt{1 - F(\rho_1^{RQ}, \rho_2^{RQ})} \log d + 2$$

when $F(\rho_1^{RQ}, \rho_2^{RQ}) > 5/9$.

This lemma will be useful in the discussion of capacity below, because the capacity is bounded by a quantity, the coherent information of a density operator ρ^Q under an operation \mathcal{E} , which may be written in terms of a conditional entropy. This quantity is defined by

$$I_c(\rho^Q, \mathcal{E}) \equiv S\left(\frac{\mathcal{E}(\rho^Q)}{\operatorname{tr} \mathcal{E}(\rho^Q)}\right) - S\left(\frac{\mathcal{I} \otimes \mathcal{E}(|\psi^{RQ}\rangle\langle\psi^{RQ}|)}{\operatorname{tr} \mathcal{E}(\rho)}\right) . (49)$$

IV. THE TYPICAL SUBSPACE AND ENTANGLEMENT FIDELITY

We now derive some interesting implications of the QAEP for entanglement fidelity. These may be summarized by the statement that in order for the entanglement fidelity to be asymptotically high, it is necessary and sufficient that the fidelity be high on the typical subspace. We will demonstrate two versions of this statement, both of which will be used later on.

Define a quantum data compression scheme for a source $\Sigma = (H_s, \rho_s^{(n)})$ to be a sequence of trace-preserving quantum operations $\mathcal{C}^{(n)}$ from $H_s^{\otimes n}$ to the ϵ -typical subspace $T_{\epsilon}^{(n)}$ of the source such that

$$C^{(n)}(\rho_s^{(n)}) = C_1^{(n)}(\rho) + C_2^{(n)}(\rho_s^{(n)}), \tag{50}$$

where $C_1^{(n)}(\rho) \equiv \Lambda_n \rho_s^{(n)} \Lambda_n$, Λ_n is the projector onto $T_{\epsilon}^{(n)}$, and tr $(\Lambda_n \rho_s^{(n)}) = 1 - \delta_n$. Then we may derive the following lemma.

Lemma 8: For any source satisfying the QAEP, any quantum data compression scheme $C^{(n)}$ and any trace-preserving operation $A^{(n)}$ from $H_s^{\otimes n}$ to $H_s^{\otimes n}$,

$$|F_e(\rho_s^{(n)}, \mathcal{A}^{(n)}\mathcal{C}^{(n)}) - F_e(\rho_s^{(n)}, \mathcal{A}^{(n)})| < 2\delta_n.$$
 (51)
This lemma is an immediate consequence of Lemma 3.

In applying this lemma, we have in mind a situation where $\mathcal{A}^{(n)}$ represents the effect of further encoding taking us from the source Hilbert space $H_s^{\otimes n}$ to the channel Hilbert space $H_c^{\otimes n}$, followed by the channel noise operation, and a decoding which takes us back to the source Hilbert space. By the QAEP, for large $n \operatorname{tr} \overline{\Lambda}^{(n)} \rho_s^{(n)} \overline{\Lambda}^{(n)} \equiv \delta_n$ becomes smaller than any predetermined positive δ ; hence the difference between the entanglement fidelity when the encoding is preceded by quantum data compression of the source, and the entanglement fidelity without such a step, is asymptotically negligible.

For some purposes, it will be more useful to compare the entanglement fidelity of a source with the entanglement fidelity of the *renormalized* projection of the source onto its typical subspace. Let Λ be the projector onto the typical subspace after n uses of the source, and $\overline{\Lambda}$ the projector onto the orthogonal subspace. For any positive ϵ and large enough n,

$$\operatorname{tr}(\overline{\Lambda}\rho_s^{(n)}\overline{\Lambda}) \le \epsilon. \tag{52}$$

Defining the renormalized restriction of the source to the typical subspace,

$$\rho_{\epsilon}^{(n)} \equiv \frac{\Lambda \rho_s^{(n)} \Lambda}{\operatorname{tr}(\Lambda \rho_s^{(n)} \Lambda)},\tag{53}$$

and applying the continuity lemma for entanglement fidelity, (5), we have the following lemma:

Lemma 9: For any trace-preserving operation \mathcal{E} and any source satisfying the QAEP,

$$|F_e(\rho_{\epsilon}^{(n)}, \mathcal{E}) - F_e(\rho_s^{(n)}, \mathcal{E})| \le \frac{4\epsilon}{(1-\epsilon)^2}.$$
 (54)

By choosing n sufficiently large, ϵ can be made arbitrarily small, and thus we see that for the entanglement fidelity for the source to be high asymptotically, it is necessary and sufficient that the entanglement fidelity be high asymptotically for the renormalized restriction of the source to the typical subspace.

V. Entanglement fidelity and minimum pure state fidelity

$A. \ Entanglement \ transmission \ implies \ pure-state \ transmission$

We will first show that if a source satisfying the QAEP can be transmitted over a channel with entanglement fidelity approaching one in the large-block limit, one can transmit a subspace which is asymptotically of dimension $2^{nS(\Sigma)}$ with minimum pure state fidelity approaching one. That is, if a channel can send entanglement at a certain rate, it can send subspaces with high pure-state fidelity at that rate also.

