Throughput and Delay Analysis of

Free Access Tree Algorithm with Mini-Slots

Technical Report 87-26

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November 1986

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Abstract

This paper analyzes throughput and delay performance of two kinds of free access tree algorithms with mini-slots. One is that binary feedback information is available in mini-slots, and the other is that ternary feedback information is available. In these algorithms Q number of mini-slots are provided within a slot to offer feedback information on the state of the channel. The maximum throughputs of binary and ternary feedback algorithms are analytically obtained. It is also shown that the highest maximum throughput 0.56714 is achieved when Q approaches infinity and mini-slot overhead goes to zero. The lower bound of the average transmission delays in these algorithms is analytically derived. The obtained lower bound is also a lower bound of the average delay of the whole class of free access algorithms.

[Key Words] tree algorithms, collision resolution algorithm, performance evaluation, throughput, average transmission delay

1. Introduction

In a multiple-access environment, where a number of geographically separated users communicate over a single shared communication channel, contention-based protocols provide a relatively efficient way of communication [KURO 84]. Contention-based access protocols are characterized by collisions and retransmissions. These protocols simultaneously offer transmission rights to a group of users in the hope that exactly one of the users has a packet to send. If, however, two or more users send packets on the channel at the same time, these messages interfere with each other and none of them will be correctly received by the destination user(s). In such cases, users retransmit packets according to a collision resolution algorithm until packets are successfully received by the destination user(s).

Among a variety of collision resolution algorithms that have been investigated, a class of tree algorithms [CAPE 79, TSYB 78] is one with the outstanding property. Tree algorithms broadly divide into two classes depending on how new packets are handled: free access [TSYB 80b, FAYO 85, MATH 85] and blocked access [TSYB 78, CAPE 79, MATH 85] tree algorithms.

Free Access (FA) tree algorithms do not distinguish between new and collided packets. Users attempt to transmit a new packet immediately after its generation. Thus, users are not required to monitor the channel continuously; They sense the channel only when they have packets to transmit (i.e, limited sensing [HUMB 86]). Due to the simplicity and ease of implementation, FA tree algorithms are of practical interest. The upper bound of the maximum throughput in the context of free access has recently been shown to be 0.567 [HUMB 86]. This is a tight bound if collided packets are retransmitted in some fashion to avoid a collision with other previously collided packets. However, no specific algorithm has been found yet to achieve the maximum throughput close to this bound.

On the other hand, blocked access (BA) tree algorithms [GALL 78, MASS 80, MOSE 85] force new packets to wait until all the outstanding collisions have been resolved, and thus require users to monitor the channel continuously. Mosely and Humblet [MOSE 85] have shown that the maximum throughput of a BA tree algorithm is 0.48776, assuming that the users distinguish between an empty, a successful and a collided slots. If users can further detect the multiplicity of a collision, i.e., the number of packets involved in a collision, the maximum throughput increases to 0.53237 [GEOR 83, TSYB 80a].

To improve the performance of a BA tree algorithm, mini-slots were introduced to provide better feedback on the channel status with network users; so called BA tree algorithms with mini-slots (BA Q-ary TA/M). In this class of algorithms, Q number of mini-slots are provided within a (large) slot to allow users to acquire additional information on the state of the channel (see Fig.1). Data sub-slot length is equal to packet transmission time. When a user sends a packet (using a data sub-slot in a large slot), he also sends a signal in a mini-slot randomly chosen. In case of a collision, the current enabled set of users (the set of users who currently have transmission right) are divided into Q number of subtrees, each corresponding to a group of users who have chosen the same mini-slot. BA Q-ary TA/Ms have been investigated under the assumption that binary (i.e., something/nothing) [SZPA 85, HUAN 85] or ternary (idle/ success/collision) [MERA 83] feedback information is available in a mini-slot, and they have been shown to provide excellent performance.

In this paper, we introduce mini-slots into a free access (FA) tree algorithm in order to improve its performance(+). We consider two types of feedback information in a mini-slot; binary (something/nothing) and ternary (idle/success/collision) feedback information. Maximum throughput and the average transmission delay are analyzed for FA TA/M.

The exact description of the algorithm is presented in section 2. Section 3 analyzes maximum throughput of the algorithm. The upper bound on the maximum throughput in the whole class of free access algorithms (including tree type algorithms and others) is also obtained as the asymptotic case where Q approaches infinity. In section 4, the lower bound on the average transmission delay of FA TA/M is analytically obtained. This lower bound is also a lower bound on the average transmission delay in the whole class of free access algorithms. In section 5, numerical examples are provided as well as the optimal value of Q (the number of mini-slots in a large slot) to achieve the highest throughput for a fixed value of a mini-slot length.

(+) Combining mini-slots with reservation scheme is another (and completely different) approach to improve the throughput performance. However, in this paper, we are interested in an FA TA/M coupled with direct channel access scheme (i.e., to send a packet directly in a data sub-slot and to use mini-slots to resolve a collision) because of its simplicity and of its practical importance. The readers may refer to [HUAN 85, VI A pp.268] for further comparison of TA/M with reservation systems.

2. Free Access Q-ary Tree Algorithms with Mini-Slots

In our analysis, we assume that the time is slotted, and a (large) slot consists of Q number of mini-slots and a data sub-slot (Fig.1). The size of a data sub-slot is equal to transmission time of a packet.

We consider a Free Access Q-ary Tree Algorithm with Mini-slots (FA TA/M). In free access algorithms, newly generated packets, regardless of if they arrive while collisions are being resolved, are transmitted immediately after their generation. We study two cases regarding the feedback information available in a mini-slot; ternary and binary feedback information.

FA TA/M with Ternary Feedback (FA TA/M-TF);

Users distinguish between an empty mini-slot (i.e, no user is sending a signal), a successful mini-slot (i.e., only one user is sending a signal) and a collided mini-slot (i.e., more than one user is sending a signal); this is often referred to as 0, 1, e-ternary feedback [BERG 84].

FA TA/M with Binary Feedback (FA TA/M-BF);

Users have limited capability to detect the signal level of a mini-slot so that users can only distinguish between an empty mini-slot (i.e., no signal detected) and a busy mini-slot (i.e., signal detected); namely, something/nothing binary feedback [BERG 84].

In Q-ary TA/M with ternary feedback, a user sends a packet (in a data sub-slot of a large slot), he also sends a signal in a mini-slot randomly chosen. In case of collision, the current enabled set of users are first partitioned into Q number of sub-sets, each corresponding to a group of users who have chosen the same mini-slot. Non-active sub-sets (i.e., sub-sets which correspond to an empty mini-slot) are deleted from the further collision resolution process. Each active sub-set with only one active user (i.e., a sub-set which corresponds to a successful mini-slot) constitutes a new subtree. Each active sub-set with more than one active user (i.e., a sub-set which corresponds to a collided mini-slot) is further divided into m subtrees, where each active user is randomly assigned to one of the m new subtrees. (This algorithm will be referred to as FA TA/M-TF(m) in the following.) Thus, this division process results in $s+m\times c$ number of new subtrees, where s and c are the numbers of successful and collided mini-slots in a large slot, respectively. One subtree is chosen from these $s+m\times c$ new subtrees for collision resolution in the subsequent (large) slot. If further collision occurs, the enabled set is continuously divided in the same manner until the collision is resolved. Note that, since we assume a free access algorithm, newly generated packets are immediately transmitted even when the system is in a collision resolution process.

In a Q-ary TA/M with binary feedback, users do not distinguish between a successful and a collided mini-slots, and hence, all the active sub-sets, regardless of how many active users there are in each of them, are treated in the same way; each active sub-set is randomly divided into m subtrees. (This algorithm will be referred to as FA TA/M-BF(m).) Thus, this division process results in $(s+c)\times m$ number of new subtrees.

Figures 2 and 3 illustrate a collision resolution process in an FA TA/M-TF(2) and an FA TA/M-BF(1), respectively. In both figures, four users (A, B, C, D) collided in slot 1. Users A and B have chosen the first mini-slot to send signal, and C and D have chosen the third mini-slot, resulting in no successful mini-slots and two collided mini-slots; i.e., s=0 and c=2. In the TA/M-TF(2) (Fig.2), the users involved in the initial collision are partitioned into 4 (i.e., $s+c\times m=4$) enabled subtrees. Note that non-active sub-set (corresponding to the 2nd minislot) has been removed from the collision resolution process. Let us assume these subtrees are; a subtree rooted at node 2 with no active users, a subtree rooted at node 3 with users A and B, a subtree at node 6 with user C, and a subtree at node 9 with user D. Since subtree 2 has no active users, the slot (slot 2) assigned to this subtree remains unused. Subtree 3 results in collision again. User C in subtree 6 collides with a new packet at user E. (Note we have assumed free access algorithm.) Subtrees 3 and 6 are again divided for further collision resolution. Subtree 9 results in a successful transmission.

TA/M-BF(1), upon detecting the initial collision, partitiones the users into two (i.e., $(s+c)\times 1=2$) subtrees, i.e., a subtree rooted at node 2 (with active users A and B) and a subtree rooted at node 6 (with active users C and D) (see Fig.3). Both subtrees 2 and 6 lead to collision again. The same collision resolution process is repeated until all the outstanding collided packets are removed from the system.

3. Analysis of Throughput Characteristics

3.1 Maximum Throughput of Q-ary FA TA/Ms

We analyze Q-ary FA TA/M-BF(m) and Q-ary FA TA/M-TF(m) to obtain their throughput characteristic (in this section) and the average transmission delay (in section 4). In this analysis, we assume that unit time is equal to a large slot and that new packet arrivals in different (large) slots are independent and follow an identical Poisson process with rate λ (packets/large slot). We further assume that the propagation delay is zero. Let h denote the ratio of the length of a mini-slot to a data sub-slot length. Since overhead due to mini-slots is hQ, the maximum throughput (per data sub-slot) $\bar{S}(Q,m)$ becomes

$$\bar{S}(Q,m) = \frac{S(Q,m)}{1+hQ}, \tag{1}$$

where S(Q,m) is the maximum throughput when there is no mini-slot overhead (i.e., when h is equal to zero). In the following, we will obtain S(Q,m).

We first consider the collision resolution time (CRT), the time required to resolve a collision, given that k number of packets are involved in the collision. Let M_k be a conditional average CRT for a given collision multiplicity k. Namely,

$$M_k = \sum_{i \geq 1} i \text{ Prob[CRT=i|Collision Multiplicity=k]}.$$

If k is zero (i.e., idle slot) or one (i.e., successful slot), only one slot is used, and hence, M_k becomes one. If k is greater than or equal to two, collision arises. In this case, collided packets are divided into Q sub-sets. A CRT to resolve the initial collision is the sum of the CRTs of these Q sub-sets. Let n_i be the random variable representing the number of packets in sub-set i. Then,

$$P[n_1=N_1,\ldots;n_i=N_i,\ldots,n_Q=N_Q]=\frac{k!}{N_1!\ldots N_Q!}\left(\frac{1}{Q}\right)^k$$

and $\Sigma_{i=1}^{Q} N_i=k$. We denote the above density function as $U(\frac{1}{Q},k)$.

