Frequency Response of Sampled-Data Systems

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***Abstract*— This paper introduces the concept of frequency response for sampled-data systems and explores some basic properties as well as its computational procedures. It is shown that 1) by making use of the lifting technique, the notion of frequency response can be naturally introduced to sampled-data systems in spite of their time-varying characteristics, 2) it represents a frequency domain steady-state behavior, and 3) it is also closely related to the original transfer function representation via an integral formula. It is shown that the computation of the frequency response can be reduced to a finite-dimensional eigenvalue problem, and some examples are presented to illustrate the results.**

I. INTRODUCTION

THE importance of the notion of frequency response for continuous-time, time-invariant systems needs no justification. It is used in various aspects of system performance evaluation and still is at the center of many design methods. This fact is only reinforced by the now-standard H^∞ control theory, and attempts have been made to generalize this design methodology to various new directions. In the setting of sampled-data systems, there are now quite a few investigations along this line—for example, [10], [7], [17], [18], [3], [27], [26], and [29], to name just a few. The difference here from the classical theory lies in the emphasis upon the importance of built-in intersample behavior in the model, so that it is part of the design specifications. As a result, in this approach the sampled-data systems are viewed as hybrid systems, and their performance is evaluated in the continuous time.

An important problem in this context of sampled-data systems is that of frequency domain analysis. In classical treatments (see, e.g., [25]) the frequency domain analysis of sampled-data systems has been carried out. The classical approach is via infinite sum formulas for sampled signals and their transforms. The mixture of continuous- and discrete-time systems introduces a time-varying periodic characteristic in sampled-data systems, and this has made the classical frequency domain treatment of sampled-data systems rather awkward. It should be noted that in the classical treatment the signals are always accompanied with (either real or fictitious) samplers, while in the modern point of view the actual continuous-time response is analyzed. Frequency domain analysis in the setting of sampled-data systems has been revisited in recent years from the modern operator theoretic standpoint in [20] and [11], and robust stability condition in the frequency domain has been analyzed in [9]. The works of [32], [1], and [2] pursue the justification of the notion of frequency response as a steady-state response; the former uses so-called lifting, and the latter impulse modulation.

Since the advent of the lifting technique [3], [4], [19], [29], [30], it has become possible to view sampled-data systems as time-invariant discrete-time systems with built-in intersample behavior. This time-invariance gives rise to the notion of the transfer function operator $G(z)$, and for stable systems it is also possible to substitute $z = e^{j\omega}$ into $G(z)$. This formal definition of frequency response, however, lacks the strong physical justification which applies to the standard linear time-invariant systems. For example, if we apply a sinusoidal input $\sin \omega t$ to an asymptotically stable sampled-data system, its response is not stationary, especially if ω is not commensurate with the sampling frequency. It turns out that this difficulty can be overcome by the steady-state analysis given in [30]. It is particularly so for the gain characteristic, and we will show that the changes induced by one sample period transition are merely a phase shift, and the total gain remains invariant in the steady state (see Section III-A).

In this paper, we take the viewpoint initiated in [32] and present a detailed analysis of frequency response of sampled-data systems. The main contributions of this paper are as follows. We first show that the above-mentioned notion of frequency response inherits some very desirable and important properties of its time-invariant, continuous-time counterpart. In Section III-B, we show that it is possible to recover the lifted transfer operator from the frequency response operator. This is a version of the well-known inverse Fourier transform formula in the setting of sampled-data systems.

Next we address the computation of the gain of the frequency response operator. Although the problem looks similar to the computation of the H^∞ norm of sampled-data systems, there is a very important and subtle difference. Since the H^∞ norm is the supremum of the gain of the frequency response operator, the positivity of a certain operator $(\gamma^2 I - \boldsymbol{D}^*\boldsymbol{D})$ is automatically satisfied for any γ that exceeds the H^∞ norm. This fact is crucial in the H^∞ norm computation for sampled-data systems, e.g., [26], [18], and [31]. On the other hand, in the computation of the gain of the frequency response operator, this positivity condition can fail in a large region of frequencies. To obtain formulas for the gain computation similar to that for the H^∞ norm computation problem given in [31], we need to guarantee that the gains can still be obtained as maximal singular values, and this requires a very different

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argument from that in [31]. This is the subject of Section IV. We will show the following:

- The gain can be characterized as the maximal singular value of the operator $G(e^{j\omega})$.
- The relevant operator singular value equation can be reduced to a finite-dimensional eigenvalue problem (Theorem 2).
- As a corollary, an H^∞ norm-equivalent finite-dimensional discrete-time problem is derived (Theorem 3).

Some examples are included to illustrate the above computation. In particular, it is seen that the obtained gain characteristic accounts for the aliasing effects as well as the frequency where they occur.

Conference versions of this paper appeared as conference papers [32], [34].

A. Notation and Convention

The notation is quite standard. $L^2[0, h]$ and $L^2[0, \infty)$ are the spaces of Lebesgue square integrable functions on $[0, h]$ and $[0, \infty)$, respectively. In general, we omit superscripts to denote the dimension of the range spaces. So we simply write $L^2[0, h]$ instead of $(L^2[0, h])^n$, etc. Likewise, l_X^2 is the space of (X-valued) square summable sequences with values in the space X. For a vector $x \in \mathbb{R}^n$, its Euclidean norm will be denoted by $|x|$ to make the distinction clear from the L^2 norm. In contrast, if we write $\|\varphi\|$, it will usually denote an L^2 (or l^2) norm or the operator norm induced by it. When precise distinction is desirable, we write $\|x\|_2$. Laplace and z-transforms are denoted by $\mathcal{L}[\varphi](s)$ and $\mathcal{Z}[\varphi](z)$, respectively. When no confusion can arise, we may also write $\hat{\varphi}(s)$, $\hat{\varphi}(z)$, depending on the context.