The argument has parallels with the classical argument that if one can transmit with low expected error (taking the expectation over messages), one can transmit with low maximal error. That argument proceeds by throwing out the highest-error half of the codewords, and then establishing a definite bound on the maximum error of the remaining codewords in terms of the average error of the initial ensemble of codewords. Both quantities go to zero together. Throwing out half the codewords reduces the rate by a bit, but asymptotically this is negligible. Here, we throw out a low-fidelity fraction of the Hilbert space dimensions, in a certain systematic way which enables us to bound the minimum fidelity of the remaining states in terms of the entanglement fidelity.

We do not expect to be able to show that a "logarithmically large" subspace of the support of an arbitrary density operator may be sent with high minimum pure-state fidelity. That would mean that the capacity for sending subspaces with high minimum fidelity would be higher than the capacity for sending entanglement, since the dimension of the support of a density matrix is typically much higher than its entropy. After all, many of the dimensions in the support of a density matrix may have negligible probability, and hence the failure to send them would be expected to have negligible impact on the entanglement fidelity. Therefore, a high entanglement fidelity would not necessarily suggest that all dimensions in the support, or even a logarithmically large subset of them, can be sent accurately. Rather, we expect to be able to show that a subspace whose dimension is a "logarithmically large" fraction of $2^{nS(\Sigma)}$ can be sent with high minimum pure-state fidelity. In fact, we expect that a logarithmically large subspace of the typical subspace can be sent with high pure-state fidelity.

Our approach, then, will be to use Lemma 9 to argue that if a source Σ generating density operators $\rho^{(n)}$ can be sent with asymptotically high entanglement fidelity, so can $\rho_{\epsilon}^{(n)}$, the renormalized restriction of $\rho^{(n)}$ to its ϵ -typical subspace. Therefore, for large enough n, $\rho_{\epsilon}^{(n)}$ can be made to have entanglement fidelity greater than $1-\eta$ for any positive η . We will indicate, without explicitly changing notation from ρ to $\rho_{\epsilon}^{(n)}$, where we first use properties of $\rho_{\epsilon}^{(n)}$.

Suppose a density operator ρ with K-dimensional support can be sent with entanglement fidelity $1-\eta$. Consider the following procedure for systematically removing dimensions from the support of the density operator. Let $|1\rangle$ be the lowest fidelity pure state in the support. We then define the (sub-normalized) positive operator ρ_1 by

$$\tilde{\rho}_1 = \rho - q_1 |1\rangle\langle 1| , \qquad (55)$$

where q_1 is the largest positive q_1 for which $\tilde{\rho}$ is still a positive operator. We continue this process recursively, defining $\tilde{\rho}_0 \equiv \rho$, and

$$\tilde{\rho}_i \equiv \tilde{\rho}_{i-1} - q_i |i\rangle\langle i| , \qquad (56)$$

where $|i\rangle$ is the state in the support of $\tilde{\rho}_{i-1}$ with the lowest pure-state fidelity, and q_i is as large as it can be subject to the constraint that $\tilde{\rho}_i$ is a positive operator.

The vectors in this set are ordered in terms of increasing pure-state fidelity; we will write f_i for the pure state fidelity $\langle i|\mathcal{E}(|i\rangle\langle i|)|i\rangle$ of $|i\rangle$.

Note that tr $\tilde{\rho}_1 = 1 - q_1$, and in general tr $\tilde{\rho}_j = \text{tr } \tilde{\rho}_{j-1} - q_j = 1 - \sum_{i=1}^j q_j$. By construction, $\text{rank}(\rho_i) = \text{rank}(\rho_{i-1}) - 1$. Hence $\rho_d = \mathbf{0}$ and $\sum_{i=1}^K q_i = 1$. Furthermore,

$$\sum_{i=1}^{K} q_i |i\rangle\langle i| = \rho , \qquad (57)$$

that is, $\{q_i, |i\rangle\}$ are a pure-state ensemble for ρ . Note that while this procedure removes dimensions from the support of the density matrix one by one, the dimensions it removes are not necessarily the one-dimensional spaces spanned by the vectors $|i\rangle$. Indeed, the vectors $|i\rangle$ will usually not be an orthonormal basis for the support of ρ although they are linearly independent.

Now

$$\langle i|\rho|i\rangle = q_i + \sum_{j\neq i} \langle j|\rho|j\rangle.$$
 (58)

Since the terms in the sum are all positive,

$$q_i < \langle i|\rho|i\rangle < \lambda_1(\rho),$$
 (59)

where $\lambda_1(\rho)$ is the largest eigenvalue of ρ . That is, any upper bound on the eigenvalues of ρ is also an upper bound on the q_i .

In particular, when $\rho = \rho_{\epsilon}^{(n)}$ then for large enough n the q_i satisfy the bounds on eigenvalues from the QAEP

$$2^{-n(S(\Sigma)+\epsilon)} \le q_i \le \frac{2^{-n(S(\Sigma)-\epsilon)}}{1-\delta} . \tag{60}$$

Now, by the convexity of entanglement fidelity in the density operator,

$$\sum_{i=1}^{n_0} q_i f_i + \left(\sum_{i=n_0+1}^n q_i\right) F_e(\rho_{n_0+1}) \ge F_e(\rho) = 1 - \eta , \quad (61)$$

where ρ_{n_0+1} is the normalized version of $\tilde{\rho}_{n_0+1}$, *i.e.*, the density operator with the lowest-fidelity n_0 dimensions of its support removed. Define $\alpha \equiv \sum_{i=1}^{n_0} q_i$. Thus we are considering the situation where we throw out n_0 of the states, leaving a fraction $(1-\alpha)$ of the total weight of the density operator.