In both binary feedback (FA TA/M-BF(m)) and ternary feedback (FA TA/M-TF(m)) cases, an empty sub-set i ($n_i=0$) among the Q sub-sets yields no new subtree, and hence, CRT for an empty sub-set is 0.

In the ternary feedback case (FA TA/M-TF(m)), sub-set i containing only one packet ($n_i=1$) results in one subtree. Let X_i^j be the number of new arrivals in the slot immediately preceding to the beginning of the collision resolution process of the j^{th} subtree (resulting from the sub-set i). In the particular case where $n_i=1$, there is only one subtree generated, and hence, j assumes only 1. Note that we have assumed that new arrivals are independent of slots, and hence, X_i^j is independent of i and j. We have further assumed that $X_i^j=X$ follows Poisson distribution, namely,

$$p(h) = \frac{\lambda^h}{h!}e^{-\lambda}.$$

Since new packets, in addition to the packet assigned to the subtree, are immediately transmitted in an FA algorithm, CRT for the original sub-set i is the time required to resolve a collision of multiplicity $1+X_i^1$. Namely, CRT in this case becomes $M_{1+X_i^1}$.

On the other hand, in the binary feedback case (FA TA/M-BF(m)), a sub-set i containing only one packet, say test packet, results in m subtrees. Of these m subtrees, CRT of the one (say the first subtree) to which the test packet has been assigned becomes $M_{1+X_i^1}$; for each of the remaining subtrees (h^{th} subtree), CRT becomes $M_{X_i^h}$. Hence, CRT for the sub-set i ($n_i=1$) becomes $M_{1+X_i^1} + \Sigma_{h=2}^{m} M_{X_i^h}$.

In both FA TA/M-TF(m) and FA TA/M-BF(m), a sub-set i containing more than one packet ($n_i\geq 2$) generates m subtrees. In this case, n_i

number of packets are randomly spread over the m subtrees. If we let r_i^j be the number of packets assigned to the j^{th} subtree (of these m subtrees), CRT of the j^{th} subtree becomes $M_{r_i^j+X_i^j}$, where X_i^j is defined above. Hence, CRT for the original sub-set i becomes $\Sigma_{j=1}^{m} M_{r_i^j+X_i^j}$. Note that $\Sigma_{j=1}^{m} r_i^j=n_i$ and that the density function of r_i^j is given by $U(\frac{1}{m},n_i)$.

From the above arguments, M_k becomes

$$M_k \begin{cases} = 1 & (k=0,\ 1) \\ = 1 + \sum\limits_{\{i|n_i=0\}} 0 + \sum\limits_{\{i|n_i=1\}} \{M_{1+X_i^1} + \delta_B \sum\limits_{h=2}^{m} M_{X_i^h}\} & \\ \qquad + \sum\limits_{\{i|n_i\geq 2\}} \sum\limits_{j=1}^{m} M_{r_i^j+X_i^j} & \text{(otherwise)}, \end{cases}$$

In the above equation, $\sum\limits_{\{i|A\}}$ represents the sum over i-s satisfying the condition A. δ_B depends on the algorithm and is

$$\delta_B = \begin{cases} 1 & \text{(TA/M-BF(m))} \\ 0 & \text{(TA/M-TF(m))}. \end{cases}$$

Thus, M_k satisfies the following equation:

$$M_k=1+\sum_{n_1+\ldots+n_Q=k} U(\tfrac{1}{Q},k) \sum_{i=1}^{Q} v_{n_i}, \qquad (k\geq 2) \tag{2}$$

where $\sum\limits_{n_1+\ldots+n_Q=k}$ represents the sum over all possible combinations of $n_1,\ldots,n_Q$ such that $n_1+\ldots+n_Q=k$, and v_{n_i} is given by

$$v_{n_i} \begin{cases} = 0 & (n_i=0) \\ = \sum\limits_{h\geq 0} M_{1+h}p(h) + \delta_B(m-1) \sum\limits_{h\geq 0} M_h p(h) & (n_i=1) \\ = \sum\limits_{r_i^1+\ldots+r_i^m=n_i} U(\frac{1}{m},n_i) \sum\limits_{h\geq 0} \sum\limits_{j=1}^{m} M_{r_i^j+h}p(h) & (n_i\geq 2). \end{cases} \tag{3}$$

To obtain Eq.(3), we have used that X_i^j is independent of i and j and follows Poisson distribution and that r_i^j has the distribution $U(\frac{1}{m},n_i)$.

Equation (2) is rewritten as (see Appendix A of [HUAN 85] for the detail)

$$M_k=1+\sum_{n=0}^{k}\binom{k}{n}(1-\frac{1}{Q})^{k-n}(\frac{1}{Q})^{n-1}v_n. \qquad (k\geq 2)$$

Furthermore, substituting Eq.(3) into the above equation, we have

$$M_k= 1+k(1-\frac{1}{Q})^{k-1}[\sum_{j\geq 0} M_{1+j}p(j) + \delta_B k(m-1)\sum_{j\geq 0} M_j p(j)]$$

$$+\sum_{n=2}^{k}\binom{k}{n}(1-\frac{1}{Q})^{k-n}(\frac{1}{Q})^{n-1}\sum_{j\geq 0} p(j)\sum_{i=0}^{n}\binom{n}{i}(1-\frac{1}{m})^{n-i}(\frac{1}{m})^{i-1}M_{i+j} \qquad (k\geq 2), \qquad (4)$$

$M_0=M_1=1.$

After some manipulation (see Appendix-A), Eq.(4) becomes

$$M_k=\begin{cases} 1+\sum_{j\geq 0}\sum_{i=0}^{k}\binom{k}{i}a^{i-1}(1-a)^{k-i}M_{i+j}p(j) \\ -\{a^{-1}(1-\frac{1}{Q})^{k}+(1-\delta_B)k(m-1)(1-\frac{1}{Q})^{k-1}\}\sum_{j\geq 0} M_j p(j) & (k\geq 2) \\ 1 & (k=0,1) \end{cases} \qquad (5)$$

where $a=(Qm)^{-1}$.

We now define the following functions:

$$M(z)=\sum_{k\geq 0} M_k \frac{z^k}{k!}, \qquad (6)$$

$$M^*(z)=e^{-z}M(z). \qquad (7)$$

Taking the derivative of these functions, we have

$$M^{(1)}(z)=\sum_{k\geq 0} M_{k+1}\frac{z^k}{k!}, \qquad (8)$$

$$M^{*(1)}(z)=e^{-z}(M^{(1)}(z)-M(z)).$$

Note that $M^*(\lambda)=\sum_{k\geq 0} M_k \frac{\lambda^k}{k!}e^{-\lambda}$ represents the average CRT under the assumption of a Poisson arrival process. Since this quantity plays a key role in the analysis, we will obtain $M^*(\lambda)$ by applying the method proposed by Mathys et al. [MATH 85].

By multiplying both sides of Eq.(5) by $z^k/k!$ and then taking sum of both sides over $k\geq 0$ (see Appendix-B), we have

$$M^*(z)-a^{-1}M^*(\lambda+az)=1+a^{-1}M^*(\lambda)f(z)+M^{*(1)}(\lambda)g(z) \tag{9}$$

where

$$f(z)=\begin{cases} -\frac{z}{Q}e^{-z}-e^{-z/Q} & \text{(TA/M-BF(m))} \\ -aze^{-z}-\{1+(\frac{1}{Q}-a)z\}e^{-z/Q} & \text{(TA/M-TF(m))}, \end{cases}$$

$$g(z)=-ze^{-z}.$$

We will solve Eq.(9) for $M^*(z)$. First, by differentiating Eq.(9) twice with respect to z, we have

$$M^{*(2)}(z)-aM^{*(2)}(\lambda+az)=a^{-1}M^*(\lambda)f^{(2)}(z)+M^{*(1)}(\lambda)g^{(2)}(z).$$

This equation has the following solution [MATH 85]:

$$M^{*(2)}(z)=a^{-1}M^*(\lambda)\sum_{i\geq 0}a^if^{(2)}(\sigma_M^{[i]}(z))+M^{*(1)}(\lambda)\sum_{i\geq 0}a^ig^{(2)}(\sigma_M^{[i]}(z)), \tag{10}$$

where

$$\sigma_M^{[i]}(z)=\lambda\frac{1-a^i}{1-a}+a^iz.$$

By integrating Eq.(10), we have (see Appendix-C for the derivation of the following equation)

$$M^{*(1)}(z)=a^{-1}M^*(\lambda)\,\Theta^{(1)}(f(.);z)+M^{*(1)}(\lambda)\,\Theta^{(1)}(g(.);z), \tag{11}$$

where

$$\Theta^{(1)}(\psi(.);z)=\int_0^z\sum_{i\geq 0}a^{-i}\psi^{(2)}(\sigma_M^{[i]}(z))dz,$$

$$\Theta(\psi(.);z)=\int_0^z\Theta^{(1)}(\psi(.);z)dz.$$

By letting $z=\lambda$ in Eq.(11), we find that $M^{*(1)}(\lambda)$ is expressed in terms of $M^*(\lambda)$ as follows:

$$\begin{aligned} M^{*(1)}(\lambda)&=a^{-1}M^*(\lambda)\Theta^{(1)}(f(.);\lambda)/\{1-\Theta^{(1)}(g(.);\lambda)\} \\ &=a^{-1}M^*(\lambda)\,w, \end{aligned} \tag{12}$$

where

$$w=\frac{\Theta^{(1)}(f(.);\lambda)}{1-\Theta^{(1)}(g(.);\lambda)}$$

$$=\begin{cases} \dfrac{e^{(1-1/Q)\mu}}{(1-\mu)Q}-\dfrac{1}{Q} & \text{(TA/M-BF(m))} \\ \dfrac{1}{1-\mu}\{a+(\frac{1}{Q}-a)\frac{\mu}{Q}\}e^{(1-1/Q)\mu}-a & \text{(TA/M-TF(m))}, \end{cases}$$

$$\mu=\frac{\lambda}{1-a}.$$

Finally, by substituting Eq.(12) into Eq.(11) and further integrating the resulting equation, we obtain the following solution to Eq.(9):

$$M^*(z)=1+a^{-1}M^*(\lambda)b^*(z), \tag{13}$$

where

$$b^*(z)=\Theta(f(.);z)+w\ \Theta(g(.);z).$$

We used $\int_0^z M^{*(1)}(z)dz=M^*(z)-M^*(0)=M^*(z)-1$ to obtain Eq.(13).