II. Model Description via Lifting

We employ the function space model of sampled-data systems via lifting, following [8], [19], [30], [29], [4], and [3]. Let h be a fixed sampling period throughout and $\mathcal{W}$ be the lifting operator that maps a function φ on $[0, \infty]$ to a function-space valued sequence $\{\varphi_k\}_{k=0}^\infty$

$$\mathcal{W}: \varphi \mapsto \{\varphi_k\}_{k=0}^\infty : \varphi_k(\theta) := \varphi(kh + \theta). \tag{1}$$

The kth element represents, in general, an intersample signal at the kth step. When considered over $L^2[0, \infty]$, this mapping gives a norm-preserving isomorphism between $L^2[0, \infty]$ and $l^2_{L^2[0,h]}$, where the latter is equipped with the norm

$$\|\{\varphi_k\}\| := \left\{\sum_{k=0}^\infty \|\varphi_k\|^2_{L^2[0,h]};\right\}^{1/2}.$$

Now consider the sampled feedback system Fig. 1 with continuous-time plant

$$\begin{aligned}\dot{x}_c(t) &= A_c x_c(t) + B_w w(t) + B_u u(t)\\ z(t) &= C_z x_c(t) + D_w w(t) + D_u u(t)\\ y(t) &= C_y x_c(t)\end{aligned} \tag{2}$$

and the discrete-time controller

$$\begin{aligned}x_{d,k+1} &= A_d x_{d,k} + B_d S y_k\\ v_k &= C_d x_{d,k} + D_d S y_k\\ u_k(\theta) &= H(\theta) v_k\end{aligned}$$

where S denotes the sampler $Sy_k := y_k(0)$. Here we have taken the direct feedthrough term from w to y to be zero to keep the closed-loop operators bounded. The feedthrough term from u to y is taken to be zero for simplicity, and it ensures well-posedness of the feedback system. It is well known that via lifting correspondence (1) this system is represented by the time-invariant discrete-time equation

$$\begin{bmatrix} x_{c,k+1} \\ x_{d,k+1}\end{bmatrix} = \begin{bmatrix} A_{cs} & A_{cd} \\ A_{ds} & A_d\end{bmatrix}\begin{bmatrix} x_{c,k} \\ x_{d,k}\end{bmatrix} + \begin{bmatrix} \boldsymbol{B} w_k(\cdot) \\ 0\end{bmatrix} \tag{3}$$

$$z_k(\theta) = [C_1(\theta) \quad C_2(\theta)]\begin{bmatrix} x_{c,k} \\ x_{d,k}\end{bmatrix} + \boldsymbol{D} w_k(\theta) \tag{4}$$

where $x_{c,k} = x_c(kh)$ and $x_{d,k}$ denote, respectively, the continuous and discrete state variables and belong to $\mathbb{C}^{n_c}$ and $\mathbb{C}^{n_d}$; matrices A_{cs}, A_{cd}, A_{ds}, $C_i(\theta)$, $K(\theta)$, $W(\theta)$ and operators $\boldsymbol{B}$, $\boldsymbol{D}$ are of the following form:

$$A_{cs} = e^{A_c h} + \int_0^h e^{A_c(h-\tau)} B_u H(\tau) D_d C_y \, d\tau$$

$$A_{cd} = \int_0^h e^{A_c(h-\tau)} B_u H(\tau) C_d \, d\tau$$

$$A_{ds} = B_d C_y$$

$$\begin{aligned}C_1(\theta) =& C_z\left(e^{A_c\theta} + \int_0^\theta e^{A_c(\theta-\tau)} B_u H(\tau) D_d C_y \, d\tau\right)\\ &+ D_u H(\theta) D_d C_y\end{aligned}$$

$$C_2(\theta) = C_z \int_0^\theta e^{A_c(\theta-\tau)} B_u H(\tau) C_d \, d\tau + D_u H(\theta) C_d$$

$$K(\theta) = e^{A_c\theta} B_w$$

$$W(\theta) = D_w \delta(\theta) + C_z e^{A_c\theta} B_w$$

$$\boldsymbol{B}\colon L^2[0, h] \to \mathbb{C}^{n_c} : w(\cdot) \mapsto \int_0^h K(h-\tau) w(\tau)\, d\tau$$

$$\boldsymbol{D}\colon L^2[0, h] \to L^2[0, h] : w(\cdot) \mapsto \int_0^\theta W(\theta-\tau) w(\tau)\, d\tau \tag{5}$$

where $\delta(\theta)$ is the delta function.

Denote (3) and (4) simply as

$$x_{k+1} = A x_k + B w_k \tag{6}$$

$$z_k = C x_k + D w_k \tag{7}$$

(note $D := \boldsymbol{D}$). Note that A is a matrix consisting of A_{cs}, A_{cd}, A_{ds}, and A_d. Now we make our fundamental assumption that

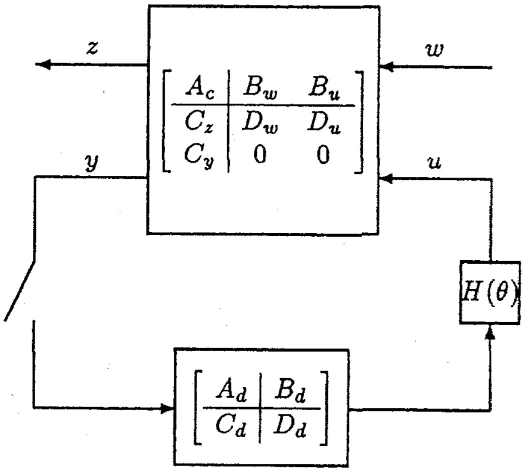

Fig. 1. Sampled feedback system.

