Mixed $\mathcal{H}_2$ and $\mathcal{H}_\infty$ Performance Objectives II: Optimal Control

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***Abstract*—This paper considers the analysis and synthesis of control systems subject to two types of disturbance signals: white signals and signals with bounded power. The resulting control problem involves minimizing a mixed $\mathcal{H}_2$ and $\mathcal{H}_\infty$ norm of the system. It is shown that the controller shares a separation property similar to those of pure $\mathcal{H}_2$ or $\mathcal{H}_\infty$ controllers. Necessary conditions and sufficient conditions are obtained for the existence of a solution to the mixed problem. Explicit state-space formulas are also given for the optimal controller.**

I. Introduction

Two performance measures in optimal control theory which have been the focus of much recent research are the $\mathcal{H}_2$ and $\mathcal{H}_\infty$ norms, defined in the frequency-domain for a stable transfer matrix $G(s)$ as

$$\|G\|_2 := \sqrt{\frac{1}{2\pi}\int_{-\infty}^{\infty} \text{Trace}[G(j\omega)^* G(j\omega)]d\omega}$$

$$\|G\|_\infty := \sup_\omega \sigma_{max}[G(j\omega)]$$

$$(\sigma_{max} := \text{maximum singular value}).$$

It is beyond the scope of this paper to review the vast literature associated with the $\mathcal{H}_2$ and $\mathcal{H}_\infty$ theory. The interested reader might consult Francis and Doyle [10], or Doyle *et al.*[5], and the references therein.

The $\mathcal{H}_\infty$ results of [5] suggest the possibility of a single theory that has the $\mathcal{H}_2$ and $\mathcal{H}_\infty$ results as special cases, and this encourages us to consider a more general problem. The basic system has the block as shown in Fig. 1 where G is the generalized plant and K is the controller. Only finite dimensional linear time-invariant (LTI) systems and controllers will be considered in this paper. The generalized Plant G contains what has been called the plant in traditional control problems as well as any weighting functions. The signals w_0 and w_1 represent all external inputs, including disturbances, sensor noise and commands.

The signal w_0 is assumed to be white, while w_1 is assumed to be bounded in power. z is an output error signal whose power is the performance objective, y represents the measured variables, and u is the control input. Let the transfer function

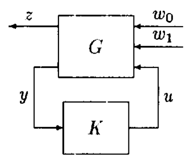

Fig. 1.

from w_0 and w_1 to z be T_{zw}. The analysis problem is, given G and K, to determine the induced norm of T_{zw}. The synthesis problem is, given G, to find a controller K which stabilizes the plant and minimizes the norm of T_{zw}. Both the analysis and the synthesis problems are referred to as "mixed" $\mathcal{H}_2$ and $\mathcal{H}_\infty$ problems. The analysis problem is considered in some detail in Zhou *et al.* [27] and the present paper is a sequel to that.

Note that if only w_0 is present, then the problem setup reduces to the standard $\mathcal{H}_2$ problem setup. Similarly, if only w_1 is present we obtain the standard $\mathcal{H}_\infty$ problem setup. Often we compare the results of this paper with those for the $\mathcal{H}_2$ and $\mathcal{H}_\infty$ problems as presented in [5], which are referred to as the "pure" $\mathcal{H}_2$ and $\mathcal{H}_\infty$ problems. The major motivation of this paper is to begin providing more flexibility in the modeling assumptions required to use optimal control methods.

The main results of this paper are presented in Sections II and IV. Specifically, Section II presents the analysis results and Section IV presents the synthesis results. The proofs of the synthesis results exploit the "separation" structure of the controller, which is reminiscent of the classical $\mathcal{H}_2$ controller and the $\mathcal{H}_\infty$ theory in [5]. Of course, there are significant differences that reflect the mixed criterion used in the problem. These differences are similar to the differences between the $\mathcal{H}_2$ and $\mathcal{H}_\infty$ separation principle discussed in [5].

It is also shown that if full state is available for feedback, then the central controller is simply a gain matrix F_∞ obtained by solving a single Riccati equation, which is the same as in the pure $\mathcal{H}_\infty$ problem. Also, the optimal estimator is an observer whose gain is obtained through the solutions to three coupled equations; this reflects the complexity of the mixed problem. In the general output feedback case, the central controller can be interpreted as an optimal estimator for $F_\infty x$.

To make the results more accessible, we have chosen to treat only a special case of the general mixed problem in this paper. This problem is similar to the problem treated in [5] and captures the essential features of the general problem. While there is some loss of generality in doing this, it relieves the proofs of serious algebraic encumbrance and makes the formulas much easier to interpret. In addition, the assumptions are common in the standard presentation of the $\mathcal{H}_2$ problem.

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Fig. 2.

Although the theory developed here follows [5], important motivation came from the work of Bernstein and Haddad [2], which uses Lagrange multiplier techniques to solve a different mixed $\mathcal{H}_2$ and $\mathcal{H}_\infty$ problem. The problem solved by Bernstein and Haddad has been shown to be a dual problem of our problem in some sense, see, e.g., Steinbuch [22] and Yeh *et al.* [25]. Rotea and Khargonekar [21] have obtained a nice solution to the dual problem for a class of state feedback problems. In Khargonekar and Rotea [14], a convex optimization approach is proposed to solve the output feedback dual problem. Maximum entropy control is a particular mixed problem and has been investigated in Glover and Mustafa [13] and Mustafa [18].

The notation and definitions in this paper are the same as in [27], and the reader is referred there for the definition of the bounded power signal space, $\mathcal{P}$ with norm $\|w_1\|_{\mathcal{P}}$, the notion of white noise that is being used, and the auto- and cross-spectra, S_{ww} and S_{zw}.

II. Systems Performance Analysis With Mixed Inputs

In this section we will examine the norms induced on G with inputs $w_0(t)$ and $w_1(t)$ as shown in Fig. 2.

We will focus on the case where $w_0(t)$ is fixed and white with unit spectral density, i.e., $S_{w_0w_0} = I$, and $w_1(t) \in \mathcal{P}$. The performance of system is measured by the power of the output $z(t)$. Thus our objective is to compute

$$\sup_{w_1 \in \mathcal{BP}} \|z\|_{\mathcal{P}}^2 \tag{1}$$

or alternatively for a given $\gamma > 0$

$$\sup_{w \in \mathcal{P}} \left\{ \|z\|_{\mathcal{P}}^2 - \gamma^2 \|w\|_{\mathcal{P}}^2 \right\}. \tag{2}$$

We note that this problem was referred to as the "mixed $\mathcal{H}_2$ and $\mathcal{H}_\infty$" problem in [27] because if we ignore w_1 then the norm induced on G from w_0 to z is the $\mathcal{H}_2$ norm; similarly, if we ignore w_0 then the norm induced on G from w_1 to z is the $\mathcal{H}_\infty$ norm. This mixed problem has an important motivation from the problem of robust $\mathcal{H}_2$ performance. It is beyond the scope of this paper, however, to give a detailed explanation. The subject of robust $\mathcal{H}_2$ performance and system analysis with various combinations of input signal classes are studied in detail in the companion paper [27], and a brief overview is given in [26]. To make the problem well-motivated physically, we will further make the following assumption.

Assumption: The signal $w_1(t)$ is allowed to be either independent of w_0 or causally dependent on w_0, i.e., either $S_{w_1w_0} = 0$ or there exists a $W(s) \in \mathcal{H}_2$ such that $w_1(t) = W(s)w_0(t)$, which implies in this case $S_{w_1w_0} = W(s)$.

Some explanation for this assumption is necessary at this point. The reason for this assumption is to restrict "the worst case signal" $w_1(t)$ to be a causal function of the system states, which in turn depends causally on the input signal w_0. This is similar to the standard $\mathcal{H}_\infty$ problem where the worst input is a causal function of the states. It is also noted from [27] that without this assumption, the worst case signal $w_1(t)$ is noncausal.

Now assume G is stable and strictly proper. Partition G compatibly with w_0 and w_1 as $[G_0 \; G_1]$, so in terms of the state-space realization, this can be represented as

$$G(s) = \left[\begin{array}{c|cc} A & B_0 & B_1 \\ \hline C & 0 & 0 \end{array}\right].$$

The remainder of this section is devoted to a time-domain solution of the mixed problem in (1) and (2). A frequency-domain solution is given in the companion paper.

Let x denote the system states. Then the system equation can be written as

$$\dot{x} = Ax + B_0 w_0(t) + B_1 w_1(t), \|x(-\infty)\| < \infty$$
$$z = Cx.$$

The following lemma is very useful in the analysis of the mixed problem.

Lemma 1:

$$R_{xw_0}(0) = \frac{1}{2} B_0.$$

Proof: Note that

$$x(s) = (sI - A)^{-1}(B_0 w_0 + B_1 w_1).$$

Hence the cross-spectral density of x and w_0 can be written as

$$\begin{aligned} S_{xw_0}(j\omega) &= (j\omega I - A)^{-1}(B_0 S_{w_0w_0} + B_1 S_{w_1w_0}) \\ &= (j\omega I - A)^{-1}(B_0 + B_1 S_{w_1w_0}) \end{aligned}$$

since $S_{w_0w_0} = I$, where $S_{w_1w_0}$ is either 0 or $W(j\omega)$. Now let Γ denote the semicircular path in the right-half plane with the radius $R > 0$ starting from jR and ending at $-jR$ $(R \to \infty)$. Then the cross-correlation at $t = 0$ can be computed as

$$\begin{aligned} R_{xw_0}(0) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} S_{xw_0}(j\omega) d\omega \\ &= \frac{1}{2\pi j} \oint S_{xw_0}(s) ds - \frac{1}{2\pi j} \int_\Gamma S_{xw_0}(s) ds \end{aligned}$$

where the contour integral is in the clockwise direction with the semicircle path Γ and the interval on imaginary axis closing the semicircle. Since A and $W(s)$ are stable

$$\frac{1}{2\pi j} \oint S_{xw_0}(s) ds = 0.$$

Note that $W(s)$ is strictly proper, so

$$\begin{aligned} R_{xw_0}(0) &= -\frac{1}{2\pi j} \int_\Gamma S_{xw_0}(s) ds \\ &= \lim_{R \to \infty} \frac{1}{2\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} S_{xw_0}(Re^{j\theta}) Re^{j\theta} d\theta \\ &= \lim_{R \to \infty} \frac{1}{2\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (Re^{j\theta} I - A)^{-1} B_0 Re^{j\theta} d\theta \\ &= \frac{1}{2} B_0 \end{aligned}$$

where the last equality is obtained by exchanging the order of limit and integration. □

We are now ready to present the main result of this section.