We now proceed to obtain the maximum throughput of FA TA/Ms. Setting $z=\lambda$ in Eq.(13) and solving the resulting equation for $M^*(\lambda)$ yields

$$M^*(\lambda)=1/(1-a^{-1}b^*(\lambda)). \tag{14}$$

where

$$b^*(\lambda)\begin{cases} =\sum\limits_{i\geq 0} a^{-i}[-e^{-x/Q}+e^{-y/Q}-a^i\frac{\lambda}{Q}e^{-y/Q}] \\ \quad +(w+\frac{1}{Q})\sum\limits_{i\geq 0} a^{-i}\{-xe^{-x}+ye^{-y}-a^i\lambda(y-1)e^{-y}\} & \text{(TA/M-BF(m))}, \\ =\sum\limits_{i\geq 0} a^{-i}[-\{1+(\frac{1}{Q}-a)x\}e^{-x/Q}+\{1+(\frac{1}{Q}-a)y\}e^{-y/Q} \\ \quad -a^i\lambda\{a+(\frac{1}{Q}-a)\frac{y}{Q}\}e^{-y/Q}] \\ \quad +(w+a)\sum\limits_{i\geq 0} a^{-i}\{-xe^{-x}+ye^{-y}-a^i\lambda(y-1)e^{-y}\}, & \text{(TA/M-TF(m))} \end{cases}$$

$$x=\mu(1-a^{i+1}),$$

$$y=\mu(1-a^i).$$

For numerical computations, we give the following by expanding the exponential functions into power series and taking the sum over i in the above equation:

$$
b^*(\lambda)\begin{cases}
=e^{-\mu/Q}\sum_{j\geq 1}\frac{(\mu/Q)^j}{j!}\frac{1}{Q(j+1)}\left(a-\frac{(1-a)j}{1-a^j}\right)\mu \\
\quad+\left(w+\frac{1}{Q}\right)e^{-\mu}\sum_{j\geq 1}\frac{\mu^{j+1}}{j!}\left(\frac{\mu}{j+1}-1\right)\left(a-\frac{(1-a)j}{1-a^j}\right) & \text{(TA/M-BF(m))} \\
=e^{-\mu/Q}\sum_{j\geq 1}\frac{(\mu/Q)^j}{j!}\left[\frac{1+(1/Q-a)\mu}{Q(j+1)}-(1/Q-a)\right]\left(a-\frac{(1-a)j}{1-a^j}\right)\mu \\
\quad+(w+a)e^{-\mu}\sum_{j\geq 1}\frac{\mu^{j+1}}{j!}\left(\frac{\mu}{j+1}-1\right)\left(a-\frac{(1-a)j}{1-a^j}\right). & \text{(TA/M-TF(m))}
\end{cases} \tag{15}
$$

As explained previously, $M^*(\lambda)$ is the unconditional average CRT assuming that new packets arrive according to a Poisson process with rate λ. Thus, the FA TA/Ms will be stable if the right hand side in Eq.(14) is positive and finite. Namely, the system is stable if an input rate satisfies the following:

$$a^{-1}b^*(\lambda)<1. \tag{16}$$

Thus, λ which satisfies $a^{-1}b^*(\lambda)=1$ gives the upper bound of the stable input rate. We denote this value of λ by S(Q,m), i.e.,

$$a^{-1}b^*(S(Q,m))=1. \tag{17}$$

S(Q,m) is the supremum of throughput.

3.2 Upper Bound of the throughput of FA TA/Ms

We will now consider the limiting case where Q is infinity and h is equal to zero. Let S* denote the throughput in this limiting case; namely, $S^*=\lim_{Q\longrightarrow\infty} S(Q,m)$. By setting $\lambda=S(Q,m)$ in Eq.(15), multiplying the resulting equation by a^{-1} and further letting Q approach infinity, we have

$$
\lim_{Q\longrightarrow\infty} a^{-1}b^*(S(Q,m)) = \begin{cases}
m/(1-S^*)\sum_{j\geq 1}\frac{S^{*j+1}}{j!}\left(1-\frac{S^*}{1+j}\right)j. & \text{(TA/M-BF(m))} \\
1/(1-S^*)\sum_{j\geq 1}\frac{S^{*j+1}}{j!}\left(1-\frac{S^*}{1+j}\right)j. & \text{(TA/M-TF(m))}
\end{cases}
$$

Since, from Eq.(17), the above equation is equal to one, we obtain the following simple expression for S*:

$$\begin{cases} mS^*e^{S^*}-(m-1)S^*-1=0 & \text{(TA/M-BF(m))} \quad (18a) \\ S^*e^{S^*}=1. & \text{(TA/M-TF(m))}. \quad (18b) \end{cases}$$

For TA/M-BF(m), we have, from Eq.(18a);

$$\frac{d\ S^*}{d\ m} = \frac{S^*(1-e^{S^*})}{me^{S^*}+mS^*e^{S^*}-m+1}$$

It can be easily shown that $\frac{d\ S^*}{d\ m}$ is negative for $S^*>0$ and $m\geq 1$, and that S^* takes its maximum when m is equal to one. The maximum throughput is given by S^* satisfying

$$S^*e^{S^*}=1. \tag{19}$$

From Eq.(16), stable condition for TA/M-BF(1) and TA/M-TF(m) becomes

$$\lim_{Q\to\infty} a^{-1}b^*(\lambda)<1.$$

After some manipulation, the above condition becomes

$$\lambda < e^{-\lambda}. \tag{20}$$

As we have seen, the maximum throughput S^* of free access algorithms with mini-slots (i.e., TA/M-BF(m) and TA/M-TF(m)) in the limiting case is given by S^* which satisfies $S^*e^{S^*}=1$. (The actual value of S^* is 0.567174(+).) In the following, we will show that this limiting case provides the perfect scheduling of collided packets, and hence, that no other free access algorithms achieve higher throughput and small delay. We will concentrate on the ternary feedback case in the following.

Let Q go to infinity in Eq.(5) ($\delta_B=0$). Note that δ_B is 0, since we are assuming ternary feedback case here. Eq.(5) becomes (see Appendix-D for the derivation)

$$M_k = \begin{cases} 1 + k \sum_{j\geq 0} M_{1+j}p(j), & (k\geq 2) \\ 1. & (k=0,1) \end{cases} \tag{21}$$

(+) Maximum throughput of the class of free access algorithms has recently been shown not to exceed 0.56714 [HUMB 86]. However, no specific algorithm has been found yet to achieve this bound.

Eq.(21) may be interpreted in the following way. The initial collision (of multiplicity k) wastes a slot. (This gives the first term "1" in Eq.(21).) k collided packets are divided up into k subtrees, each having exactly one packet, for collision resolution. Since we are assuming a free access algorithm, each subtree suffers further collision with j number of newly arriving packets. Therefore, each subtree requires $\sum_{j\geq 0} M_{1+j}p(j)$ to resolve a collision. (This gives the second term "$k \sum_{j\geq 0} M_{1+j}p(j)$" in Eq.(21).)

In other words, in the limiting case, the packets involved in the initial collision will be isolated for collision resolution; no two packets will be assigned to the same subtree. Collided packets are optimally scheduled in the sense that there will be no further collisions among packets in the initial collision. It will take only k slots to transmit k collided packets successfully, if there are no new arrivals. Therefore, this limiting case provides the perfect scheduling of collided packets. We further note that, in the limiting case, we let Q go to infinity, keeping hQ (mini-slot overhead) zero (see section 3.1). Hence, there is not scheduling overhead. For this reason, S* given by Eq.(19) is the highest maximum throughput in the whole class of free access algorithms.

From Eq.(21), M_k (conditional average CRT) of FA TA/M-TF(m) in the limiting case becomes (see Appendix E for the derivation);

$$M_k = 1 + k/(e^{-\lambda} - \lambda) \qquad (k \geq 2), \tag{22}$$

$$M_0 = M_1 = 1.$$

Let S_k be the conditional throughput (per slot) over a collision resolution interval (CRI) which has been initiated by a collision of multiplicity k. Note that the average length of this CRI is equal to the average CRT M_k. The average number of successfully transmitted packets during this time interval is $k+\lambda(M_k-1)$, i.e., sum of k packets involved in the initial collision and $\lambda(M_k-1)$ number of new packets arriving during the CRI. Thus, S_k is give by

$$S_k = \{k+ \lambda(M_k-1)\}/M_k = e^{-\lambda} - \frac{(e^{-\lambda}-\lambda)e^{-\lambda}}{k+(e^{-\lambda}-\lambda)}$$

S_k is an increasing function of k and approaches $e^{-\lambda}$ when k goes to infinity. In other words, no more than $e^{-\lambda}$ number of packets will be successfully transmitted. This provides an intuitive explanation of the stable condition $\lambda < e^{-\lambda}$ (see Eq.(20)). This system will be stable if an input rate λ is less than the system capacity $e^{-\lambda}$.

4. Average Transmission Delay in the Limiting Case

Let's define the transmission delay of a packet as the time interval beginning at the packet generation and ending with the completion of its successful transmission. In this section, we obtain the lower bound of the average transmission delay E[D] of the free access TA/Ms with mini-slot.

As we saw in the previous section, FA TA/M-BF(1) and FA TA/M-TF(m) achieve the lower delay bound when Q (the number of mini-slots in a large slot) goes to zero. Further, TA/M-BF(1) is equivalent to TA/M-TF(1). Hence, in the following, we focus on the delay of the FA TA/M-TF(m) in the limiting case.

The transmission delay of a packet divides into two elements; D_1, the time from packet generation to the beginning of its initial transmission, and D_2, the time from the initial transmission to the end of its successful transmission. The mean $E[D_1]$ of the first element D_1 is 1/2, and thus we have

$$E[D] = 1/2 + E[D_2]. \tag{23}$$

In the following, we will obtain the average of D_2, $E[D_2]$.