A is a power stable matrix, i.e., $A^n \to 0$ as $n \to \infty$. This is equivalent to the eigenvalues of A all inside the unit circle.

Introducing the z-transform

$$\mathcal{Z}[\{\varphi_k\}_{k=0}^{\infty}] := \sum_{k=0}^{\infty} \varphi_k z^{-k} \tag{8}$$

we can also define the transfer function operator of (6) and (7) as $G(z) := D + C(zI - A)^{-1}B$. While this definition primarily makes sense as a formal power series (with z being an indeterminate), it also admits the Neumann series expansion

$$G(\lambda) = D + \sum_{k=1}^{\infty} CA^{k-1}B\lambda^{-k} =: \boldsymbol{D} + G_0(\lambda) \tag{9}$$

at least for sufficiently large complex λ. In fact, since A is stable, this series is uniformly convergent for $|\lambda| \geq 1$ and is analytic there. In general, poles of $G(\lambda)$ are contained in the spectrum of A. Hence if (even without the stability assumption on A) $G(\lambda)$ is analytic in $|\lambda| \geq 1$, we will say that $G(z)$ is stable. (For a detailed discussion on the correspondence of stability, see, e.g., [7], [30], etc). By the continuity of B, C, and D, $G(\lambda)$ gives a bounded linear operator on $L^2[0, h]$ at least for each fixed $|\lambda| \geq 1$. Furthermore, by the uniform convergence, $G(\lambda)$ is uniformly bounded for $|\lambda^{-1}| \leq 1$, so that [22] its H^∞-norm

$$\begin{aligned}\|G\|_\infty &:= \sup_{|\lambda^{-1}|\leq 1} \left\{ \sup_{v \in L^2[0,h]} \frac{\|G(\lambda)v\|_2}{\|v\|_2} \right\} \\ &= \sup_{0\leq\omega\leq 2\pi} \left\{ \sup_{v \in L^2[0,h]} \frac{\|G(e^{j\omega})v\|_2}{\|v\|_2} \right\}\end{aligned} \tag{10}$$

is finite. The second equality follows from the maximum modulus principle. It is also known that this norm is equal to the L^2-induced norm in the time domain. Also, for each fixed λ with $|\lambda| \geq 1$, $G_0(\lambda)$ in (9) converges in norm because $A^k \to 0$. Since B is a compact operator as an integral operator with L^2 kernel function $K(\theta)$ as above, each CA^kB is also compact, so that as a uniform limit of compact operators, $G_0(z)$ is compact (but $\boldsymbol{D}$, and therefore $G(\lambda)$, is never compact unless D_w is zero).

III. Frequency Response—Basic Properties

Taking the viewpoint initiated in [32], we now introduce the notion of frequency response for (3) and (4). We review some basic facts as well as derive a new formula that gives a lifted transfer operator from the frequency response.

A. Frequency Response as Steady-State Response

Let $G(z) = \sum_{n\geq 0} G_n z^{-n}$ be the transfer function operator of this system introduced in the previous section. As noted above, for each fixed real ω, substitution $z = e^{j\omega h}$ also makes sense, and one might call the resulting operator $G(e^{j\omega h})$, acting on $L^2[0, h]$, regarded as a function of ω, the frequency response of this system. This formal definition by itself, however, lacks the highly physical steady-state interpretation similar to that for continuous-time systems. Nonetheless, it is still possible to associate a very natural steady-state interpretation to this concept.

We begin by recalling the following lemma from [30].

Lemma 1: Let $G(z)$ be the transfer operator of (3) and (4), and let the input u be such that

$$u_k(\theta) := \lambda^k v(\theta), \quad |\lambda| \geq 1, \quad k = 0, 1, \cdots.$$

Then the output y asymptotically approaches

$$y(kh + \theta) = \lambda^k (G(\lambda)v)(\theta)$$

as $k \to \infty$. See [30] for a proof.

Now observe that a sinusoidal function $u(t) = \exp(j\omega t)v_0$ can be expressed as a power function via lifting as follows:

$$\{u_k(\theta)\}_{k=0}^{\infty} := \{(e^{j\omega h})^k v(\theta)\}_{k=0}^{\infty}, \qquad v(\theta) = e^{j\omega\theta}v_0 \tag{11}$$

with z-transform

$$\mathcal{Z}(\{(e^{j\omega h})^k v(\theta)\}_{k=0}^{\infty}) = \frac{zv(\theta)}{z - e^{j\omega h}}.$$

Then, by Lemma 1, the output asymptotically approaches $(e^{j\omega h})^k G(e^{j\omega h})v$. While this is never in "steady state" in the strict sense unless $\lambda = 1$, its modulus $|(G(e^{j\omega h})v)(\theta)|$ remains the same. In other words, the essential part of the asymptotic response is $(G(e^{j\omega h})v)(\theta)$, and each particular response $(e^{j\omega h})^k G(e^{j\omega h})v$ at the kth step is obtained by the phase shift with successive multiplication by $e^{j\omega h}$.

In view of this observation, it is natural to call this operator $G(e^{j\omega h})\colon L^2[0, h] \to L^2[0, h]$ the frequency response operator.

