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The Rearrangement Process in a Two-Stage Broadcast Switching Network

SØREN B. JACOBSEN

Abstract—**This paper considers the rearrangement process in the two-stage broadcast switching network presented by F. K. Hwang and G. W. Richards in IEEE TRANSACTIONS ON COMMUNICATIONS, October 1985. By defining a certain function it is possible to calculate an upper bound on the number of connections to be moved during a rearrangement. When each inlet channel appears twice, the maximum number of connections to be moved is found. For a special class of inlet assignment patterns in the case where each inlet channel appears three times, the maximum number of connections to be moved is also found. In the general case, an upper bound is given when the number of outlets at each second-stage switch is kept below a certain bound.**

I. THE KNOWN PROPERTIES OF THE NETWORK

THE network to be considered here (see Fig. 1) is identical to the one presented in [1], and it is described by the three parametres n_1, n_2, and M where

- n_1 is the number of inlet channels at each first-stage switch,
- n_2 is the number of outlets at each second-stage switch, and
- M is the number of times each inlet channel appears in the first stage.

The number of crosspoints in the network divided by the number of crosspoints in the corresponding rectangular switch is called the reduced number of crosspoints and is given by

$$C_{\text{red}} = M(1/n_1 + 1/n_2). \tag{1.1}$$

To minimize C_{red}, the fraction M/n_2 has to be made as close to zero as possible but the rearrangement requirement puts a lower bound on the fraction.

Hall's theorem on a system of distinct representatives [2] ensures that the network is rearrangeable, if and only if, the following condition is fulfilled.

The Rearrangement Condition: For any $n \le n_2$, there are at least n first-stage switches containing appearance of any n inlet channels.

To ensure that the n_1^2 inlet channels are effectively rotated in the M blocks the following condition is assumed to be fulfilled.

The Pair Condition: No pair of inlet channels appears on the same first-stage switch more than once throughout the Mn_1 first-stage switches.

All the inlet assignment patterns presented in [1] fulfill the pair condition, but instead of working with some explicit patterns, it is more advantageous in a general approach just to assume the pair condition to be fulfilled.

II. AN UPPER BOUND ON THE NUMBER OF CONNECTIONS TO BE MOVED DURING A REARRANGEMENT BY MEANS OF THE FUNCTION S_{M,n_1}

As it will be seen later, the rearrangement condition as well as an upper bound on the number of connections to be moved during a rearrangement can be given by means of a function whose values are in general unknown. The function will be called S_{M,n_1}, and it is defined by means of another function T_{M,n_1}. Let E be a subset of the set of inlet channels. Then,

$$T_{M,n_1}(E) := \text{The number of first-stage switches having elements from } E \text{ amongst its inlet channels.} \tag{2.1}$$

Now S_{M,n_1} is defined by

$$S_{M,n_1}(n) := \min \{ T_{M,n_1}(E) \mid E \text{ has } n \text{ elements} \}. \tag{2.2}$$

S_{M,n_1} is an increasing function, it depends on the inlet assignment pattern chosen, and $S_{M,n_1}(n)$ denotes the smallest number of first-stage switches that n inlet channels can appear on.

In terms of S_{M,n_1}, we have

The network is rearrangeable if and only if

$$n \le S_{M,n_1}(n) \text{ for all } n \le n_2.$$

This means that the optimal choice for n_2 is

$$n_2 = \max \{ n \mid n \le S_{M,n_1}(n) \}. \tag{2.3}$$

An upper bound on the number of connections to be moved during a rearrangement can be found in terms of a sequence

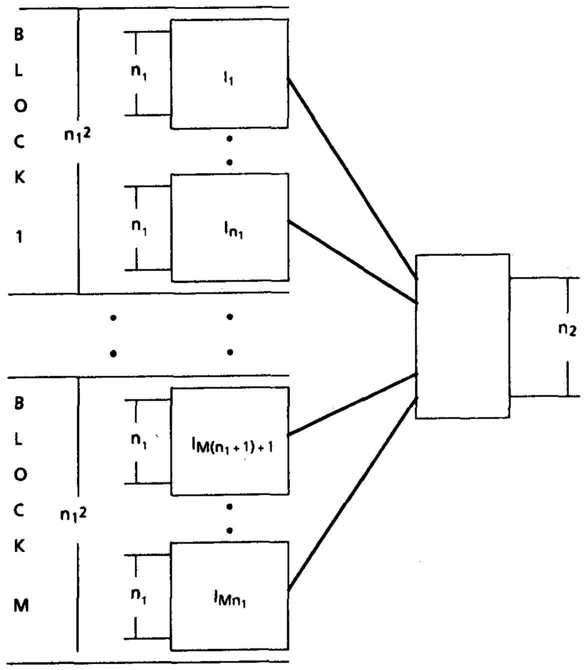

Fig. 1. The network presented in [1]. Every inlet channel has exactly one appearance in each of the M blocks.

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$\{s_k\}$ defined recursively from S_{M,n_1} by

$$s_1 := S_{M,n_1}(1) \text{ and } s_{k+1} := S_{M,n_1}(s_k+1). \tag{2.4}$$

The main result of this section is as follows.

Result 2.1: Let r denote the integer with the property: $s_r \leq n_2 - 1$ and $s_{r+1} \geq n_2$. Then the number of connections to be moved during a rearrangement will never exceed r.

Proof: To prove the result, consider the rearrangement algorithm given in [1]. It is characterized by the blocking relationship tree and its associated levels. The development of the tree stops at the first level where an idle switch arises. See Fig. 2. The important fact, given in [1], is

> The number of connections to be moved is one less than the number of the level where the first idle switch arises.

Now the idea is to step through all levels of the blocking relationship tree, and in each level, keep an eye on the total number of inlet channels that have appeared so far, and to see how many first-stage switches they necessarily have appearance on. Sooner or later a level is reached where so many inlet channels have appeared, that the number of switches they demand, exeed the number of busy lines, which is, at most, $n_2 - 1$. When this occurs, at least one switch is idle.

The sequence $\{s_k\}$ is defined so that s_k is a lower bound on the number of first-stage switches that have appeared in the first k levels of the blocking relationship tree. This fact is explained in Fig. 3. Therefore, an idle switch must arise in the first level where $s_k \geq n_2$, which according to the definition of r is level $r + 1$. This concludes the proof of Result 2.1.