Let a_k and d_k denote the expected number of new packet arrivals in a CRI and the sum of the transmission delays of these a_k packets, respectively, given that the CRI started with a collision of multiplicity k. Let us define the following functions:

$$A(z)= \sum_{k\geq 1} a_k \frac{z^k}{k!}$$

$$A^*(z)=e^{-z}A(z)$$

$$D(z)= \sum_{k\geq 1} d_k \frac{z^k}{k!}$$

$$D^*(z)=e^{-z}D(z).$$

Since the time points when a CRI starts form a renewal process in the free access TA/M, $E[D_2]$ is given by the following equation (see [FAYO 85]):

$$E[D_2] = \frac{D^*(\lambda)}{A^*(\lambda)}. \tag{24}$$

Since that a_k is the (conditional) average of the number of packets transmitted in a CRI, given that the CRI has started with a collision of multiplicity k, and that those k packets involved in the initial collision followed Poisson arrivals, the denominator of Eq.(24),

$A^*(\lambda)=e^{-\lambda} \sum_{k\geq 1} a_k \frac{\lambda^k}{k!}$, is the unconditional average of the number of packets transmitted in a CRI. Therefore, we have $A^*(\lambda)= \lambda M^*(\lambda)$, where $M^*(\lambda)$ is the average CRT. $M^*(\lambda)$ is given by (see Eq.(E.7) in Appendix-E)

$$M^*(\lambda)=(1-\lambda)/(1-\lambda e^{\lambda}),$$

and hence, we have

$$A^*(\lambda) = \lambda(1-\lambda)/(1-\lambda e^{\lambda}). \tag{25}$$

The numerator $D^*(\lambda)$ of Eq.(24) is the unconditional average of the accumulated transmission delay experienced by all the packets transmitted in a CRI. In order to evaluate $D^*(\lambda)$, we first derive the recurrence equation for d_k, the sum of the accumulated transmission delays of the packets in the CRI which started with a collision of multiplicity k.

In the limiting case, colliding k packets will be divided into k subtrees, each with only one active user (see Eq.(21)). The accumulated transmission delay in a subtree contains three elements: the delay due to an initial collision, the time interval from the initial collision to the beginning of a collision resolution process of the subtree, and the accumulated transmission delay of (1+j) packets, where j denotes the number of new packets joining the subtree. Since the initial collision wastes a slot, the first element is 1 slot. The second element is the sum of all preceding CRTs; for the i^{th} subtree, this element becomes the sum of the preceding (i-1) CRTs. The third element is given by d_{1+j}. Therefore, d_k satisfies

$$d_k=\sum_{i=1}^{k}\{1+\sum_{j\geq 0}(i-1)M_{1+j}p(j)+\sum_{j\geq 0}d_{1+j}p(j)\} \quad (k\geq 2).$$

Thus, we have

$$d_k=\begin{cases} k+\sum_{i=1}^{k}\sum_{j\geq 0}(i-1)M_{1+j}p(j)+k\sum_{j\geq 0}d_{1+j}p(j) & (k\geq 2)\\ 1 & (k=1)\\ 0 & (k=0).\end{cases} \tag{26}$$

Define the following functions:

$$D^{(1)}(z)=\frac{dD(z)}{d\ z}=\sum_{k\geq 0}d_{1+k}\frac{z^k}{k!}$$

$$D^{*(1)}(z)=\frac{dD^*(z)}{d\ z}.$$

By multiplying both sides of Eq.(26) by $e^{-z}z^k/k!$ and summing over $k\geq 0$, we have

$$D^*(z)=e^{-z}\sum_{k\geq 1}k\frac{z^k}{k!}+e^{-z}\sum_{k\geq 2}\frac{z^k}{(k-1)!}[\frac{k-1}{2}M^{(1)}(\lambda)+D^{(1)}(\lambda)]e^{-\lambda}$$

$$=z+z(1-e^{-z})D^{(1)}(\lambda)e^{-\lambda}+z^2M^{(1)}(\lambda)\frac{e^{-\lambda}}{2}. \tag{27}$$

Since $D^{*(1)}(\lambda)=e^{-\lambda}D^{(1)}(\lambda)-D^*(\lambda)$, Eq.(27) becomes

$$D^*(z)=z+z(1-e^{-z})\{D^{*(1)}(\lambda)+D^*(\lambda)\}+z^2M^{(1)}(\lambda)\frac{e^{-\lambda}}{2}. \tag{28}$$

Taking the derivative of Eq.(28), we obtain the following equation

$$D^{*(1)}(z)=1+(1+(z-1)e^{-z})(D^{*(1)}(\lambda)+D^*(\lambda))+ze^{-\lambda}M^{(1)}(\lambda). \tag{29}$$

By letting $z=\lambda$ in Eq.(29), we have

$$D^{*(1)}(\lambda)=[e^{\lambda}+\{e^{\lambda}+\lambda-1\}D^*(\lambda)+\lambda M^{(1)}(\lambda)]\frac{1}{1-\lambda}. \tag{30}$$

Substituting Eq.(30) into Eq.(28) and letting $z=\lambda$ in the resulting equation, we obtain

$$D^*(\lambda)=[\lambda(e^{\lambda}-\lambda)+\{2-(1+\lambda)e^{-\lambda}\}\ \lambda^2M^{(1)}(\lambda)\frac{1}{2}]\frac{1}{1-\lambda e^{\lambda}}. \tag{31}$$

From Eqs.(8) and (22) (or see Eqs.(E.5) and (E.7) in Appendix-E), we have

$$M^{(1)}(\lambda)=e^{2\lambda}/(1-\lambda e^{\lambda}).$$

Substitution of $M^{(1)}(\lambda)$ into Eq.(31) yields

$$D^{*}(\lambda)=[\lambda(e^{\lambda}-1)+\{2-(1+\lambda)e^{-\lambda}\}\frac{\lambda^{2}e^{2\lambda}}{(1-\lambda e^{\lambda})2}]\frac{1}{1-\lambda e^{\lambda}} \tag{32}$$

Consequently, from Eqs.(23), (24), (25) and (32), we have

$$E[D]=\{2-(1+\lambda)e^{-\lambda}\}\frac{\lambda e^{2\lambda}}{(1-\lambda e^{\lambda})(1-\lambda)2}+\frac{e^{\lambda}-\lambda}{1-\lambda}+\frac{1}{2}. \tag{33}$$

Note that E[D] becomes 1.5 as $\lambda \longrightarrow 0$ and becomes infinity as $\lambda \longrightarrow 0.56714$, which is the supremum of throughput.

5. Numerical Results

In Tables 1, 2 and Fig.4, mini-slot length is assumed to be zero. The maximum throughput $\bar{S}(Q,m)$ is given by $\bar{S}(Q,m)=S(Q,m)/(1+hQ)=S(Q,m)$ (see Eq.(1) in section 3.1).

Table 1 shows the maximum throughput S(Q,m) for FA TA/M-BF(m)s. When m is one or two, the maximum throughput is an increasing function of Q; on the contrary, for $m \geq 3$, S(Q,m) is a decreasing function of Q. Among various values of m, m=1, namely TA/M-BF(1), achieves the highest maximum throughput for $Q \geq 3$. In this table (and in Table 2), the values of limiting case maximum throughput $S^{*}=\lim_{Q \to \infty} S(Q,m)$ are also shown. Table 2 shows S(Q,m) for FA TA/M-TF(m)s. For all m, S(Q,m) is an increasing function of Q. TA/M-TF(2) achieves the highest throughput for $Q \geq 3$.

Figure 4 compares the throughput performance of FA TA/M-BF(1) and TA/M-TF(2), namely the best algorithms in binary feedback and ternary feedback cases. Figure 4 also shows the value of limiting case maximum throughput $S^{*}=0.56714$. It is seen that S(Q,m) of TA/M-BF(1) and TA/M-TF(2) approaches fairy quickly to the upper bound S^{*} as Q increases.

Tables 3 and 4 assume non-zero mini-slot length (h) and show the effect of mini-slot overhead on the throughput performance. For $h \neq 0$, the maximum throughput $\bar{S}(Q,m)$ is given by $\bar{S}(Q,m)=S(Q,m)/(1+hQ)$. Table 3 shows $\bar{S}(Q,m)$ as a function of Q. h is assumed to be 0.001 in this table. Both S(Q,m) and 1+hQ are increasing functions of Q (see Fig.4 for the monotonic increase of S(Q,m)). Therefore, there is an optimum value of Q (Q_{opt}) which maximizes $\bar{S}(Q,m)$ for a given value of h. For instance, Q_{opt} is 23 in TA/M-BF(1) and 19 in TA/M-TF(2), when h is 0.001.

Table 4 shows Q_{opt} and the corresponding values of $\bar{S}(Q,m)$ for various values of h. It is seen that reasonably small h achieves the throughput very close to the upper bound 0.56714. For instance, TA/M-TF(2) achieves the throughput of 0.56482 when h is 0.001. Noting that a 2000 bit data packet and a 2 bit mini-slot give h=0.001, TA/Ms achieve throughput close to the upper bound in practical systems.

Figures 5 and 6 show simulation results for the average transmission delays in TA/M-BF(1) and TA/M-TF(2), respectively. In both figures, mini-slot length h is assumed to be zero. Theoretical lower bound of transmission delays (given by Eq.(33)) is also shown in these figures. The average delay approaches the lower bound as Q increases in both figures.

Figure 7 assumes that h is equal to 0.001 and shows, through simulations, the average transmission delays of the optimum TA/M-BF(1) and TA/M-TF(2); i.e., TA/M-BF(1) with Q=23 and TA/M-TF(2) with Q=19 (see Table 2 for optimum values of Q). This figure shows that, if h is reasonably small (i.e., h=0.001), both TA/M-BF(1) and TA/M-TF(2) provide the average transmission delay close to the lower bound.

As seen in Figs. 4, 5, 6 and 7, there is not significant difference in both throughput and delay characteristics between TA/M-BF(1) and TA/M-TF(2). Since that TA/M-BF(1) only requires something/nothing binary feedback in mini-slot and that TA/M-BF(1) is less complex than TA/M-TF(2), we may conclude that TA/M-BF(1) is the more practical from an implementation view point.

6. Conclusions

In this paper, we have studied two kinds of free access tree algorithms with mini-slots. One (i.e., TA/M-BF) assumed that the binary feedback was available in a mini-slot, and the other (i.e., TA/M-TF) assumed that the ternary feedback was available. For both TA/M-BF and TA/M-TF, our analysis provides the following three performance measures:

(1) the maximum throughput,

(2) the upper bound of throughput, and

(3) the lower bound of the average transmission delay.

We also presented simulation results of the average transmission delay.

As explained in section 3, the lower bound of the average transmission delay obtained in this paper for TA/M-BF and TA/M-TF is also the lower bound of the average delay in the whole class of free access algorithms. There is not significant difference in performance between TA/M-BF and TA/M-TF, and both algorithms give a practical way to achieve throughput and delay performance very close to the best of the whole class of the free access algorithms.