Definition 1: Let $G(z)$ be the transfer operator of the lifted system as above. Let $\omega_s := 2\pi/h$. The frequency response operator is the operator

$$G(e^{j\omega h})\colon L^2[0, h] \to L^2[0, h] \tag{12}$$

regarded as a function of $\omega \in [0, \omega_s)$. Its gain at ω is defined to be

$$\|G(e^{j\omega h})\| = \sup_{v\in L^2[0,h]} \frac{\|G(e^{j\omega h})v\|}{\|v\|}. \tag{13}$$

By (10), the least upper bound of the gain $\|G(e^{j\omega h})\|$ as ω ranges from 0 to ω_s is precisely the H^∞ norm of G. We also note that although we have considered frequency response on the interval $[0, \omega_s)$,[1] it is also possible to extend this function periodically over $(-\infty, \infty)$. This is justified because $e^{j(\omega+n\omega_s)h} = e^{j\omega h}$ for any integer n. This convention will be employed in Subsection B.

We next remark on aliasing and the equality $e^{j(\omega+n\omega_s)h} = e^{j\omega h}$. Suppose that our input is $e^{j\omega t}$, with $\omega > \omega_s$. It is expressible as $w_k(\theta) = (e^{j\omega' h})^k(e^{j\omega\theta})$ with some ω' satisfying $0 \le \omega' < \omega_s$ and $\omega = \omega' + n\omega_s$ for some integer n. This means that the effect of this high-frequency input $e^{j\omega t}$ appears at the frequency $e^{j\omega' h} = e^{j\omega h}$ as an alias effect. The only difference between $e^{j\omega t}$ and $e^{j\omega' t}$ is that the initial intersample signal $e^{j\omega\theta}$ is different from $e^{j\omega'\theta}$. Definition (13) thus takes all such aliasing effects into account by taking the supremum over all $v \in L^2[0, h]$ on the right-hand side.

B. Recovery of Transfer Operators from Frequency Response

We have given a definition of the frequency response operator $G(e^{j\omega h})$. Recall that for standard linear time-invariant systems, transfer functions can always be recovered from the frequency response. It is then natural to ask: How can the lifted transfer matrix operator $G(z)$ be recovered from the knowledge of $G(e^{j\omega h})$? We also recall that in the standard lifting setup the system is specified in terms of the state-space representations, and transfer operators are defined using them. From the purely external point of view this is awkward, and it should be possible to give a formula for a lifted transfer operator without going through state-space representations. Here we give an answer based on the frequency response. To this end, we will need some material from [33].

Lemma 2: Fix any $\omega \in [0, \omega_s]$, and let $\omega_n := \omega + n\omega_s$. Then every $\varphi \in L^2[0, h]$ can be expanded in terms of $\{e^{j\omega_n\theta}\}_{n=-\infty}^{\infty}$ as

$$\varphi(\theta) = \sum_{n=-\infty}^{\infty} a_n e^{j\omega_n\theta} \tag{14}$$

with

$$a_n = \frac{1}{h}\int_0^h e^{-j\omega_n\tau}\varphi(\tau)\,d\tau = \frac{1}{h}\hat{\varphi}(j\omega_n) \tag{15}$$

where $\varphi \in L^2[0, h]$ is embedded in $L^2[0, \infty]$ as a function having support contained in $[0, \infty)$. Furthermore, the L^2 norm $\|\varphi\|$ is given by

$$\|\varphi\|^2 = h\sum_{n=-\infty}^{\infty} |a_n|^2. \tag{16}$$

Proof: Expand $e^{-j\omega\theta}\varphi(\theta)$ in terms of $e^{2nj\pi\theta/h}$ into Fourier series. This readily yields (14). Since $\|e^{-j\omega\theta}\varphi\| = \|\varphi\|$, identity (16) follows from Parseval's identity. □

Now let $G(z)$ be a stable lifted transfer function, and let $e^{j(\omega+l\omega_s)t}$, $0 \le \omega < \omega_s$ be our input to G. According to Lemma 2, we have the following expansion:

$$G(e^{j\omega h})[e^{j\omega_l\theta}] = \sum_{n=-\infty}^{\infty} g_n^l(\omega)e^{j\omega_n\theta} \tag{17}$$

where $g_n^l(\omega)$ are determined by

$$\begin{aligned} g_n^l(\omega) &:= \frac{1}{h}\int_0^h e^{-jn\omega_s\tau}e^{-j\omega\tau}(G(e^{j\omega h})e^{j\omega_l\theta})(\tau)\,d\tau \\ &= \frac{1}{h}\int_0^h e^{-j\omega_n\tau}(G(e^{j\omega h})e^{j\omega_l\theta})(\tau)\,d\tau. \end{aligned}$$

Remark 1: Another notion of frequency response based upon a quantity equivalent to $g_n^l(\omega)$ is studied by [1] and [2]. It is also used by [9] for the analysis of robust stability. An advantage of such an approach is that it is often possible to derive a formula for $g_n^l(\omega)$ without going through state-space representations of $G(e^{j\omega h})$. We also note that the equivalence of these two notions of frequency response is recently shown by [33]. Therefore, once we establish the formula for lifted transfer operators in terms of $g_n^l(\omega)$ as given below, it can be obtained without recourse to the state-space representations as in (9).