Theorem 1: Suppose $\gamma > \|G_1\|_\infty$. Then

$$\sup_{w_1 \in \mathcal{P}} \{\|z\|_\mathcal{P}^2 - \gamma^2 \|w_1\|_\mathcal{P}^2\} = \text{Trace}(B_0' X_\gamma B_0)$$

with a worst-case signal $\tilde{w}_1 = \gamma^{-2} B_1' X_\gamma x$, where X_γ is the solution to the Riccati equation

$$A' X_\gamma + X_\gamma A + \gamma^{-2} X_\gamma B_1 B_1' X_\gamma + C'C = 0$$

and $A + \gamma^{-2} B_1 B_1' X_\gamma$ is stable.

Proof: Let X_γ be the Riccati solution and differentiate $x' X_\gamma x$ along the trajectory of a solution to get

$$\begin{aligned}\frac{d}{dt}(x' X_\gamma x) &= \dot{x}' X_\gamma x + x' X_\gamma \dot{x} \\ &= x'(A' X_\gamma + X_\gamma A)x \\ &\quad + 2w_1' B_1' X_\gamma x + 2w_0' B_0' X_\gamma x \\ &= -x'(\gamma^{-2} X_\gamma B_1 B_1' X_\gamma + C'C)x \\ &\quad + 2w_1' B_1' X_\gamma x + 2w_0' B_0' X_\gamma x \\ &= -\|z\|^2 + \gamma^2 \|w_1\|^2 - \|\gamma w_1 - \frac{1}{\gamma} B_1' X_\gamma x\|^2 \\ &\quad + 2w_0' B_0' X_\gamma x.\end{aligned}$$

In the above, we used the Riccati equation to substitute for $A' X_\gamma + X_\gamma A$, and then completed the square. Integrating from $-T$ to T and taking the average, we have

$$\begin{aligned}\lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} \frac{d}{dt}(x' X_\gamma x)\, dt = -\|z\|_\mathcal{P}^2 + \gamma^2 \|w_1\|_\mathcal{P}^2 - \|\gamma w_1 \\ - \frac{1}{\gamma} B_1' X_\gamma x\|_\mathcal{P}^2 \\ + 2\text{Trace}(B_0' X_\gamma R_{xw_0}(0)).\end{aligned} \tag{3}$$

The left-hand side of (3) is zero from the assumption on the external input signals. Now by making use of the relation from Lemma 1 we get

$$\begin{aligned}\|z\|_\mathcal{P}^2 - \gamma^2 \|w_1\|_\mathcal{P}^2 = \text{Trace}(B_0' X_\gamma B_0) \\ - \gamma^2 \|w_1 - \gamma^{-2} B_1' X_\gamma x\|_\mathcal{P}^2.\end{aligned}$$

Note that for fixed γ, an input which maximizes the quantity $\|z\|_\mathcal{P}^2 - \gamma^2 \|w_1\|_\mathcal{P}^2$ is $\tilde{w}_1 = \gamma^{-2} B_1' X_\gamma x$. In fact, any input w_1 which causes $\gamma w_1 - \frac{1}{\gamma} B_1' X_\gamma x$ to be in $\mathcal{L}_2$ yields the same maximum, but $\tilde{w}_1$ is in some sense the most natural. In contrast with the $\mathcal{L}_2$ case in [5], the minimizing w_1 is not unique.

Hence

$$\sup_{w_1 \in \mathcal{P}} \{\|z\|_\mathcal{P}^2 - \gamma^2 \|w_1\|_\mathcal{P}^2\} = \text{Trace}(B_0' X_\gamma B_0)$$

with worst-case signal $\tilde{w}_1 = \gamma^{-2} B_1' X_\gamma x \in \mathcal{P}$ since $A + \gamma^{-2} B_1 B_1' X_\gamma$ is stable. □

Assume now that the input to the system is $\tilde{w}_1$, for fixed γ. Then the system equations become

$$\begin{aligned}\dot{x} &= \left(A + \frac{1}{\gamma^2} B_1 B_1' X_\gamma\right)x + B_0 w_0(t), \|x(-\infty)\| < \infty \\ z &= Cx.\end{aligned}$$

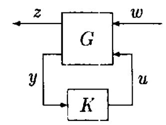

Fig. 3.

Let P_γ be the solution to the Lyapunov equation

$$\left(A + \frac{1}{\gamma^2} B_1 B_1' X_\gamma\right) P_\gamma + P_\gamma \left(A + \frac{1}{\gamma^2} B_1 B_1' X_\gamma\right)' + B_0 B_0' = 0$$

then

$$\|z\|_\mathcal{P}^2 = \text{Trace}(C P_\gamma C').$$

Note that for small γ, $\tilde{w}_1$ may not be in the unit ball $\mathcal{BP}$. Hence to compute (1), we have to find a suitable γ such that $\tilde{w}_1 \in \mathcal{BP}$. This is given in the following theorem.

Theorem 2: Let γ_0 be such that $\|\gamma_0^{-2} B_1' X_{\gamma_0} x\|_\mathcal{P} = 1$. Then

$$\sup_{w_1 \in \mathcal{BP}} \|z\|_\mathcal{P}^2 = \text{Trace}(C P_{\gamma_0} C') = \text{Trace}(B_0' X_{\gamma_0} B_0) + \gamma_0^2.$$

The condition for the existence of such γ_0 is given in [27]. Hence, computing the power norm of z involves iterations on γ, as in the pure $\mathcal{H}_\infty$ case. As an aside, note that the optimal γ level almost always satisfies $\gamma > \|G_1\|_\infty$ when $w_0 \neq 0$.

III. Review of Standard $\mathcal{H}_2$ and $\mathcal{H}_\infty$ Theory

This section reviews some standard results in $\mathcal{H}_2$ and $\mathcal{H}_\infty$ control theory. All results presented here are essentially taken from [5], and the proofs can be found therein. Consider the system described by Fig. 3. Both G and K are real-rational and proper. The pure $\mathcal{H}_2$ or $\mathcal{H}_\infty$ problem is concerned with how to pick K to minimize the $\mathcal{H}_2$ or $\mathcal{H}_\infty$ norm of T_{zw}, the transfer matrix from w to z, where K is constrained to provide internal stability. Internal stability in state space means that the states of G and K go to zero from all initial values when $w = 0$. A controller which provides internal stability is said to be admissible.

The realization of the transfer matrix G is taken to be of the form

$$G(s) = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & 0 & D_{12} \\ C_2 & D_{21} & 0 \end{array}\right]$$

and the following assumptions are made:

1) (A, B_2) is stabilizable and (C_2, A) is detectable.
2) D_{12} has full column rank with $[D_{12} \quad D_{12}^\perp]$ unitary and D_{21} has full row rank with $\begin{bmatrix} D_{21} \\ D_{21}^\perp \end{bmatrix}$ unitary.
3) $\begin{bmatrix} A - j\omega I & B_2 \\ C_1 & D_{12} \end{bmatrix}$ has full column rank for all $\omega \in \mathbb{R}$.
4) $\begin{bmatrix} A - j\omega I & B_1 \\ C_2 & D_{21} \end{bmatrix}$ has full row rank for all $\omega \in \mathbb{R}$.

Two additional assumptions that are implicit in the assumed realization for $G(s)$ are that $D_{11} = 0$ and $D_{22} = 0$. Relaxing these assumptions complicates the formulas substantially, as can be seen in Glover and Doyle [11].

The following lemma is essentially from [15].

Lemma 2: Suppose D has full column rank and denote $R = D'D > 0$. Let H has the form

$$H = \begin{bmatrix} A - BR^{-1}D'C & -BR^{-1}B' \\ -C'(I - DR^{-1}D')C & -(A - BR^{-1}D'C)' \end{bmatrix}.$$

Then $H \in dom(Ric)$ iff (A, B) is stabilizable and $\begin{bmatrix} A - j\omega I & B \\ C & D \end{bmatrix}$ has full column rank for all ω. Furthermore, $X = Ric(H) \geq 0$ and $\mathrm{Ker}(X) = 0$ if and only if $(D'_{\perp}C, A - BR^{-1}D'C)$ has no stable unobservable modes.

A. $\mathcal{H}_2$ Problem

The pure $\mathcal{H}_2$ problem is to find an admissible controller K which minimizes $\|T_{zw}\|_2$. It is easy to see from Lemma 2 that the Hamiltonian matrices

$$H_2 := \begin{bmatrix} A - B_2D'_{12}C_1 & -B_2B'_2 \\ -C'_1D^{\perp}_{12}(D^{\perp}_{12})'C_1 & -(A - B_2D'_{12}C_1)' \end{bmatrix}$$

$$J_2 := \begin{bmatrix} (A - B_1D'_{21}C_2)' & -C'_2C_2 \\ -B_1(D^{\perp}_{21})'D^{\perp}_{21}B'_1 & -(A - B_1D'_{21}C_2) \end{bmatrix}$$

belong to dom(Ric) and, moreover, $X_2 := Ric(H_2)$ and $Y_2 := Ric(J_2)$ are positive semi-definite. Define $F_2 := -(D'_{12}C_1 + B'_2X_2)$, $L_2 := -(B_1D'_{21} + Y_2C'_2)$, and

$$A_{F_2} := A + B_2F_2, C_{1F_2} := C_1 + D_{12}F_2$$
$$A_{L_2} := A + L_2C_2, B_{1L_2} := B_1 + L_2D_{21}$$
$$\hat{A}_2 := A + B_2F_2 + L_2C_2$$
$$G_c(s) := \left[\begin{array}{c|c} A_{F_2} & I \\ \hline C_{1F_2} & 0 \end{array}\right], \quad G_f(s) := \left[\begin{array}{c|c} A_{L_2} & B_{1L_2} \\ \hline I & 0 \end{array}\right].$$

Theorem 3: The unique optimal controller is

$$K_{opt}(s) := \left[\begin{array}{c|c} \hat{A}_2 & -L_2 \\ \hline F_2 & 0 \end{array}\right].$$

Moreover, $\min \|T_{zw}\|_2^2 = \|G_cB_1\|_2^2 + \|F_2G_f\|_2^2 = \|G_cL_2\|_2^2 + \|C_1G_f\|_2^2$.