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Appendix-A **Derivation of Eq.(5)**

Eq.(4) can be rewritten as

$$M_k = 1+k(1-\frac{1}{Q})^{k-1}\sum_{j\geq 0} M_{1+j}p(j)+\delta_B k(1-\frac{1}{Q})^{k-1}(m-1)\sum_{j\geq 0} M_j p(j)$$
$$+Q\sum_{n=2}^{k}\binom{k}{n}(1-\frac{1}{Q})^{k-n}(\frac{1}{Q})^n\sum_{j\geq 0}p(j)\{m\sum_{i=0}^{n}\binom{n}{i}(1-\frac{1}{m})^{n-i}(\frac{1}{m})^i M_{i+j}\}.$$
$$= 1+k(1-\frac{1}{Q})^{k-1}\sum_{j\geq 0} M_{1+j}p(j)+\delta_B k(1-\frac{1}{Q})^{k-1}(m-1)\sum_{j\geq 0} M_j p(j)$$
$$+Qm\sum_{j\geq 0}p(j)\{\sum_{n=0}^{k}\binom{k}{n}(1-\frac{1}{Q})^{k-n}(\frac{1}{Q})^n\sum_{i=0}^{n}\binom{n}{i}(1-\frac{1}{m})^{n-i}(\frac{1}{m})^i M_{i+j}\}$$
$$-\{Qm(1-\frac{1}{Q})^k+(m-1)k(1-\frac{1}{Q})^{k-1}\}\sum_{j\geq 0} M_j p(j)$$
$$-k(1-\frac{1}{Q})^{k-1}\sum_{j\geq 0} M_{1+j}p(j). \qquad (A.1)$$

Let us define the following:

$$F=\sum_{n=0}^{k}\binom{k}{n}(1-\frac{1}{Q})^{k-n}(\frac{1}{Q})^n\sum_{i=0}^{n}\binom{n}{i}(1-\frac{1}{m})^{n-i}(\frac{1}{m})^i M_{i+j}.$$

Since that $\binom{k}{n}\binom{n}{i}=\binom{k-i}{n-i}\binom{k}{i}$, F becomes

$$F=\sum_{n=0}^{k}\sum_{i=0}^{n}\binom{k-i}{n-i}\binom{k}{i}(1-\frac{1}{Q})^{k-n}(1-\frac{1}{m})^{n-i}(\frac{1}{Qm})^i(\frac{1}{Q})^{n-i}M_{i+j}$$

$$=\sum_{i=0}^{k}\binom{k}{i}M_{i+j}\sum_{h=0}^{k-i}\binom{k-i}{h}(1-\frac{1}{Q})^{k-i-h}\{(1-\frac{1}{m})\frac{1}{Q}\}^h(\frac{1}{Qm})^i \quad (h=n-i)$$

$$=\sum_{i=0}^{k}\binom{k}{i}M_{i+j}(\frac{1}{Qm})^i(1-\frac{1}{Qm})^{k-i}. \qquad (A.2)$$

Finally, substituting Eq.(A.2) into Eq.(A.1), we obtain Eq.(5):

$$M_k = 1+\sum_{j\geq 0}\sum_{i=0}^{k}\binom{k}{i}(\frac{1}{Qm})^{i-1}(1-\frac{1}{Qm})^{k-i}M_{i+j}p(j)$$
$$-\{Qm(1-\frac{1}{Q})^k+(1-\delta_B)(m-1)k(1-\frac{1}{Q})^{k-1}\}\sum_{j\geq 0} M_j p(j). \quad (k\geq 2)$$

Appendix-B Derivation of Eq.(9)

By multiplying both sides of Eq.(5) by $z^k/k!$ and taking sum of both sides over $k \geq 0$, we have

$$\sum_{k\geq 0} M_k \frac{z^k}{k!} = \sum_{k\geq 0} \frac{z^k}{k!} + \sum_{k\geq 2} \sum_{i=0}^{k} \sum_{j\geq 0} \binom{k}{i} a^{i-1}(1-a)^{k-i} \frac{z^k}{k!} M_{i+j} p(j)$$

$$- \sum_{k\geq 2} \{a^{-1}(1-\frac{1}{Q})^k + (1-\delta_B)(m-1)k(1-\frac{1}{Q})^{k-1}\} \frac{z^k}{k!} \sum_{j\geq 0} M_j p(j). \quad (B.1)$$

The second term of the right hand side in Eq.(B.1) becomes

$$\sum_{k\geq 2} \sum_{i=0}^{k} \sum_{j\geq 0} \binom{k}{i} a^{i-1}(1-a)^{k-i} \frac{z^k}{k!} M_{i+j} p(j)$$

$$= \sum_{k\geq 0} \sum_{i=0}^{k} \sum_{j\geq 0} \binom{k}{i} a^{i-1}(1-a)^{k-i} \frac{z^k}{k!} M_{i+j} p(j)$$

$$- \sum_{j\geq 0} \{a^{-1}(1-a)M_j + M_{1+j}\} z p(j) - \sum_{j\geq 0} a^{-1} M_j p(j).$$

Let F denote the first term in the right hand side of the above equation. Then, we have

$$F = a^{-1} \sum_{k\geq 0} \sum_{i=0}^{k} \sum_{j\geq 0} \frac{1}{i!(k-i)!} (az)^i \{(1-a)z\}^{k-i} M_{i+j} \frac{\lambda^j}{j!} e^{-\lambda}$$

$$= a^{-1} \sum_{k\geq 0} \sum_{i=0}^{k} \sum_{j\geq 0} \binom{i+j}{i} (az)^i \lambda^j \frac{\{(1-a)z\}^{k-i}}{(k-i)!} M_{i+j} \frac{1}{(i+j)!} e^{-\lambda}$$

$$= a^{-1} \sum_{n\geq 0} \sum_{i=0}^{n} \binom{n}{i} (az)^i \lambda^{n-i} \sum_{h\geq 0} \frac{\{(1-a)z\}^h}{h!} M_n \frac{1}{n!} e^{-\lambda}$$

$$= a^{-1} \sum_{n\geq 0} e^{z-(\lambda+az)} M_n \frac{(\lambda+az)^n}{n!}.$$

Thus, from the definition of $M^*(z) = \sum_{k\geq 0} M_k \frac{z^k}{k!} e^{-z}$, F becomes

$$F=a^{-1}e^{z}M^{*}(\lambda+az).$$

Therefore, we can rewrite Eq.(B.1) as

$$\begin{aligned}M(z)=&e^{z}+a^{-1}e^{z}M^{*}(\lambda+az)\\&-a^{-1}\{e^{z(1-1/Q)}-z(1-\frac{1}{Q})-1\}M^{*}(\lambda)\\&-a^{-1}(1-\delta_B)\{(1-\frac{1}{m})\frac{z}{Q}e^{z(1-1/Q)}-(1-\frac{1}{m})\frac{z}{Q}\}M^{*}(\lambda)\\&-a^{-1}(1-a)zM^{*}(\lambda)-zM^{(1)}(\lambda)e^{-\lambda}-a^{-1}M^{*}(\lambda).\end{aligned}$$

Multiplying the above equation by e^{-z}, we have, from the definition of $M^{*(1)}(\lambda)=M^{(1)}(\lambda)e^{-\lambda}-M^{*}(\lambda)$,

$$M^{*}(z)-a^{-1}M^{*}(\lambda+az)=\begin{cases}1+a^{-1}M^{*}(\lambda)[-\frac{z}{Q}e^{-z}-e^{-z/Q}]+M^{*(1)}(\lambda)(-ze^{-z}). & \text{(TA/M-BF(m))}\\ 1+a^{-1}M^{*}(\lambda)[-aze^{-z}-\{(1-\frac{1}{m})\frac{z}{Q}+1\}e^{-z/Q}] \\ \quad +M^{*(1)}(\lambda)(-ze^{-z}). & \text{(TA/M-TF(m))}\end{cases}$$

Thus, defining the functions f(z) and g(z) as follows:

$$f(z)=\begin{cases}-\frac{z}{Q}e^{-z}-e^{-z/Q} & \text{(TA/M-BF(m))}\\ -aze^{-z}-\{(1-\frac{1}{m})\frac{z}{Q}+1\}e^{-z/Q}, & \text{(TA/M-TF(m))}\end{cases}$$

$$g(z)=-ze^{-z},$$

we obtain Eq.(9):

$$M^{*}(z)-a^{-1}M^{*}(\lambda+az)=1+a^{-1}M^{*}(\lambda)f(z)+M^{*(1)}(\lambda)g(z).$$

<u>Appendix-C</u> Derivation of Eq.(11)

For a function $\sum_{i\geq 0} a^{-i}\,{}^{(2)}(\sigma_M^{[i]}(z))$, we define the following functions:

$$\Theta^{(1)}(\psi(.);z)=\int_0^z \sum_{i\geq 0} a^{-i}\psi^{(2)}(\sigma_M^{[i]}(z))dz,$$

$$\Theta(\psi(.);z)=\int_0^z \Theta^{(1)}(\psi(.);z)dz.$$

The following results were obtained in [MATH 85] (see [MATH 85] for the details). First, the functions defined above are given by

$$\Theta^{(1)}(\psi(.);z)=\sum_{i\geq 0}\{\psi^{(1)}(\sigma_M^{[i]}(z))-\psi^{(1)}(\sigma_M^{[i]}(0))\},$$

$$\Theta(\psi(.);z)=\sum_{i\geq 0} a^{-i}\{\psi(\sigma_M^{[i]}(z))-\psi(\sigma_M^{[i]}(0))-a^{i}z\,\psi^{(1)}(\sigma_M^{[i]}(0))\}.$$

Note that the function $\sigma_M^{[i]}(z)$ satisfies the following three equations:

$$\sigma_M^{[0]}(z)=z,$$

$$\sigma_M^{[i]}(\lambda)=\lambda\frac{1-a^{i}}{1-a}+\lambda a^{i}$$

$$=\sigma_M^{[i+1]}(0),$$

$$\lim_{i\to\infty}\sigma_M^{[i]}(z)=\frac{\lambda}{1-a}.$$

Next, defining

$$\mu=\frac{\lambda}{1-a},$$

we have

$$\Theta^{(1)}(\psi(.);\lambda)=\sum_{i\geq 0}[\psi^{(1)}(\sigma_M^{[i+1]}(0))-\psi^{(1)}(\sigma_M^{[i]}(0))]$$

$$=\psi^{(1)}(\lambda)-\psi^{(1)}(0).$$

Thus, the integral of Eq.(10) is expressed in terms of $\Theta^{(1)}(\psi(.);z)$ as follows (i.e., Eq.(11)):

$$M^{*(1)}(z)=a^{-1}M^{*}(\lambda)\,\Theta^{(1)}(f(.);z)+M^{*(1)}(\lambda)\,\Theta^{(1)}(g(.);z).$$

Appendix-D Derivation of Eq.(21)

