# Technical Notes and Correspondence 

# The Use of a Bilinear Transformation of the Shift Operator in Subspace Model Identification 

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#### Abstract

We propose a mechanism which can improve the numerica robustness of a subspace based system identification method, the PI scheme [14, Part III], when the unknown system has poles situated close to $:=1$, a condition that often arises in applications where the sampling rate is too high. The PI method is capable of solving a deterministic MIMO identification problem in which the output can be corrupted by a very general perturbation including arbitrarily colored noise, transients due to nonzero initial conditions, and a deterministic zero bias. By performing a bilinear transformation on the shift operator we are able to move the poles away from the point $z=1$ and a more robust identification results. The implementation of this transformation gives rise to a series of anticausal filters applied to the input/output data. Estimation accuracy is further improved by taking the unknown end conditions of the anticausal filters into account, particularly when only short data records are available. A numerical simulation highlights the improvements realized by our new algorithms.


## I. Introduction

In linear system identification, the generality of an identification problem depends on the restrictions on the structure of the model to be identified and on the type of perturbations permitted on the input and output data. In this paper, we focus on the following general identification problem:

Given a linear finite dimensional multiple-input, multiple-output (MIMO) state-space model that determines the linear stable operator $G$ by

$$
g=G(u):\left\{\begin{array}{l}
x_{k+1}=A x_{k}+B u_{k}  \tag{1}\\
g_{k}=C x_{k}+D u_{k}
\end{array}\right.
$$

and the measured output $y_{k}$ is related to the output of $G$ by the expression

$$
\begin{equation*}
y_{k}=g_{k}+\nu_{k} \tag{2}
\end{equation*}
$$

the task is to estimate this linear operator using the sequences of input and output data: $\left[u_{1}, u_{2}, \cdots, u_{N}\right]$ and $\left[\begin{array}{llll}y_{1}, & y_{2}, & \cdots, & y_{N}\end{array}\right]$ under the following constraints a) the input ${ }^{\prime}{ }_{k}$ is zero-mean, sufficiently rich ${ }^{1}$ (persistently exciting), and exactly known, ${ }^{2}$ b) a nonzero initial condition may be present, c) the additive perturbations $\nu_{k}$ in (2) can be the sum of: c1) zero-mean stochastic process of arbitrary color and (statistically)

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${ }^{1}$ For a discussion of the persistency of excitation condition in the context of the PI scheme analyzed in this paper, we refer to [14, Part III].
${ }^{2}$ Because of the linearity of the operator $G$ in (1), this assumption does not exclude the presence of zero-mean error of arbitrary color introduced by the transducers [15].
independent from the input $u_{k}$, and c 2 ) a deterministic (constant, periodic, etc.) zero bias such that $\nu_{k}$ is statistically independent from $u_{k}$ or such that $\lim _{N \rightarrow \infty} \sum_{j=1}^{N} u_{k+j} \nu_{\ell+j}^{T}=0$ for $\forall k, \ell$ in case both quantities are deterministic.
In many applications it is the linear operator, $G$ in (1), that is the chief quantity of interest. If the disturbance, $\nu_{k}$ in (2), is noise produced by the data acquisition apparatus, or perhaps by a nearby but unrelated system, then its modeling is of secondary, if any, importance. In some applications the goal is to obtain insight into the system, rather than to predict its output. Examples of this include the identification of the dynamics of a distillation column in the process industry [13] as well as those of the human ankle joint [6]. There are several other applications within the field of biomedical engineering [17].

Despite the fact that the key quantity of interest in the above identification problem is a linear operator, it is well known (see, e.g., [8], [12], or [17]) that even simplified versions of the above output-error identification problem easily leads to difficult iterative optimization problems. Certainly when little (a priori) knowledge is available about the linear operator, a noniterative approach is by far more preferred.

One classical noniterative solution that solves the above identification problem consistently is to apply the Ho-Kalman realization scheme [4] and make use of estimated Markov parameters. Such an approach was evaluated using real data in [5]. Apart from the fact that this approach requires the underlying linear operator to be represented accurately by a finite impulse response (FIR), preferably of low order, this approach suffers from a number of pitfalls:
-The unknown initial conditions can only be taken into account asymptotically because the transient response in the data has to be discarded prior to estimating the FIR. For short data length batches this might lead to a drastic loss of information, especially when the underlying (discrete-time) system has its poles close to the unit circle.
-Depending on the length of the FIR and the nature of the input signal, the estimation of the FIR can easily give rise to ill-conditioned least squares problems.
In [14, Part II] it is was shown that a better (numerically more reliable) alternative to the above classical approach is via subspace model identification (SMI). This is confirmed in the papers describing other, but related SMI algorithms, such as e.g., [7] and [11]. One variant of these SMI algorithms, namely the PI scheme, allows the deterministic identification problem be addressed consistently, overcoming the difficulties with the classical scheme mentioned previously.

Despite the improved performances of the PI scheme over the classical approach, it was shown experimentally in [16] that the sensitivity of the calculations increases when the poles of the deterministic system approach very close to the unit circle, which can happen, for example, if the sampling rate is too high. In such situations the sensitivity of algorithms solving various system analysis and control design problems can be improved by the use of the so-called $\delta$ operator [9] or Laguerre polynomials [1]. Application of the latter polynomials in estimating the FIR of a linear operator is demonstrated in [18]. In this paper, we study the use of a bilinear transformation of the shift operator, which corresponds to the elementary Laguerre
polynomials used in [18], in improving the numerical sensitivity of the PI scheme.

In the next section, we present the operator theoretic framework used throughout the paper. We then consider the representation of a linear state-space model under a bilinear transformation of the shift operator. Section III establishes its use in the PI scheme, paying special attention to the realistic case in which only short data records are available. The presence of the perturbations $\nu_{k}$ is considered in Section III-C, where we study the consistency properties of the algorithms. In Section IV, a realistic simulation study is used to illustrate the improvements that can be obtained with the newly developed algorithms.