The controller K_{opt} has the well-known separation structure, clearly shown in the theorem.

B. $\mathcal{H}_\infty$ Problem

The problem considered here is the suboptimal $\mathcal{H}_\infty$control problem: to find an admissible K such that $\|T_{zw}\|_\infty < \gamma$. Clearly, γ must be greater than the $\mathcal{H}_\infty$-optimal level. Optimal $\mathcal{H}_\infty$ controllers are more difficult to characterize than suboptimal ones, and this is one major difference between the $\mathcal{H}_\infty$ and $\mathcal{H}_2$ results.

The $\mathcal{H}_\infty$ solution involves other two Hamiltonian matrices

$$H_\infty := \begin{bmatrix} A - B_2D'_{12}C_1 & \gamma^{-2}B_1B'_1 - B_2B'_2 \\ -C'_1D^{\perp}_{12}(D^{\perp}_{12})'C_1 & -(A - B_2D'_{12}C_1)' \end{bmatrix}$$

$$J_\infty := \begin{bmatrix} (A - B_1D'_{21}C_2)' & \gamma^{-2}C'_1C_1 - C'_2C_2 \\ -B_1(D^{\perp}_{21})'D^{\perp}_{21}B'_1 & -(A - B_1D'_{21}C_2) \end{bmatrix}$$

The following theorem can be found essentially in [5]. The detailed proof of the theorem can be found in [12].

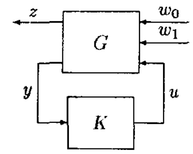

Fig. 4.

Theorem 4: There exists an admissible controller such that $\|T_{zw}\|_\infty < \gamma$ iff the following three conditions hold:

i) $H_\infty \in$ dom(Ric) and $X_\infty := Ric(H_\infty) \geq 0$

ii) $J_\infty \in$ dom(Ric) and $Y_\infty := Ric(J_\infty) \geq 0$

iii) $\rho(X_\infty Y_\infty) < \gamma^2$.

Moreover, when these conditions hold, one such controller is

$$K_{sub}(s) := \left[\begin{array}{c|c} \hat{A}_\infty & -Z_\infty L_\infty \\ \hline F_\infty & 0 \end{array}\right]$$

where

$$\hat{A}_\infty := A + \gamma^{-2}B_1B'_1X_\infty + B_2F_\infty + Z_\infty L_\infty C_2$$
$$F_\infty := -(D'_{12}C_1 + B'_2X_\infty),$$
$$L_\infty := -(B_1D'_{21} + Y_\infty C'_2),$$
$$Z_\infty := (I - \gamma^{-2}Y_\infty X_\infty)^{-1}.$$

The $\mathcal{H}_\infty$ controller displayed in Theorem 4 has certain obvious similarities to the $\mathcal{H}_2$ controller as well as some important differences. Although it is not as apparent as in the $\mathcal{H}_2$ case, the $\mathcal{H}_\infty$ controller also has an interesting separation structure. Furthermore, each of the conditions in the theorem can be given a system-theoretic interpretation in terms of this separation. These interpretations are given in [5].

IV. Mixed $\mathcal{H}_2$ and $\mathcal{H}_\infty$ Synthesis

In this section, we consider the synthesis problem when the system is subjected to mixed disturbance signals. Specifically, consider the system described by Fig. 4, where again the Plant G and controller K are assumed to be real-rational and proper. The significance of various signals shown in the diagram is as follows: $w_0 \in \mathbb{R}^{m_0}$ and $w_1 \in \mathbb{R}^{m_1}$ are the disturbances $u \in \mathbb{R}^{m_2}$ is the control input, $z \in \mathbb{R}^{p_1}$ is the error or controlled output, and $y \in \mathbb{R}^{p_2}$ is the measurement output.

Problem (G): Given the Plant G, a constant γ, exogenous signals w_0, with $S_{w_0w_0} = I$, and $w_1 \in \mathcal{P}$ which is either independent of w_0 or dependent causally on w_0. The mixed $\mathcal{H}_2$ and $\mathcal{H}_\infty$ optimal control problem is to find a controller K such that

$$\sup_{w_1 \in \mathcal{P}} \inf_K \left\{ \|z\|_{\mathcal{P}}^2 - \gamma^2 \|w_1\|_{\mathcal{P}}^2 \right\}$$

is solved, where the minimization is constrained to those K providing internal stability.

The phrase "Problem (G)" means the minimization problem corresponding to the Plant "G." As mentioned earlier, when $w_0 = 0$ or $w_1 = 0$, the induced norm becomes the $\mathcal{H}_\infty$ or $\mathcal{H}_2$ norm, respectively. Thus, Problem (G) is solvable only if the corresponding pure $\mathcal{H}_2$ and $\mathcal{H}_\infty$ problems are solvable.

It is also interesting to note that

$$\sup_{w_1\in\mathcal{P}}\inf_K\left\{\|z\|_{\mathcal{P}}^2-\gamma^2\|w_1\|_{\mathcal{P}}^2\right\}\overset{\gamma\to\infty}{\longrightarrow}\inf_K\left\{\|z\|_{\mathcal{P}}^2\Big|w_1=0\right\}$$

i.e., the mixed problem becomes a standard $\mathcal{H}_2$ problem when $\gamma\to\infty$. It should be pointed out that this is fundamentally different from the situation in the standard $\mathcal{H}_\infty$ problem where the *central solution* approaches to the $\mathcal{H}_2$ solution as $\gamma\to\infty$.

In this paper, we do not usually address the issue of the optimal mixed controller and only discuss optimality in terms of a given γ, restricting γ to be greater than the corresponding $\mathcal{H}_\infty$ optimal level, γ_∞. Thus, optimal controller means optimal for a given γ level. Clearly, any mixed optimal controller is a suboptimal pure $\mathcal{H}_\infty$ controller, but the converse need not be true.

Lemma 3: Problem (G) is solvable only if there exists a K such that $\|T_{zw_1}\|_\infty<\gamma$, i.e., the corresponding suboptimal $\mathcal{H}_\infty$ problem ($w_0=0$) is solvable.

The results in this paper show that the condition in the lemma is not only necessary, but may also be sufficient.

Assumptions on the Plant G: The system has the following realization

$$G(s)=\left[\begin{array}{c|ccc}A & B_0 & B_1 & B_2\\\hline C_1 & 0 & 0 & D_{12}\\C_2 & D_{20} & D_{21} & 0\end{array}\right].$$

The following assumptions are made:

i) (A,B_2) is stabilizable and (C_2,A) is detectable.

ii) D_{12} has full column rank with $[D_{12}\quad D_{12}^\perp]$ unitary.

iii) D_{20} has full row rank with $R_0:=D_{20}D_{20}'>0$ and $R_1:=D_{21}D_{21}'$.

iv) $\begin{bmatrix}A-j\omega I & B_2\\C_1 & D_{12}\end{bmatrix}$ has full column rank for all $\omega\in\mathbb{R}$.

v) $\begin{bmatrix}A-j\omega I & B_0\\C_2 & D_{20}\end{bmatrix}$ has full row rank for all $\omega\in\mathbb{R}$.

Assumption i) is clearly necessary for internal stability. The essential assumption in ii) is that D_{12} has full column rank, while the second part of the assumption is only made to simplify the formulas in our solution. There is no loss of generality in making a such assumption, since a transformation can be applied first to bring it to this standard form. The significance of Assumption iii) is that it insures that the corresponding $\mathcal{H}_2$ problem is nonsingular. Assumptions iv) and v) are made for the same reason as in the $\mathcal{H}_2$ problem: to guarantee that the Riccati equation associated with the pure $\mathcal{H}_2$ problem has a stabilizing solution.

There is no loss of generality in assuming $D_{22}=0$, since the controller for the $D_{22}\neq0$ case can be found from the controller for $D_{22}=0$ case by a linear fractional transformation, see, e.g., [11]. On the other hand, the solution for the $D_{11}\neq0$ case is much more complicated, as can be seen from [11] for $\mathcal{H}_\infty$ problem. The formulas in this paper should generalize in the same way.

A. Separation Principle for Mixed $\mathcal{H}_2$ and $\mathcal{H}_\infty$ Problems

The following theorem is one of the main results in this paper. It shows that the solution to Problem (G) shares a kind of separation principle, i.e., the controller can be constructed from full information control (or state feedback) and optimal estimation of the full information feedback.

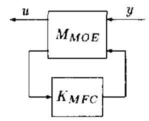

Fig. 5.

Theorem 5: There exists an admissible controller K which solves the following optimization problem

$$\sup_{w_1\in\mathcal{P}}\inf_K\left\{\|z\|_{\mathcal{P}}^2-\gamma^2\|w_1\|_{\mathcal{P}}^2\right\}$$

iff the following conditions hold:

i) $H_\infty\in\mathrm{dom}(Ric)$ and $X_\infty:=Ric(H_\infty)\geq0$;

ii) There exists a controller K_{MFC} which solves Problem $(\hat{G}_{MFC})$ (called the Mixed Full Control (MFC) Problem) with

$$\hat{G}_{MFC}(s)=\left[\begin{array}{c|ccc}A_{tmp} & B_0 & B_1 & [I\ \ 0]\\\hline -F_\infty & 0 & 0 & [0\ \ I]\\C_2+\gamma^{-2}D_{21}B_1'X_\infty & D_{20} & D_{21} & [0\ \ 0]\end{array}\right]$$

where $A_{tmp}=A+\gamma^{-2}B_1B_1'X_\infty$ and $F_\infty=-(D_{12}'C_1+B_2'X_\infty)$.