We will denote by $1-\gamma$ the pure state fidelity of $|n_0+1\rangle$,

$$f_{n_0+1} \equiv 1 - \gamma \; ; \tag{62}$$

this is the lowest pure-state fidelity of any of the remaining vectors $|i\rangle$ for $i \geq n_0$, and by construction also the lowest pure-state fidelity of any state in the subspace they span. Then

$$(1 - \gamma)\alpha + (1 - \alpha) \ge 1 - \epsilon , \qquad (63)$$

so that

$$\gamma \le \frac{\eta}{\alpha}.\tag{64}$$

Thus the reciprocal of α is the factor by which error is increased when the first n_0 dimensions are removed from the support of ρ by the above procedure. Since

$$2^{-n(S(\Sigma)+\epsilon)} \le q_i , \qquad (65)$$

$$n_0 2^{-n(S(\Sigma) + \epsilon)} \le \alpha . {(66)}$$

Thus, for a fixed α , our procedure leaves us with a subspace having dimensionality $D \equiv K - n_0$ of pure states which can be sent with fidelity at least $1 - \eta/\alpha$. Now,

$$1 - \alpha = \sum_{i=n_0+1}^{K} q_i \le D2^{-n(S-\epsilon)}$$
 (67)

SO

$$D \ge (1 - \alpha)2^{n(S - \epsilon)} \tag{68}$$

and the rate

$$\frac{\log D}{n} \ge \frac{\log (1 - \alpha)}{n} + S(\Sigma) - \epsilon . \tag{69}$$

That is, for any fixed η and α strictly between zero and one, for large enough n the size in qubits of a subspace with minimum fidelity $1-\eta/\alpha$ approaches $n(S(\Sigma)-\epsilon)$. Hence all rates less than $S(\Sigma)$ may be achieved. Since Σ was any density operator source that could be sent with high entanglement fidelity, this implies that any rate less than the capacity for sending entanglement may be achieved for sending subspaces with high minimum entanglement fidelity. Thus we have

Theorem 1: $Q_s \geq Q_e$.

B. Pure state transmission implies entanglement transmission

We now show that the entanglement fidelity of a density operator under an operation cannot be too much less than the minimum pure-state fidelity of states in the density operator's support. As minimum pure-state fidelity approaches one, so does entanglement fidelity, so that *any* density operator with support entirely in this subspace can be sent with high entanglement fidelity. Specifically, we prove the following theorem (see also [41]). The argument makes no use of the notion of typical subspace, and hence is not limited to sources satisfying the QAEP.

Theorem 2: Suppose all pure states $|\psi\rangle$ in a subspace S have pure state fidelity $\langle \psi | \mathcal{E}(|\psi\rangle \langle \psi|) | \psi\rangle$ greater than or equal to $1-\eta$. Then any density operator ρ whose support lies entirely in that subspace has entanglement fidelity $F_e(\rho, \mathcal{E}) \geq 1 - \frac{3}{2}\eta$.

For applications to asymptotic channel capacity what is important is that the error for sending entanglement goes to zero if the maximum error for density operators in the subspace does, and that the relationship between the two fidelities involves no factors of the dimension of Hilbert space, which could cause trouble in taking the large block limit. This means that if we can transmit Hilbert-space dimensions with minimum fidelity approaching one at a rate C, we can also reliably transmit the entanglement of any source Σ with entropy $S(\Sigma) < C$.

Proof: We Schmidt decompose $|\Psi^{RQ}\rangle$:

$$|\Psi^{RQ}\rangle = \sum_{k} \sqrt{\lambda_k} |k^R\rangle |k^Q\rangle . \tag{70}$$

In the Schmidt decomposition $|k^R\rangle$ and $|k^Q\rangle$ are the diagonal bases of the density operators ρ^R and ρ^Q , labeled according to their common eigenvalues λ_k .

Then

$$\rho^{RQ'} \equiv (\mathcal{I} \otimes \mathcal{E})(|\psi^{RQ}\rangle\langle\psi^{RQ}|)$$

$$= \sum_{kl} |k^R\rangle\langle l^R| \otimes \mathcal{E}(|k^Q\rangle\langle l^Q|). \tag{71}$$

The entanglement fidelity becomes (omitting the superscripts R and Q to reduce clutter):

$$F_{e}(\rho, \mathcal{E}) = \sum_{mnkl} \sqrt{\lambda_{m} \lambda_{n} \lambda_{k} \lambda_{l}} \langle m|n \rangle \langle l|n \rangle \langle m|\mathcal{E}(|k \rangle \langle l|)|n \rangle$$
$$= \sum_{kl} \lambda_{k} \lambda_{l} \langle k|\mathcal{E}(|k \rangle \langle l|)|l \rangle . \tag{72}$$