## II. Model and Data Representation

We will represent the LTI (linear time-invariant) finite dimensional system in an operator theoretic framework. Here, we denote a timesequence $r$ as a doubly infinite row vector sequence with entries in $\mathbf{R}^{\prime \prime}$

$$
x=\left[\begin{array}{lllll}
\cdots & x_{-1} & x_{0} & x_{1} & \cdots
\end{array}\right]
$$

The element in the square box is the element at time zero. Applying the shift operator $Z$

$$
x^{Z} Z=\left[\begin{array}{lllll}
\cdots & x_{0} & x_{1} & x_{2} & \cdots
\end{array}\right]
$$

Clearly, the shift operator $Z$ represents an anticausal system. In general, an operator representing an anticausal linear system has a lower triangular matrix representation. To denote finite segments of these double infinite time-sequences, we introduce the notation $[.]_{j, k}$ for $k>j$. In this way

$$
\left[\begin{array}{llll}
{[r]_{j, k}} & =\left[\begin{array}{llll}
x_{j}, & x_{j+1} & \cdots & x_{k}
\end{array}\right]
\end{array}\right.
$$

and $[x]_{k}$ equals $x_{k}$. This notation can also be used to take a finite time-window out of a matrix.

Let $u$ and $y$ denote $t_{2}$ sequences, with elements in $\mathbb{R}^{m}$ and $\mathbb{R}^{l}$, respectively; then we are able to denote the classical discrete-time state-space model as

$$
\begin{align*}
r Z & =A x+B u \\
y & =C x+D u \tag{3}
\end{align*}
$$

If $A$ is asymptotically stable then $x$ will also be an $l_{2}$-sequence. Next, consider a bilinear transformation in the complex plane

$$
\begin{equation*}
w=\frac{z-a}{1-a z} \quad \Leftrightarrow \quad z=\frac{u+a}{1+a u} \quad-1<a<1 \quad u^{2}: \in \in \mathbb{C} \tag{4}
\end{equation*}
$$

which maps the unit circle unto itself. In operator format, we have [17]

$$
\mathbb{W}=(Z-a I)(I-a Z)^{-1} \quad \text { and } \quad Z=(W+a I)(I+a W)^{-1}
$$

Theorem 1: Let the discrete-time and asymptotically stable statespace model induced by the shift operator $Z$ be given as in (3). let the bilinear transformation $w$ be given as in (4), and let the state-space model induced by this operator $W^{-}$be denoted as

$$
\begin{align*}
x W & =F x+G u(I+a W) \\
y(I+a W) & =H x+J u(I+a W) \tag{5}
\end{align*}
$$

Then:

1) The system matrix $F$ is asymptotically stable. and
2) The quadruple of system matrices $\left[\begin{array}{ll}A & B \\ C & D\end{array}\right]$ and $\left[\begin{array}{cc}F & C \\ H & j\end{array}\right]$ are related as follows

$$
\begin{align*}
A & =(F+a I)(I+a F)^{-1} \\
B & =\left(1-a^{2}\right)(I+a F)^{-1} G \\
C & =H(I+a F)^{-1} \\
D & =J-a H(I+a F)^{-1} G  \tag{6}\\
F & =(I-a A)^{-1}(A-a I) \\
G & =(I-a A)^{-1} B \\
H & =\left(1-a^{2}\right) C(I-a A)^{-1} \\
J & =D+a C(I-a A)^{-1} B
\end{align*}
$$

Proof: The proof is elementary and given in [17].
The observability and controllability of the state-space model (5) is defined in a similar way as with ordinary state-space models induced by the shift operator $Z$. Based on these definitions, it is easy to show that the minimality of the state-space quadruple $[F, G . H . J]$ is equivalent to that of the original system $[A, B . C, D][17]$.

## III. The Modification of the PI Scheme Under the Bilinear Transformation

In this section we derive the PI realization method, under bilinear transformation (4). This new algorithm will be termed the PIBTZ scheme, as it is a variant of the PI scheme under a bilinear transformation of the shift operator $\underline{Z}$. For a description of the PI scheme we refer to [14, Part III].

The first step in the derivation of a subspace model identification type of algorithm is the organization of the input-output data into structured (Hankel) matrices. For state-space model (3), this organization leads to the following data equation, given in operator form

$$
\begin{align*}
& {\left[\begin{array}{c}
y \\
y Z \\
\vdots \\
y Z^{s-1}
\end{array}\right]=\left[\begin{array}{c}
C \\
C A \\
\vdots \\
C A^{s-1}
\end{array}\right] x+} \\
& {\left[\begin{array}{ccc}
D \\
C B & D & \\
\vdots & \ddots & \\
C A^{s-2} B & C B & D
\end{array}\right]\left[\begin{array}{c}
u \\
u Z \\
\vdots \\
u Z^{s-1}
\end{array}\right]} \tag{7}
\end{align*}
$$

Similarly, the state-space model induced by the bilinear transformation (5) gives rise to

$$
\begin{align*}
& {\left[\begin{array}{c}
y(I+a W) \\
y(I+a W) W \\
\vdots \\
y(I+a W) W^{s-1}
\end{array}\right]=\left[\begin{array}{c}
H \\
H F \\
\vdots \\
H F^{s-1}
\end{array}\right] x+} \\
& {\left[\begin{array}{ccc}
J \\
H G & J & \\
\vdots & \ddots & \\
H F^{s-2} G & H G & J
\end{array}\right]\left[\begin{array}{c}
u(I+a W) \\
u(I+a W) W \\
\vdots \\
u(I+a W) W^{s-1}
\end{array}\right]} \tag{8}
\end{align*}
$$

Applying finite length time-windows to (7) and (8) results in relationships between finite sized matrices, such as (7) in [14, part I].