To use Result 2.1 on a given network, it is sufficient to know $S_{M,n_1}(n)$ for $n = 1, 2, \cdots, n_2 + 1$. In the case $M = 2$, S_{M,n_1} is independent of n_1 and for $n \leq 6$ it takes the following values:

$$S_2(1)=2,\quad S_2(2)=3,\quad S_2(3)=4,\quad S_2(4)=4,$$
$$S_2(5)=5,\quad S_2(6)=5.$$

This means that the optimal choice for n_2 is 5, as given in [1]. The sequence $\{s_k\}$ takes for $k \leq 3$, the values

$$s_1=2,\quad s_2=S_2(3)=4,\quad s_3=S_2(5)=5=n_2$$

from which we see that the integer r defined in Result 2.1 equals 2 proving.

Result 2.2: The number of connections to be moved during a rearrangement will never exeed 2, when $M = 2$.

Fig. 2 shows a situation where two connections must be moved. This means that the maximum number of connections to be moved is two.

III. The Maximum Number of Connections to be Moved During a Rearrangement for a Special Class of Inlet Assignment Patterns when $M = 3$

In this section, we consider the case $M = 3$. Fix an inlet assignment pattern and let the switches in each of the 3 blocks be numbered $0, 1, \cdots, n_1 - 1$. We will show that this inlet assignment pattern induces a latin square of order n_1. Define the $n_1 \times n_1$ matrix Z by

> *ij'te element in Z is the number of the switch in block 3, containing the common element of switch i from block 1 and switch j from block 2.*

This definition is taken from [3] and the pair condition ensures that Z is a latin square. We restrict the calculation of S_{3,n_1} to inlet assignment patterns where the induced latin square is the multiplication table of a group. Result 4.1 applied to the case

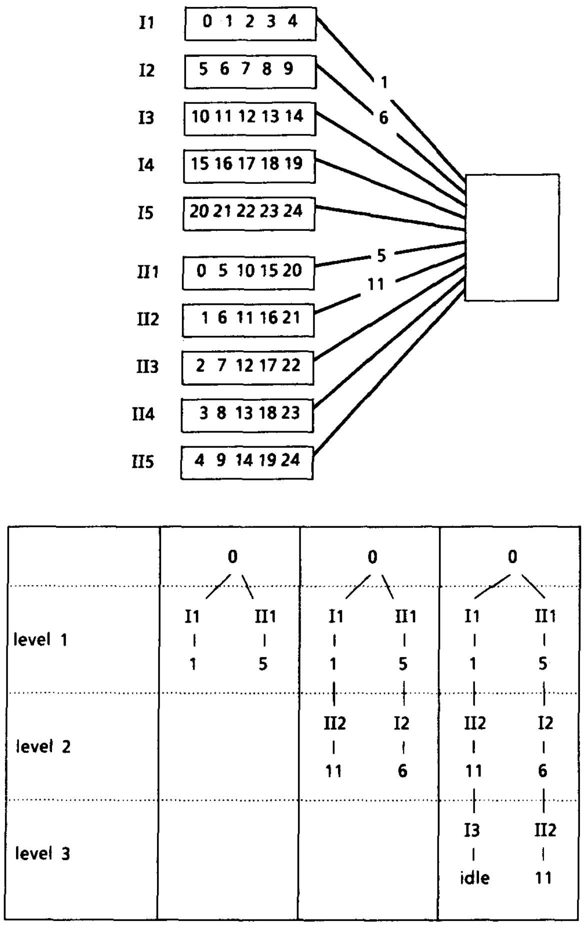

Fig. 2. The development of the blocking relationship tree. Here the tree stops in level 3 and the rearrangement is done by moving channel 11 to switch I3 and channel 1 to switch II2.

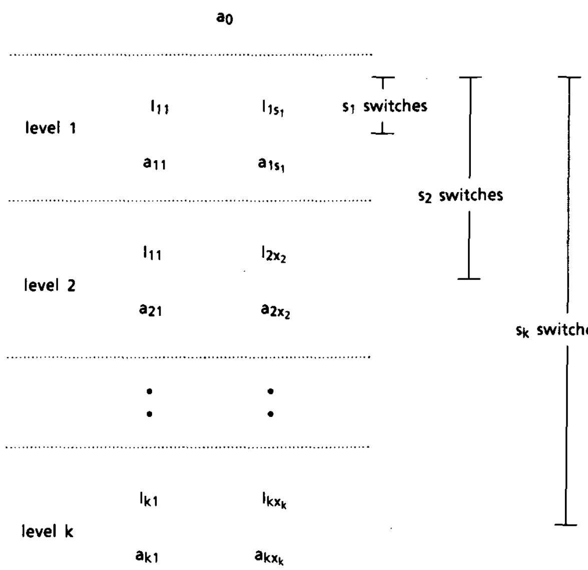

Fig. 3. Inlet channel a_0 is present at $s_1 = S_{M,n_1}(1)$ first-stage switches. None of them are idle meaning that $s_1 + 1$ channels are present. They have appearance on at least $s_2 = S_{M,n_1}(s_1 + 1)$ first-stage switches. If none of them are idle we now have a total of $s_2 + 1$ channels present, etc.

$M = 3$ together with the upper bound U_3 given in Appendix A yields

$$S_{3,n_1}(1)=3, \quad S_{3,n_1}(2)=5, \quad S_{3,n_1}(3)=6,$$
$$S_{3,n_1}(5)=8, \quad S_{3,n_1}(7)=9, \quad S_{3,n_1}(10)=11,$$
$$S_{3,n_1}(4)=6 \text{ or } 7, \quad S_{3,n_1}(6)=8 \text{ or } 9, \quad S_{3,n_1}(8)=9 \text{ or } 10,$$
$$S_{3,n_1}(9)=9, \ 10 \text{ or } 11, \quad S_{3,n_1}(11)=11 \text{ or } 12,$$
$$S_{3,n_1}(13)=12 \text{ or } 13 \quad S_{3,n_1}(14)=12 \text{ or}^1 13.$$