Our objective here is to derive a formula for lifted transfer operator based upon the knowledge of $g_n^l(\omega)$. Let

$$G(\lambda) = \sum_{k=0}^{\infty} G_k\lambda^{-k} \tag{18}$$

be the Neumann series expansion of $G(\lambda)$. Under the hypothesis of exponential stability, this series converges uniformly at least for $|\lambda^{-1}| \le 1$. Substitute $\lambda = e^{j\omega h}$ into (18), multiply both sides by $e^{j\omega kh}$, and then integrate on the unit circle to obtain

$$\begin{aligned} G_k &= \frac{h}{2\pi}\int_0^{\omega_s} G(e^{j\omega h})e^{j\omega kh}\,d\omega \\ &= \frac{1}{2\pi j}\oint G(\lambda)\lambda^{k-1}\,d\lambda. \end{aligned}$$

Take any $f \in L^2[0, h]$ with expansion

$$f(\theta) = \sum_l \alpha_l(\omega)e^{j\omega_l\theta}$$

according to Lemma 2. Here we have emphasized the dependence of α_l on ω. By (15), $\alpha_l(\omega)$ is given by

$$\alpha_l(\omega) = \frac{1}{h}\int_0^h e^{-j\omega_l\tau}f(\tau)\,d\tau = \frac{1}{h}\hat{f}(j\omega_l)$$

where $\hat{f}(s)$ is the finite Laplace transform

$$\hat{f}(s) = \int_0^h f(\theta)e^{-s\theta}\,d\theta.$$

It follows that

$$\begin{aligned} G_k f &= \frac{h}{2\pi}\int_0^{\omega_s}\sum_l G(e^{j\omega h})\alpha_l e^{j\omega_l\theta}e^{j\omega kh}\,d\omega \\ &= \frac{1}{2\pi}\sum_l\int_0^{\omega_s} G(e^{j\omega h})e^{j\omega kh}e^{j(\omega+l\omega_s)\theta}\hat{f}(j(\omega+l\omega_s))\,d\omega. \end{aligned}$$

Introduce the change of variable $\sigma := \omega_l = \omega + l\omega_s$ and note $e^{j\omega h} = e^{j\omega_l h}$ to obtain

$$\begin{aligned} G_k f &= \frac{1}{2\pi}\sum_l\int_{l\omega_s}^{(l+1)\omega_s} G(e^{j\sigma h})e^{j\sigma kh}e^{j\sigma\theta}\hat{f}(j\sigma)\,d\sigma \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} G(e^{j\sigma h})e^{j\sigma(kh+\theta)}\hat{f}(j\sigma)\,d\sigma. \end{aligned} \tag{19}$$

[1] Some authors take $(-\omega_s/2, \omega_s/2)$ instead.

By (17), we have

$$G(e^{j\sigma h})e^{j\sigma\theta} = \sum_{n=-\infty}^{\infty} g_n^l(\sigma - l\omega_s)e^{j\omega_n\theta} \tag{20}$$

where $l = [\sigma/\omega_s]$. This yields the following theorem.

Theorem 1: Let $G(z)$ and f be as above. Then the kth coefficient $G_k f$ of the lifting $G(z)f$ is given by

$$G_k f = \frac{1}{2\pi}\int_{-\infty}^{\infty} G(e^{j\sigma h})e^{j\sigma(kh+\theta)}\hat{f}(j\sigma)\,d\sigma \tag{21}$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\sum_{n=-\infty}^{\infty} g_n^{[\sigma/\omega_s]}(\sigma_r) \cdot e^{j(\sigma_r+n\omega_s)(kh+\theta)}\hat{f}(j\sigma)\,d\sigma \tag{22}$$

where $[\sigma/\omega_s]$ is the integer part of σ/ω_s and $\sigma_r := \sigma - [\sigma/\omega_s]\,\omega_s$.

Proof: The first formula is precisely (19). The second one is obtained by substituting (20) into (19). Observe that $l = [\sigma/\omega_s]$, $\omega = \sigma - l\omega_s$, and $\omega_n = \omega + n\omega_s = \sigma + (n - [\sigma/\omega_s])\omega_s$. □

The formula above gives the response at $kh + \theta$ via the inverse Fourier transform. In general, the formula becomes involved due to the correction factor $e^{j(n-[\sigma/\omega_s])\omega_s t}$ arising from aliasing. For the lifted transfer function of a continuous-time plant $G_c(s)$, however, the relationship is particularly simple, as shown in the following.

Corollary 1: Let $G_c(s)$ be a stable continuous-time transfer function. Then its lifted transfer function $G(z)$ or its kth coefficient operator G_k is given by

$$(G_k f)(\theta) = \frac{1}{2\pi}\int_{-\infty}^{\infty} G_c(j\sigma)e^{j\sigma t}\hat{f}(j\sigma)\,d\sigma \tag{23}$$

where $t = kh + \theta$.

Proof: If we apply an input $e^{j\sigma t}$ to $G_c(s)$, we get the output $G_c(j\sigma)e^{j\sigma t}$ in the steady state. Hence

$$G(e^{j\sigma h})(e^{j\sigma\theta}) = G_c(j\sigma)e^{j\sigma\theta}.$$

In other words

$$g_n^l(\sigma) = \begin{cases} G_c(j\sigma), & n = l \\ 0, & n \neq l. \end{cases}$$

Substituting these into (21) or (22) yields (23). □

Remark 2: Combining the formula above with the formula for the sampler will again yield the general case (22) since the frequency response defined here is clearly multiplicative.

We here give an example to assure that (22) indeed recovers the lifted transfer function $G(e^{j\omega h})$.

Example 1: Consider the system depicted in Fig. 2. If the input is $w(t) = \exp(j\omega_l t)$, then

$$y(t) = \frac{e^{j\omega_l t}}{j\omega_l + 1}$$

in the steady state. It turns out that [33]

$$g_n^l = \frac{1 - e^{-jh\omega_n}}{jh\omega_n} \cdot \frac{1}{j\omega_l + 1}. \tag{24}$$

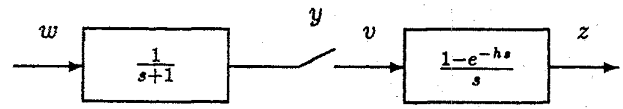

Fig. 2. Sampled first-order system.