## A. The PI-BTZ Realization Scheme

Let us define the matrices $\left[L_{u}^{-}\right]_{p}$ and $\left[L_{u}\right]_{f}$ as

$$
\begin{aligned}
& {\left[U_{u}\right]_{F}=\left[\begin{array}{c}
u\left(I+a W^{\prime}\right) \\
\vdots \\
u\left(I+a W^{\prime}\right) W^{s-1}
\end{array}\right]_{1 \ldots}} \\
& {\left[{U_{u}^{-}}^{-}\right]_{f}=\left[\begin{array}{c}
u(I+a W) W^{-s} \\
\vdots \\
u(I+a W) W^{2 s-1}
\end{array}\right]_{1 \ldots}}
\end{aligned}
$$

and let $\left[Y_{\mu}\right]_{f}$ be defined as $\left[L_{w}^{*}\right]_{f}$ but from the sequence $y$. We can then write data equation (8), multiplied on the right by $W^{-s}$, as

$$
\begin{equation*}
\left[Y_{u} \cdot\right]_{f}=\Gamma_{u} \cdot\left[x W^{-s}\right]_{1 . N}+H_{u}\left[L_{u} \cdot\right]_{f} . \tag{9}
\end{equation*}
$$

The rows of $\left[C_{u}^{-}\right]_{f}$ can be obtained by filtering the input sequence with an anticausal filter. Thus, the sequence $[u(I+a W)]_{1 . N}$ is generated by the filter

$$
\begin{equation*}
\zeta_{k}^{0}=a \zeta_{k+1}^{0}+u_{k}\left(1-a^{2}\right) \quad \text { with end condition } \quad \zeta_{v+1} \tag{10}
\end{equation*}
$$

Similarly, $\left[y\left(I+a W^{-}\right)\right]_{1, N}$ is generated by the filter

$$
\begin{equation*}
\xi_{k}^{0}=a \xi_{k+1}^{0}+y_{k}\left(1-a^{2}\right) \quad \text { with end condition } \xi_{N+1} \tag{11}
\end{equation*}
$$

The sequences $\left[u\left(I+a W^{*}\right) W^{-j}\right]_{1, N}$ and $\left[y\left(I+a W^{j}\right) W^{j}\right]_{1 \ldots}$ are generated by the following series of filter operations

$$
\begin{align*}
& \zeta_{k}^{i}=a \zeta_{k+1}^{i}+\zeta_{k+1}^{i-1}-a \zeta_{k}^{i-1} \\
& \quad \text { with end condition } \zeta_{N+1}^{i}  \tag{12}\\
& \xi_{k}^{i}=a \xi_{k+1}^{i}+\xi_{k+1}^{i-1}-a \xi_{k}^{i-1} \\
& \quad \text { with end condition } \xi_{N+1}^{i} \text { for } i=1: j \tag{13}
\end{align*}
$$

These time sequences cannot be computed explicitly since the end conditions of the anticausal filters are unknown. As stated in [18], the effect of these end conditions will vanish with increasing record length. As illustrated in [17], however, when dealing with short data records the end conditions need to be taken into account, otherwise extreme biases can result. To cope with the end condition in the PI-BTZ scheme, we split the anticausal filter outputs into two components: a "forced response," due to the input, and a "transient response," due to the end conditions. We shall use a hat to denote terms in the forced response. Then, the output of the filter (10) can be written as

$$
\begin{array}{rllll}
{\left[\begin{array}{llllll}
\zeta_{1}^{0} & \cdots & \zeta_{Y-2}^{0} & \zeta_{Y-1}^{0} & \zeta_{N}^{0}
\end{array}\right]=} & & {\left[\begin{array}{llllll}
\hat{\zeta}_{1}^{0} & \cdots & \hat{\zeta}_{N-2}^{0} & \hat{\zeta}_{N-1}^{0} & \hat{\zeta}_{N}^{0}
\end{array}\right]} \\
& +\zeta_{N+1}^{0}\left[\begin{array}{lllll}
a^{N} & \cdots & a^{3} & a^{2} & a
\end{array}\right]
\end{array}
$$

and the output of the anticausal filter (11) is

$$
\begin{aligned}
{\left[\begin{array}{lllll}
\xi_{1}^{0} & \cdots & \xi_{N-2}^{0} & \xi_{N-1}^{0} & \xi_{N}^{0}
\end{array}\right]=} & {\left[\begin{array}{lllll}
\xi_{1}^{0} & \cdots & \xi_{N-2}^{0} & \hat{\xi}_{N-1}^{0} & \hat{\xi}_{N}^{0}
\end{array}\right] } \\
& +\xi_{N+1}^{0}\left[\begin{array}{lllll}
a^{N} & \cdots & a^{3} & a^{2} & a
\end{array}\right]
\end{aligned}
$$

To define the output of filters (12) and (13), we define the sequence $\left[\begin{array}{lllll}a^{N} & \cdots & a^{3} & a^{2} & a\end{array}\right]$ as $\dot{\iota}_{a}^{0}$ and define $u_{a}^{i}$ as the output of the filter

$$
\begin{equation*}
\iota_{a}^{i}=w u_{a}^{i-1} \quad \text { for } i \geq 1 \tag{14}
\end{equation*}
$$

with zero end conditions at time instant $N+1$. Then we can write the output of anticausal filters (12) and (13) when nonzero end conditions $\zeta_{V+1}^{i}$ and $\xi_{V+1}^{i}$ are present, as

$$
\zeta_{k}^{i}=\hat{\zeta}_{k}^{i}+\sum_{j=0}^{i} \zeta_{N+1}^{(i-j)} \psi_{a, k}^{\prime j}
$$

and

$$
\xi_{k}^{i}=\hat{\xi}_{k}^{i}+\sum_{j=0}^{i} \xi_{N+1}^{(i-j)} \psi_{a, k}^{j}
$$

for $k=1:-1: 1$.