Equation (5) for TA/M-TF(m) is

$$M_k = 1 + \sum_{j\geq 0} \sum_{i=0}^{k} \binom{k}{i} a^{i-1}(1-a)^{k-i} M_{i+j}\, p(j)$$
$$-\{Qm(1-\frac{1}{Q})^k + k(m-1)(1-\frac{1}{Q})^{k-1}\} \sum_{j\geq 0} M_j\, p(j). \qquad (k\geq 2) \qquad \text{(D.1)}$$

First, we rewrite Eq.(D.1) to

$$M_k = 1 + \sum_{j\geq 0} \sum_{i=2}^{k} \binom{k}{i} a^{i-1}(1-a)^{k-i} M_{i+j}\, p(j)$$
$$+ \sum_{j\geq 0} \binom{k}{0} a^{-1}(1-a)^k M_j\, p(j)$$
$$+ \sum_{j\geq 0} \binom{k}{1} a^{0}(1-a)^{k-1} M_{1+j}\, p(j)$$
$$-\{Qm(1-\frac{1}{Q})^k + k(m-1)(1-\frac{1}{Q})^{k-1}\} \sum_{j\geq 0} M_j\, p(j). \qquad \text{(D.2)}$$

Let Q go to infinity in Eq.(D.2). Then, the second term approaches zero and the fourth term approaches $k \sum_{j\geq 0} M_{1+j}\, p(j)$. Let R denote the sum of the third and fifth terms; R is given by

$$R = \sum_{j\geq 0} M_j\, p(j)[Qm(1-\frac{1}{Qm})^k - Qm(1-\frac{1}{Q})^k - k(m-1)(1-\frac{1}{Q})^{k-1}].$$

The term in the brackets [ ] becomes

$$Qm \sum_{i=0}^{k} \binom{k}{i}(-\frac{1}{Qm})^i - Qm \sum_{i=0}^{k} \binom{k}{i}(-\frac{1}{Q})^i - k(m-1) \sum_{i=0}^{k-1} \binom{k-1}{i}(-\frac{1}{Q})^i$$
$$= Qm + Qmk(-\frac{1}{Qm}) + Qm \sum_{i=2}^{k} \binom{k}{i}(-\frac{1}{Qm})^i$$
$$-\{Qm + Qmk(-\frac{1}{Q}) + Qm \sum_{i=2}^{k} \binom{k}{i}(-\frac{1}{Q})^i\}$$
$$-\{k(m-1) + k(m-1) \sum_{i=1}^{k-1} \binom{k-1}{i}(-\frac{1}{Q})^i$$
$$= \sum_{i=2}^{k} \binom{k}{i}(-\frac{1}{Qm})^{i-1} + m \sum_{i=2}^{k} \binom{k}{i}(-\frac{1}{Q})^{i-1} - k(m-1) \sum_{i=1}^{k-1} \binom{k-1}{i}(-\frac{1}{Q})^i$$

Thus, $\lim_{Q\to\infty} R = 0$. Therefore, we obtain

$$\lim_{Q\to\infty} M_k = 1 + k \sum_{j\geq 0} M_{1+j}\, p(j).$$

Appendix-E Derivation of Eq.(22)

Using Eq.(8), Eq.(21) becomes

$$M_k=1+kM^{(1)}(\lambda)e^{-\lambda} \qquad (k\geq 2) \tag{E.1}$$

$$M_0=M_1=1.$$

By multiplying Eq.(E.1) by $z^k/k!$ and taking the sum of both sides over $k\geq 0$, we have

$$M(z)=e^z+z(e^z-1)\{M^{*(1)}(\lambda)+M^*(\lambda)\}. \tag{E.2}$$

Multiplying Eq.(E.2) by e^{-z} yields

$$M^*(z)=1+z(1-e^{-z})\{M^{*(1)}(\lambda)+M^*(\lambda)\}. \tag{E.3}$$

Substituting $z=\lambda$ into Eq.(E.3), we have

$$M^{*(1)}(\lambda)=M^*(\lambda)[\frac{1}{\lambda(1-e^{-\lambda})}-1]-\frac{1}{\lambda(1-e^{-\lambda})}. \tag{E.4}$$

Furthermore, taking the derivative of Eq.(E.3) and substituting $z=\lambda$ into the resulting equation, we obtain

$$M^{*(1)}(\lambda)=[\frac{1}{(1-\lambda)e^{-\lambda}}-1]M^*(\lambda). \tag{E.5}$$

Substituting Eq.(E.5) into Eq.(E.2), we have

$$M(z)=e^z+\frac{z(e^z-1)}{(1-\lambda)e^{-\lambda}}M^*(\lambda). \tag{E.6}$$

From Eqs.(E.4) and (E.5), $M^*(\lambda)$ becomes

$$M^*(\lambda)=\frac{(1-\lambda)e^{-\lambda}}{e^{-\lambda}-\lambda}. \tag{E.7}$$

Substituting Eq.(E.7) into Eq.(E.6), we get

$$M(z)=e^z+\frac{z(e^z-1)}{e^{-\lambda}-\lambda}.$$

By expanding the exponential function into power series, we have

$$M(z)=\sum_{k\geq 0}\frac{z^k}{k!}+\frac{1}{e^{-\lambda}-\lambda}\sum_{k\geq 2}k\frac{z^k}{k!}$$

Noting that $M(z)=\sum_{k\geq 0} M_k \frac{z^k}{k!}$ and equating the coefficients of $z^k/k!$ of both sides of the above equation, we finally obtain