Now let $f \in L^2[0, h]$. If this f is applied to system Fig. 2, the corresponding $y(t)$ is given by

$$y(t) = \int_0^h e^{-(t-\tau)}f(\tau)\,d\tau.$$

Hence we readily have

$$(G_k f)(\theta) = y(kh) = \int_0^h e^{-(kh-\tau)}f(\tau)\,d\tau. \tag{25}$$

Let us see that this is also obtained via (22). Indeed, from (24) we have

$$(G_k f)(\theta) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\sum_{n=-\infty}^{\infty}\frac{1 - e^{jh\omega_n}}{jh\omega_n} \cdot \frac{1}{j\sigma + 1}e^{j\sigma t}e^{j(n-l)\omega_s t}\hat{f}(j\sigma)\,d\sigma$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\sum_{n=-\infty}^{\infty}\frac{1 - e^{-jh\omega_n}}{jh\omega_n} \cdot \frac{1}{j\sigma + 1}e^{jn\omega_s\theta}e^{j\omega t}\hat{f}(j\sigma)\,d\sigma$$

where $t = kh + \theta$, $\sigma = \omega + l\omega_s$, and $e^{nj\omega_s t} = e^{nj\omega_s\theta}$. By Lemma 2 we have

$$\sum_{n=-\infty}^{\infty}\frac{1 - e^{-jh\omega_n}}{jh\omega_n}e^{jn\omega_s\theta} = e^{-j\omega\theta}$$

so that

$$(G_k f)(\theta) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{e^{-j\omega\theta}}{j\sigma + 1}e^{j\omega(kh+\theta)}\hat{f}(j\sigma)\,d\sigma$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{1}{j\sigma + 1}e^{j\sigma kh}\hat{f}(j\sigma)\,d\sigma$$

because $e^{j\omega kh} = e^{j\sigma kh}$. By the inverse Fourier transform formula, the last term clearly agrees with (25).

IV. Computation of Frequency Response

The frequency response operator introduced here is infinite dimensional. How can we compute the gain of this operator? An answer to this question will lead to the analog of the Bode magnitude plot for standard linear time-invariant systems. In this section, we give a procedure computing the gain of the frequency response operator. This is done by reducing the problem to a finite-dimensional eigenvalue problem. Although the procedure is apparently similar to the computation of H^∞ norm of sampled-data systems [17], [18], [31], there is a very important difference. In the case of H^∞ norm, $\|G\|_\infty \geq \|\boldsymbol{D}\|$ always holds, and this simplifies the whole procedure. On the other hand, in the present context the norm $\|G(e^{j\omega h})\|$ actually can be less than $\|\boldsymbol{D}\|$, so that reduction to an eigenvalue problem is nontrivial. This problem is the subject of this section.

A. Characterization as Singular Values

Let $G(e^{j\omega h})$ be the frequency response operator as introduced in the previous section. Its gain is the norm induced from that of $L^2[0, h]$. If we resort to an analogy to the ordinary finite-dimensional case, we may attempt to compute this norm via the singular value equation

$$(\gamma^2 I - G^* G(e^{j\omega h}))w = 0. \tag{26}$$

However, in the present context the operator $G(e^{j\omega h})$ is infinite dimensional, and when $D_\omega \neq 0$, it is not even compact. As a result, the induced norm $\|G(e^{j\omega h})\|$ need not be attained as the maximal singular value that satisfies (26). To remedy this, we need the following developments.

Let T be an operator in a Hilbert space X. Its spectrum and essential spectrum are denoted by $\sigma(T)$, $\sigma_e(T)$, respectively [23] and [16]. Also, their radii $r(T)$, $r_e(T)$ are defined by

$$r(T) := \sup\{|\lambda|;\ \lambda \in \sigma(T)\}$$

$$r_e(T) := \sup\{|\lambda|;\ \lambda \in \sigma_e(T)\}.$$

Since $\sigma_e(T) \subset \sigma(T)$, $r_e(T) \leq r(T)$. The key lemma is the following fact on perturbations by compact operators.

Lemma 3: [16] Let $T = T_0 + T_1$ be an operator in a Hilber space where T_1 is compact. Then, $\sigma_e(T) = \sigma_e(T_0)$ and $r_e(T) = r_e(T_0)$. In other words, perturbation by a compact operator does not change the essential spectrum. Furthermore, if $\sigma_e(T)$ is at most a countable set, then every point $\lambda \in \sigma(T)\backslash\sigma_e(T)$ is an eigenvalue.

Now let us return to the sampled-data transfer function $G(z)$ given by (9). Note that the operator $\boldsymbol{D}$ can be decomposed as $D_w + \boldsymbol{D}_0$ where D_w is the multiplication operator by the matrix D_w and $\boldsymbol{D}_0$ is an integral operator with L^2 kernel function $W_0(\theta) = C_z e^{A_c\theta} B_w$, and hence compact. This implies that for each fixed λ ($|\lambda| \geq 1$), $G(\lambda)$ can be decomposed as

$$G(\lambda) = D_w + G_1(\lambda)$$

where $G_1(\lambda) = \boldsymbol{D}_0 + G_0(\lambda)$ is a compact operator. Since the composition of a compact operator with a bounded operator is again compact, $V(\lambda) := G^*(\lambda)G(\lambda)$ admits the decomposition

$$V(\lambda) = D_w^* D_w + V_1(\lambda)$$

where $V_1(\lambda)$ is compact. Clearly, $\|V(\lambda)\| = \|G(\lambda)\|^2$, and since $V(\lambda)$ is self-adjoint, its norm is given as the spectral radius, i.e., $\|V(\lambda)\| = r(V(\lambda))$ [28]. We then have the following result.