A first attempt at a proof might split up the sum as:

$$F_{e} = \sum_{k} \lambda_{k}^{2} \langle k | \mathcal{E}(|k\rangle \langle k|) | k \rangle$$

$$+ \sum_{k \neq l} \lambda_{k} \lambda_{l} \langle k | \mathcal{E}(|k\rangle \langle l|) | l \rangle . \tag{73}$$

We see that the first sum here can certainly be bounded below using the fact that pure state fidelities for vectors in the basis $|k\rangle$ are greater than $1-\eta$, but the second term has cross-terms that are more difficult to deal with. The proof will have to use the fact that not only vectors in the basis $|k\rangle$, but arbitrary superpositions of them, have high fidelity, and the pure state fidelities of these superpositions will contain such cross-terms. Since the expressions we want to bound contain the probabilities λ , we will consider superpositions with amplitudes $\sqrt{\lambda}$ and all possible phase factors $e^{i\phi_k}$:

$$|\psi(\phi_1, ..., \phi_k)\rangle \equiv \sum_k \sqrt{\lambda_k} e^{i\phi_k} |k\rangle ,$$
 (74)

The pure state fidelity for this is:

$$= \sum_{mnkl}^{\langle \psi | \mathcal{E}(|\psi\rangle\langle\psi|)|\psi\rangle} \sqrt{\lambda_m \lambda_n \lambda_k \lambda_l} \langle m | \mathcal{E}(|k\rangle\langle l|) | n \rangle e^{i(\phi_k + \phi_n - \phi_m - \phi_l)} .$$
(75)

The m = k, n = l terms will give the entanglement fidelity in the form (72) (since the phases appear in complex conjugate pairs in those terms, they disappear). But there are other terms in (75) which we need to argue are small, or somehow get rid of, in order to argue that the high fidelity of these pure states implies that the terms constituting the entanglement fidelity are high. We do this by averaging the entanglement fidelity for these superpositions over all phases from zero to 2π . We still get the desired terms, but many of the cross terms will disappear. Only those with four indices identical, or with indices identical in complex conjugate pairs, will remain; the rest will contain integrals like $\int_0^{2\pi} d\phi_k e^{i\phi_k}$ (from indices whose value is not equal to that of some other index), or $\int_0^{2\pi} d\phi_k e^{2i\phi_k}$ (from pairs of identical indices that are not complex conjugates). The average is of course still greater than $1-\eta$; and the remaining terms are:

(71)
$$\overline{f} = \sum_{kl} \lambda_k \lambda_l \langle k | \mathcal{E}(|k\rangle \langle l|) | l \rangle + \sum_{km, k \neq m} \langle m | \mathcal{E}(|k\rangle \langle k|) | m \rangle$$

$$= F_e + \sum_{km, k \neq m} \langle m | \mathcal{E}(|k\rangle \langle k|) | m \rangle \ge 1 - \eta . \tag{76}$$

(The same result obtains if the average is done not by integration over all possible values of the complex phase factor, but only over the phases $\pm 1, \pm i$.) We need to upper bound the terms that do not appear in the entanglement fidelity. These terms all contain the fidelity of the output state $\mathcal{E}(|k\rangle\langle k|)$ to a state $|m\rangle$ orthogonal to the input state $|k\rangle$. Since $\mathcal{E}(|k\rangle\langle k|)$ has high fidelity to input k, one expects its fidelity to states orthogonal to $|k\rangle$ will be small. In fact, since \mathcal{E} is trace-preserving, taking the trace in the $|m\rangle$ basis gives:

$$\sum_{m} \langle m | \mathcal{E}(|k\rangle\langle k|) | m \rangle = 1 , \qquad (77)$$

and the fact that $\langle k|\mathcal{E}(|k\rangle\langle k|)|k\rangle \geq 1-\eta$ then gives:

$$\sum_{m \neq k} \langle m | \mathcal{E}(|k\rangle\langle k|) | m \rangle \le \eta \ . \tag{78}$$

Let us assume the eigenvalues have been ordered from largest to smallest, $\lambda_1 \equiv \lambda_1(\rho) \geq \lambda_2 \equiv \lambda_2(\rho)$ etc. Then when k = 1, we have $\lambda_m \leq \lambda_2$, so the k = 1 term is bounded:

$$\lambda_{1} \sum_{m \neq 1} \lambda_{m} \langle m | \mathcal{E}(|1\rangle\langle 1|) | m \rangle \leq \lambda_{1} \lambda_{2} \sum_{m \neq 1} \langle m | \mathcal{E}(|1\rangle\langle 1|) | m \rangle$$
$$\leq \lambda_{1} \lambda_{2} \eta . \tag{79}$$

When $k \neq 1$, we must use the looser bound $\lambda_m \leq \lambda_1$ in a similar fashion, giving:

$$\sum_{k \neq 1} \lambda_k \sum_{m \neq k} \lambda_m \langle m | \mathcal{E}(|k\rangle \langle k|) | m \rangle$$

$$\leq \sum_{k \neq 1} \lambda_k \lambda_1 \sum_{m \neq k} \langle m | \mathcal{E}(|k\rangle \langle k|) | m \rangle$$

$$\leq \sum_{k \neq 1} \lambda_k \lambda_1 \eta = (1 - \lambda_1) \lambda_1 \eta . \tag{80}$$

Thus

$$F_e \ge 1 - (1 + \lambda_1 \lambda_2 + (1 - \lambda_1) \lambda_1) \eta$$
 (81)

For given λ_1 , this is minimized where $\lambda_2 = (1 - \lambda_1)$. The resulting bound,

$$F_e \ge 1 - (1 + 2\lambda_1(1 - \lambda_1))\eta$$
, (82)

is clearly minimized when $\lambda_1 = \lambda_2 = \frac{1}{2}$, giving

$$F_e(\rho, \mathcal{E}) \ge 1 - \frac{3}{2}\eta \ . \tag{83}$$

Corollary 1: $Q_e \geq Q_s$.