Moreover, when these conditions hold, one such controller is given by the transfer matrix from y to u in Fig. 5.

$$M_{MOE}(s)=\left[\begin{array}{c|cc}A+\gamma^{-2}B_1B_1'X_\infty+B_2F_\infty & 0 & [I\ \ -B_2]\\\hline -F_\infty & 0 & [0\ \ I]\\C_2+\gamma^{-2}D_{21}B_1'X_\infty & I & [0\ \ 0]\end{array}\right]$$

Notice that i) corresponds to the condition for full information control and ii) corresponds to the condition for the optimal estimation of $F_\infty x$. Thus, the separation principle of mixed controllers is now evident and is similar to the separation principle for $\mathcal{H}_\infty$ controllers given in [5]: The mixed $\mathcal{H}_2$ and $\mathcal{H}_\infty$ output feedback controller is the output estimator of the full information control law in the presence of a worst-case disturbance $w_{1_{\text{worst}}}=\gamma^{-2}B_1'X_\infty x$.

Note that (A_{tmp},B_1) is stabilizable since (A,B_1) is and $(-F_\infty,A_{tmp})$ is detectable since $A_{tmp}+B_2F_\infty$ is stable by the condition $H_\infty\in\mathrm{dom}(Ric)$. For the MFC Problem to be solvable, it is also necessary to require $(C_2+\gamma^{-2}D_{21}B_1'X_\infty,A_{tmp})$ be detectable. This condition will be satisfied implicitly if there is an admissible controller solving the MFC Problem.

The proof of Theorem 5 is given in Section IV-C. The proof in Section IV-C uses the following lemma and the result in Section IV-B.

Lemma 4: Suppose $H_\infty\in\mathrm{dom}(Ric)$ and $X_\infty=\mathrm{Ric}(H_\infty)\geq0$. Then there exists an admissible controller $K(s)$ such that $K(s)$ solves Problem (G) iff $K(s)$ solves

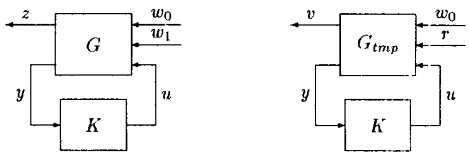

Fig. 6.

Problem (G_{tmp}), where

$$G_{tmp}(s)=\left[\begin{array}{c|ccc} A_{tmp} & B_0 & B_1 & B_2 \\ \hline -F_\infty & 0 & 0 & I \\ C_2+\gamma^{-2}D_{21}B_1'X_\infty & D_{20} & D_{21} & 0 \end{array}\right].$$

See Fig. 6.

Proof: Since $H_\infty \in \mathrm{dom}(Ric)$, $X_\infty = \mathrm{Ric}(H_\infty) \geq 0$. Hence X_∞ satisfies

$$\begin{aligned}(A-B_2D_{12}'C_1)'X_\infty &+ X_\infty(A-B_2D_{12}'C_1)\\ &+\gamma^{-2}X_\infty B_1B_1'X_\infty - X_\infty B_2B_2'X_\infty\\ &+C_1'D_{12}^\perp(D_{12}^\perp)'C_1 = 0. \qquad (4)\end{aligned}$$

Denote

$$\begin{aligned}A_{F_\infty} &= A + B_2F_\infty\\ C_{1F_\infty} &= C_1 + D_{12}F_\infty\end{aligned}$$

and define new disturbance and control variables

$$\begin{aligned}r &:= w_1 - \gamma^{-2}B_1'X_\infty x\\ v &:= u + (D_{12}'C_1 + B_2'X_\infty)x = u - F_\infty x.\end{aligned}$$

Then

$$\begin{bmatrix} z \\ \gamma r\end{bmatrix} = \left[\begin{array}{c|ccc} A_{F_\infty} & B_0 & \gamma^{-1}B_1 & B_2 \\ \hline C_{1F_\infty} & 0 & 0 & D_{12} \\ -\gamma^{-1}B_1'X_\infty & 0 & I & 0\end{array}\right]\begin{bmatrix} w_0 \\ \gamma w_1 \\ v\end{bmatrix} =: \hat{P}\begin{bmatrix} w_0 \\ \gamma w_1 \\ v\end{bmatrix}$$

where

$$\hat{P} = \left[\begin{array}{c|cc} P_{10} & P_{11} & P_{12} \\ P_{20} & P_{21} & P_{22}\end{array}\right] = [P_0 \quad P]$$

and it is also easy to see that

$$\begin{bmatrix} v \\ y\end{bmatrix} = G_{tmp}\begin{bmatrix} w_0 \\ r \\ u\end{bmatrix}.$$

The above transformation is depicted by Fig. 7.

It is shown in [5] that $P \in \mathcal{RH}_\infty$, $P_{21}^{-1} \in \mathcal{RH}_\infty$, and P is inner. Setting $w_0 = 0$ we also see from Lemma 15 of [5] that $K(s)$ internally stabilizes $G(s)$ and $\|T_{zw_1}\|_\infty < \gamma$ iff $K(s)$ internally stabilizes G_{tmp} and $\|T_{vr}\|_\infty < \gamma$.

Finally, to show that Problem (G) is equivalent to Problem (G_{tmp}), differentiate $x'X_\infty x$ along a trajectory of the state to get

$$\begin{aligned}\frac{d}{dt}(x'X_\infty x) &= \dot{x}'X_\infty x + x'X_\infty \dot{x}\\ &= x'(A'X_\infty + X_\infty A)x + 2\langle w_0, B_0'X_\infty x\rangle\\ &\quad + 2\langle w_1, B_1'X_\infty x\rangle + 2\langle u, B_2'X_\infty x\rangle\end{aligned}$$

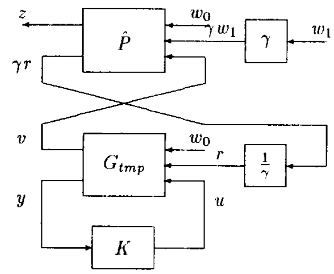

Fig. 7.

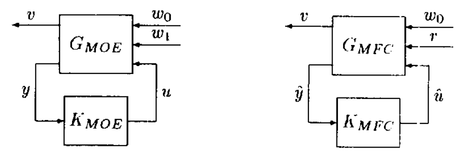

Fig. 8.

where $\langle , \rangle$ denotes the inner product. Using (4) to substitute for $A'X_\infty + X_\infty A$ gives

$$\begin{aligned}\frac{d}{dt}(x'X_\infty x) &= -\|(D_{12}^\perp)'C_1x\|^2 - \gamma^{-2}\|B_1'X_\infty x\|^2\\ &\quad + \|B_2'X_\infty x\|^2 + 2\langle B_2'X_\infty x, D_{12}'C_1x\rangle\\ &\quad + 2\langle w_0, B_0'X_\infty x\rangle + 2\langle w_1, B_1'X_\infty x\rangle\\ &\quad + 2\langle u, B_2'X_\infty x\rangle.\end{aligned}$$

Finally, completing the squares gives the key equation

$$\begin{aligned}\frac{d}{dt}(x'X_\infty x) &= -\|z\|^2 + \gamma^2\|w_1\|^2 - \gamma^2\|w_1 - \gamma^{-2}B_1'X_\infty x\|^2\\ &\quad + \|u + (D_{12}'C_1 + B_2'X_\infty)x\|^2\\ &\quad + 2\langle w_0, B_0'X_\infty x\rangle.\end{aligned}$$

Assuming $x(-\infty)$ is bounded, taking the time average on both sides of the above equation yields

$$\begin{aligned}\|z\|_{\mathcal{P}}^2 - \gamma^2\|w_1\|_{\mathcal{P}}^2 &= \|u + (D_{12}'C_1 + B_2'X_\infty)x\|_{\mathcal{P}}^2\\ &\quad - \gamma^2\|w_1 - \gamma^{-2}B_1'X_\infty x\|_{\mathcal{P}}^2\\ &\quad + \mathrm{Trace}(B_0'X_\infty B_0).\\ &= \|v\|_{\mathcal{P}}^2 - \gamma^2\|r\|_{\mathcal{P}}^2 + \mathrm{Trace}(B_0'X_\infty B_0).\end{aligned}$$

From the above, it is obvious that

$$\begin{aligned}\sup_{w_1\in\mathcal{P}}\inf_K\left\{\|z\|_{\mathcal{P}}^2 - \gamma^2\|w_1\|_{\mathcal{P}}^2\right\} &= \mathrm{Trace}(B_0'X_\infty B_0)\\ &\quad + \sup_{r\in\mathcal{P}}\inf_K\left\{\|v\|_{\mathcal{P}}^2 - \gamma^2\|r\|_{\mathcal{P}}^2\right\}.\end{aligned}$$

Hence a controller $K(s)$ solving Problem (G) will solve Problem (G_{tmp}) and vice versa. □

B. Mixed Output Estimation

As we have mentioned earlier in Theorem 5, we will call the mixed control problem having the structure of G_{MFC} the MFC problem, while the problem having the structure of G_{tmp} will be called the mixed output estimation (MOE) problem. In the following, we show how to reduce the MOE problem to

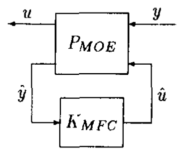

Fig. 9.

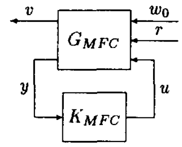

Fig. 10.

the MFC problem. Instead of using G_{tmp} directly, we use an arbitrary Plant G_{MOE} having the structure of G_{tmp}. Consider Fig. 8 where

$$G_{MOE}(s) = \left[\begin{array}{c|ccc} A_t & B_0 & B_1 & B_2 \\ \hline E_1 & 0 & 0 & I \\ E_2 & D_{20} & D_{21} & 0 \end{array}\right]$$

$$G_{MFC}(s) = \left[\begin{array}{c|ccc} A_t & B_0 & B_1 & [I \;\; 0] \\ \hline E_1 & 0 & 0 & [0 \;\; I] \\ E_2 & D_{20} & D_{21} & [0 \;\; 0] \end{array}\right]$$

and we assume that $A_t - B_2 E_1$ in the realization of G_{MOE} is stable.

Note that Assumptions i)-v) on Problem (G) are not needed in obtaining the reduction from the MOE problem to the MFC. This will be clear from the procedure.

Let T_{MOE} and T_{MFC} denote the closed-loop transfer matrices from (w_0, r) to v for the MOE problem and the MFC problem, respectively. The following result is obvious.

Proposition 1: The controller K_{MOE} internally stabilizes G_{MOE} iff $K_{MFC} = \begin{bmatrix} B_2 \\ I \end{bmatrix} K_{MOE}$internally stabilizes G_{MFC}. Furthermore, in this case $T_{MOE} = T_{MFC}$.