We will now determine when $S_{3,n_1}(4) = 6$. If $S_{3,n_1}(4) = 6$ then there must exist four inlet channels appearing in two switches in block 1, in two switches in block 2, and in two switches in block 3. (If all four inlet channels appeared on the same switch in one of the blocks then they would appear on four different switches in the remaining two blocks.) In the language of latin squares this means we can find four entries appearing in two rows and two columns so that these four entries contain only two different element which we call x_1 and x_2. Since the latin square corresponds to the multiplication table of a group, the two rows correspond to two group elements a_1, a_2, and the two columns correspond to two group elements b_1, b_2 so that the multiplication table for $\{a_1, a_2\} \times \{b_1, b_2\}$ is

$$\begin{array}{c|cc} & b_1 & b_2 \\ \hline a_1 & x_1 & x_2 \\ a_2 & x_2 & x_1 \end{array}.$$

This yields $a_1 = x_1 b_1^{-1}$ and $a_2 = x_2 b_1^{-1}$ and therefore $x_1 b_1^{-1} b_2 = x_2$ and $x_2 b_1^{-1} b_2 = x_1$ implying $(b_1^{-1} b_2)^2 = 1$. Since an element of order 2 exist, if and only if, 2 divides the order of the group, we have proved

$$S_{3,n_1}(4) = \begin{cases} 6 \text{ when 2 divides } n_1 \\ 7 \text{ else} \end{cases}.$$

In Appendix A, similar methods are used to find $S_{3,n_1}(n)$ when $n = 11$ and 13. The remaining values can be calculated the same way. We therefore have the following.

Result 3.1: Assume $M = 3$ and $n_l \geq 4$. For all inlet assignment patterns where the induced latin square is the multiplication table of a group, we have:

1) *S_{3,n_1} takes the values in Table I.*

2) *The optimal choice for n_2 (assuming $S_{3,n_1}(14) \leq 13$ for all inlet assignment patterns), and the maximum number of connections to be moved during a rearrangement are the numbers given in Table II.*

When n_1 is a multiple of four or five, it is easy to construct states where a rearrangement requires 3 connections to be moved. Figs. 4 and 5 show a state where 4 (5) connections has to be moved. The upper bounds given in Table II are therefore the maximum number of connections to be moved.

Can the results in this section be extended so that they include arbitrary latin squares? The answer is no. Consider the following latin square:

$$\begin{array}{ccccc} \underline{0} & \underline{1} & 2 & 3 & 4 \\ \underline{1} & \underline{0} & 3 & 4 & 2 \\ 2 & 4 & 0 & 1 & 3 \\ 3 & 2 & 4 & 0 & 1 \\ 4 & 3 & 1 & 2 & 0 \end{array}.$$

Select the four entries in the two upper rows and the two left columns and conclude that $S_{3,5}(4) = 6$ in this case where 2 does not divides $n_1 (=5)$.

[1] For latin squares corresponding to some special groups of order a power of 3, it is in doubt whether $S_{3,n_1}(14) = 13$ *or* 14. If $S_{3,n_1}(14) = 14$ there exist inlet patterns where best n_2 is 14.

TABLE I
THE VALUES OF THE FUNCTION S_{3,n_1} FOR $1 \leqslant n \leqslant 13$

n	$S_{3,n_1}(n)$
1	3
2	5
3	6
4	6 when 2 divides n_1; 7 else
5	8
6	8 when 3 divides n_1; 9 else
7	9
8	9 when 3 divides n_1; 10 else
9	9 when 3 divides n_1; 10 when 4 but *not* 3 divides n_1; 11 else
10	11
11	11 when 4 divides n_1; 12 else
12	11 when 4 divides n_1; 12 else
13	12 when 4 or 5 divides n_1; 13 else

IV. An Upper Bound on the Number of Connections to be Moved During a Rearrangement when M is Arbitrary

When n_1 and M grow, it becomes very time consuming to calculate the values of S_{M,n_1}. It would, therefore, be advantageous if upper and lower bounds could be given. In Appendix A we prove the following.

Result 4.1: For all inlet assignment patterns fulfilling the pair condition the following estimate is valid:

$$S_{M,n_1}(n) \geq G_M(n) \quad \text{for any } n \leq n_1^2 \tag{4.1}$$

where

$$G_M(n) = \begin{cases} (p+1)M - 1 \text{ when } p^2+1 \leq n \leq p^2+p \\ (p+1)M \text{ when } p^2+p+1 \leq n \leq (p+1)^2. \end{cases} \tag{4.2}$$

Since $G_M(n) \leq S_{M,n_1}(n)$, the network is rearrangeable as long as $n_2 \leq \max \{n | n \leq G_M(n)\}$. But (4.2) gives that max $\{n | n \leq G_M(n)\} = M(M+1) - 1$ so

$$n_2 = M(M+1) - 1 \tag{4.3}$$

well known from [1].

To get an upper bound on the number of connections to be moved during a rearrangement the following sequence $\{g_k\}$ [compare to (2.4)] is defined

$$g_1 := G_M(1) \text{ and } g_{k+1} = G_M(g_k + 1). \tag{4.4}$$

g_k has the following two obvious properties:

1) $g_k \leq s_k$ for any k and g_k is therefore a lower bound on the number of first-stage switches that have appeared in the first k levels of the blocking relationship tree.

2) Let m be the integer with the property $g_m \leq n_2 - 1$ and $g_{m+1} \geq n_2$. Then m is an upper bound on the number of connections to be moved during a rearrangement.

It is now easy to verify the following.