Proof: The theorem implies, as noted in [16], that if there is a sequence of encodings, decodings, and subspaces $H^{(n)}$ that achieves rate R for subspace transmission, the sequence of uniform density operators on these subspaces $I^{(n)}/\dim(H^{(n)})$ will also have limiting entanglement fidelity one under the same transmission operations. Since the entropy rate of this source is R, the same rate is achievable for entanglement transmission.

C. Consequences for Capacity

The results of the two previous sections immediately imply that the capacities for pure-state transmission and for entanglement transmission are equal. They also imply that if a source can be sent on a given channel with high entanglement fidelity, so can any source with lower entropy which satisfies the QAEP. Hence

Theorem 3: Any source Σ with $S(\Sigma) < C(\mathcal{N})$ may be transmitted with high entanglement fidelity over the channel \mathcal{N} .

VI. Encodings

In [16], we conjectured that an expression for the quantum capacity was:

$$\lim_{n \to \infty} \max_{H_s, \rho^{(n)} \in H_s, \mathcal{E}^{(n)}: B(H_s) \to B(H_c)} I_c(\rho^{(n)}, \mathcal{N}^{\otimes n} \mathcal{E}^{(n)}) . \tag{84}$$

and showed that this expression was no smaller than the capacity. This involves a maximization over input density operators and trace–preserving completely positive encoding maps. However, we also conjectured that the maximization over encodings was not necessary. Rather, the analogous expression with the maximization over encodings removed, and the density operator maximization done over density operators on the channel input Hilbert space H_c , instead of operators on the source space, was conjectured to also be a correct expression for the capacity. This would make the situation more similar to the classical one, where no maximization over encodings appears in the expression for channel capacity. In [16], we showed that if encodings could be restricted to be unitary, then indeed the maximization over encodings could be dropped entirely.

A. Partially isometric encodings

Our strategy for removing the maximization over encodings will be to show that we may restrict our attention to partially isometric encodings, that is, encodings of the form

$$\mathcal{E}(\rho) = V \rho V^{\dagger}, \tag{85}$$

where V is a partial isometry from the source space to the channel space. An encoding corresponding to a partial isometry from a source space to a smaller channel space (as in noiseless data compression, for instance) will be trace-decreasing for density operators having support outside the subspace that is unitarily mapped into the source space. In our definition of channel capacity, we required that encodings be trace-preserving. But trace-decreasing encodings are relevant to our problem because they may be embedded in trace-preserving ones with no loss of fidelity. We say a trace-decreasing operation $\mathcal F$ is embedded in a trace-preserving operation $\mathcal A$ if

$$\mathcal{A} = \mathcal{F} + \mathcal{G} \tag{86}$$

for some trace-decreasing \mathcal{G} . Since

$$F_e(\rho, \mathcal{F} + \mathcal{G}) = F_e(\rho, \mathcal{F}) + F_e(\rho, \mathcal{G}) , \qquad (87)$$

the entanglement fidelity of a trace-decreasing operation is a lower bound on the entanglement fidelity of any tracepreserving operation into which the trace-decreasing one has been embedded. This is what makes partially isometric encodings relevant to physical situations in which a tracepreserving encoding is used; we will use this in IX.

VII. RESTRICTING THE ENCODINGS

We will show that if there exists a general encoding that achieves high fidelity transmission for a given source, there is also a partially isometric encoding achieving fidelity not much lower for that source, where "not much lower" will be quantified in such a way that if the general encoding has fidelity approaching one, the lower bound on fidelity with partially isometric encoding also approaches one, and there is no dimensional dependence in the relation between the fidelities that would cause difficulty with the large block limit (in which the Hilbert space dimension grows exponentially).

A. Perfect transmission

The intuition behind the argument may be illustrated for the case of transmission with fidelity precisely one. It is frequently easy to show something for fidelity exactly one, but more difficult to extend it to fidelities which are merely very close to one, as is necessary for channel capacity arguments, and that is the case here. If the operation of encoding followed by noise followed by decoding achieves perfect transmission for some ρ , this implies that the encoding operation is perfectly reversible for ρ , since it is reversed by the composition of noise with decoding. As noted in [42], an operation \mathcal{A} that is perfectly reversible for a density operator may, when restricted to the subspace C (with dimension d_C) supporting that density operator, be written in the form of unitaries from the support into mutually orthogonal d_C -dimensional subspaces of the output space, randomly applied with probabilities p_i . That is

$$\mathcal{A} \sim \{\sqrt{p_i}U_i\} ,$$

$$U_i^{\dagger}U_j = \delta_{ij}P_C$$
(88)

where P_C is the projector onto C. If there exists an operation which reverses this with perfect fidelity for some input ρ , it must reverse each of the unitaries U_i with fidelity one. Hence we may remove the factor $\sqrt{p_i}$ from any of the operators in the canonical decomposition of the encoding, and use it as an encoding, which will achieve perfect transmission when the same decoding is used.