To complete the equivalence, suppose that we have a controller for the MFC problem, denoted by K_{MFC} and let K_{MOE} be the transfer matrix generated by Fig. 9.

$$P_{MOE} = \left[\begin{array}{c|cc} A_t - B_2 E_1 & 0 & [I \;\; -B_2] \\ \hline E_1 & 0 & [0 \;\; I] \\ E_2 & I & [0 \;\; 0] \end{array}\right].$$

Proposition 2: The controller K_{MFC} internally stabilizes G_{MFC} iff K_{MOE} given above internally stabilizes G_{MOE}. Furthermore, in this case $T_{MOE} = T_{MFC}$.

Proof: Let x and $\hat{x}$ denote the states of G_{MOE} and P_{MOE}, respectively. Then the overall equations in terms of $e := x + \hat{x}$ and $\hat{x}$ are

$$\begin{aligned} \dot{\hat{x}} &= (A_t - B_2 E_1)\hat{x} + [I \;\; -B_2]\hat{u} \\ \dot{e} &= A_t e + B_0 w_0 + B_1 r + [I \;\; 0]\hat{u} \\ v &= E_1 e + [0 \;\; I]\hat{u} \\ \hat{y} &= E_2 e + D_{20} w_0 + D_{21} r \\ \hat{u} &= K_{MFC}\hat{y}. \end{aligned}$$

Written in matrix notation, this is

$$\begin{bmatrix} v \\ \hat{y} \end{bmatrix} = \left[\begin{array}{cc|ccc} A_t - B_2 E_1 & 0 & 0 & 0 & [I \;\; -B_2] \\ 0 & A_t & B_0 & B_1 & [I \;\; 0] \\ \hline 0 & E_1 & 0 & 0 & [0 \;\; I] \\ 0 & E_2 & D_{20} & D_{21} & [0 \;\; 0] \end{array}\right] \begin{bmatrix} w_0 \\ r \\ \hat{u} \end{bmatrix}$$

and

$$\hat{u} = K_{MFC}\hat{y}.$$

Since $A_t - B_2 E_1$ is stable, the above transfer matrix is equivalent to

$$\begin{bmatrix} v \\ \hat{y} \end{bmatrix} = \left[\begin{array}{c|ccc} A_t & B_0 & B_1 & [I \;\; 0] \\ \hline E_1 & 0 & 0 & [0 \;\; I] \\ E_2 & D_{20} & D_{21} & [0 \;\; 0] \end{array}\right] \begin{bmatrix} w_0 \\ r \\ \hat{u} \end{bmatrix}$$

which is the form of the MFC problem. □

C. Proof of Theorem 5

Since a controller solving Problem (G) is also a suboptimal $\mathcal{H}_\infty$ controller, it is obvious that i) is necessary. Hence, if the problem is solvable, then $H_\infty \in \text{dom}(Ric)$ and $X_\infty = \text{Ric}(H_\infty) \geq 0$. Now using Lemma 4, the original problem is equivalent to Problem (G_{tmp}). The theorem then follows by applying Propositions 1 and 2 to G_{tmp} and note the fact that $A_{tmp} - B_2(-F_\infty) = A + \gamma^{-2} B_1 B_1' X_\infty - B_2 D_{12}' C_1 - B_2 B_2' X_\infty$ is stable by the condition $H_\infty \in \text{dom}(Ric)$. □

D. Mixed Full Control Problem

In Theorem 5, we have seen that the mixed synthesis problem, Problem (G), can be reduced to an MFC problem with $G_{MFC} = \hat{G}_{MFC}$. This section is devoted to the solution of this problem. We will give explicit necessary and sufficient conditions for the solvability of this problem. For the simplicity of notation, we shall consider the following generalized system and associated block diagram as shown in Fig. 10

$$G_{MFC}(s) = \left[\begin{array}{c|ccc} A_t & B_0 & B_1 & [I \;\; 0] \\ \hline E_1 & 0 & 0 & [0 \;\; I] \\ E_2 & D_{20} & D_{21} & [0 \;\; 0] \end{array}\right].$$

It will be assumed that G_{MFC} satisfies, in additional to the assumption iii) for the general output feedback problem, the assumption that (E_2, A_t) is detectable.

MFC Problem: Find an admissible controller K_{MFC} such that K_{MFC} internally stabilizes G_{MFC} and minimizes

$$\sup_{r \in \mathcal{P}} \inf_{K_{MFC}} \left\{ \|v\|_{\mathcal{P}}^2 - \gamma^2 \|r\|_{\mathcal{P}}^2 \right\}$$

where w_0 is white noise and has unit spectral density and r is allowed to be either independent of w_0 or dependent causally on w_0.

The next lemma follows from standard min-max optimization theory.

Lemma 5: Suppose the Plant G_{MFC} is given as above. Then

$$\begin{aligned}\sup_{r\in\mathcal{P}}&\left\{ \|v\|_{\mathcal{P}}^2 - \gamma^2\|r\|_{\mathcal{P}}^2 \,\middle|\, K_{MFC}\,\text{given}\right\}\\ &\geq \sup_{r\in\mathcal{P}}\inf_{K_{MFC}}\left\{\|v\|_{\mathcal{P}}^2-\gamma^2\|r\|_{\mathcal{P}}^2\right\}\\ &\geq \inf_{K_{MFC}}\left\{\|v\|_{\mathcal{P}}^2-\gamma^2\|r\|_{\mathcal{P}}^2 \,\middle|\, r\,\text{given}\right\}.\end{aligned}$$

The solution to the MFC problem involves following equations in unknowns L, $Y \geq 0$, and $P \geq 0$

$$(L_Y)\quad Y\big(LR_0 + B_0D_{20}' + PE_2' + \gamma^{-2}PYLR_1 + \gamma^{-2}PYB_1D_{21}'\big) = 0$$

$$(Y)\quad Y(A_t+LE_2)+(A_t+LE_2)'Y + \gamma^{-2}Y(B_1+LD_{21})(B_1 + LD_{21})'Y + E_1'E_1 = 0$$

$$A_t+LE_2+\gamma^{-2}(B_1+LD_{21})(B_1+LD_{21})'Y \text{ is stable}$$

$$\begin{aligned}(P)\quad &\{A_t + LE_2 + \gamma^{-2}(B_1 + LD_{21})(B_1 + LD_{21})'Y\}P\\ &+P\{A_t+LE_2+\gamma^{-2}(B_1+LD_{21})(B_1+LD_{21})'Y\}'\\ &+ (B_0 + LD_{20})(B_0 + LD_{20})' = 0.\end{aligned}$$

Note that since $A_t + LE_2 + \gamma^{-2}(B_1 + LD_{21})(B_1 + LD_{21})'$ is stable, it follows that $(A_t + LE_2, \gamma^{-1}(B_1 + LD_{21})'Y)$ is detectable. This, in turn, implies that $A_t + LE_2$ is stable since $Y \geq 0$.

The following theorem gives necessary and sufficient conditions for the inequalities in Lemma 5 to be equalities under an assumption on the optimal controller structure.

Theorem 6: There exists an optimal MFC controller in the form of $K_{MFC} = \begin{bmatrix} L\\ 0\end{bmatrix}$ if and only if there exist $Y \geq 0$ and $P \geq 0$, which, together with L, satisfy (L_Y), (Y), and (P). Furthermore if L, $Y \geq 0$, and $P \geq 0$ satisfy (L_Y), (Y), and (P), then see (5) at the bottom of the page where x is the state of the system G_{MFC} and $\hat{x}$ is obtained from

$$\dot{\hat{x}} = (A_t + LE_2)\hat{x} - Ly + [I \quad 0]u.$$

Proof: (Necessity) If $u = \begin{bmatrix} L\\ 0\end{bmatrix}y$ is an optimal control law, then the closed-loop system can be written as

$$\begin{aligned}\dot{x} &= (A_t + LE_2)x + (B_0 + LD_{20})w_0 + (B_1 + LD_{21})r\\ v &= E_1x\end{aligned}$$

with $A_t + LE_2$ stable.

From the previous analysis results in Section II, we know that there exists a $Y \geq 0$ such that (Y) is satisfied and the cost is

$$\begin{aligned}J &:= \sup_{r\in\mathcal{P}}\left\{ \|v\|_{\mathcal{P}}^2 - \gamma^2\|r\|_{\mathcal{P}}^2 \mid K_{MFC} = \begin{bmatrix} L\\ 0\end{bmatrix}\right\}\\ &= \text{Trace}\{Y(B_0 + LD_{20})(B_0 + LD_{20})'\}.\end{aligned}$$

This is a constrained minimization problem with cost function J and constraint (Y), we may apply Lemma 7 in the Appendix to this problem. To apply the lemma, we need to check the regularity condition first. Let $\Psi := [Y \quad L]$, and

$$\begin{aligned}G(\Psi) &:= Y(A_t + LE_2) + (A_t + LE_2)'Y\\ &\quad+ \gamma^{-2}Y(B_1 + LD_{21})(B_1 + LD_{21})'Y + E_1'E_1\end{aligned}$$

and denote

$$\begin{aligned}\tilde{A}_t &:= A_t + \gamma^{-2}B_1B_1'Y + \gamma^{-2}B_1D_{21}'L'Y\\ \tilde{E}_2 &:= E_2 + \gamma^{-2}D_{21}B_1'Y + \gamma^{-2}R_1L'Y.\end{aligned}$$

Then

$$\frac{\partial}{\partial\Psi}\text{Trace}\{G(\Psi)\Pi\}=\left[\tfrac{\partial}{\partial Y}\text{Trace}\{G(\Psi)\Pi\}\ \tfrac{\partial}{\partial L}\text{Trace}\{G(\Psi)\Pi\}\right]$$

and

$$\frac{\partial}{\partial Y}\text{Trace}\{G(\Psi)\Pi\} = (\tilde{A}_t + L\tilde{E}_2)\Pi + \Pi(\tilde{A}_t + L\tilde{E}_2)' = 0$$

has a unique solution $\Pi = 0$ since $\tilde{A}_t + L\tilde{E}_2 = A_t + LE_2 + \gamma^{-2}(B_1+LD_{21})(B_1+LD_{21})'Y$ is stable. Hence the regularity condition is satisfied. Let P be the Lagrange multiplier and let

$$\begin{aligned}\hat{J} = \text{Trace}\{&Y(B_0 + LD_{20})(B_0 + LD_{20})'\\ &+ [Y(A_t + LE_2) + (A_t + LE_2)'Y\\ &+ \gamma^{-2}Y(B_1 + LD_{21})(B_1 + LD_{21})'Y + E_1'E_1]P\}.\end{aligned}$$

Now using Lemma 7, the necessary conditions for L, Y, and P being admissible are

$$\frac{\partial\hat{J}}{\partial L} = 0,\quad \frac{\partial\hat{J}}{\partial P} = 0,\ \text{and}\ \frac{\partial\hat{J}}{\partial Y} = 0.$$

These derivatives generate exactly (L_Y), (Y), and (P).

