Let us discuss the efficiency of the algorithm in the case of the second degree kernel contribution. The number of operations using (1) is proportional to

$$NO \approx 2\left[2 \sum_{i=1}^{N-1} i^2 + N^2\right] = 2\,\frac{2N^3 + N}{3}. \tag{27}$$

Our derived algorithm requires in the case of one processor with FFT structure

$$NO \approx \frac{N^2 + N}{2} + N \cdot [2 \cdot [2N \log_2 2N]] + N \cdot 2N + 2N \log_2 2N \tag{28}$$

operations.

If we use N processors with FFT structure we get the efficiency expressed by the number of operations of one processor as

$$NO \approx N + 2[2N \log_2 2N] + 2N + 2N \log_2 2N. \tag{29}$$

Comparing (28) and (27) one observes that the number of the operations required to calculate the contribution of the second degree kernel can be considerably reduced when we use the processor with FFT structure. Another enormous reduction of the necessary execution time given by (29) is possible through the use of a parallel architecture of processors.

To estimate the efficiency of the described procedure we can continue in the same way also in the case of higher degree kernels. It is evident that the number of the required processors working in parallel to calculate the contribution of the kth degree kernel is

$$NP = \sum_{i=1}^{N} i^{k-1}. \tag{30}$$

Then the time to perform the calculation is given by (29), i.e., the same as in the case of the second degree kernel.

V. Conclusion

The derived algorithm of nonlinear digital filtering considerably reduces the execution time, which is achieved thanks to the following facts:

1) We itemized the calculation process to independent subprocesses. It makes the use of a parallel SIMD architecture of processors possible.

2) The individual subprocesses represent the operations of linear convolutions (see (13)). One can apply to these subprocesses the well-known, today already classical, FFT method or any other from the variety of algorithms, which emphasize the speed as well as the precision of calculation.

As regards the precise linear filtering methods we recall the usage of the number-theoretical transforms (Fermat, Mersenne) and the polynomial transforms [8].

The method presented in this correspondence demonstrates that the fast computation of the output signal of the digital filter is possible not only in the case of linear systems but also in nonlinear systems, which is of great importance mainly in their real-time applications.

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Correction to "The PDF of Adaptive Beamforming Weights"

Allan O. Steinhardt

In the above correspondence,[1] on page 1233, (6) is missing, on the right-hand side, a factor of

$$\exp\,(j \arg\,((\sqrt{R}^{-1}\, d)_{1,1})).$$

This missing phase factor eventually cancels and the final published pdf is correct as it stands. (Alternatively, one can absorb the missing phase into Q, whereupon Q is no longer Hermitian.)

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Discrete Hartley Transform in Error Control Coding

Ja-Ling Wu and Jiun Shiu

***Abstract*—A new class of real-valued linear code obtained by using the discrete Hartley transform (DHT) is defined in this correspondence. We have derived the limitation on the choice of parity frequencies so as to define DHT codes with the cyclic-shift property. Then, by introducing the well-established encoding/decoding algorithm for cyclic codes in error control coding, we have constructed the encoder/decoder for the DHT cyclic codes.**

I. Introduction

As noticed by Marshall [1] in the theory of error control coding (ECC), an (N, K) linear code is simply a K dimensional subspace of an N dimensional vector space which is defined by the set of N-tuples over a field which is finite conventionally. It is possible,

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in spite of the common assumption of binary source codewords given in the theory of ECC, to define error control codes for real-valued data.

Since the rows of an N-point discrete transform matrix are a set of N linearly independent vectors with N components, they form a basis of the N dimensional vector space over the infinite field of real numbers and any K rows of the matrix span a K dimensional subspace which is, by definition, an (N, K) linear code. According to such a methodology, two classes of real-valued linear codes using the Hadamard transform (HT) and the discrete Fourier transform (DFT) had been defined by Marshall in [1]. In this correspondence, we use the discrete Hartley transform (DHT) to define a new class of real-valued linear codes DHT codes, and derive the cyclic-shift property of them which leads us to the condition for constructing cyclic DHT codes.

II. DHT Linear Codes

The DHT of a data sequence

$$x = [x_0, x_1, \cdots, x_{N-1}]$$

is defined as [2]

$$X_f = \frac{1}{N} \sum_{i=0}^{N-1} x_i \operatorname{cas}\left(\frac{2\pi}{N} if\right), \qquad f = 0, 1, \cdots, N-1 \tag{1}$$

where $\operatorname{cas} t = \cos t + \sin t$.