In this way the data matrices $\left[L_{u \cdot}^{-}\right]_{f},\left[U_{u x}^{-}\right]_{j}$, and $\left[Y_{u}\right]_{f}$ become equal to

$$
\begin{aligned}
& {\left[\begin{array}{l}
{\left[\tilde{C}_{u}^{-}\right]_{p}} \\
{\left[\hat{U}_{u}^{-}\right]_{f}} \\
{\left[\hat{Y}_{u}\right]_{f}}
\end{array}\right]=\left[\begin{array}{l}
{\left[\hat{U}_{u}\right]_{\mu}} \\
{\left[\hat{U}_{u}^{*}\right]_{f}} \\
{\left[\hat{\mathrm{Y}}_{u}\right]_{f}}
\end{array}\right]}
\end{aligned}
$$

If we denote the underbraced term as

$$
\left[\begin{array}{cc}
E_{\zeta .11} & 0 \\
E_{\zeta, 21} & E_{\zeta, 22} \\
E_{\xi .11} & E_{\xi, 12}
\end{array}\right] \quad\left[\begin{array}{l}
\Psi_{1} \\
\Psi_{2}
\end{array}\right]
$$

then (9) can be rewritten as
$\widehat{\left.\mathrm{Y}_{u}\right]_{f}}=\Gamma_{u}\left[r \mathrm{H}^{-s}\right]_{1 . \lambda}$

$$
+\left[H_{u} \mid\left[H_{u} E_{\zeta, 21}-E_{\xi, 11} \mid H_{u} E_{\zeta, 22}-E_{\xi, 12}\right]\right]\left[\begin{array}{l}
\left.\widehat{l_{u}}\right]_{f} \\
{\left[\begin{array}{c}
\Psi_{1} \\
\Psi_{2}
\end{array}\right]}
\end{array}\right] .
$$

Based on this data equation and the RQ factorization considered in the PI scheme [14, Part I], we will consider the following RQ factorization

$$
\left[\begin{array}{c}
{\left[\widehat{\mathcal{L}_{u}}\right]_{f}}  \tag{16}\\
{\left[\begin{array}{|}
\Psi_{1} \\
\Psi_{2}
\end{array}\right]} \\
{\left[\widehat{\left[\bar{L}_{u}\right.}\right]_{p}} \\
{\left[\widehat{\hat{Y}_{u}}\right]_{f}}
\end{array}\right]=\left[\begin{array}{cccc}
R_{11} & 0 & 0 & 0 \\
R_{21} & R_{22} & 0 & 0 \\
R_{31} & R_{32} & R_{33} & 0 \\
R_{41} & R_{42} & R_{43} & R_{44}
\end{array}\right]\left[\begin{array}{l}
Q_{1} \\
Q_{2} \\
Q_{3} \\
Q_{4}
\end{array}\right] .
$$

By the last two expressions, we have that

$$
\begin{aligned}
{\left[\hat{Y}_{u}\right]_{f}=} & \Gamma_{u!}\left[u\left[W^{-s}\right]_{1 \ldots,}+\left[H_{u} \mid\right.\right. \\
& {\left.\left[H_{u} \cdot E_{\zeta, 21}-E_{\varepsilon, 11} \mid H_{u!} E_{\zeta, 22}-E_{\varepsilon, 12}\right]\right] } \\
{\left[\begin{array}{cc}
R_{11} & 0 \\
R_{21} & R_{22}
\end{array}\right]\left[\begin{array}{l}
Q_{1} \\
Q_{2}
\end{array}\right]=} & R_{41} Q_{1}+R_{42} Q_{2}+R_{43} Q_{3}+R_{41} Q_{4} .
\end{aligned}
$$

In addition assume the following RQ factorization to be given

$$
\left[\begin{array}{c}
{\left[\widehat{U_{u x}}\right]_{f}}  \tag{17}\\
{\left[\begin{array}{c}
\Psi_{1} \\
\Psi_{2}
\end{array}\right]} \\
{\left[\widehat{U_{u}}\right]_{p,}} \\
{\left[x W^{s}\right]_{1, N}}
\end{array}\right]=\left[\begin{array}{cccc}
R_{11} & 0 & 0 & 0 \\
R_{21} & R_{22} & 0 & 0 \\
R_{31} & R_{32} & R_{333} & 0 \\
R_{x 1} & R_{s 22} & R_{x 3} & R_{x 4}
\end{array}\right]\left[\begin{array}{c}
Q_{1} \\
Q_{2} \\
Q_{3} \\
Q_{x}
\end{array}\right] .
$$

From these two expressions, we readily obtain

$$
\begin{align*}
& \Gamma_{u} R_{r 1}+H_{u} R_{11} \\
& \quad+\left[H_{u} E_{\zeta, 21}-E_{\xi, 11} \mid H_{u} E_{\zeta .22}-E_{\xi, 12}\right] R_{21}=R_{41} \\
& \Gamma_{u} R_{x 2}+\left[H_{u} E_{\zeta, 21}-E_{\xi, 11} \mid H_{u} E_{\zeta, 22}-E_{\xi .12}\right] R_{22}=R_{42} \tag{19}
\end{align*}
$$

$$
\begin{equation*}
\Gamma_{u} \cdot R_{r 3}=R_{43} . \tag{20}
\end{equation*}
$$