Proposition 1: Fix any λ with $|\lambda| \geq 1$ and let $\gamma := \|G(\lambda)\|$. Then, $\gamma^2 = r(V(\lambda)) \geq r_e(V(\lambda))$. Moreover, only one of the following two possibilities can occur:

1) either $\gamma^2 = r_e(V(\lambda)) = \|D_w\|^2$, or
2) $\gamma^2 > r_e(V(\lambda))$ and it is an eigenvalue of $V(\lambda)$.

Proof: Let us first prove that

$$\sigma_e(V(\lambda)) = \{\sigma_i^2;\ i = 1, \cdots, p\} \tag{27}$$

where σ_i, $i = 1, \cdots, p$ are the singular values of the matrix D_w. To this end, let us first observe that $\sigma_e(V(\lambda)) = \sigma_e(D_w^* D_w)$ by Lemma 3. Since $D_w^* D_w$ is a Hermitian matrix, we may assume, with a suitable change of basis, that it is a diagonal matrix

$$D_w^* D_w = \operatorname{diag}[\sigma_1^2, \cdots, \sigma_p^2].$$

It is seen easily that $\ker(\sigma_i^2 I - D_w^* D_w)$ is infinite dimensional, so that $\sigma_e(D_w^* D_w) = \{\sigma_i^2;\ i = 1, \cdots, p\}$. This shows (27). This also implies $r_e(D_w^* D_w) = \max\{\sigma_i^2;\ i = 1, \cdots, p\} = \|D_w\|^2$. Hence if $\gamma^2 = r_e(V(\lambda))$, it is also equal to $\|D_w\|^2$.

Now suppose $\gamma^2 > r_e(V(\lambda)) = \|D_w\|^2$. As noted above, $\|V(\lambda)\|$ is attained as the spectral radius $r(V(\lambda))$. Also, since $\sigma(V(\lambda))$ is a closed set, γ^2 must belong to $\sigma(V(\lambda))\backslash\sigma_e(V(\lambda))$. By (27), $\sigma_e(V(\lambda))$ is a finite set, so that again by Lemma 3, γ^2 must be an eigenvalue of $V(\lambda)$. This yields Case 2), completing the proof. □

This proposition shows the following:

- $\|D_w\|$ gives a lower bound for $\|G(z)\|$.
- If $\|G(z)\| > \|D_w\|$, it can be found as the maximal singular value.

Therefore, we can essentially resort to an eigenvalue problem for computing the frequency response of $G(z)$.

B. Reduction to a Finite-Dimensional Eigenvalue Problem

We have seen that when $\|G(e^{j\omega h})\| > \|D_w\|$ it is characterized as the maximal singular value of $G(e^{j\omega h})$. So we are led to solving the singular value equation

$$(\gamma^2 I - G^* G(e^{j\omega h}))w = 0.$$

We now have the following theorem.

Theorem 2: Assume $\gamma > \|D_w\|$ and γ is not a singular value of $\boldsymbol{D}$. Define

$$R_\gamma = (\gamma^2 I - \boldsymbol{D}^* \boldsymbol{D}).$$

There exists a nontrivial solution w to the equation

$$(\gamma^2 I - G^* G(e^{j\omega h}))w = 0 \tag{28}$$

if and only if

$$\det(e^{j\omega h}\mathcal{E} - \mathcal{A}) = 0 \tag{29}$$

where $\mathcal{E}$ and $\mathcal{A}$ are given by

$$\mathcal{E} := \begin{bmatrix} I & 0 & \mathcal{E}_{13} & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & \mathcal{E}_{33} & A_{ds}^* \\ 0 & 0 & \mathcal{E}_{43} & A_d^* \end{bmatrix}, \qquad \mathcal{A} := \begin{bmatrix} \mathcal{A}_{11} & \mathcal{A}_{12} & 0 & 0 \\ \mathcal{A}_{ds} & \mathcal{A}_d & 0 & 0 \\ \mathcal{A}_{31} & \mathcal{A}_{32} & I & 0 \\ \mathcal{A}_{41} & \mathcal{A}_{42} & 0 & I \end{bmatrix} \tag{30}$$

$$\mathcal{E}_{13} = -\boldsymbol{B} R_\gamma^{-1} K^*(h - \cdot)$$

$$\mathcal{E}_{33} = A_{cs}^* + \int_0^h C_1^*(\theta) \boldsymbol{D} R_\gamma^{-1} K^*(h - \cdot)\, d\theta$$

$$\mathcal{E}_{43} = A_{cd}^* + \int_0^h C_2^*(\theta) \boldsymbol{D} R_\gamma^{-1} K^*(h - \cdot)\, d\theta$$

$$\begin{aligned}\mathcal{A}_{11} &= A_{cs} + \boldsymbol{B}R_\gamma^{-1}\boldsymbol{D}^* C_1(\cdot)\\ \mathcal{A}_{12} &= A_{cd} + \boldsymbol{B}R_\gamma^{-1}\boldsymbol{D}^* C_2(\cdot)\end{aligned}$$

$$\mathcal{A}_{31} = -\int_0^h C_1^*(\theta)(I + \boldsymbol{D}R_\gamma^{-1}\boldsymbol{D}^*)C_1(\theta)\,d\theta$$

$$\mathcal{A}_{32} = -\int_0^h C_1^*(\theta)(I + \boldsymbol{D}R_\gamma^{-1}\boldsymbol{D}^*)C_2(\theta)\,d\theta$$

$$\mathcal{A}_{41} = -\int_0^h C_2^*(\theta)(I + \boldsymbol{D}R_\gamma^{-1}\boldsymbol{D}^*)C_1(\theta)\,d\theta$$

$$\mathcal{A}_{42} = -\int_0^h C_2^*(\theta)(I + \boldsymbol{D}R_\gamma^{-1}\boldsymbol{D}^*)C_2(\theta)\,d\theta. \qquad (31)$$