Result 4.2: If $n_2 \leq M(M+1) - 1$ then the number of

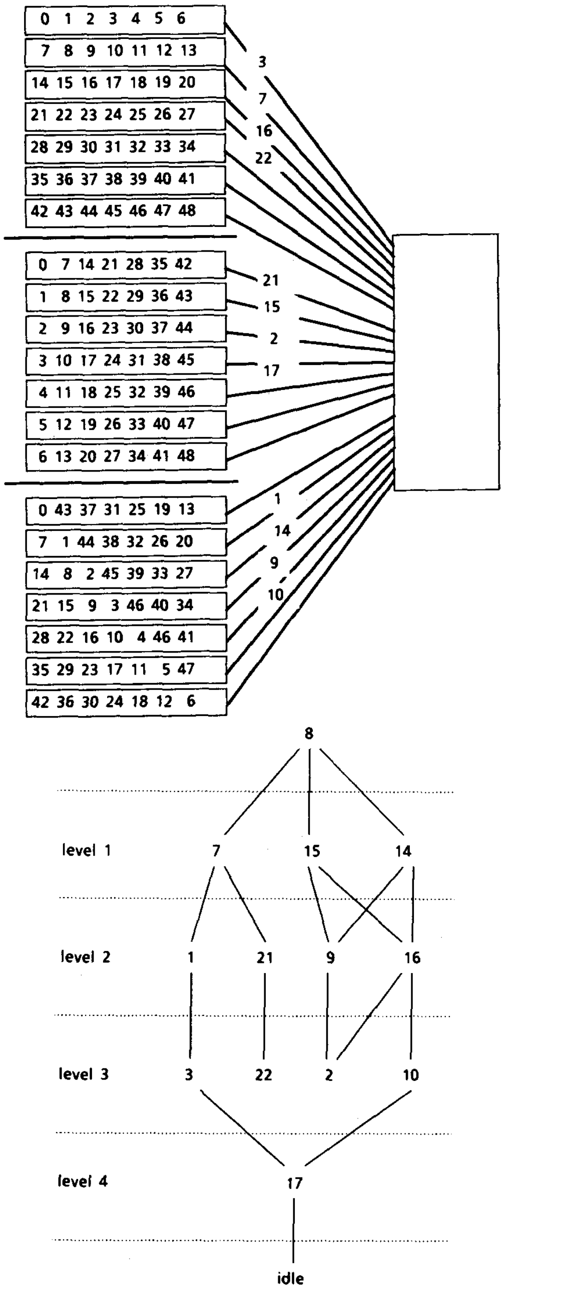

Fig. 4. In this network having a total of 49 inlet channels, a request is made for channel 8 and four connections have to be moved.

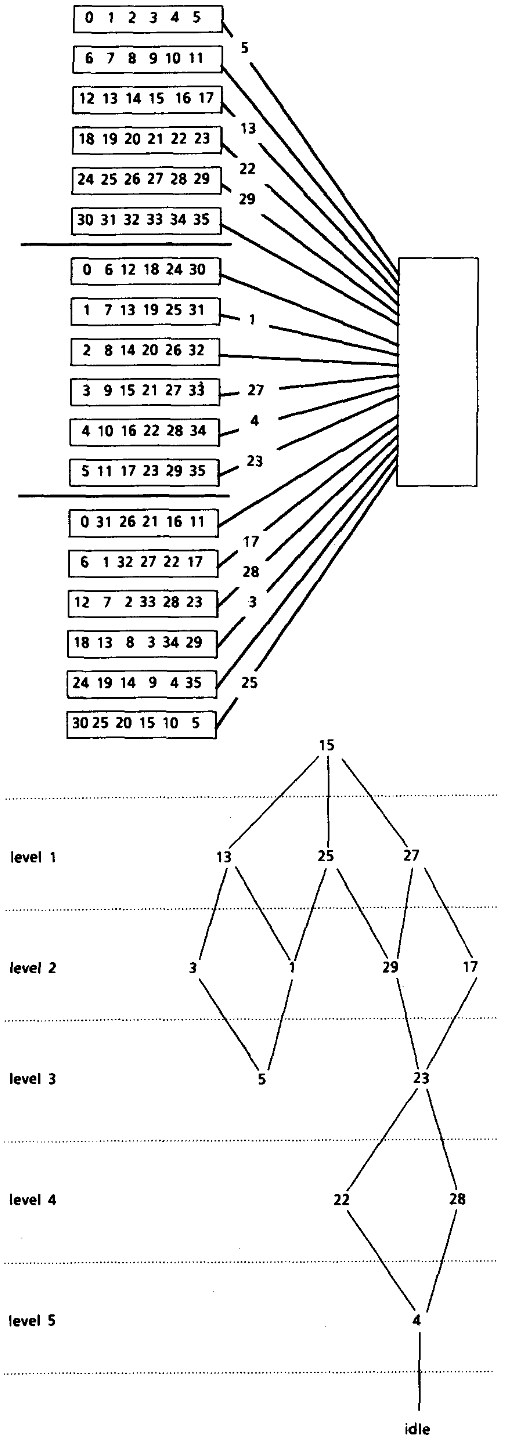

Fig. 5. In this network having a total of 36 inlet channels a request is made for channel 15 and five connections have to be moved.

connections to be moved during a rearrangement will for $M \leq 21$ never exceed the numbers given in Table III.

In Appendix C it is proven that:

Result 4.3: If $n_2 \leq M(M + 1) - 1$, then the number of connections to be moved during a rearrangement will never exceed $3 + \lceil (1/\ln 2)\ln(M^2 \ln M) \rceil$ where $\lceil x \rceil$ is the smallest integer bigger than or equal to x.

Result 4.3 is not the best obtainable but it shows that the number of connections to be moved grow at most logarithmic in M.

V. The Inlet Assignment Pattern and Finite Geometry

In this section, results from the theory of finite geometries is used to examine the inlet assignment pattern.

According to the definition [4, p. 251], a geometric k net is a set of points together with a set of lines appearing in k different parallel classes such that

1) each point belongs to exactly one line of each parallel class
2) if l_1 and l_2 are lines of different parallel classes, then l_1 and l_2 have exactly one point in common
3) there are at least two points on each line.

TABLE II
THE MAXIMUM NUMBER OF OUTLETS AT EACH SECOND STAGE SWITCH AND THE MAXIMUM NUMBER OF CONNECTIONS TO BE MOVED DURING A REARRANGEMENT

	Optimal value for n_2	The maximum number of connections to be moved during a rearrangement
n_1 is a multiple of 4	11	3
n_1 is a multiple of 5 but not 4	12	3
n_1 is not a multiple of 4 or 5	13	5
n_1 is not a multiple of 2, 3 or 5	13	4

TABLE III
AN UPPER BOUND ON THE NUMBER OF CONNECTIONS TO BE MOVED DURING A REARRANGEMENT WHEN $n_2 \leqslant M(M+1)-1$

	$M=2$	$M=3$	$M=4$	$M=5$	$M=6,..,9$	$M=10,..,14$	$M=15,..,21$
the upper bound m	2	3	4	4	5	6	7

Consider a network with n_1, M, and an inlet assignment pattern given. The n_1^2 inlet channels correspond to the set of points. Each of the M blocks corresponds to a parallel class, and each switch in a block corresponds to a line in this parallel class. Since each inlet channel is present exactly once in each block, 1) is fulfilled. The pair condition ensures that 2) is fulfilled, and for $n_1 \geq 2$ the inlet assignment pattern given is a geometric M-net of order n_1.

