### Maximum Flatness Spectral Modeling

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Abstract—The maximum entropy method obtains the flattest spectrum consistent with the given autocorrelation values, according to a specific flatness measure. In this correspondence, the suitability of the maximum flatness criterion for spectral estimation is discussed, concluding that it offers useful insight into the spectral models arising from the optimization approach.

## I. INTRODUCTION

In Burg's optimization or variational approach to spectral estimation [1] the objective is to maximize the functional

$$I = \frac{1}{2\pi} \int_{-\pi}^{\pi} F[S(\omega)] d\omega \tag{1}$$

subject to the autocorrelation constraints

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}S(\omega) e^{j\omega n} d\omega = r_n, \quad n = 0, \pm 1, \cdots, \pm M \quad (2)$$

where the area of  $S(\omega)$  will be normalized here to unity, i.e.,  $r_0 = 1$ .

Using Lagrange multipliers to solve the constrained maximization problem, the implicit spectral model

$$F'[S(\omega)] = P(\omega) \tag{3}$$

is obtained, where F'(S) denotes the derivative of F(S) with respect to S, and  $P(\omega)$  is a trigonometric polynomial whose coefficients depend on the data  $r_n$ ,  $n = 0, \pm 1, \dots, \pm M$ . Given a spectral model and a set of autocorrelations, those coefficients and, consequently, the spectral estimate are determined by means of an algorithm that solves the set of equations (2). Since these equations are generally nonlinear, the algorithm has to be iterative [1], [2].

Thus, once the constraints (2) are specified, the performance of the spectral estimation method arising from the above approach is completely characterized by the spectral model or, equivalently, by the objective function F(S), so it can be useful to consider the problem of designing a proper function F(S). As will be shown in the following section, this problem may be regarded as one of measuring flatness, i.e., the similarity between the shape of the spectrum estimate  $S(\omega)$  and the flat shape of the constant spectrum  $S(\omega) = 1$ .

### II. MAXIMUM ENTROPY VERSUS MAXIMUM FLATNESS

First, we shall consider the maximum entropy method [1] (referred to here as MEM1) whose objective function is  $F(S) = \log S$ . It maximizes the entropy of the zero-mean Gaussian process determined by the second moment  $S(\omega)$ , resulting in the most unpredictable process consistent with the constraints. Hence, the MEM1 aims at obtaining an estimate  $S(\omega)$  maximally close to the white noise (flat) spectrum. For this reason, it has been claimed that it yields the flattest spectrum among those matching the given autocorrelations [1], [3].

On one hand, the flatness concept is understood in its usual geometrical sense. On the other, it is a common claim that the MEM1 estimate is a high-resolution spectrum on account of its ability to show sharp peaks. Therefore, as Makhoul observes [3], "the two notions (flatness and high-resolution) are clearly contradictory."

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Fig. 1. (a) Exact spectrum; (b) spectrum estimates, for M = 3, obtained with  $F_y(S)$ , for g = 0, 1, 2, and 3.

Makhoul further discusses the high-resolution quality of the MEM1. Conversely, we will question its flatness quality.

In fact, whiteness or unpredictability is related to geometrical flatness and, in the extreme case  $S(\omega) = 1$ , both reach their maximum point. However, the peaky spectra yielded by the MEMI can be hardly conceived, in general, as the flattest alternatives. Let us consider, for example, the spectrum shown in Fig. 1(a), that was formed by adding two Gaussian functions of opposite sign to unity. The four estimates of this spectrum which are shown in Fig. 1(b) match its first four autocorrelation values and result from the same general objective function  $F_g(S)$  for g = 0, 1, 2, and 3. This objective function, as well as its associated spectral model (3), can be properly described by means of its second derivative with respect to S [4], which is given by

$$F_{e}''(S) = -S^{-g}$$
(4)

where g is a real parameter that equals 2 for the MEM1.