Writing these N equations in matrix form, one gets

$$\begin{bmatrix} X_0 \\ X_1 \\ \vdots \\ X_{N-1} \end{bmatrix} = \frac{1}{N} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & \phi(1) & \phi(2) & \phi(N-1) \\ 1 & \phi(N-1) & \phi(2(N-1)) & \phi((N-1)^2) \end{bmatrix} \cdot \begin{bmatrix} x_0 \\ x_1 \\ x_{N-1} \end{bmatrix} \tag{2}$$

where

$$\phi(\theta) = \operatorname{cas} \frac{2\pi}{N} \theta.$$

The $N \times N$ matrix in (2) is the so-called N-point DHT matrix.

Select any K rows of the N-point DHT matrix, say $j_0, j_1, \cdots, j_{K-1}$, as the rows of a $K \times N$ matrix G. Then G will be a matrix of rank K and will generate an (N, K) linear code [3]. The encoding procedure is the same as that described in [1].

Let the $(N - K) \times N$ matrix H consist of the remaining $(N - K)$ rows, say $j_K, j_{K+1}, \cdots j_{N-1}$ which are called the parity frequencies. Because the DHT matrix is an orthonormal matrix [2], any two rows of it are orthogonal. Therefore, it is easy to verify that

$$GH^T = 0$$

$$GG^T = I_k$$

where I_K is the Kth order identity matrix.

Thus, matrix H is the parity check matrix [3] of the code, and matrix G^T is the right inverse of the generator matrix G.

III. DHT-Based Cyclic Codes

By introducing a limitation on the selection of parity frequencies, we can define, via the DHT matrix, cyclic codes [3] which are linear code with the following cyclic-shift property:

If $y = [y_0, y_1, \cdots, y_{N-1}]$ is a code vector, then $y' = [y_{N-1}, y_0, y_1, \cdots, y_{N-2}]$, a cyclic-shifted version of y, is also a code vector.

Theorem (cyclic shift property of DHT codes): An (N, K) DHT linear code is a cyclic code if and only if the following condition is satisfied:

If "p" is a parity frequency, then "$(N - p) \bmod N$" must be one of the parity frequencies.

The proof of this theorem is given in the Appendix. Now we will construct a (16, 13) DHT code as a concrete example to clarify what we have done.

By selecting rows 0, 1, and 15 of a 16-point DHT matrix as the parity frequencies, we have the generator matrix G and parity check matrix H as follows:

generator matrix G_{13*16} row

$$\begin{array}{cc}
\begin{bmatrix}
a & b & a & 0 & -a & -b & -a & 0 & a & b & a & 0 & -a & -b & -a & 0 \\
a & c & 0 & c & -a & d & b & d & -a & c & 0 & c & a & d & b & d \\
a & a & -a & -a & a & a & -a & -a & a & a & -a & -a & a & a & -a & -a \\
a & d & -b & d & a & c & 0 & c & -a & d & b & d & -a & c & 0 & c \\
a & 0 & -a & b & -a & 0 & a & -b & a & 0 & -a & b & -a & 0 & a & -b \\
a & d & 0 & d & -a & c & -b & c & -a & d & 0 & d & a & c & b & c \\
a & -a & a & -a & a & -a & a & -a & a & -a & a & -a & a & -a & a & -a \\
a & c & b & c & a & d & 0 & d & -a & c & -b & c & -a & d & 0 & d \\
a & -b & a & 0 & -a & b & -a & 0 & a & -b & a & 0 & -a & b & -a & 0 \\
a & c & 0 & c & -a & d & b & d & -a & c & 0 & c & a & d & -b & d \\
a & -a & -a & a & a & -a & -a & a & a & -a & -a & a & a & -a & -a & a \\
a & d & -b & d & a & c & 0 & c & -a & d & b & d & -a & c & 0 & c \\
a & 0 & -a & -b & -a & 0 & a & b & a & 0 & -a & -b & -a & 0 & a & b
\end{bmatrix}
&
\begin{array}{c} 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ 9 \\ 10 \\ 11 \\ 12 \\ 13 \\ 14 \end{array}
\end{array}$$

parity check matrix H_{3*16} row

$$\begin{array}{cc}
\begin{bmatrix}
a & d & 0 & -d & -a & -c & -b & -c & -a & -d & 0 & d & a & c & b & c \\
a & a & a & a & a & a & a & a & a & a & a & a & a & a & a & a \\
a & c & b & c & a & d & 0 & -d & -a & -c & -b & -c & -a & -d & 0 & d
\end{bmatrix}
&
\begin{array}{c} 15 \\ 0 \\ 1 \end{array}
\end{array}$$

where $a = 0.25 \quad b = 0.353553$
$c = 0.326641 \quad d = 0.135299.$

Since G generates a cyclic code, it can be put into the "cyclic form" [3] by using some elementary row operations:

$$\begin{bmatrix}
1 & -a' & a' & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 1 & -a' & a' & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 1 & -a' & a' & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 1 & -a' & a' & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 1 & -a' & a' & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 1 & -a' & a' & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 1 & -a' & a' & -1 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -a' & a' & -1 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -a' & a' & -1 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -a' & a' & -1 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -a' & a' & -1 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -a' & a' & -1 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -a' & a' & -1
\end{bmatrix}$$

where $a' = 2.84776$.