This connection to k net can be used to find the highest possible value of M before the pair condition is violated. For $n_1 \leq 9$, we have ([4, ch. 8]).

TABLE IV
THE MAXIMUM NUMBER OF TIMES EACH INLET CHANNEL CAN APPEAR IN THE FIRST STAGE BEFORE THE PAIR CONDITION IS VIOLATED

n_1	2	3	4	5	6	7	8	9	p prime p^n
Highest possible value of M	3	4	5	6	3	8	9	10	p^n+1

The function S_{M,n_1} is known in general to depend on the inlet assignment pattern chosen. To be more precise an equivalence relation is introduced. Let P_1 and P_2 be two inlet assignment patterns in the same network, i.e., M and n_1 is fixed. Then,

$$P_1 \sim P_2 \Leftrightarrow S_{M,n_1,P_1} = S_{M,n_1,P_2}.$$

The equivalence relation splits the set of inlet assignment patterns for the network into classes and in order to obtain the best network an inlet pattern that makes S_{M,n_1} as big as possible has to be chosen.

When $M = 2$, there is for any n_1 only one class, and it is therefore impossible to improve the network by using inlet patterns different from the one used in the Fig. 2.

When $M = 3$, the results in Section III prove that S_{3,n_1} depends on n_1 and for $n_1 \geq 5$ there are in general more than one class. From lemma 1 in [3], it can be seen that for n_1 a prime all inlet assignment patterns made from the subarrays given in [1] are contained in only one class.

For a general M and n_1, it seems very difficult to determine the classes. Then it seems more practical to find an useful upper bound U_M on S_{M,n_1}, which can be used to decide whether or not a given inlet pattern makes S_{M,n_1} big enough. In [4] and [5], geometric nets are used to construct projective planes, and it is not unlikely that methods and results there can be helpful in finding an useful upper bound.

VI. CONCLUSIONS

In this paper, an upper bound on the number of connections to be moved during a rearrangement in a two-stage broadcast switching network is found. In general, the bound is given in terms of the function S_{M,n_1}, which means that when the values of S_{M,n_1} are known, then the upper bound is easily calculated.

When $M = 2$ the function S_{M,n_1} is independent of n_1 and of the inlet assignment pattern, and two is the maximum number of connections to be moved during a rearrangement.

When $M = 3$ the function S_{M,n_1} depends on n_1. For a special class of inlet patterns the values of S_{3,n_1} is found, and the optimal choice for n_2 is 11 when n_1 is a multiple of 4, it is 12 when n_1 is a multiple of 5 but not 4, and it is 13 in the other cases. The maximum number of connections to be moved during a rearrangement is 3 when n_1 is a multiple of 4 or 5. When n_1 is not a multiple of 4 or 5, the maximum number of connections to be moved is 5 and when n_1 is a prime it is 4.

In the case where M is arbitrary the pair condition is used to find a lower bound G_M on S_{M,n_1}, and this lower bound yields that the number of connections to be moved during a rearrangement grows, at most, logarithmic as a function of M when the number of outlets at each second-stage switch is not exceeding $M(M + 1) - 1$.

Finally, the close connection between the inlet assignment pattern and finite geometry is considered.

APPENDIX A

THE CALCULATIONS OF S_{3,n_1}

We first find an upper bound U_3 for S_{3,n_1}. Let for $1 \leq n \leq 14$, U_3 be defined by Table V.

Result A.1: Assume $n_1 \geq 4$. For all inlet assignment patterns where the induced latin square is the multiplication table of a group we have $S_{3,n_1}(n) \leq U_3(n)$, $n = 1, 2, \cdots, 13$. $S_{3,n_1}(14) \leq 13$ when the induced latin square

TABLE V
AN UPPER BOUND ON THE FUNCTION S_{3,n_1} INDEPENDENT OF n_1

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14
$U_3(n)$	3	5	6	7	8	9	9	10	11	11	12	12	13	13

corresponds to a group not isomorfic to groups of the form $Z_3 \times \cdots \times Z_3$.

Proof: Assume $1 \leq n \leq 13$. By explicitly selecting the rows, columns, and entries in the latin square the way described in [3, Sect. II], we get the result when the latin square is the multiplication table of Z_{n_1} (the cyclic group of order n_1). For the groups $Z_2 \times Z_2$, $Z_3 \times Z_3$, and Σ_3 (the permutation group of three elements), it is also easy by explicit selection to show that U_3 is upper bound in these cases.

To prove the result in the general case, assume that G is an arbitrary finite group of order n_1. In the rest of the proof, we will use the result from the standard group theory, see for example, [6, chap. I.6]. Let p denote the greatest prime that divides n_1. Then G has an element of order p and, therefore, a cyclic subgroup of order p. When $p \geq 5$, we get the result by choosing the rows, columns, and entries among the elements of this subgroup. If n_1 is a power of 2, G is a two-group and therefore it has a subgroup of order 4, which is either Z_4 or $Z_2 \times Z_2$. Conclude by choosing rows, columns, and entries among this subgroup. When n_1 is a power of 3, G is a three-group and therefore it contains a subgroup of order 9 (either Z_9 or $Z_3 \times Z_3$). Finally, if $n_1 = 2^k 3^l$, then G has a subgroup of order 2^k and a subgroup of order 3^l leaving only $n_1 = 6$ as unsolved. But Z_6 and Σ_3 are the only groups of order 6 and U_3 is known to be an upper bound in these cases.

For $n = 14$ and the latin square equal to $Z_3 \times Z_3$, the author has only been able to select 14 entries in a way that proves $S_{3,n_1,Z_3\times Z_3}(14) \leq 14$. When the latin square corresponds to a group of the form $Z_3 \times \cdots \times Z_3$, it is therefore in doubt whether $S_{3,n_1}(14) = 13$ or 14.