If $s>n$, the pair $[F H]$ is observable, and

$$
\begin{equation*}
p^{( }\left(R_{r 3}\right)=n \tag{21}
\end{equation*}
$$

then we can retrieve the column space $\Gamma_{u}$ via an SVD of the matrix $R_{43}$. Furthermore, when

$$
\begin{equation*}
R_{22} \text { is invertible } \tag{22}
\end{equation*}
$$

knowledge of the column space of $\Gamma_{u}$ allows us to reduce (19) to

$$
\begin{aligned}
& \left(\Gamma_{u}^{\perp}\right)^{T} R_{12} R_{22}^{-1} \\
& \quad=\left(\Gamma_{u}^{\perp}\right)^{T}\left[H_{u} E_{\zeta .21}-E_{\xi .11} \mid H_{u} E_{\zeta .22}-E_{\xi .12}\right]
\end{aligned}
$$

and therefore when

$$
\begin{equation*}
R_{11} \text { is invertible } \tag{23}
\end{equation*}
$$

(18) is reduced to

$$
\begin{equation*}
\left(\Gamma_{u^{\prime}}^{\perp}\right)^{T}\left[R_{41}-R_{42} R_{22}^{-1} R_{21}\right] R_{11}^{-1}=\left(\Gamma_{u}^{\perp}\right)^{T} H_{u} \tag{24}
\end{equation*}
$$

The basis of the PI-BTZ scheme, summarized in this section, are (20) and (24), which were derived from the triple set of (18)-(20). The calculation of the column space of $\Gamma_{u}$ from (20) and the formulation of (24) is feasible, when conditions (21)-(23) are satisfied. The crucial condition (21) is analyzed in Theorem 2.

Theorem 2: Let the state-space model (5) be controllable, let $s>n$. let $F^{s} \equiv 0$, and let the matrix $R_{33}$ be invertible, then condition (21) is satisfied.

Proof: With the state-space model (5) and the expression for $\left[U_{u}\right]_{p}$ given in (15) we can write $\left[x W^{-s}\right]_{1, N}$ as

$$
\left.\left.\begin{array}{l}
{\left[x W^{+s}\right]_{1 \ldots}=F^{s}[x]_{1 \ldots}+\left[F^{s-1} G\right.} \\
\cdots
\end{array}\right] \quad F G \quad G\right]\left[\begin{array}{ll}
{\left[\Psi_{i}\right.} \\
\left.\left[\widehat{U_{w}}\right]_{p}+\left[\begin{array}{ll}
E_{\zeta, 11} & 0
\end{array}\right]\left[\begin{array}{l}
\Psi_{2}
\end{array}\right]\right]
\end{array}\right.
$$

With $F^{s} \equiv 0$ and the RQ factorizations (16) and (17), we obtain the following equality

$$
\begin{aligned}
& R_{r 1} Q_{1}+R_{r 2} Q_{2}+R_{r 3} Q_{3}+R_{r-4} Q_{x} \\
& \quad=\left[\begin{array}{llll}
F^{s-1} G & \cdots & F G & G
\end{array}\right] \\
& \quad\left(R_{31} Q_{1}+R_{32} Q_{2}+R_{33} Q_{3}\right) \\
& \quad+\left[\begin{array}{llll}
F^{s-1} G & \cdots & F G & G
\end{array}\right]\left[\begin{array}{lll}
E_{\zeta .11} & 0
\end{array}\right] \\
& \quad \times\left(\begin{array}{ll}
\left.R_{21} Q_{1}+R_{22} Q_{2}\right)
\end{array}\right.
\end{aligned}
$$

Multiplying on the right with $Q_{3}^{T}$ and using the orthogonality of the different $Q_{\text {, }}$ matrices yields

$$
R_{s: 3}=\left[\begin{array}{llll}
F^{*-1} G & \cdots & F G & G
\end{array}\right] R_{33} .
$$

Therefore, the conditions of the theorem guarantee that (21) is satisfied.

One condition stipulated in Theorem 2 is $F^{*} \equiv 0$, which is generally not satisfied. The goal of the bilinear transformation, however, is to map the poles of the original system close to the origin. In which case we may assume that $F^{s}=O(\varepsilon)$, i.e., $\left\|F^{s}\right\| \leq \varepsilon$, and the theorem would still hold, since

$$
R_{x 3}=\left[\begin{array}{llll}
F^{s-1} G & \cdots & F G & G
\end{array}\right] R_{33}+O(\Xi)
$$

The experimental simulation study in Section V supports the above assertion.

Theorem 2 shows that the successful operation of the PI-BTZ scheme can be checked by evaluating the condition numbers of matrices like $R_{11}$, etc, defined in (16), which can be retrieved from the data matrices. By selecting the nature of the input signals, we can influence the condition numbers of the matrices $R_{11}$ and $R_{33}$. The condition number of $R_{22}$ is harder to control since this matrix mainly depends on the deterministic sequence $\iota^{,}, \frac{a}{a}$, defined in (14). This is not so critical, as we shall see, because we need not compute its inverse explicitly. Consider the following SVD

$$
\left[R_{21} \quad R_{22}\right]=C^{\prime} S^{\prime \prime}\left[\begin{array}{ll}
\mathrm{V}_{1} & 1!
\end{array}\right]
$$

where $I_{i} \in \mathbb{R}^{2 s \times m s}$ and $T_{2} \in \mathbb{R}^{2 * \times 2 s}$. Then, the term $R_{22}^{-1} R_{21}$ in (24) can be written as

$$
R_{22}^{-1} R_{21}=\left(U^{-} S^{\prime \prime}\left[\mathrm{I}_{2}^{-}\right]\right)^{-1}\left(U^{-} S^{\prime \prime}\left[\mathrm{V}_{1}^{\prime}\right]\right)
$$

or alternatively

$$
U^{\prime \prime}\left[\mathrm{I}_{2}\right]\left(R_{22}^{-1} R_{21}\right)=U S^{\prime \prime}\left[\mathrm{V}_{1}\right]
$$