$$M_k=1+\frac{k}{e^{-\lambda}-\lambda} \qquad (k\geq 2),$$
$$M_0=M_1=1.$$

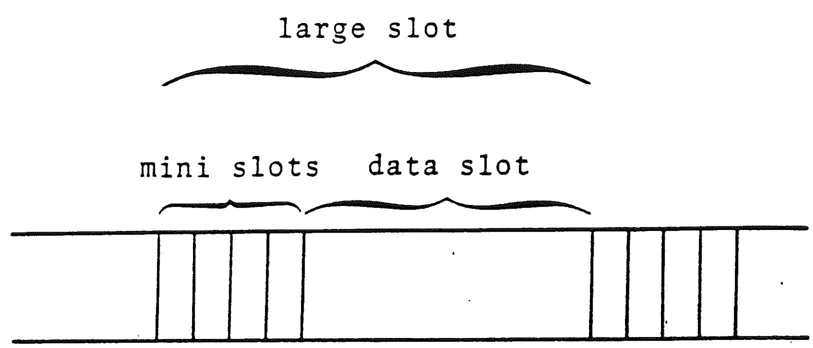

Figure 1 Slot configuration

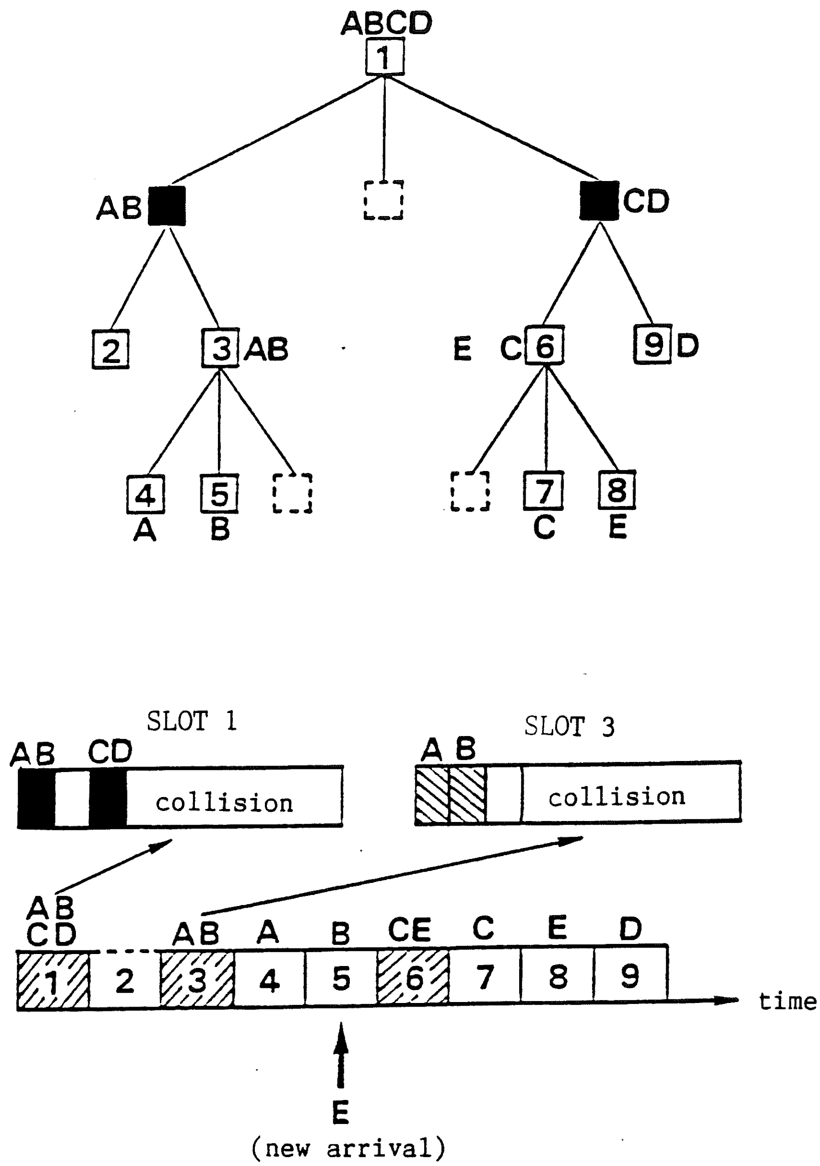

Figure 2 Transmission example in FA TA/M-TF(2) with Q=3

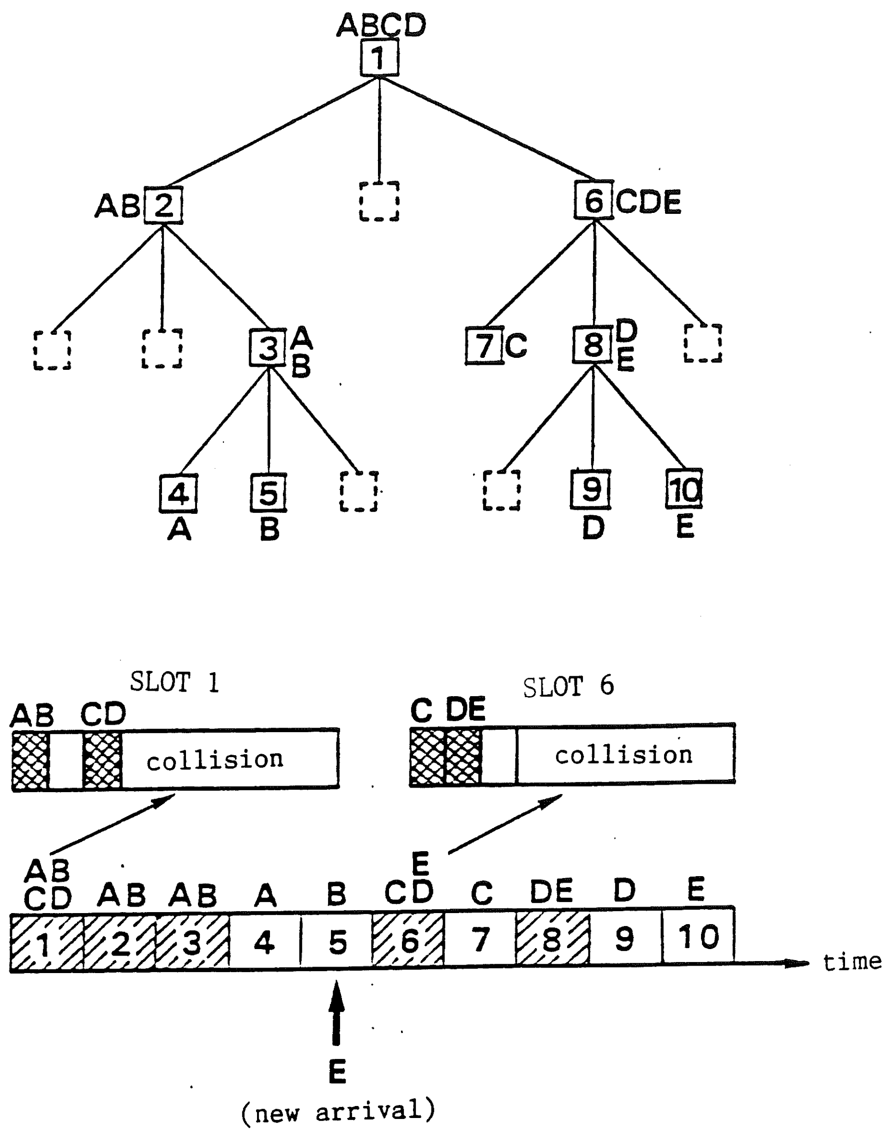

Figure 3 Transmission example in FA TA/M-BF(1) with Q=3

Table 1

S(Q,m) in FA Q-ary TA/M-BF(m)

Q	S(Q,1)	S(Q,2)	S(Q,3)	S(Q,4)
2	0.37682	0.42912	0.40600	0.37999
3	0.45555	0.44241	0.40517	0.37402
4	0.48848	0.44787	0.40455	0.37118
5	0.50646	0.45084	0.40414	0.36953
6	0.51778	0.45269	0.40385	0.36845
7	0.52554	0.45396	0.40364	0.36769
8	0.53120	0.45489	0.40348	0.36713
9	0.53551	0.45559	0.40335	0.36669
10	0.53890	0.45614	0.40325	0.36635
11	0.54163	0.45659	0.40317	0.36607
12	0.54388	0.45695	0.40309	0.36583
13	0.54576	0.45726	0.40303	0.36563
14	0.54737	0.45752	0.40298	0.36547
15	0.54875	0.45775	0.40294	0.36532
16	0.54995	0.45794	0.40290	0.36519
17	0.55100	0.45811	0.40286	0.36508
18	0.55193	0.45826	0.40283	0.36498
19	0.55276	0.45840	0.40280	0.36489
20	0.55351	0.45852	0.40278	0.36481
∞	0.56714	0.46073	0.40229	0.36332

Table 2

S(Q,m) in FA Q-ary TA/M-TF(m)

Q	S(Q,1)	S(Q,2)	S(Q,3)	S(Q,4)
2	0.37682	0.47105	0.47258	0.46055
3	0.45555	0.50377	0.49996	0.48798
4	0.48848	0.51978	0.51488	0.50394
5	0.50646	0.52931	0.52432	0.51446
6	0.51778	0.53564	0.53085	0.52195
7	0.52554	0.54015	0.53564	0.52755
8	0.53120	0.54353	0.53932	0.53191
9	0.53551	0.54616	0.54222	0.53540
10	0.53890	0.54826	0.54457	0.53826
11	0.54163	0.54998	0.54652	0.54065
12	0.54388	0.55141	0.54815	0.54267
13	0.54576	0.55262	0.54955	0.54440
14	0.54737	0.55366	0.55075	0.54590
15	0.54875	0.55456	0.55180	0.54722
16	0.54995	0.55535	0.55273	0.54838
17	0.55100	0.55604	0.55354	0.54941
18	0.55193	0.55666	0.55427	0.55034
19	0.55276	0.55721	0.55493	0.55117
20	0.55351	0.55771	0.55552	0.55193
∞		0.56714		

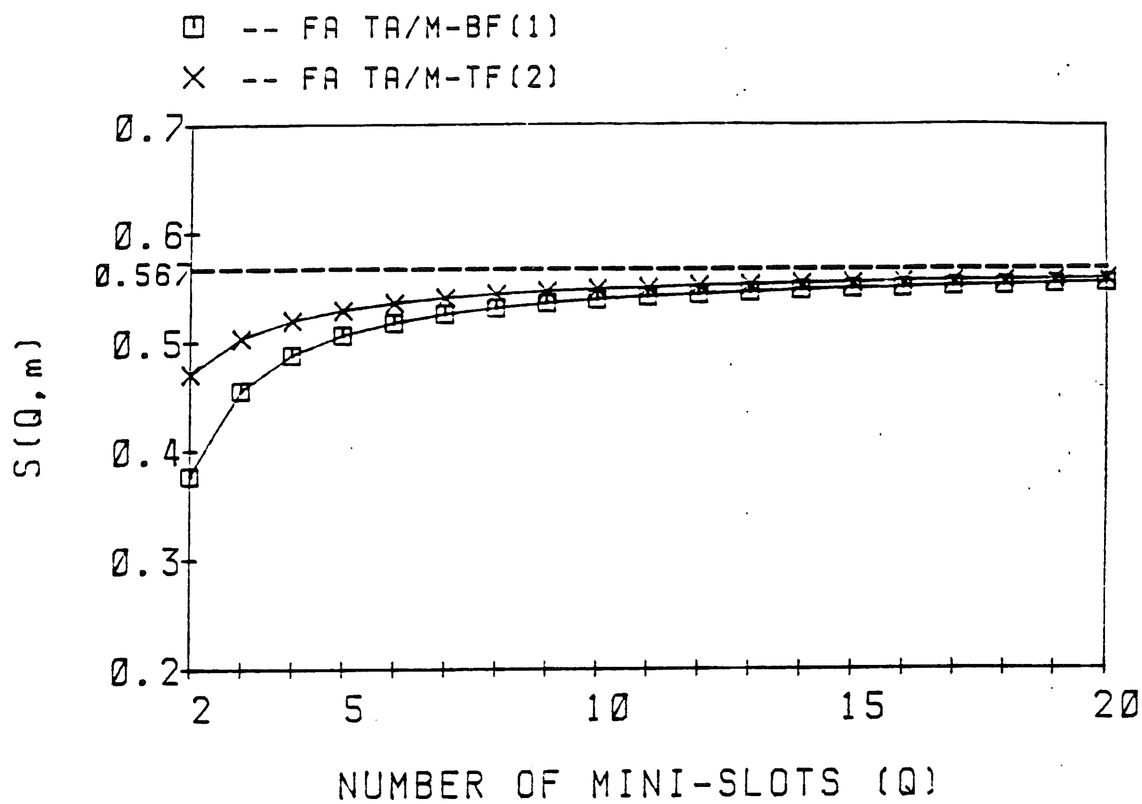

Figure 4 S(Q,m) and its upper bound in FA TA/M-BF(1) and TA/M-TF(2)

Table 3

$\bar{S}(Q,m)$ in FA TA/M-BF(1) and TA/M-TF(2) (h=0.001)

Q	TA/M-BF(1)	TA/M-TF(2)
2	0.37607	0.47011
3	0.45419	0.50226
4	0.48653	0.51771
5	0.50394	0.52668
6	0.51469	0.53245
7	0.52189	0.53640
8	0.52698	0.53922
9	0.53073	0.54129
10	0.53356	0.54283
11	0.53574	0.54400
12	0.53743	0.54487
13	0.53876	0.54553
14	0.53981	0.54602
15	0.54064	0.54637
16	0.54129	0.54660
17	0.54179	0.54675
18	0.54217	0.54682
19	0.54245	0.54682
20	0.54266	0.54678
21	0.54278	0.54668
22	0.54284	0.54654
23	0.54285	0.54637
24	0.54281	0.54617
25	0.54274	0.54594
26	0.54263	0.54569
27	0.54248	0.54542
28	0.54232	0.54514
29	0.54212	0.54483
30	0.54190	0.54452

Table 4

Optimum values of Q and $\bar{S}(Q_{opt},m)$ in FA TA/M-BF(1) and TA/M-TF(2)

h	TA/M-BF(1)		TA/M-TF(2)	
	Q_{opt}	$\bar{S}(Q_{opt},1)$	Q_{opt}	$\bar{S}(Q_{opt},2)$
0.001	23	0.54285	19	0.54682
0.002	16	0.53290	13	0.53862
0.003	14	0.52531	11	0.53241
0.004	12	0.51897	9	0.52718
0.005	11	0.51339	9	0.52264
0.006	10	0.50840	8	0.51864
0.007	9	0.50377	7	0.51492
0.008	9	0.49954	7	0.51151
0.009	8	0.49552	6	0.50820
0.010	8	0.49185	6	0.50532
0.011	8	0.48824	6	0.50248
0.012	7	0.48482	6	0.49966
0.013	7	0.48170	5	0.49700
0.014	7	0.47863	5	0.49468
0.015	7	0.47560	5	0.49238
0.016	7	0.47261	5	0.49010
0.017	6	0.46985	5	0.48784
0.018	6	0.46731	5	0.48561
0.019	6	0.46479	5	0.48339
0.020	6	0.46230	4	0.48128