Result 4.1 in the case $M = 3$ and Result A.1 determine $S_{3,n_1}(n)$ when $n = 1, 2, 3, 5, 7, 10$. Since the calculation of S_{3,n_1} proceeds the same way for $n = 4, 6, 8, 9, 11, 12, 13, 14$, we only consider the two most difficult cases $n = 11$ and $n = 13$ here. We start with the following.

Lemma A.1: Let G be a finite group of order at least 4. Let 1, a, b, c be four distinct elements of G where 1 denotes the identity element. Assume that $a^2 = 1$. *Consider the multiplication table for* $\{1, a, b, c\}$. *Put for* $j = 1, 2, 3, 4$,

$C_j :=$ the set of elements in column j of the multiplication table

$$\begin{array}{c|cccc} & 1 & a & b & c \\ \hline 1 & 1 & a & b & c \\ a & a & 1 & ab & ac \\ b & b & ba & b^2 & bc \\ c & c & ca & cb & c^2 \end{array}.$$

If $|C_1 \cap C_j| \geq 3$ for $j = 2, 3$ or $j = 2, 4$, then G contains a subgroup of order 4.

Proof: Since $|C_1 \cap C_2| \geq 3$, we must have $ba = c$ or $ca = b$ since no element appears more than once in a row or a column. Since $a^2 = 1$ this implies that $ba = c$ and $ca = b$. The multiplication table then looks as follows:

$$\begin{array}{c|cccc} & 1 & a & b & c \\ \hline 1 & 1 & a & b & c \\ a & a & 1 & - & - \\ b & b & c & - & - \\ c & c & b & - & - \end{array}$$

We only prove the lemma in the case where $|C_1 \cap C_j| \geq 3$ for $j = 2, 3$, Then there are three possibilities to consider: 1) ab is the unknown element, i.e., we do not know whether or not $ab \in \{1, a, b, c\}$, 2) b^2 is the unknown, or 3) cb is the unknown element.

Assume that ab is the unknown element: Then $b^2 = a$ or $b^2 = 1$. If $b^2 = a$ then b has order 4 and the subgroup generated by b is a subgroup of order 4. If $b^2 = 1$ we have $cb = a$ implying $c = cb^2 = ab$. But then $a^2 = 1$, $b^2 = 1$, $c = ab = ba$ and therefore $\{1, a, b, c\}$ is the Klein Four Group ($Z_2 \times Z_2$).

Assume that b^2 is the unknown element; then $ab = c$. Multiplying by a from left gives $b = ac$. Now $cb = 1$ or $cb = a$. If $cb = 1$ then $a = acb = b^2$ implying that b has order four. If $cb = a$ then $1 = a^2 = acb = b^2$ and then $ab = ba = c$ implying that $\{1, a, b, c\}$ is the Klein Four Group.

Assume that cb is the unknown element; then $ab = c$ and therefore $b^2 = 1$ or $b^2 = a$. If $b^2 = 1$, $\{1, a, b, c\}$ is the Klein Four Group and if $b^2 = a$, b has order 4. Since we have now covered all cases the proof of lemma A.1 is completed.

We now proceed with the calculation of S_{3,n_1}. The fact that no element appears more than once in a row (column) will be used without comment.

n = 11: Assume $S_{3,n_1}(11) = 11$. If the 11 inlet channels appeared on only two switches in one of the blocks then they would have to appear on at least six switches in each of the remaining two blocks and the 11 inlet channels would then appear on at least 14 switches. Because of the symmetry in rows and columns, we only have to consider the following two cases: 1) when the 11 inlet channels appear on four switches in block 1, four switches in block 2, and three switches in block 3; 2) when the 11 channels appear on four switches in block 1, three switches in block 2, and four switches in block 3.

Case 1): In the language of latin squares we have 11 entries containing only three different elements (x_1, x_2, x_3) and appearing in four rows and four columns. The four rows (columns) correspond to four group elements a_1, a_2, a_3, a_4 (b_1, b_2, b_3, b_4). Since it is 11 entries containing only x_1, x_2, x_3, three of the rows and three of the columns contain all the elements x_1, x_2, x_3 and the remaining row and column contains two of these three elements. After possible renaming of the group elements we may assume that we have the following table:

$$\begin{array}{c|cccc} & b_1 & b_2 & b_3 & b_4 \\ \hline a_1 & x_1 & x_2 & x_3 & - \\ a_2 & x_2 & - & - & - \\ a_3 & x_3 & - & - & - \\ a_4 & - & - & - & - \end{array}$$

where row 1, 2, and 3 and column 1, 2, and 3 or 4, contain x_1, x_2, x_3 while row 4 and column 4 or 3 contain two of these three elements. Multiply the row elements by a_1^{-1} from the left and the column elements by b_1^{-1} from the right and obtain

$$\begin{array}{c|cccc} & 1 & a & b & d \\ \hline 1 & 1 & a & b & d \\ a & a & - & - & - \\ b & b & - & - & - \\ c & c & - & - & - \end{array}$$

where $a := a_1^{-1}a_2 = b_2b_1^{-1}$, $b := a_1^{-1}a_3 = b_3b_1^{-1}$, $c := a_1^{-1}a_4$, and $d := b_4b_1^{-1}$ and where it is row 4 and column 3 or 4 that have only two of the elements 1, a, b.

Assume $a^2 \notin \{1, a, b\}$. Since row 2 and column 2 both contains 1, a, b, we have $ba = 1$, $ca = b$, and $ab = 1$ and $ad = b$. This implies $b = a^{-1}$ and $c = d = a^{-2}$. Since 1, a, b, c are distinct a has order at least four. Use that row 3 has all three elements 1, a, a^{-1} to conclude that a has order 4 and, therefore, 4 divides n_1.

Assume $ba \notin \{1, a, b\}$; then $a^2 = 1$ or $a^2 = b$. If $a^2 = 1$ then $ca = b$ and therefore $c = ba$. Since row 2 contains 1, a, b, we get $ad = b$ and therefore $d = ab$. The same argument is applied to row 3 gives $b^2 = 1$ and $bd = a$ or $b^2 = a$ and $bd = 1$. If $b^2 = a$ then b is an element of order 4. If $b^2 = 1$ then $d = b^2d = b(bd) = ba = c$ and then $\{1, a, b, c\} = \{1, a, b, d\}$ is the Klein Four Group.