Therefore, when $S^{\prime \prime}$ is nonsingular the matrix $\left(R_{22}^{-1} R_{21}\right)$ equals

$$
\left(R_{22}^{-1} R_{21}\right)=\left[\mathrm{T}_{2}\right]^{-1}\left[\mathrm{~V}_{1}\right]
$$

Furthermore, when $S^{\prime \prime}$ contains small singular values, we are able to approximate the quantity ( $R_{22}^{-1} R_{21}$ ). To explain this, let the SVD of the matrix $\left[\begin{array}{ll}R_{21} & R_{22}\end{array}\right]$ now be denoted as

$$
\left[\begin{array}{ll}
R_{21} & R_{22}
\end{array}\right]=L^{-} \underbrace{\left[\begin{array}{ll}
S & \\
& S^{\prime}
\end{array}\right]}_{S^{\prime \prime}}\left[\begin{array}{ll}
V_{11} & V_{12} \\
V_{21} & V_{22}
\end{array}\right]
$$

where $\left\|S^{\prime}\right\| \leq \equiv\|S\|$ (for $\equiv$ a small positive number), then

$$
\left[\begin{array}{ll}
R_{21} & R_{22}
\end{array}\right]=L^{\prime} S\left[\mathrm{~V}_{11}+O(\Xi) \quad \mathrm{V}_{12}+O(\Xi)\right]
$$

and

$$
\begin{equation*}
\left(R_{22}^{-1} R_{21}\right) \approx V_{12}^{\ddagger} V_{11} \tag{25}
\end{equation*}
$$

where $[.]^{\ddagger}$ denotes the right pseudoinverse of the matrix [.], $V_{12} V_{12}^{\ddagger}=I$. Hence, instead of checking the condition number of the matrix $R_{22}$ in monitoring the operation of the PI-BTZ scheme, we have to check whether

$$
\begin{equation*}
\text { the matrix } V_{12} \text { has full row rank } \tag{26}
\end{equation*}
$$

and this can be done by evaluating its condition number again. We refer to the simulation study section to evaluate the usefulness of this alternative strategy for constructing (24).

## B. PI-BTZ Algorithm

Given: The following quatities are specified as input quantities to the algorithm: a) an estimate of the underlying system order $n$, b) a dimension parameter $s$, satisfying $s>n, \mathrm{c}$ ) the parameter $a$ determining the bilinear transformation $w$ in (4), with $a \in \mathbb{R}$ and $a<1$, and d) the input and output data sequences, $[u]_{1, \lambda}$ and $[y]_{1, \lambda}$ with $\Gamma \gg m . s$.
 $\left[\begin{array}{l}\Psi_{1} \\ \Psi_{2}\end{array}\right]$ as in (15).
Step 2: Calculate the RQ factorization as indicated in (16).
Step 3: Compute the SVD of the matrix $R_{43}$ as $R_{43}=\left[l_{n} \mid\right.$ $\left.L_{n}^{-\perp}\right]\left[\begin{array}{cc}S_{n} & 0 \\ 0 & 0\end{array}\right] \mathbb{V}^{-T}$ with $U_{n} \in R^{\ell s \times n}$ and singular values ordered in descending order. Then calculate the pair $\left[\begin{array}{ll}F_{T} & H_{T}\end{array}\right]$ as follows

$$
\begin{aligned}
H_{T} & =U_{n}(1: \ell,:) \\
\zeta_{n}\left(1:((s-1) .:) F_{T}\right. & =\zeta_{n}(\ell+1: \ell s,:)
\end{aligned}
$$

where the matrix partitioning used is conformal with the notation used in MATLAB [10].
Step 4: Construct the (24), with ( $R_{22}^{-1} R_{21}$ ) computed either using the pseudoinverse of the matrix $R_{22}$, which we shall call the $\mathrm{C}-1$ variant, or using (25), the $\mathrm{C}-2$ variant.

The matrix pair $\left[G_{T} \quad J\right]$ appears linearly in this equation when the matrices $F_{T}$ and $H_{T}$ are known. An explicit reformulation of this equation such that $\left[\begin{array}{ll}G_{T} & J\end{array}\right]$ appears linearly can exactly be obtained in the same way as the overdetermined set of equations (45) of [14, Part I] was obtained in the PI scheme. For the sake of brevity we refrain stating this set of equations explicitly for the present scheme.

The conditions for solving the equations in Steps 3 and 4 of the PIBTZ scheme are exactly the same as their counterpart in the original PI scheme. These are discussed in [14, Parts I and III].

## C. The PI-BTZ Identification Scheme

In the identification problem defined in the introduction, we assumed the output quantity $y$ to be perturbed by the additive error quantity $\nu$; see Fig. 1. As a consequence, the underlying system to be identified is modeled as

$$
\begin{align*}
x Z & =A x+B u \\
y & =C x+D u+\nu \tag{27}
\end{align*}
$$

and in terms of the shift operator $W$ as

$$
\begin{align*}
x W & =F x+G u(I+a W) \\
y(I+a W) & =H x+J u(I+a W)+\nu(I+a W) . \tag{28}
\end{align*}
$$

Therefore, we see that our basic data (9) changes into

$$
\begin{equation*}
\left[Y_{w}\right]_{f}=\Gamma_{w}\left[x W^{s}\right]_{1, N}+H_{w}\left[U_{u}\right]_{1, N}+\left[V_{u}\right]_{f} \tag{29}
\end{equation*}
$$

where the matrix $\left[V_{u}\right]_{f}$ is similarly defined as $\left[U_{w}\right]_{f}$ to be equal to

$$
\left[\begin{array}{c}
\nu(I+a W) W^{s} \\
\vdots \\
\nu(I+a W) W^{2 s-1}
\end{array}\right]_{1, N} .
$$

The effect of these errors $\nu_{k}$ on the consistency of the PI-BTZ scheme is studied in our final theorem. We will examine the stochastic processes within the ergodic-algebraic framework proposed in [14, Part I]. In this framework, the sequences $u$ and $n$ are treated as realizations of ergodic stochastic processes, $\mathbf{u}$ and $\mathbf{v}$ respectively, and limits are probablistic, and hold with probability one.