In this way, we get the "generator polynomial" $g(x)$ for this (16, 13) DHT cyclic code as

$$g(x) = g_0 + g_1x + g_2x^2 + g_3x^3$$

$$\text{where} \quad g_0 = -g_3 = 1.00000$$

$$-g_1 = g_2 = 2.84776.$$

By the well-established encoding/decoding algorithms for cyclic codes [3], we can construct the encoder/decoder circuits for this (16, 13) DHT cyclic code as given in [3, ch. 6].

IV. Conclusions and Discussions

We have defined a new class of real-valued linear codes called DHT codes by using the discrete Hartley transform (DHT). In order to construct DHT cyclic codes we have derived a restriction on the selection of parity frequencies. The generator matrix G has been transformed by some elementary row operations to get the equivalent codes which can be described by a generator polynomial $g(x)$. The encoders and decoders are essentially IIR filters which implement the divide$-g(x)$ circuits.

Since the proposed DHT codes have nearly the same parameters as the DFT codes defined in [1], the following question arises naturally: Whether the DHT codes have any advantages or disadvantages as compared with the DFT codes. In the following, our view to this question will be presented:

1) The limitation on selection of parity frequencies of the DHT codes is equivalent to the symmetric index constraint of the DFT codes [1], which is required to form a real-number DFT code. Thus, the ranges of parameters permitted in both codes are the same.

2) It was established in [1] that real-number maximum distance separable DFT codes exist for all choices of parameters. Since there are not specific restrictions in the construction of the DHT codes, it follows that the DHT codes exist for all (N, K) and the codes can detect errors equal in number to the rank of G, the maximum permitted by the Singleton bound [3].

3) The dynamic range of the codewords and the encoding accuracy in the presence of noise provide an basis for comparing real-number codes. Generally speaking, the DHT codes will outperform the complex codes in these areas because only real arithmetics are involved for the former [2], [4].

4) In transform coding applications, there are many consecutive zeros above the cutoff frequency, due to the energy packing property of orthogonal discrete transforms [5]. These consecutive zeros in the transform domain can be used for error control as suggested by the definition of transform-based real-number codes. Since the DHT has better performance in the transform coding application than the DFT [6], it is our belief that the DHT provides a good opportunity and a useful tool for unifying the problems of source coding and channel coding.

Appendix

In this Appendix we will prove the cyclic shift property of DHT codes. Let us first take 4 lemmas.

Lemma A.1:

$$\phi(Nf) = 1, \qquad f \in Z.$$

Lemma A.2:

$$\phi((i + l)f) = \cos\frac{2\pi f}{N}\phi(if) + \sin\frac{2\pi f}{N}\phi(-if).$$

Lemma A.3:

$$\phi(-if) = \phi(if) - 2\sin\frac{2\pi if}{N}.$$

Lemma A.4:

$$\sum_{i=0}^{N-1}\cos\frac{2\pi if}{N} = \begin{cases} N, & f = 0 \pmod N\\ 0, & \text{otherwise}\end{cases}$$

$$\sum_{i=0}^{N-1}\sin\frac{2\pi if}{N} = 0.$$

Now we are going to derive the necessary limitation on the selection of a set of parity frequencies $f_K, f_{K+1}, \cdots, f_{N-1}$ to construct a DHT with cyclic-shift property, that is:

If $\boldsymbol{y} = [y_0, y_1, \cdots, y_{N-1}]$ is a code vector, then $\boldsymbol{y}' = [y_{N-1}, y_0, y_1, \cdots, y_{N-2}]$, a cyclic-shifting version of $\boldsymbol{y}$, is also a code vector.

Since $\boldsymbol{y}$ is a code vector, there must be a K-tuple

$$\boldsymbol{x} = [x_0, x_1, \cdots, x_{K-1}]$$

such that $\boldsymbol{y} = \boldsymbol{x}G$, or

$$\sqrt{N}y_i = \sum_{m=0}^{K-1} x_m\phi(if), \qquad i = 0, 1, \cdots, n-1. \qquad \text{(A.1)}$$