(Sufficiency) We only need to show that (5) holds if L, Y, and P satisfy (L_Y), (Y), and (P) respectively. This is done by actually computing the costs on each side of (5) and showing that they are equal. Note first that by using (P) and (Y), the cost J can be rewritten as

$$\begin{aligned}\sup_{r\in\mathcal{P}}&\left\{ \|v\|_{\mathcal{P}}^2 - \gamma^2\|r\|_{\mathcal{P}}^2 \,\middle|\, K_{MFC} = \begin{bmatrix} L\\ 0\end{bmatrix}\right\}\\ &= \text{Trace}(PE_1'E_1) - \gamma^{-2}\\ &\quad\ \text{Trace}\{PY(B_1 + LD_{21})(B_1 + LD_{21})'Y\}.\end{aligned}$$

To compute the right-hand side of (5), let $e := x - \hat{x}$, then

$$\begin{aligned}\dot{e} &= \{\tilde{A}_t + L\tilde{E}_2\}e + (B_0 + LD_{20})w_0\\ v &= E_1x + [0 \quad I]u.\end{aligned}$$

$$\begin{aligned}\sup_{r\in\mathcal{P}} l&\left\{ \|v\|_{\mathcal{P}}^2 - \gamma^2\|r\|_{\mathcal{P}}^2 \,\middle|\, K_{MFC} = \begin{bmatrix} L\\ 0\end{bmatrix}\right\} = \sup_{r\in\mathcal{P}}\inf_{K_{MFC}}\left\{\|v\|_{\mathcal{P}}^2-\gamma^2\|r\|_{\mathcal{P}}^2\right\} = \inf_{K_{MFC}}\left\{\|v\|_{\mathcal{P}}^2-\gamma^2\|r\|_{\mathcal{P}}^2 \mid r\right.\\ &\left.= \gamma^{-2}(B_1 + LD_{21})'Y(x - \hat{x})\right\} \qquad (5)\end{aligned}$$

So $e \in \mathcal{P}$ and e is independent of the control action u! Hence

$$\inf_{K_{MFC}} \left\{ \|v\|_{\mathcal{P}}^2 - \gamma^2 \|r\|_{\mathcal{P}}^2 \,|\, r = \gamma^{-2}(B_1 + LD_{21})'Y(x - \hat{x}) \right\}$$

$$= \inf_{K_{MFC}} \left\{ \|E_1 x + [0 \quad I]u\|_{\mathcal{P}}^2 \right\} - \gamma^{-2} \|(B_1 + LD_{21})'Ye\|_{\mathcal{P}}^2. \tag{6}$$

It is easy to see that

$$\|(B_1 + LD_{21})'Ye\|_{\mathcal{P}}^2 = \mathrm{Trace}\{PY(B_1 + LD_{21})(B_1 + LD_{21})'Y\}.$$

Thus it suffices to show that

$$\inf_{K_{MFC}} \left\{ \|E_1 x + [0 \quad I]u\|_{\mathcal{P}}^2 \right\} \geq \mathrm{Trace}(PE_1'E_1). \tag{7}$$

It is clear that the above problem is a standard estimation problem. The solution is

$$u = K_{FMC}(s)y = -\begin{bmatrix} 0 \\ I \end{bmatrix} E_1 \tilde{x}$$

where $\tilde{x}$ is an optimal estimate of x in the sense that $\|E_1(x - \tilde{x})\|_{\mathcal{P}}$ is minimized. Now define

$$\tilde{e} := \tilde{x} - \hat{x}$$

and

$$\tilde{y} := y + E_2 \hat{x}.$$

Then the system equations can be written as

$$\dot{e} = \{\tilde{A}_t + L\tilde{E}_2\}e + (B_0 + LD_{20})w_0 \tag{8}$$

$$v = E_1 e - E_1 \tilde{e} \tag{9}$$

$$\tilde{y} = \tilde{E}_2 e + D_{20} w_0. \tag{10}$$

Note that $\hat{x}$ is constructed from the measurement y. Hence both $\hat{x}$ and $\tilde{y}$ are known quantities. Then the estimation problem becomes finding an optimal estimate $\tilde{e}$ such that $\|E_1(e - \tilde{e})\|_{\mathcal{P}}$ is minimized.

To compute $\|E_1(e - \tilde{e})\|_{\mathcal{P}}$, we shall consider two cases:

a) $(E_1, A_t + LE_2)$ is not completely observable: Partition e and $\tilde{e}$ as

$$e = \begin{bmatrix} e_1 \\ e_2 \end{bmatrix}, \qquad \tilde{e} = \begin{bmatrix} \tilde{e}_1 \\ \tilde{e}_2 \end{bmatrix}$$

and matrices A_t, L, B_0, B_1, and E_2 correspondingly as

$$A_t = \begin{bmatrix} A_{t11} & A_{t12} \\ A_{t21} & A_{t22} \end{bmatrix}, \; L = \begin{bmatrix} L_1 \\ L_2 \end{bmatrix}, \; B_0 = \begin{bmatrix} B_{01} \\ B_{02} \end{bmatrix},$$

$$B_1 = \begin{bmatrix} B_{11} \\ B_{12} \end{bmatrix}, \quad E_2 = [E_{21} \quad E_{22}].$$

We can assume without loss of generality (by a similarity transformation) that

$$A_t + LE_2 = \begin{bmatrix} \hat{A}_{t11} & 0 \\ \hat{A}_{t21} & \hat{A}_{t22} \end{bmatrix} := \begin{bmatrix} A_{t11} + L_1 E_{21} & 0 \\ A_{t21} + L_2 E_{21} & A_{t22} + L_2 E_{22} \end{bmatrix}$$

and

$$E_1 = [E_{11} \quad 0]$$

with $(E_{11}, \hat{A}_{t11})$ completely observable. It is easy to see that

$$\ker(Y) = \text{unobservable subspace of } (E_1, A_t + LE_2)$$

and

$$Y = \begin{bmatrix} Y_{11} & 0 \\ 0 & 0 \end{bmatrix}$$

where $Y_{11} > 0$ is the solution of the Riccati equation

$$Y_{11}\hat{A}_{t11} + \hat{A}_{t11}'Y_{11} + \gamma^{-2}Y_{11}(B_{11} + L_1 D_{21}) \cdot (B_{11} + L_1 D_{21})'Y_{11} + E_{11}'E_{11} = 0.$$

Now let

$$\tilde{E}_{21} := E_{21} + \gamma^{-2}(B_{12} + L_2 D_{21})(B_{11} + L_1 D_{21})'Y_{11}$$

$$\bar{A}_{t11} := \hat{A}_{t11} + \gamma^{-2}(B_{11} + L_1 D_{21})(B_{11} + L_1 D_{21})'Y_{11}$$

$$\bar{A}_{t21} := \hat{A}_{t21} + \gamma^{-2}(B_{12} + L_2 D_{21})(B_{11} + L_1 D_{21})'Y_{11}.$$

Then

$$\tilde{E}_2 = [\tilde{E}_{21} \quad E_{22}]$$

$$\tilde{A}_t + L\tilde{E}_2 = A_t + LE_2 + \gamma^{-2}(B_1 + LD_{21})(B_1 + LD_{21})'Y = \begin{bmatrix} \bar{A}_{t11} & 0 \\ \bar{A}_{t21} & \hat{A}_{t22} \end{bmatrix}$$

and it is easy to see that $\hat{A}_{t11}$, $\bar{A}_{t11}$, and $\hat{A}_{t22}$ are all stable from the stability condition of (Y).

Let P be partitioned compatible as

$$P = \begin{bmatrix} P_{11} & P_{12} \\ P_{12}' & P_{22} \end{bmatrix} \geq 0.$$

Then the corresponding equations for L_1 and P_{11} from (L_Y) and (Y) can be simplified as

$$L_1 R_0 + B_{01}D_{20}' + P_{11}\tilde{E}_{21}' = 0 \tag{11}$$

$$\bar{A}_{t11}P_{11} + P_{11}\bar{A}_{t11}' + (B_{01} + L_1 D_{20})(B_{01} + L_1 D_{20})' = 0. \tag{12}$$

Now note that $E_1(e - \tilde{e}) = E_{11}(e_1 - \tilde{e}_1)$

Hence we only need to show that

$$\|E_1(e_1 - \tilde{e}_1)\|_{\mathcal{P}}^2 \geq \mathrm{Trace}(P_{11}E_{11}'E_{11}).$$

To show that, let us consider the equation for e_1

$$\dot{e_1} = \bar{A}_{t11}e_1 + (B_{01} + L_1 D_{20})w_0$$

$$v = E_{11}e_1 - E_{11}\tilde{e}_1$$

$$\tilde{y}_1 := \tilde{y} - E_{22}e_2 = \tilde{E}_{21}e_1 + D_{20}w_0.$$

Since the cost function does not depend on e_2 and the equation for e_1 is decoupled from e_2, estimating e_1 and e_2 together will not reduce the cost function comparing with estimating e_1 alone and assuming e_2 is known. Now suppose that e_2 is known. Then $\tilde{y}_1$ is known and from the standard Kalman filtering theory, we know that an optimal estimate for e_1 is given by

$$\dot{\tilde{e}}_1 = \bar{A}_{t11}\tilde{e}_1 + \tilde{L}_1(\tilde{E}_{21}\tilde{e}_1 - \tilde{y}_1)$$

where $\tilde{L}_1$ together with some $\tilde{P}_{11} \geq 0$ satisfies the following equations

$$\tilde{L}_1 R_0 + (B_{01} + L_1 D_{20})D'_{20} + \tilde{P}_{11}\tilde{E}'_{21} = 0 \quad (13)$$

$$(\bar{A}_{t11} + \tilde{L}_1\tilde{E}_{21})\tilde{P}_{11} + \tilde{P}_{11}(\bar{A}_{t11} + \tilde{L}_1\tilde{E}_{21})' + ((B_{01} + L_1 D_{20}) + \tilde{L}_1 D_{20}) ((B_{01} + L_1 D_{20}) + \tilde{L}_1 D_{20})' = 0 \quad (14)$$

and the optimal cost $\|E_{11}(e_1 - \tilde{e}_1)\|_{\mathcal{P}}^2 = \text{Trace}\left(\tilde{P}_{11}E'_{11}E_{11}\right)$. Compare the above two equations with (11) and (12), we have $\tilde{L}_1 = 0$ and $\tilde{P}_{11} = P_{11}$. Hence the conclusion follows.