Theorem 3: Let the LTI system to be identified be given by (27) and let

$$
\begin{equation*}
E\left[\mathbf{u}_{\mathbf{j}} \mathbf{v}_{l}^{T}\right]=0 \quad \forall \mathbf{j}, \mathbf{l} \tag{30}
\end{equation*}
$$

and let the RQ factorizations (16) and (17) be given with the dependency on $N$ of the different submatrices of the RQ factorization denoted explicitly by an additional superscript $N$. let the matrices $R_{11}^{N}$ and $R_{33}^{N}$ be invertible for $N \geq N_{0}$, and let the assumption of ergodicity hold, then

$$
\begin{aligned}
\lim _{N \rightarrow \infty} \frac{1}{\sqrt{N}} R_{41}^{N} & =\lim _{N \rightarrow \infty}\left(\frac{1}{\sqrt{N}} \Gamma_{w} R_{x 1}^{N}+\frac{1}{\sqrt{N}} H_{w} R_{11}^{N}\right) \\
\lim _{N \rightarrow \infty} \frac{1}{\sqrt{N}} R_{43}^{N} & =\lim _{N \rightarrow \infty} \frac{1}{\sqrt{N}} \Gamma_{w} R_{x 3}^{N}
\end{aligned}
$$

Proof: The proof of this theorem is again elementary and for the sake of brevity not included in this paper. The interested reader is referred to [17].

The theorem shows that when condition (30) is satisfied, the quantities $\frac{1}{\sqrt{N}} R_{41}^{N}$ and $\frac{1}{\sqrt{N}} R_{43}^{N}$, computed by the PI-BTZ scheme, are consistent estimates of the two quantities $\frac{1}{\sqrt{N}} \Gamma_{w} R_{i 1}^{N}+\frac{1}{\sqrt{N}} H_{w} R_{11}^{N}$ and $\frac{1}{\sqrt{N}} \Gamma_{w} R_{x 3}^{N}$, respectively. These two quantities are equal to the left-hand side of (18) and (20), respectively, and we remark that the derivation of the PI-BTZ scheme is completely based on knowledge of these two quantities. Because in the theorem the case $N \rightarrow \infty$ is considered. the effect of the unknown end conditions in (18) and (20)
has vanished since $a$ was assumed to be smaller then one. Condition (30) is satisfied for the class of perturbations indicated by item cl ) and $c 2$ ) in the problem statement in the introduction. Finally, we remark at this point that the presence of unknown initial conditions is automatically taken into account in the use of data equation (8).

As a consequence, the quantities, such as the column space of $\Gamma_{u}$ and the quadruple of system matrices $\left[A T, B_{T}, C_{T}, D\right]$, which are derived from these two key equations, can also be obtained in a statistically consistent manner.

## IV. Identifying Phugoid Aircraft Dynamics

In this section, we report the results of a realistic simulation study to highlight the usefulness of the algorithms and their consistency analysis presented in this paper. For a discussion of additional simulation studies, we refer the interested reader to [17].

The dynamics of the longitudinal long motion, or phugoid, of an F-8 aircraft [2], flying at an altitude of 20.000 ft , an airspeed of 620 $\mathrm{ft} / \mathrm{sec}$ and an angle of attack 0.078 rad , the discrete-time secondorder system that describes the phugoid dynamics of this airplane when using a sampling period of 0.05 sec . is

$$
\left.\begin{array}{rl}
x_{k+1}= & {\left[\begin{array}{rr}
1.0021 e+00 & 3.3948 e-01 \\
-1.7189 e-04 & 9.9116 e-01
\end{array}\right] x_{k}} \\
& +\left[\begin{array}{r}
-2.5375 e+00 \\
-1.7173 e+00
\end{array}\right] \delta_{t, k} \\
{\left[\begin{array}{l}
u \\
\theta
\end{array}\right]_{k}=} & {\left[\begin{array}{r}
-3.1312 e-01 \\
4.913 \pi e-01 \\
5.9204 e-04
\end{array} 5.1454 e-02\right.} \tag{31}
\end{array}\right] x_{k} .
$$

where $u$ is the horizontal component of the airspeed, $\theta$ is the pitch angle, and $\delta_{t}$ the deterministic elevator deflection angle. The poles of system (31) are equal to ( $0.9966 \pm 0.0053 j$ ) (with $j$ now equal to $\sqrt{-1}$ ).

In the simulation, the input $\delta, k$ was taken a zero-mean white noise sequence with standard deviation 0.4 and the measured output was

$$
\left[\begin{array}{l}
u  \tag{32}\\
\theta
\end{array}\right]_{m, k}=\left[\begin{array}{l}
u \\
\theta
\end{array}\right]_{k}+\left[\begin{array}{l}
r_{1} \\
v_{2}
\end{array}\right]_{k}
$$

where $c_{1}$ and $c_{2}$ are additive zero-mean white noise sequences independent from $\delta_{e}$ and with standard deviation 4 and 0.15 , respectively.
The objectives of this simulation were to evaluate the sensitivity of the eigenvalues of the $A_{T}$ matrix under both the PI and PI-BTZ schemes, as well as the sensitivity of the estimated impulse response, using the C-1 and C-2 variants described in Section III-B. In the computation of the pseudoinverse in the $\mathrm{C}-1$ variant, we discarded all singular values of the matrix $R_{22}$ smaller than $10^{-4}$ [3]. The same threshold was used for $s$ in (24) of the C-2 variant. Note that these two variants only differ in the calculation of the $B_{T}$ and $D$ matrices.