If $a^2 = b$ then $ba = 1$ or $ca = 1$. $ba = 1$ implies $a^3 = 1$ and $b^2 = a$ and makes it impossible for row 4 and column 4 to contain more than one of the elements 1, a, b. Therefore, $ca = 1$ and in row 2, we get $ad = 1$. Then $c = a^{-1} = d$ and $\{1, a, b, c\} = \{1, a, b, d\} = \{1, a, a^2, a^{-1}\}$ and it follows as before that a has order 4.

Assume $ca \notin \{1, a, b\}$. Look at column 2 and conclude that $ba = 1$ and therefore $a^2 = b$. Row 3 implies that $b^2 = a$ or $bd = a$. $b^2 = a$ contradicts the assumption that row 4 and column 4 contains two of the elements 1, a, b. Therefore, $bd = a$. Since $1 = ba = ab$, it must be column 3 that contains all the elements 1, a, b and by looking at row 3, we get that $cb = a$ which implies $c = a^2$. Multiply $bd = a$ by a from the left and get $d = a^2 = c$. We now have the elements 1, a, a^2, a^{-1} and as before, we conclude that a has order 4.

Case 2): Assume that we have 11 inlet channels appearing on four switches in block 1, on three switches in block 2, and on four switches in block 3. In the induced latin square, we can find 11 entries appearing in four rows and four columns so that the 11 entries contains only four different elements. By appropriate multiplication as in the former case, we may assume that the row elements are 1, a, b, c and the column elements are 1, a, b. Since only four different elements are present in the 11 entries, we have $C_1 = C_2$ or $C_1 = C_3$. and $|C_1 \cap C_j| \geq 3$ for $j = 2, 3$. We assume that $C_1 = C_2$. If $a^2 = 1$, we conclude by lemma A.1 that 4 divides n_1. If $a^2 = b$, then $ba = 1$ or $ba = c$. Since $ba = 1$ forces $ca = c$, we conclude that $ba = c$. Then $ca = 1$ yielding $1 = ba^2 = a^4$ and $c = a^3$. Therefore, a is an element of order 4 and 4 divides n_1. When $a^2 = c$ the same argument yields that a has order 4.

$n = 13$: Assume $S_{3,n_1}(13) = 12$. If the 13 inlet channels appeared on only three switches in one of the blocks this would force the inlet channels to appear on at least five switches in each of the remaining blocks. We may, therefore, assume that in the induced latin square, there exist 13 entries containing only four different elements x_1, x_2, x_3, x_4 and appearing in four rows and four columns. There exists at least one row and one column each containing all the elements x_1, x_2, x_3, x_4. By appropriate multiplication as in Case 1) of $n = 11$, we may assume that the four rows and the four columns correspond to the elements 1, a, b, c. If there exist two columns both containing these four elements, we can proceed exactly as in Case 2) of $n = 11$ and conclude that 4 divides n_1. We may, therefore, assume that $|C_1 \cap C_j| = 3$ for $j = 2, 3, 4$. From this we see that if $a^2 = 1$ then by lemma A.1, we conclude that 4 divides n_1.

$$\begin{array}{c|cccc} & 1 & a & b & c \\ \hline 1 & 1 & a & b & c \\ a & a & 1 & - & - \\ b & b & - & - & - \\ c & c & - & - & - \end{array}.$$

Assume that a^2 is unknown, i.e., we do not know whether or not $a^2 \in C_1$. Then $ba = 1$ or $ba = c$. If $ba = 1$ then $ca = b$. Then we see $b = a^{-1}$ and $c = a^{-2}$. Therefore, we only have to consider the multiplication table of 1, a, a^{-1}, a^{-2}. Since $|C_1 \cap C_4| = 3$, we get $a^{-3} \in \{1, a\}$ or $a^{-4} \in \{1, a\}$ yielding a has order 4 or 5. The conditions $|C_1 \cap C_j| = 3$ for $j = 2, 4$ give no further information.

If $ba = c$ then $ca = b$ or $ca = 1$. $ca = b$ implies $ca^2 = c$ implying $a^2 = 1$ and by lemma A.1, we conclude that 4 divides n_1. $ca = 1$ implies $c = a^{-1}$ and $b^2a = 1$ and we have once more the multiplication table of 1, a, a^{-1}, a^{-2} from which we conclude that 4 or 5 divides n_1.

Assume that ab is unknown. Then $a^2 = 1$ or $a^2 = c$. If $a^2 = 1$ lemma A.1 yields that 4 divides n_1. If $a^2 = c$ then $ca = 1$ or $ca = b$. $ca = 1$ implies $a^3 = 1$ and by using $|C_1 \cap C_3| = 3$ this implies $b = c$ or column 3 contains the same element twice; in both cases a contradiction. Therefore, $ca = b$ implying $b = a^3$, i.e., $\{1, a, b, c\} = \{1, a, a^2, a^3\}$. As in the case a^2 unknown, this yields 4 divides n_1 or 5 divides n_1.

Since ac unknown proceeds the same way and gives the same result, it will be omitted. The calculation of $S_{3,n_1}(13)$ is therefore completed.

Appendix B

A Lower Estimate for S_{M,n_1}

In this Appendix, a proof of Result 4.1 is given. The M blocks in the first stage is denoted by $B_1, B_2, \cdots, B_M$ and the Mn_1 first-stage switches are denoted $I_1, \cdots, I_{Mn_1}$.

Let E be of subset of the set of inlet channels, assume that E has n elements and put

$k_i(E) :=$ The number of switches in B_i having elements from E among their inlet channels.

$$k(E) := \min \{k_i(E) | i = 1, 2, \cdots, M\}.$$

$a_j(E) :=$ The number of elements from E appearing on I_j.

$a(E) := \max \{a_j(E) | j = 1, 2, \cdots, Mn_1\}$.