b) $(E_1, A_t + LE_2)$ is observable: then $Y > 0$ and $e_1 = e$. The conclusion follows from part a). □

It should be pointed out that, contrary to our early assertion in [26], the existence of solutions to (L_Y), (Y) and (P) does not necessarily imply the existence of solutions to (L), (Y), and (P), where

$$(L) \quad LR_0 + B_0 D'_{20} + PE'_2 + \gamma^{-2}PYLR_1 + \gamma^{-2}PYB_1D'_{21} = 0$$

As a counterexample, let $\gamma = 1$ and

$$A_t = \begin{bmatrix} -2.5 & 0 \\ -2\sqrt{3} & -2 \end{bmatrix}, B_0 = \begin{bmatrix} \sqrt{3} \\ 2 \end{bmatrix}, B_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$E_1 = [2 \quad 0], E_2 = [-\sqrt{3} - 1 \quad -1], D_{20} = 1, D_{21} = 1.$$

Then it is easy to check that

$$L = \begin{bmatrix} 0 \\ L_2 \end{bmatrix}, Y = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, P = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

for any $L_2 > -2$ satisfy (L_Y), (Y), and (P). Moreover

$$A_t + LE_2 = \begin{bmatrix} -2.5 & 0 \\ -2\sqrt{3} - (\sqrt{3}+1)L_2 & -(2 + L_2) \end{bmatrix}$$

and

$$A_t + LE_2 + \gamma^{-2}(B_1 + LD_{21})(B_1 + LD_{21})'Y = \begin{bmatrix} -1.5 & 0 \\ -\sqrt{3}(2 + L_2) & -(2 + L_2) \end{bmatrix}$$

are both stable as required.

On the other hand, the solutions to (L), (Y), and (P) are given by

$$L = \begin{bmatrix} 0 \\ -2 \end{bmatrix}, \quad Y = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad P = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

which make both $A_t + LE_2$ and $A_t + LE_2 + \gamma^{-2}(B_1 + LD_{21})(B_1 + LD_{21})'Y$ have an eigenvalue at origin. Hence the strict stability requirement in (Y) is not satisfied.

It can indeed be shown using the same partition as in the proof of the above theorem that the existence of solutions to (L_Y), (Y), and (P) implies the existence of solutions to (L), (Y), and (P) with weakened stability condition, i.e., $A_t + LE_2 + \gamma^{-2}(B_1 + LD_{21})(B_1 + LD_{21})'Y$ is only required to have all the eigenvalues in the closed left-half plane. We will not pursue that further here.

Note that the necessary conditions in Theorem 6 are obtained by assuming that the MFC controller has a particular structure. The following lemma shows when the MFC controller has that form.

Lemma 6: Suppose $\gamma > \gamma_\infty$ ($\mathcal{H}_\infty$ optimal level). Let $\mathcal{L} := \{L_f\}$ be the set of $\mathcal{H}_\infty$ full control constant controllers, i.e., L_f is such that $A_t + L_fE_2$ is stable and there exists a $Y_f \geq 0$ to

$$Y_f(A_t + L_fE_2) + (A_t + L_fE_2)'Y_f + \gamma^{-2}Y_f(B_1 + L_fD_{21})(B_1 + L_fD_{21})'Y_f + E'_1E_1 = 0$$

and $A_t + L_fE_2 + \gamma^{-2}(B_1 + L_fD_{21})(B_1 + L_fD_{21})'Y_f$ is stable. Now define

$$J(L_f) := \sup_{r \in \mathcal{P}} \left\{ \|v\|_{\mathcal{P}}^2 - \gamma^2\|r\|_{\mathcal{P}}^2 \,\middle|\, K_{MFC} = \begin{bmatrix} L_f \\ 0 \end{bmatrix} \right\} = \text{Trace}\{Y_f(B_0 + L_fD_{20})(B_0 + L_fD_{20})'\}.$$

Let

$$J_{opt} = \inf_{L_f \in \mathcal{L}} J(L_f)$$

Now if J_{opt} is achieved by L in the interior of $\mathcal{L}$, then there exist L, $Y \geq 0$, and $P \geq 0$ satisfying (L_Y), (Y), and (P).

Proof: Since J_{opt} is achieved by L in the interior of $\mathcal{L}$, the Lagrange multiplier technique in the proof of Theorem 6 can be used to reach the desired conclusion. □

Now the question is whether $J(L_f)$ is minimized at an interior point. It is noted that the boundary of the set $\mathcal{L}$ is the set of L_f that make the J_∞-Hamiltonian associated with pure $\mathcal{H}_\infty$ problem have $j\omega$ eigenvalues. Thus the feasible set of L_f is an open set and will be typically unbounded as well. In the above example, one could form a sequence of L_f that converge to the optimal J_{opt} and to the boundary. One problem is that $J(L_f)$ does not become unbounded as L_f tends to infinity or as L_f tends to the boundary because at the limiting values the ARE for Y_f will still have a solution (not strictly stabilizing solution). The exact conditions for which $J(L_f)$ is minimized at an interior point are as yet not known.

Combining Theorem 6 and Lemma 6 we get the main result of this section.

Theorem 7: Suppose that $\inf_{L_f \in \mathcal{L}} J(L_f)$ is achieved by an interior point L. Then there exists a controller solving the MFC problem if and only if there exist constant matrices L, $Y \geq 0$, and $P \geq 0$ solving (L_Y), (Y), and (P). Moreover, in this case the optimal controller is given by $K_{MFC} = \begin{bmatrix} L \\ 0 \end{bmatrix}$.

E. Explicit State Space Formulas for Mixed Control

In this section, we give an explicit formulas for mixed norm synthesis. The formulas are obtained from combining Theorem 5 and Theorem 7. Our purpose is to get some explicit comparisons with $\mathcal{H}_2$ and $\mathcal{H}_\infty$ results.

Theorem 8: Given $\gamma > 0$ and Plant G, there exists a controller $K(s)$ which solves Problem (G) only if the following condition hold:

i) $H_\infty \in \mathrm{dom}(Ric)$ and $X_\infty := Ric(H_\infty) \geq 0$. Furthermore, the controller exists if the following additional condition is satisfied:

ii) There exist L, Y, and P which satisfy

$$Y(LR_0 + B_0D_{20}' + PC_2' + \gamma^{-2}PX_\infty B_1D_{21}' + \gamma^{-2}PYLR_1 + \gamma^{-2}PYB_1D_{21}') = 0$$

$$YA_{ml} + A_{ml}'Y + \gamma^{-2}Y(B_1 + LD_{21})(B_1 + LD_{21})'Y + F_\infty' F_\infty = 0$$

$Y \geq 0$ and $A_{ml} + \gamma^{-2}(B_1 + LD_{21})(B_1 + LD_{21})'Y$ is stable

$$\{A_{ml} + \gamma^{-2}(B_1 + LD_{21})(B_1 + LD_{21})'Y\}P + P\{A_{ml} + \gamma^{-2}(B_1 + LD_{21})(B_1 + LD_{21})'Y\}' + (B_0 + LD_{20})(B_0 + LD_{20})' = 0.$$

Condition ii) is also necessary if $J(L_f)$ defined in the last subsection is minimized in the interior of the feasible set.

Moreover, whenever Conditions i) and ii) hold, one such controller is given by

$$K(s) := \left[\begin{array}{c|c} A_{ml} + B_2F_\infty & -L \\ \hline F_\infty & 0 \end{array}\right]$$

where $A_{ml} = A + \gamma^{-2}B_1B_1'X_\infty + L(C_2 + \gamma^{-2}D_{21}B_1'X_\infty)$ and $F_\infty = -(D_{12}'C_1 + B_2'X_\infty)$.

It is easy to see from conditions i) and ii) that the solution reduces to standard $\mathcal{H}_2$ solution when $\gamma \to \infty$.

We have noted before that the controllers characterized here and in previous sections are only optimal for a given $\gamma > \gamma_\infty$, the pure $\mathcal{H}_\infty$ optimal γ-level. To find a truly optimal mixed controller which satisfies

$$\sup_{w_1 \in \mathcal{BP}} \inf_K \|z\|_\mathcal{P}$$

we must pick an appropriate γ_{mixed} to design for. One way of obtaining this γ_{mixed} is through the following iteration: pick $\gamma > \gamma_\infty$ and compute a controller as above. Apply the analysis in Section II to the closed-loop system and determine the power of the worst case signal, $w_{1_{\text{worst}}}$. Increase or decrease γ according to whether $\|w_{1_{\text{worst}}}\|_\mathcal{P}$ is greater than or less than 1, respectively, and repeat the process. The optimal γ_{mixed} occurs when $\|w_{1_{\text{worst}}}\|_\mathcal{P} = 1$.