Notice that in Fig. 1(b) the MEM1 spectrum does not have the flattest aspect among the plotted estimates, since it shows a too sharp peak. In fact, as (4) suggests, the MEM1 flatness measure is just an item (g = 2) among an infinite sequence of measures. As it appears in Fig. 1(b), for g > 0, the measures give more emphasis to the peaks than to the valleys and they proportionally increase the amplitude range of spectral values as g grows [4]. If we ask which spectrum is the flattest, the answer will depend on whether we pay more attention to the peak than to the valley or vice versa. In fact, geometrical flatness is an arbitrary concept, so many criteria can be used to define it. In other words, the entropy measure that characterizes the MEM1 is just a way of measuring flatness.

For example, there exists an alternative version of the maximum entropy principle (we will refer to it as MEM2) whose functional J is an entropy measure of the instantaneous frequency of the as-

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	TABLE I
CHARACTERISTIC FU	NCTIONS OF FOUR SPECTRAL ESTIMATORS
CORRESPONDING TO	CONSECUTIVE INTEGER VALUES OF THE
REAL PARAMETER	g. P is a Trigonometric Polynomial
	OF THE VARIABLE $\omega$

Method	g	-F"g(S)	F <sub>g</sub> (S)	Model S=
втм	0	1	0.5(S-1) <sup>2</sup>	Р
MEM2	1	1/S	-S log S	eP
MEM1	2	1/S <sup>2</sup>	log S	1 / P
_	3	1/S <sup>3</sup>	-(S <sup>1/2</sup> -S <sup>-1/2</sup> ) <sup>2</sup>	1/P1/2

sociated (not necessarily Gaussian) random process [5]. The probability density function of this random variable is just the spectrum of the process and its entropy is a measure of closeness to the flat density function [2]. This flatness measure corresponds to  $F_g(S)$ for g = 1 and can be seen in Table I. Note that in the MEM2 spectrum depicted in Fig. 1(b), the peak is not as emphasized as in the MEM1 spectrum, and that the opposite occurs at the valley. As another example, at the other side of MEM1, the case g = 3 results from the asymptotic symmetrized Kullback divergence [7] between a Gaussian process and white noise when the sign is reversed. Its objective function and its spectrum are also shown in Table I and Fig. 1(b), respectively.

Taking another viewpoint, we can avoid any reference to the entropy of the random process or other information measures, and just pay attention to the spectrum itself. Then, the Euclidean measure of closeness to the constant spectrum appears as a sensible flatness measure. It corresponds to the selection g = 0, also indicated in Table I. The result is the classical Blackman-Tukey method (BTM) with rectangular window that extrapolates with zeros the given autocorrelations. Since this method assumes a finite autocorrelation sequence, it actually involves an all-zero or moving-average (MA) modeling of the process (if the estimate  $S(\omega)$  is nonnegative), in the same way as the MEM1 involves an autoregressive (AR) modeling. The BTM spectrum is also depicted in Fig. 1(b); for this particular example, it may reasonably be considered as the best estimate among those in Fig. 1(b). However, the BTM estimate may show negative values. In fact, the positivity of  $S(\omega)$ is only guaranteed for g > 0 [4].

Let us now consider another example. The exact spectrum, which is plotted in Fig. 2(a), was also formed by using two Gaussian functions of opposite sign; however, in this case, the amplitudes of the peak and the valley are logarithmically equivalent, so the peak is noticeably higher than in the foregoing example. Fig. 2(b) shows the BTM spectrum and the MEM1 spectrum obtained using the first 20 autocorrelations as constraints. Note that the MEM1 again yields a sharper peak. Moreover, its spectrum is not so influenced by the leakage phenomenon as the BTM spectrum. Due to these two reasons, it is obvious that the MEM1 performs better than the BTM in this case.

Hence, there can exist many criteria for measuring flatness, i.e., many maximum flatness methods, and each of them will be useful as far as its assumed spectral model (3) is close to the actual spectral model of the underlying random process.