B. Isometric encoding suffices

Theorem 4: Given a trace-preserving map A and a map \mathcal{E} with tr $\mathcal{E}(\rho) = 1$ and

$$F_e(\rho, \mathcal{AE}) > 1 - \eta$$
, (89)

there exists a partial isometry W such that

$$F_e(\rho, \mathcal{A}W) > 1 - 2\eta \ . \tag{90}$$

In applying this theorem, we will take \mathcal{E} to be the encoding map, and \mathcal{A} to be the concatenation of noise and decoding.

The proof proceeds via the following two lemmas:

Lemma 10: There exist operator decompositions of \mathcal{A} and \mathcal{E} such that $F_e(\rho, \mathcal{A}\mathcal{E}) \leq F_e(\rho, A_1E_1/\sqrt{\operatorname{tr}(E_1\rho E_1^{\dagger})})$.

(Note that $E_1/\sqrt{\operatorname{tr}(E_1\rho E_1^{\dagger})}$ is not necessarily trace decreasing.)

Proof: Let $\{A_i\}$ and $\{E_i\}$ be operator decompositions of \mathcal{A} and \mathcal{E} . Let X be the matrix with elements tr $(A_iE_j\rho)$. Then $F_e = \sum_{ij} |(X)_{ij}|^2$. The singular value decomposition ensures that by changing the operator decomposition of \mathcal{A} and \mathcal{E} , we can transform to a representation where X is diagonal; assume without loss of generality that A_i and E_j are already such representations. Then $F_e(\rho, \mathcal{A}\mathcal{E}) = \sum_k \operatorname{tr} (A_k E_k \rho)^2$ (since $\operatorname{tr} (A_k E_j \rho) = 0$ if $k \neq j$). Let $\lambda_k = \operatorname{tr} (E_k \rho E_k^{\dagger})$. Then $\sum_k \lambda_k (\operatorname{tr} (A_k E_k \rho)^2 / \lambda_k) = F_e(\rho, \mathcal{A}\mathcal{E})$ and $\sum_k \lambda_k = 1$, so there exists a k such that $\operatorname{tr} (A_k E_k \rho)^2 / \lambda_k \geq F_e(\rho, \mathcal{A}\mathcal{E})$.

Lemma 11: Let

$$E : S \to C$$

$$A : C \to S$$

$$(91)$$

be linear operators, $\rho \in B(C)$ a density matrix. If $F_e(\rho, AE) \geq 1 - \eta$, $A^{\dagger}A \leq I$ and tr $(E\rho E^{\dagger}) = 1$, then there is a maximal partial isometry $W: S \to C$ such that $F_e(\rho, AW) \geq 1 - 2\eta$.

Proof: Let UD_AV be a singular value decomposition of A. Here we can take D_A to have matrix elements proportional to the Kronecker delta in a (fixed) basis for C, V unitary on C and $U:C\to S$ a maximal partial isometry. Consider the maximal partial isometry $W:S\to C$ defined by $W=V^\dagger U^\dagger$. Then $U=(VW)^\dagger$ and

$$|\operatorname{tr} (AE\rho)|^{2} = |\operatorname{tr} (\rho^{1/2}UD_{A}^{1/2}VWUD_{A}^{1/2}VE\rho^{1/2})|^{2}$$

$$\leq \operatorname{tr} ((UD_{A}^{1/2}VW)^{\dagger}\rho UD_{A}^{1/2}VW)$$

$$\times \operatorname{tr} (UD_{A}^{1/2}VE\rho E^{\dagger}(UD_{A}^{1/2}V)^{\dagger}) \quad (92)$$

$$\leq \operatorname{tr} (UD_{A}^{1/2}VW(VW)^{\dagger}D_{A}^{1/2}U^{\dagger}\rho) \quad (93)$$

$$= \operatorname{tr} (UD_{A}U^{\dagger}\rho)$$

$$= \operatorname{tr} (UD_{A}VW\rho). \quad (94)$$

(The first inequality is an operator Schwarz inequality, while the second is due to the fact that $A^{\dagger}A$, and therefore D_A , is less than or equal to I, and the fact that if $B \geq 0$ and $I \geq C \geq 0$, tr $BC \leq \operatorname{tr} B$.) It follows that $\operatorname{tr} (AW\rho) \geq 1 - \eta$, hence $|\operatorname{tr} (AW\rho)|^2 = F_e(\rho, AW) \geq 1 - 2\eta$.

To obtain Theorem 4 as a corollary, apply Lemma 10 and the premise of the theorem to get:

$$F_e(\rho, A_1 E_1 / \sqrt{\operatorname{tr}(E_1 \rho E_1^{\dagger})}) \ge F_e(\rho, \mathcal{A}\mathcal{E}) \ge 1 - \eta$$
. (95)

Lemma 11, with

$$E = E_1 / \text{tr} \left(E_1 \rho E_1^{\dagger} \right) \tag{96}$$

then gives the result.