We performed a Monte-Carlo simulation study where we generated a new perturbation sequences, $\iota_{1}$ and $l_{2}$, as well as a new input sequence, $\delta_{\kappa}$, for each run. The $a$ parameter was set equal to 0.97 and the dimension parameter $s=10$. In the PI scheme we also set $s$ equal to 10 . Fifty trials using 1200 point records were performed.

The results of this experiment are:

1) The eigenvalues of the matrix $A_{T}$ and $F_{T}$ estimated with the PI-BTZ scheme (see Fig. 1(a)) and related singular values (see Fig. 1(b)) which allows one to anticipate the accuracy of the calculated column space of $\Gamma_{u}$.
2) The eigenvalues of the matrix $A_{T}$ estimated with the PI scheme (see Fig. 2(a)) and related singular values (see Fig. 2(b)). Note that in the singular value plots of both Figs. 1 and 2, only the four dominant singular values (sv's) were included for the


Fig. 1. Estimated eigenvalues (a) of the $F(+)$ and $A$-matrix ( 0 ) and relevant singular values (b), as computed by the PI-BTZ scheme for $a=0.97$. The eigenvalues of $F_{T}$ have been shifted by $b=\left[\begin{array}{ll}-0.2354 & 0.2354\end{array}\right]^{T} \cdot j$.


Fig. 2. Estimated eigenvalues of the $A$-matrix computed by PI scheme and relevant singular values.
sake of clarity. The rest are of the same order of magnitude compared as the third and fourth sv's in both cases.
3) The impulse response of the quadruple of system matrices $\left[A_{T}, B_{T}, C_{T}, D\right]$ estimated by the two variants $\mathrm{C}-1$ and $\mathrm{C}-2$ of the PI-BTZ scheme is plotted in our final figure.
In this experiment, the PI-BTZ scheme was able to identify the phugoid dynamics, despite the complete failure of the PI scheme. Specifically, we note that the following:

1) There is a clear gap between the second and third singular values plotted on the right-hand side of Fig. 1, suggesting that the underlying system might be of second order [14, Part III]. The singular values computed using the PI scheme, shown in Fig, 2(b), show no such gap. As stated in [14, Part III], the actual decision on the system order should be made based on additional measures, such as the prediction errors.
2) In support of Theorem 3, we note the eigenvalues of $F_{T}$, shown as ' + '-signs in Fig. 1(a), are unbiased estimates of the true eigenvalues, indicated by the center of the dashed cross. Furthermore. the eigenvalues of the $A_{i}$ matrix cluster very
closely around their true value, and all of the estimated systems remained stable.
3) Although the PI-BTZ scheme is able to accurately recover the poles of the phugoid, the identification of the full dynamics can be sensitive. The left and right of Fig. 3 show the impulse responses estimated using the $\mathrm{C}-1$ and $\mathrm{C}-2$ variants, respectively. Clearly, the $\mathrm{C}-2$ variant is the more robust of the two. For the $\mathrm{C}-1$ variant, the condition number of the approximation used in computing the pseudo inverse of $R_{22}$ is $O\left(10^{4}\right)$, whereas that of the $V_{12}$ matrix, used in the $\mathrm{C}-2$ variant, is $O(1)$.

## V. Concluding Remarks

In this paper we have shown that the use of a bilinear transformation of the shift operator $Z$ in the complex plane can lead to dramatic improvements in the accuracy of the PI subspace model identification algorithm, originally proposed in [14, Part III]. This is especially true when the poles of the deterministic system to be identified cluster around the point $:=1$. This situation can easily arise in


Fig. 3. Estimated impulse responses by the PI-BTZ scheme using the pseudoinverse of $R_{22}$ (variant C 1 ) and using the pseudoinverse of $V_{12}$ (variant C2).
practice, particularly if the sampling rate too high. Furthermore, in this situation one can easily end up with short length data batches, where the length is compared to the (dominant) time constants. Also for this situation the outline strategy of this paper allows to improve the accuracy of the estimates.

An important feature of the PI-BTZ algorithms, developed in this note, is the ability to check the reliability of the identification by evaluating certain condition numbers, which can be calculated from the experimental data. Thus unlike previous SMI schemes, [14, part I], [7] and [11], the user is able to monitor the performance of the algorithm explicitly.

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# Stability of Nonlinear Parameter-Varying Digital 2-D Systems 

Jerzy E. Kurek

Abstract-A general state-space model for nonlinear parameter-varying digital two-dimensaional (2-D) systems is proposed. Sufficient conditions for system stability are given. Presented theorem can be considered as a generalization of the Lyapunov stability theorem for one-dimensional nonlinear systems. Given results can be useful in stability analysis for systems described by 2-D models.

## I. Introduction

The Lyapunov stability theorem is one of the most powerful tools used in control theory. It enables one to test stability of linear timeinvariant and time-varying systems as well as nonlinear systems. Approaches based on the Lyapunov equation are also often used in analysis and design of robust control systems.

Two-dimensional (2-D) systems have found many applications, e.g., in analysis of time delay one-dimensional (1-D) systems [7]. One of the main problems in analysis and design of 2-D systems, however, is stability. Whereas they may be viewed as a generalization of 1-D systems, the extension of 1-D stability tests to 2-D systems is, unfortunately, rather difficult.

The first formulation of the Lyapunov-type equation for a linear 2-D system was presented in [14]. Then, a number of papers were published where the stability problem for linear 2-D systems was considered based on a Lyapunov equation approach. There were two different approaches to the problem. The first one was based on 2-D Lyapunov equation with real matrices [1]-[4], [11], [12]. The second one, [5], employed polynomial matrices in Lyapunov equation.

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