### III. IS THE MAXIMUM FLATNESS CRITERION SUITED FOR SPECTRAL ESTIMATION?

When the spectral area  $r_0$  is the only constraint (i.e., M = 0),  $P(\omega)$  in (3) is a constant, so  $S(\omega)$  equals 1 whatever F(S) is. Therefore, maximum flatness is a common tendency of the methods arising from the optimization approach, a tendency that is only bounded by the constraints. For this reason, it was used as a unifying principle for spectral estimation [6]. Furthermore, a comparative investigation based on both the flatness concept and the function F''(S) was revealed to be useful in explaining the salient trends of the various spectral models, even making possible a generalized



Fig. 2. (a) Exact spectrum; (b) MEM1 spectrum (----) and BTM spectrum (----), for M = 19.

approach in which any kind of flatness measure, either analytic or numeric, could be envisaged [4].

Thus, the maximum flatness criterion appears well suited to the optimization approach. Nevertheless, there may actually exist sensible flatness measures that can hardly be useful for spectral estimation. In order to illustrate this fact, we shall now consider a new family of measures obtained from  $F_g(S)$  that will allow us to make some additional observations. The second derivatives of their corresponding objective functions  $\overline{F}_g(S)$  are defined by symmetrizing  $F_v^{r}(S)$  in (4) with respect to g, namely

$$\overline{F_g''(S)} = -0.5(S^g + S^{-g}) = -\cosh(g \cdot \log S).$$
 (5)

Note that  $\overline{F}_g(S)$  reduces to the Blackman-Tukey function for g = 0. Due to the fact that, as shown in [4], for g > 0,  $F_g(S)$  flattens valleys (see Fig. 1(b)) whereas  $F_{-g}(S)$  has the same effect with peaks, both peaks and valleys are flattened by the family  $\overline{F}_g(S)$ , so it actually aims at minimizing the spectral amplitude range.

This fact is illustrated in Fig. 3, which shows the spectral estimates obtained from  $\overline{F}_2(S)$  for the two above mentioned examples (Figs. 1 and 2); these estimates will be referred to as the symmetrized MEM1 (SMEM1) spectra. Comparing them with the BTM spectra in the same Fig. 3, a smaller amplitude range than that of the  $F_g(S)$  family, as well as a more accentuated effect when g grows, are observed. Both examples illustrate the fact that the greater g is, the farther the spectral estimate obtained with  $\overline{F}_g(S)$ is from the exact spectrum. This is especially true for the last example, where the decrease of the peak amplitude is accompanied by a more accentuated effect of leakage to the extent that, surprisingly, the valley is better matched by the MEM1 than by its symmetrized version, the SMEM1.

Note from the autocorrelation functions of the last example, which are depicted in Fig. 4, that the MEM1 takes into account the trend contained in the first 20 autocorrelations to obtain, in this case, a reasonable or "natural" extrapolation. Conversely, the



Fig. 3. SMEM1 spectra (----) and BTM spectra (----) corresponding to the examples of (a) Fig. 1 and (b) Fig. 2.

SMEM1 spectral model uses this knowledge of the trend to perform an extrapolation of reversed sign which allows it to lower the amplitude of the spectral peak more than with a zero extrapolation.

### **IV.** CONCLUSIONS

Taking advantage of an infinite sequence of spectral estimators characterized by a real parameter g and including the BTM, MEM1, and MEM2 as particular cases, the maximum flatness property usually attributed to the MEM1 spectrum has been discussed, observing that there can exist many ways of seeking maximum flatness and the MEM1 is just one of them.

Finally, if we are given a set of constraints extracted from the signal samples which incorporate some sort of information about the spectrum, the actual aim must be to succeed in a proper spectral model for the underlying random process rather than to obtain an estimate according to some *a priori* criterion as maximum flatness or maximum entropy. Nevertheless, the criterion on which a given objective F(S) is based offers useful insight into the corresponding spectral model arising from the optimization approach. This is especially true for the flatness concept since in addition to furnishing an interpretation of the various spectral models that result from the optimization approach it serves as a basis to compare them.

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### Using the Sphericity Test for Source Detection with Narrow-Band Passive Arrays

DOUGLAS B. WILLIAMS AND DON H. JOHNSON

Abstract—A prerequisite for many high resolution bearing estimation algorithms is an accurate estimate of the number of sources

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