It follows that if there exists a coding scheme which transmits the entanglement of a source reliably, there exists a coding scheme using partially isometric encodings which transmits it reliably.

VIII. FORWARD CLASSICAL COMMUNICATION DOESN'T HELP

Bennett, DiVincenzo, Smolin and Wootters (BDSW) [17] showed that a forward classical channel, from encoder to decoder, cannot help one achieve perfect transmission. They did this by constructing, from any fidelity-one coding scheme for such a channel, a fidelity-one coding scheme which makes no use of the classical channel. This is another example of a result which is apparently hard to extend to asymptotically high fidelity transmission.

An argument virtually identical to the proof of Theorem 4 can be used to extend their result to the asymptotically high fidelity situation, for the problem of sending the entanglement of a uniform source with asymptotically high fidelity (cf. also [21]). (By results in Section (V), this is equivalent to the problem BDSW considered, of sending every state in a source space with high fidelity.) We may do this by modeling a classical forward channel in a manner analogous to the model of the observed channel in [16] except that the decoder takes into account classical information about the encoding rather than the noise. We take the encoding to be a set $\{\mathcal{E}_m\}$ of trace-nonincreasing operations which sum to a trace-preserving operation. The value of the index m represents classical information available to the encoder (as a measurement result, say) which may be sent to the decoder and used in decoding, so we allow the decoder to use one of a collection of trace-preserving decodings \mathcal{D}_m . Formally, we define a coding scheme with classical forward communication to be a sequence of such collections of encodings and decodings, $[\mathcal{E}_m^{(n)}, \mathcal{D}_m^{(n)}]$, where m takes values $1...M^{(n)}$ so that the number of available encoding operations may depend on n, and

$$\sum_{m=1}^{M^{(n)}} \mathcal{E}_m^{(n)} \equiv \mathcal{E}^{(n)} \tag{97}$$

is trace-preserving, while each of $\mathcal{E}_m^{(n)}$ is trace-nonincreasing. We say a source may sent reliably over this channel with classical forward communication, if there exists a coding scheme such that

$$\lim_{n \to \infty} \sum_{m} F_e(\rho^{(n)}, \mathcal{D}_m^{(n)} \mathcal{N}^{\otimes n} \mathcal{E}_m^{(n)}) = 1.$$
 (98)

We define the capacity of a channel for entanglement transmission with forward classical communication, $Q_e^{(fc)}$, to be the supremum of the entropy rates of sources that can be sent reliably on the channel.

Now suppose the entanglement fidelity of a density operator ρ sent through such a channel is high (omit the superscripts $^{(n)}$ for clarity),

$$\sum_{m} F_e(\rho, \mathcal{D}_m \mathcal{N} \mathcal{E}_m) > 1 - \eta. \tag{99}$$

Then there exists a value of j of the index m for which

$$\hat{F}_e(\rho, \mathcal{D}_i \mathcal{N} \mathcal{E}_i) > 1 - \eta. \tag{100}$$

Now

$$\hat{F}_e(\rho, \mathcal{D}_j \mathcal{N} \mathcal{E}_j) = F_e(\rho, \mathcal{D}_j \mathcal{N} \frac{\mathcal{E}_j}{\operatorname{tr} \mathcal{E}_j(\rho)}), \qquad (101)$$

so that by Theorem 4, there is a partial isometry W such that

$$F_e(\rho, \mathcal{D}_i \mathcal{N} W) > 1 - 2\eta \ . \tag{102}$$

The partially isometric encoding can be extended to a trace-preserving encoding with no loss of fidelity, hence the same source can be sent without using the forward classical channel. Hence

Theorem 5: $Q_e^{(fc)} = Q_e$.

IX. AN UPPER BOUND ON CAPACITY

We will now treat an issue raised in section VI-A. We will show that the fact that partially isometric encodings suffice to achieve the channel capacity implies that we may omit the maximization over encodings from the expression that upper bounds the capacity.

Since the entanglement fidelity of any trace-preserving encoding into which a (possibly trace-decreasing) partially isometric encoding $\mathcal V$ might be embedded is bounded below by the unrenormalized entanglement fidelity of $\mathcal V$, we consider trace-preserving encodings $\mathcal F \equiv \mathcal V + \mathcal A$, where $\mathcal V$ is partially isometric. Polar decompose V into a maximal partial isometry W and a positive Γ (which will be a projector), so that $V = W\Gamma$.