We shall now illustrate the use of the theorem by a simple example. Consider a generalized system given by

$$G(s) = \left[\begin{array}{c|ccc} A & B_0 & B_1 & B_2 \\ \hline C_1 & 0 & 0 & D_{12} \\ C_2 & D_{20} & D_{21} & 0 \end{array}\right] = \left[\begin{array}{cc|ccc} -2.5 & 0 & \sqrt{3} & 1 & 0 \\ -2\sqrt{3} & -2 & 2 & 0 & 0 \\ \hline 2 & 0 & 0 & 0 & 1 \\ -\sqrt{3}-1 & -1 & 1 & 1 & 0 \end{array}\right].$$

Let $\gamma = 1$. It is easy to check that $X_\infty = 0$ is the stabilizing solution for the X_∞ Riccati equation and, furthermore, from the last example

$$L = \begin{bmatrix} 0 \\ L_2 \end{bmatrix}, \quad Y = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad P = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

for any $L_2 > -2$ satisfy (L_Y), (Y), and (P). Hence we have the optimal controller given by

$$K_{opt} = \left[\begin{array}{cc|c} -2.5 & 0 & 0 \\ -2\sqrt{3} - (\sqrt{3}+1)L_2 & -(2 + L_2) & -L_2 \\ \hline -2 & 0 & 0 \end{array}\right] = 0.$$

Note that the results by Bernstein and Haddard in [2] can not be used here although our problem is dual of theirs in some sense as shown in [25]. The reason is that their results are obtained by assuming the optimal controller is minimal, which is clearly not true for this example.

V. Conclusions

In this paper, we have formulated and obtained a solution to a mixed $\mathcal{H}_2$ and $\mathcal{H}_\infty$ problem. This problem is an interesting generalization of existing $\mathcal{H}_2$ and $\mathcal{H}_\infty$ theory. An interesting feature of this problem formulation is that no stochastic concepts have been used, i.e., the problem is approached from a completely deterministic viewpoint.

From an application point of view, a major problem concerns solving the coupled Riccati equations. To that end, homotopy methods such as those used in the algorithms developed in [20] and [17] may prove useful. Since our equations are much simpler than those appearing in the oblique projection method, it is possible that special properties may be exploited and an efficient algorithm developed. This is another subject for future research.

Appendix

In this Appendix we are going to review some results from mathematical programming. The results presented here are basically the matrix version of Theorem 13.3 in Luenberger (1973).

Let $\Psi \in \mathbb{R}^{r_1 \times r_2}$ and suppose $F(\Psi) \in \mathbb{R}^{r_3 \times r_3}$, $G(\Psi) \in \mathbb{R}^{r_3 \times r_3}$ are continuously differentiable functions. $\Psi \in \mathbb{R}^{r_1 \times r_2}$

is called a regular point of the constraint $G(\Psi) = 0$ if Ψ has the following property: let $\Pi \in \mathbb{R}^{r_3 \times r_3}$, then

$$\frac{\partial}{\partial \Psi}\text{Trace}\{G(\Psi)\Pi\} = 0 \tag{15}$$

if and only if $\Pi = 0$.

Then the following theorem holds

Lemma 7: Let $\Psi \in \mathbb{R}^{r_1 \times r_2}$ be a local extremum of $\text{Trace}\{F(\Psi)\}$ subject to the constraints $G(\Psi) = 0$. Furthermore, assume Ψ is a regular point of the constraint. Then there is a $\Gamma \in \mathbb{R}^{r_3 \times r_3}$ such that

$$\frac{\partial}{\partial \Psi}\text{Trace}\{F(\Psi) + G(\Psi)\Gamma'\} = 0. \tag{16}$$

$$\frac{\partial}{\partial \Gamma}\text{Trace}\{F(\Psi) + G(\Psi)\Gamma'\} = G(\Psi) = 0. \tag{17}$$

The partial differential of a matrix trace satisfies the properties

$$\frac{\partial}{\partial \Psi}\text{Trace}\{A\Psi B\} = A'B'$$
$$\frac{\partial}{\partial \Psi}\text{Trace}\{A\Psi' B\} = BA$$
$$\frac{\partial}{\partial \Psi}\text{Trace}\{A\Psi B\Psi\} = A'\Psi'B' + B'\Psi'A'$$
$$\frac{\partial}{\partial \Psi}\text{Trace}\{A\Psi B\Psi'\} = A'\Psi B' + A\Psi B.$$

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REFERENCES

[1] B.D.O. Anderson, "An algebraic solution to the spectral factorization problem," *IEEE Trans. Automat. Contr.*, vol. AC-12, pp. 410-414, 1967.
[2] D. S. Bernstein and W.M. Haddad, "LQG control with an $\mathcal{H}_\infty$ performance bound: A Riccati equation approach," *IEEE Trans. Automat. Contr.*, vol. 34, pp. 293-305, 1989.
[3] S. Boyd, V. Balakrishnan, and P. Kabamba, "A bisection method for computing the $\mathcal{H}_\infty$ norm of a transfer matrix and related problems," *Math. Contr., Sig., Syst.*, vol. 2, no. 3, pp. 207-220, 1989.
[4] J. C. Doyle, "Analysis of feedback systems with structured uncertainties", in *IEE Proc.*, vol. 129, Part D, no. 6, pp. 242-250, 1982.
[5] J. C. Doyle, K. Glover, P.P. Khargonekar, and B.A. Francis, "State-space solutions to standard $\mathcal{H}_2$ and $\mathcal{H}_\infty$ control problems," *IEEE Trans. Automat. Contr.*, vol. 34, no. 8, pp. 831-847, 1989.
[6] J. C. Doyle, J. Wall, and G. Stein, "Performance and robustness analysis for structured uncertainty," in *Proc. 21st IEEE Conf. Decision Contr.*, 1982, pp. 629-636.
[7] J. C. Doyle, K. Zhou, and B. Bodenheimer, "Optimal Control with mixed $\mathcal{H}_2$ and $\mathcal{H}_\infty$ performance objective," in *Proc. 1989 Amer. Contr. Conf.*, Pittsburgh, PA., 1989, pp. 2065-2070.
[8] M. K. H. Fan and A.L. Tits, "Characterization and Efficient Computation of the Structured Singular Value," *IEEE Trans. Automat. Contr.*, vol. AC-31, no. 8, pp. 734-743, 1986.
[9] B. A. Francis, *A Course In $\mathcal{H}_\infty$ Control Theory*, (Lecture Notes in Control and Information Sciences) vol. 88. Berlin: Springer-Verlag, 1987.
[10] B. A. Francis and J.C. Doyle, "Linear control theory with an $\mathcal{H}_\infty$ optimality criterion," *SIAM J. Contr. Opt.*, vol. 25, pp. 815-844, 1987.
[11] K. Glover and J. Doyle, "State-space formulas for all stabilizing controllers that satisfy an $\mathcal{H}_\infty$ norm bound and relations to risk sensitivity," *Syst. Contr. Lett.*, vol. 11, pp. 167-172, 1988.
[12] K. Glover and J. C. Doyle, "A state-space approach to $\mathcal{H}_\infty$ optimal control," in *Three Decades of Mathematical Systems Theory: A Collection of Surveys at the Occasion of the 50th Birthday of Jan C. Willems*, H. Nijmeijer and J. M. Schumacher (Eds.), (Lecture Notes in Control and Information Sciences) vol. 135. Berlin: Springer-Verlag, 1989.
[13] K. Glover and D. Mustafa, "Derivation of the maximum entropy $\mathcal{H}_\infty$ controller and a state-space formula for its entropy," *Int. J. Contr.*, vol. 50, pp. 899, 1989.
[14] P. P. Khargonekar and M. A. Rotea, "Mixed $\mathcal{H}_2\mathcal{H}_\infty$ control: A convex optimization approach," *IEEE Trans. Automat. Contr.*, vol. 36, no. 7, pp. 824-837, July 1991.
[15] V. Kucera, "A contribution to matrix quadratic equations," *IEEE Trans. Automat. Contr.*, vol. AC-17, no. 3, pp. 344-347, 1972.
[16] D. G. Luenberger, *Introduction to Linear and Nonlinear Programming*. Reading, MA: Addison-Wesley, 1973.
[17] M. Mariton and R. Bertrand, "A homotopy algorithm for solving coupled Riccati equations," *Optimal. Contr. Appl. Meth.*, vol. 6, pp. 351-357, 1985.
[18] D. Mustafa, "Relations between maximum entropy / $\mathcal{H}_\infty$ control and combined $\mathcal{H}_\infty/LQG$ control," *Syst. Contr. Lett.*, vol. 12, no. 3, pp. 193, 1989.
[19] A. K. Packard, "What's new with μ: Structured uncertainty in multivariable control," Ph.D. dissertation, University of California, Berkeley, 1988.
[20] S. L. Richter, "A homotopy algorithm for solving the optimal projection equations for fixed-order dynamic compensation: Existence, convergence and global optimality," 1987 Amer. Contr. Conf., Minneapolis, MN, June 1987.
[21] M. A. Rotea and P. P. Khargonekar, "$\mathcal{H}_2$ optimal control with an $\mathcal{H}_\infty$ constraint: The state feedback case," in *Proc. 1990 Amer. Contr. Conf.*, 1990, pp. 2380-2384.
[22] M. Steinbuch and O. H. Bosgra, "Robust performance in $\mathcal{H}_2$ / $\mathcal{H}_\infty$ optimal control," in *Proc. 30th CDC*, Brighton, England, 1991, pp. 549-550.
[23] J. C. Willems, "Least-squares stationary optimal control and the algebraic Riccati equation," *IEEE Trans. Auto. Contr.*, vol. AC-16, pp. 621-634, 1971.
[24] W. M. Wonham, *Linear Multivariable Control: A Geometric Approach*, 3rd ed. New York: Springer-Verlag, 1985.
[25] H. Yeh, S. Banda, and B. Chang, "Necessary and sufficient conditions for mixed $\mathcal{H}_2$ and $\mathcal{H}_\infty$ control," *IEEE Trans. Automat. Contr.*, vol. 37, no. 3, pp. 355-358, 1992.
[26] K. Zhou, J.C. Doyle, K. Glover, and B. Bodenheimer, "Mixed $\mathcal{H}_2$ and $\mathcal{H}_\infty$ Control," Amer. Contr. Conf., 1990.
[27] ——, "Mixed $\mathcal{H}_2$ and $\mathcal{H}_\infty$ performance objectives I: robust performance analysis," *IEEE Trans. Automat. Control*, submitted.

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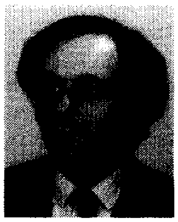

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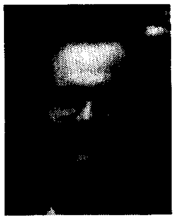

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