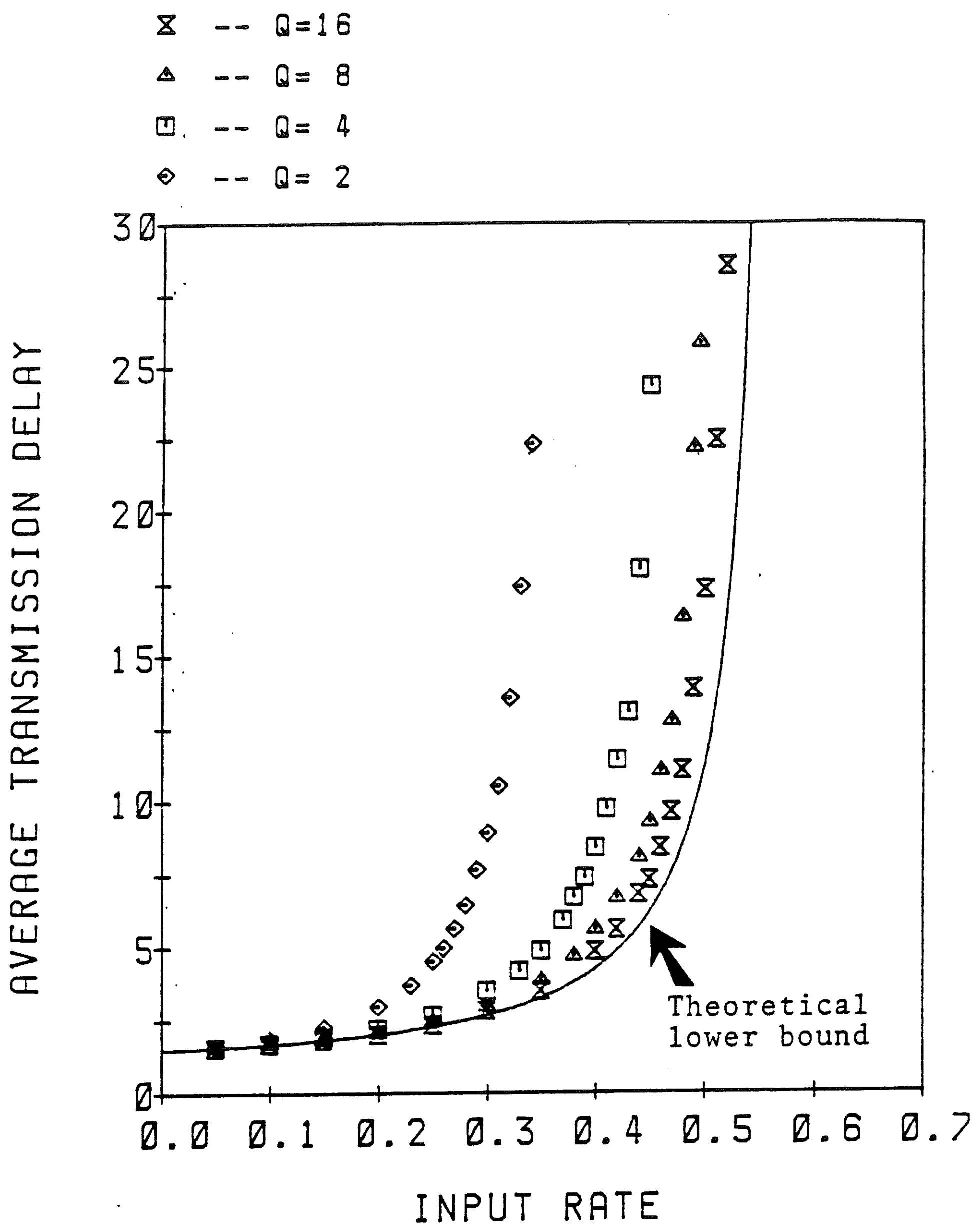

Figure 5 Average transmission delay of

FA TA/M-BF(1) (h=0)

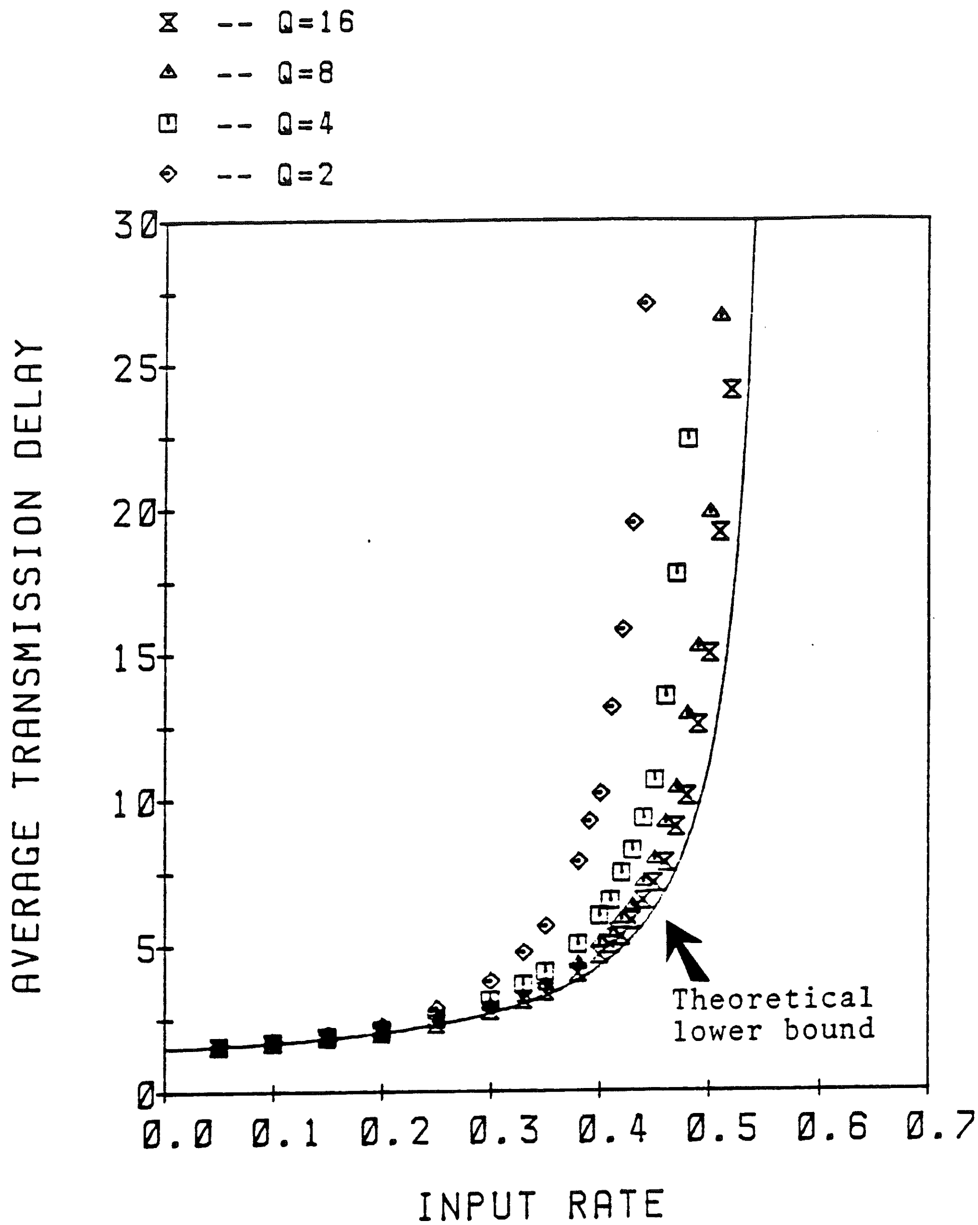

Figure 6 Average transmission delay of

FA TA/M-TF(2) (h=0)

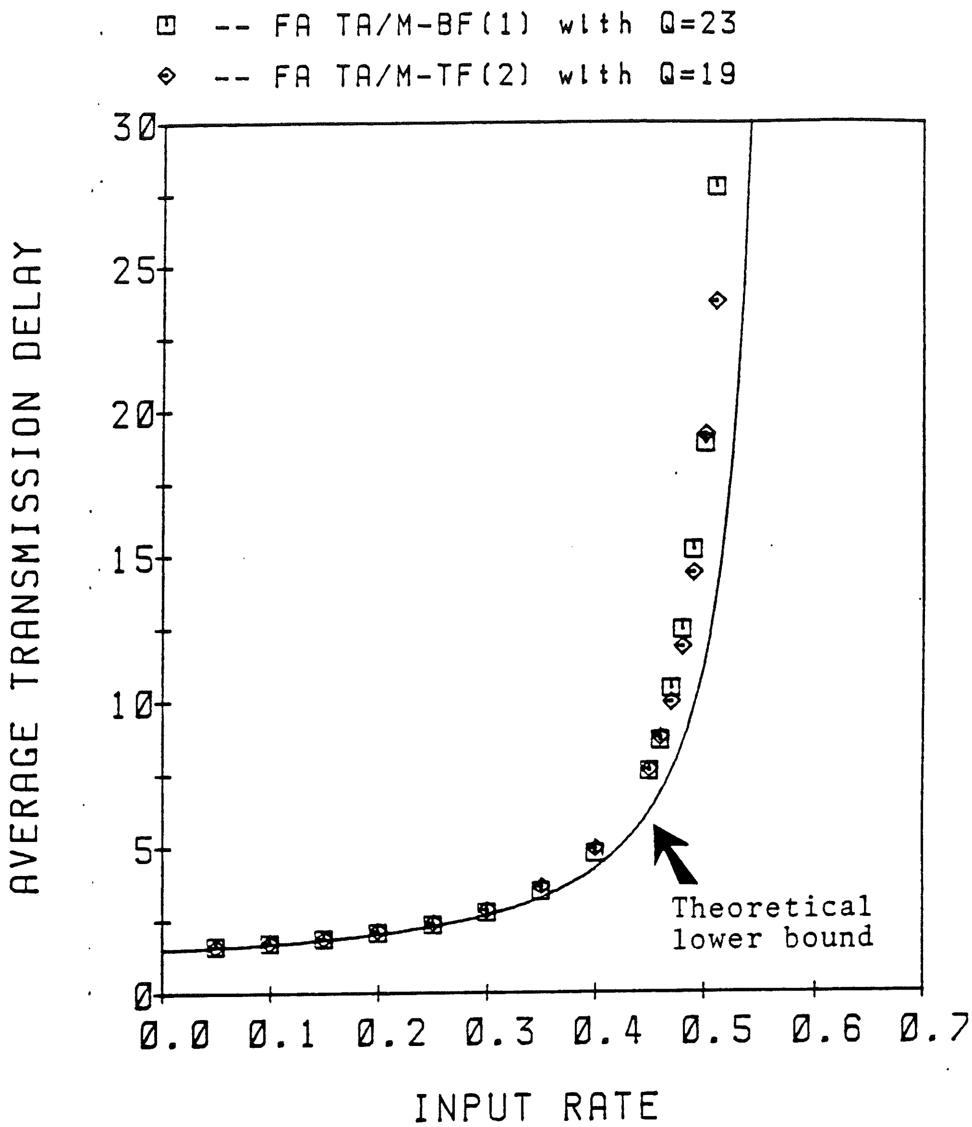

Figure 7 Average transmission delay of

FA TA/M-BF(1) and TA/M-TF(2) (h=0.001)