We know from Theorem 4 that, given a sequence of general encodings $\mathcal E$ and decodings $\mathcal D$ that sends a given source (so that overall entanglement fidelity goes to one with increasing block size), there exists a sequence of partially isometric encodings $\mathcal V^{(n)}$ that (when used with the same decodings as before), sends that source with the unrenormalized entanglement fidelity approaching one with increasing block size. Hence the entanglement fidelity when some sequence of trace-preserving extensions $\mathcal F^{(n)} = \mathcal V^{(n)} + \mathcal A^{(n)}$ is used to encode goes to one with increasing block size as well. More precisely, if for a given ϵ and large enough n, we have

$$F_e(\rho^{(n)}, \mathcal{D}^{(n)} \mathcal{N}^{\otimes n} \mathcal{E}^{(n)}) > 1 - \epsilon$$
 (103)

then by Theorem 4 for large enough n

$$F_e(\rho^{(n)}, \mathcal{D}^{(n)} \mathcal{N}^{\otimes n} \mathcal{F}^{(n)}) > 1 - 2\epsilon$$
 (104)

Now let us consider the fidelity of the output states $\rho^{RQ''} \equiv \mathcal{D}^{(n)} \mathcal{N}^{\otimes n} \mathcal{E}^{(n)}(\rho^{(n)})$ and $\sigma^{RQ''} \equiv \mathcal{D}^{(n)} \mathcal{N}^{\otimes n} \mathcal{F}^{(n)}(\rho^{(n)})$ obtained by using the different encodings. By Lemma 4,

$$F(\rho^{RQ''}, \sigma^{RQ''}) > 1 - 3\epsilon . \tag{105}$$

To obtain the upper bound on capacity in [16], we used the following fact:

$$S(\rho) \le I_c(\rho^{(n)}, \mathcal{N}^{\otimes n} \mathcal{E}^{(n)}) + 2$$
$$+4(1 - F_e(\rho, \mathcal{D}^{(n)} \mathcal{N}^{\otimes n} \mathcal{E}^{(n)})) \log d_c^n , \qquad (106)$$

where all operations involved are trace-preserving (d_c is the dimension of the channel). In particular, this holds for the operations $\mathcal{F}^{(n)}$. We now consider the coherent information with such encoding operations.

Recall the representation of the coherent information I_c as a conditional entropy and apply Lemma 7, the continuity of conditional entropy, to obtain:

$$|I_c(\rho^{(n)}, \mathcal{N}^{\otimes n} \mathcal{E}^{(n)}) - I_c(\rho^{(n)}, \mathcal{N}^{\otimes n} \mathcal{F}^{(n)})|$$

$$< 6\sqrt{3\epsilon} \log d_c^n + 2.$$
(107)

Hence

$$\lim_{n \to \infty} \left| \max_{\rho_1^{(n)}} \frac{I_c(\rho_1^{(n)}, \mathcal{N}^{\otimes n} \mathcal{E}^{(n)})}{n} - \max_{\rho_2^{(n)}} \frac{I_c(\rho^{(n)}, \mathcal{N}^{\otimes n} \mathcal{F}^{(n)})}{n} \right| = 0.$$
 (108)

So, the coherent information bound with general encodings is the same as the bound for encodings restricted to have the form \mathcal{F} .

We now show that this bound implies that with the maximization over channel input density operators alone. In earlier work [16], we defined the coherent information of a non-trace-preserving operation as:

$$I_c(\rho, \mathcal{E}) \equiv S\left(\frac{\mathcal{E}(\rho)}{\operatorname{tr} \mathcal{E}(\rho)}\right) - S\left(\frac{\mathcal{I} \otimes \mathcal{E}(|\psi^{RQ}\rangle\langle\psi^{RQ}|)}{\operatorname{tr} \mathcal{E}(\rho)}\right) , (109)$$

the conditional entropy using the renormalized output state of the system and entangled reference. Now, the coherent information of the channel \mathcal{F} is bounded above by the coherent information of the observed channel $\mathcal{N}\{\mathcal{V},\mathcal{A}\}$ in which we know which of \mathcal{V} and \mathcal{A} occurred. The latter is given [16] by:

$$I_{c}(\rho, \mathcal{N}^{\otimes n} \{ \mathcal{V}, \mathcal{A} \}) = (\operatorname{tr} \Gamma \rho) I_{c}(\rho, \mathcal{N}^{\otimes n} \mathcal{V})$$
$$+ (1 - \operatorname{tr} \Gamma \rho) I_{c}(\rho, \mathcal{N}^{\otimes n} \mathcal{A}) .$$
(110)

A straightforward calculation shows that the first term is equal to

$$\operatorname{tr} \Gamma \rho I_c(W \widehat{\Gamma \rho \Gamma} W^{\dagger}, \mathcal{N}^{\otimes n}) .$$
 (111)

(We use the notation $\hat{A} \equiv A/\text{tr } A$.) Since $1 \geq \text{tr } \Gamma \rho \geq F_e(\rho, \mathcal{DN}^{\otimes n}\mathcal{V})$ and the latter approaches one in the large-n limit, so does tr $\Gamma \rho$, and hence:

$$S(\Sigma) \le \lim_{n \to \infty} \frac{I_c(W\widehat{\Gamma}\widehat{\rho}\widehat{\Gamma}W^{\dagger}, \mathcal{N}^{\otimes n})}{n} . \tag{112}$$

The inequality still holds when we maximize over ρ . The ability to maximize over ρ followed by projection using Γ ,

normalization, and placing the density operator in some subspace of the channel via W just allows us to access some of the possible channel input density matrices. Hence the RHS of (112) is bounded above by:

$$\lim_{n \to \infty} \max_{\rho^{(n)}} \frac{I_c(\rho^{(n)}, \mathcal{N}^{\otimes n})}{n} . \tag{113}$$

This is the promised upper bound on the quantum capacity.

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