And by the definition of syndrome vector [3], we have

$$\boldsymbol{S} = [S_0, S_1, \cdots, S_{N-K-1}] = \boldsymbol{y}HT = 0$$

or

$$\sqrt{N} S_r = \sum_{i=0}^{N-1} y_i \phi(if_{K+r})$$

$$= \sum_{i=0}^{N-1} y_i \phi(ip_r) = 0, \quad p_r = f_{K+r},$$

$$r = 0, 1, \cdots, N-K-1. \tag{A.2}$$

The condition for y' to be a code vector is equivalent to that for $S' = y'H^T = 0$, that is,

$$\sqrt{N} S'_r = y_{N+1} + \sum_{i=0}^{N-2} y_i \phi((i+1)p_r), \quad \text{by Lemma A.1}$$

$$= \sum_{i=0}^{N-1} y_i \phi((i+1)p_r), \quad \text{by Lemma A.2}$$

$$= \cos \frac{2\pi p_r}{N} \sum_{i=0}^{N-1} y_i \phi(ip_r) + \sin \frac{2\pi p_r}{N} \sum_{i=0}^{N-1} y_i \phi(-ip_r)$$

by (A.2) and Lemma A.3

$$= \sin \frac{2\pi p_r}{N} \left[\sum_{i=0}^{N-1} y_i \phi(ip_r) - 2 \sum_{i=0}^{N-1} y_i \sin \frac{2\pi i p_r}{N} \right]$$

by (A.2) and (A.1)

$$= -2 \sin \frac{2\pi p_r}{N} \sum_{i=0}^{N-1} \left[\sum_{m=0}^{K-1} x_m \phi(if_m) \right] \sin \frac{2\pi i p_r}{N}$$

$$= -2 \sin \frac{2\pi p_r}{N} \sum_{m=0}^{K-1} x_m \sum_{i=0}^{N-1} \phi(if_m) \sin \frac{2\pi i p_r}{N}.$$

Since $x = [x_0, x_1, \cdots, x_{K-1}]$ is an arbitrary K-tuple, the equation above will be zero only when the inner summation is zero. By this observation, we can derive further

$$\sum_{i=0}^{N-1} \phi(if_m) \sin \frac{2\pi i p_r}{N}$$

$$= \sum_{i=0}^{N-1} \cos \frac{2\pi i f_m}{N} \sin \frac{2\pi i p_r}{N} + \sum_{i=0}^{N-1} \sin \frac{2\pi i f_m}{N} \sin \frac{2\pi i p_r}{N}$$

$$= \frac{1}{2} \left[\sum_{i=0}^{N-1} \sin \frac{2\pi i(f_m + p_r)}{N} - \sum_{i=0}^{N-1} \sin \frac{2\pi i(f_m - p_r)}{N} \right.$$

$$\left. + \sum_{i=0}^{N-1} \cos \frac{2\pi i(f_m - p_r)}{N} - \sum_{i=0}^{N-1} \cos \frac{2\pi i(f_m + p_r)}{N} \right],$$

by Lemma A.4

$$= -\frac{1}{2} \sum_{i=0}^{N-1} \cos \frac{2\pi i(f_m - p_r)}{N}.$$

By Lemma A.4, we know that the above equation will be zero if

$$f_m + p_r = f_m + f_{K+r} \neq 0 \bmod N$$

for all $m = 0, 1, \cdots, K$, $r = 0, 1, \cdots, N-K-1$. It is obvious that this condition is exactly equivalent to that in our theorem.

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Constrained Notch Filtering of Nonuniformly Spaced Samples for Enhancement of an Arbitrary Signal Corrupted by a Strong FM Interference

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***Abstract*—The novelty of the proposed method is based on exploring the concept of constrained notch filtering (CNF) as applied to any given arbitrary signal with time varying parameters. First, it is shown that any signal with a constant envelope such as FM may be transformed to a discrete sinusoidal one by applying nonuniform sampling strategy. Second, a signal buried under a strong FM interference is retrieved by applying CNF in the transformed time domain. The main assumption made is that there exists an auxiliary input which provides information about the instantaneous frequency of the interference.**

I. Introduction

The problem of retrieving a signal from an additive mixture is of a general interest. It is made possible by the existence of some additional information about the signal and/or interference. The classical situation is where the signal and interference possess different but known power spectral densities, leading to Wiener filtering. Another situation assumes that there exists an auxiliary input which is correlated with the interference and uncorrelated with the signal. This approach is related to the adaptive filtering first proposed by Widrow *et al.* [1]. In such a case it is possible to retrieve the signal even if the interference is more powerful than the signal. There exists another approach, based on constrained adaptive notch filters (CANF), where sinusoidal signals are retrieved in the presence of background noise [2]. It has also been shown in [2] that this approach can be used even if the frequencies of the sinusoidal signals vary slowly with time.

In this correspondence we propose an algorithm for the retrieval of an arbitrary signal corrupted by a strong interference, which may be modeled by a sinusoidal signal with arbitrary frequency varia-

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