If x is a real number $\lceil x \rceil$ denoted the smallest integer not less than x. In this notation, a switch exist, which has at least $\lceil n/k(E) \rceil$ elements from E among its inlet channels. Therefore,

$$a(E) \geq \lceil n/k(E) \rceil. \tag{B.1}$$

The definition of $k(E)$ and T_{M,n_1} (2.1) ensures

$$T_{M,n_1}(E) \geq k(E)M. \tag{B.2}$$

The pair condition gives

$$T_{M,n_1}(E) \geq a(E)(M-1) + k(E). \tag{B.3}$$

From 2.2, B.1, B.2, and B.3, we get

$$S_{M,n_1}(n) \geq \min \left\{ \max \left[k(E)M, \left\lceil \frac{n}{k(E)} \right\rceil \cdot (M-1) + k(E) \right] \middle| E \text{ has } n \text{ elements} \right\}. \tag{B.4}$$

When E runs through all subsets with n elements, $k(E)$ runs through a subset of $\{1, 2, \cdots n_1\}$ yielding

$$S_{M,n_1}(n) \geq \min \left\{ \max \left[kM, \left\lceil \frac{n}{k} \right\rceil \cdot (M-1) + k \right] \middle| k = 1, 2, \cdots, n_1 \right\}. \tag{B.5}$$

Put $f_1(k) := kM$ and $f_{2,n}(k) := \lceil n/k \rceil (M-1) + k$. To finish the proof of Result 4.1, it is enough to show that

$$\min_k \{\max [f_1(k), f_{2,n}(k)]\} = G_M(n) \quad \text{for any } n \leq n_1^2.$$

Choose p as the integer having the property

$$n = p^2 + x \quad \text{where } 1 \leq x \leq 2p + 1. \tag{B.6}$$

First assume that $1 \le x \le p$. Then for $k = 0, 1, \cdots, p-1$, we have

$$
\begin{aligned}
f_{2,n}(p-k) &= \left\lceil \frac{p^2+x}{p-k} \right\rceil (M-1)+p-k \\
&\ge (p+k+1)(M-1)+p-k \\
&= (p+1)M+k(M-2)-1 \\
&\ge (p+1)M-1.
\end{aligned}
$$

Since $f_{2,n}(p) = (p+1)M - 1$ and $f_1(k) \ge (p+1)M$ for $k \ge p+1$, it is now proved that

$$
\min_k \{\max [f_1(k), f_{2,n}(k)]\} = (p+1)M-1
$$

$$
\text{when } n=p^2+x \text{ and } 1 \le x \le p. \tag{B.7}
$$

Assume now that $p+1 \le x \le 2p+1$. Then for $k = 0, 1, \cdots, p-1$ we have

$$
\begin{aligned}
&f_{2,n}(p-k) \\
&= \left\lceil \frac{(p-k)(p+k+1)+k^2+k+x-p}{p-k} \right\rceil (M-1)+p-k \\
&\ge (p+k+2)(M-1)+p-k \\
&= (p+1)M+(k+1)(M-2) \\
&\ge (p+1)M.
\end{aligned}
$$

Since $f_{2,n}(p+1) = (p+1)M$ and $f_1(k) \ge (p+1)M$ for $k \ge p+1$ it is proved that

$$
\min_k \{\max [f_1(k), f_{2,n}(k)]\} = (p+1)M
$$

$$
\text{when } n=p^2+x \text{ and } p+1 \le x \le 2p+1. \tag{B.8}
$$

B.5, B.7, and B.8 concludes the proof of Result 4.1.

Appendix C

Proof of Result 4.3

In this Appendix, we prove Result 4.3. It is enough to show if $k \ge 3 + 1/(ln2ln)(M^2 lnM)$ then $g_k \ge M^2$. The following lemma will be helpful.

Lemma C.1: For $k \ge 2$ and $M \ge 2$ the following estimate is valid:

$$
g_k \ge M^{2-2^{1-k}} \left[1+\left(\sum_{i=0}^{k-2} 3^i \right) 3^{1-k} M^{-2} \right]. \tag{C.1}
$$

Proof: We have $g_2 = G_M(M+1) \ge M(M+1)^{1/2} \ge M^{3/2}(1 + 1/3M) \ge M^{3/2}(1 + 1/3M^2)$. This proves the lemma for $k = 2$. Assume the lemma is true for some k. Then,

$$
\begin{aligned}
g_{k+1} &= G_M(g_k+1) \ge M(g_k+1)^{1/2} \\
&\ge M\left(M^{2-2^{1-k}} \left(1+\left(\sum_{i=0}^{k-2} 3^i \right) \cdot 3^{1-k} M^{-2} \right) +1 \right)^{1/2} \\
&= M^{2-2^{k}} \left(1+\left(\sum_{i=0}^{k-2} 3^i \right) \cdot 3^{1-k} M^{-2} + M^{2^{1/2-k}-2} \right) \\
&\ge M^{2-2^{-k}} \left(1+\left(\sum_{i=0}^{k-2} 3^i \right) \cdot 3^{-k} M^{-2} + \frac{1}{3} M^{-2} \right) \\
&= M^{2-2^{1-(k+1)}} \left(1+\left(\sum_{i=0}^{k-1} 3^i \right) \cdot 3^{1-(k+1)} M^{-2} \right)
\end{aligned}
$$

which concludes the proof of the lemma.

Now assume $k \ge 3 + (1/ln2)ln(M^2 lnM)$. By taken exp, we obtain

$$
2^{k-3} \ge M^2 lnM.
$$

Since $k \ge 2$, we have $(3^{k-1}-1)/3^{k-1} \ge 2/3$ yielding

$$
2^{k-3} \frac{3}{2} \frac{3^{k-1}-1}{3^{k-1}} \ge M^2 lnM \Leftrightarrow \frac{3}{4} \frac{3^{k-1}-1}{2 \cdot 3^{k-1}} M^{-2} \ge 2^{1-k} lnM.
$$

Since $ln(1+x) \ge (3/4)x$ for x small, this implies

$$
ln \left[1+\frac{3^{k-1}-1}{2 \cdot 3^{k-1}} M^{-2} \right] \ge 2^{1-k} lnM.
$$

Take exp and multiply by $M^{2-2^{1-k}}$ and obtain

$$
M^{2-2^{1-k}} \left(1+\frac{3^{k-1}-1}{2 \cdot 3^{k-1}} M^{-2} \right) \ge M^2.
$$

But,

$$
\begin{aligned}
&M^{2-2^{1-k}} \left(1+\frac{3^{k-1}-1}{2 \cdot 3^{k-1}} M^{-2} \right) \\
&\quad = M^{2-2^{1-k}} \left(1+\left(\sum_{i=0}^{k-2} 3^i \right) 3^{1-k} M^{-2} \right) \\
&\quad = g_k.
\end{aligned}
$$

and the proof is completed.

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