Outline of Proof: To express (28) in terms of the state-space equations, write down $v = G(e^{j\omega h})w$ and $r = G^*(e^{j\omega h})v$, $w, v, r \in L^2[0, h]$, and set $r = \gamma^2 w$. If $G(z)$ is represented by (3) and (4), then by the standard duality theory its dual system is given by

$$\begin{aligned}p_k &= A^* p_{k+1} + C^* v_k \qquad (32)\\ r_k &= B^* p_{k+1} + \boldsymbol{D}^* v_k. \qquad (33)\end{aligned}$$

Therefore, the singular value equation (28) admits a nontrivial solution w if and only if there exist w, v, r, not all zero, such that

$$\begin{aligned}G(e^{j\omega h})w:&\quad e^{j\omega h}x = Ax + Bw, \quad v = Cx + \boldsymbol{D}w \qquad (34)\\ G^*(e^{j\omega h})v:&\quad p = e^{j\omega h}A^* p + C^* v\\ &\quad r = \gamma^2 w = e^{j\omega h}B^* p + \boldsymbol{D}^* v \qquad (35)\end{aligned}$$

where $x := [x_c^T, x_d^T]^T$, $p := [p_c^T, p_d^T]^T$. Combining (34) and (35) leads to

$$\gamma^2 w = \boldsymbol{D}^*\boldsymbol{D}w + e^{j\omega h}B^* p + \boldsymbol{D}^* Cx$$

so that

$$R_\gamma w = (\gamma^2 I - \boldsymbol{D}^*\boldsymbol{D})w = e^{j\omega h}B^* p + \boldsymbol{D}^* Cx.$$

By the discussion in Subsection A, any number $\gamma^2 > \|D_w\|^2$ in the spectrum of $\boldsymbol{D}^*\boldsymbol{D}$ must be its eigenvalue. Since γ is not a singular value of $\boldsymbol{D}$, R_γ becomes invertible and w can be solved as

$$w(\theta) = R_\gamma^{-1}(e^{j\omega h}B^* p + \boldsymbol{D}^* Cx).$$

Substituting this for w in (34) and (35) and computing the precise dual operators A^*, B^*, C^*, $\boldsymbol{D}^*$ in (32) and (33) as in [31] implies that (28) holds if and only if the generalized eigenvalue problem

$$e^{j\omega h}\mathcal{E}\xi = \mathcal{A}\xi \qquad (36)$$

admits a nontrivial solution ξ. This is precisely (29). (The detailed computation of dual operators and (31) can be found in [31].) □

Remark 3: Some remarks are in order on computational aspects. For γ to be a singular value of $G(e^{j\omega h})$, characteristic equation (29) must be satisfied precisely for each frequency ω. This is in marked contrast to the H^∞ norm computation, where the same equation may be satisfied for some frequency. This in turn leads to an inequality condition $\gamma > \|G(e^{j\omega h})\|$ from which a bisection type algorithm can be derived [31], [24]. In the present case, checking (29) is numerically a much more delicate problem. Fortunately, it has been found recently that one can give fairly good upper and lower bound estimates for $\|G(e^{j\omega h})\|$ as well as a bisection algorithm. We refer the reader to [13] and [21] for details.

As an application of Theorem 2 obtained above, let us now derive a finite-dimensional discrete-time plant $(\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D})$ whose H^∞ control problem is equivalent to that of the original sampled-data system in Fig. 1. This problem has been extensively studied, and several solutions have been obtained [17], [18], [3], [15]. We here show that once the generalized eigenvalue problem (Theorem 2) is obtained, it is straightforward to obtain such a norm-equivalent system.

We start with the following lemma which is an easy consequence of the computation above.

Lemma 4: Let $\overline{G}(z) = \left[\begin{array}{c|c}\overline{A} & \overline{B}\\ \hline \overline{C} & 0\end{array}\right]$ be a finite-dimensional discrete-time system, and let $\gamma \geq \|\overline{G}(e^{j\omega})\|$ for some $0 \leq \omega < 2\pi$. Then, $\|\overline{G}\|_\infty < \gamma$ if and only if there exists no λ of modulus 1 such that

$$\det\left(\lambda\begin{bmatrix}I & -\overline{B}\,\overline{B}^*/\gamma^2\\ 0 & \overline{A}^*\end{bmatrix} - \begin{bmatrix}\overline{A} & 0\\ -\overline{C}^*\overline{C} & I\end{bmatrix}\right) = 0. \qquad (37)$$

This is an easy counterpart of the well-known continuous-time result [5]; see also [12], [31], and [24] for details.

Observe that (29) has the same form as above, with

$$\overline{A} = A + BR_\gamma^{-1}\boldsymbol{D}^*[C_1(\theta) \quad C_2(\theta)]$$

$$\overline{B}\,\overline{B}^*/\gamma^2 = \boldsymbol{B}R_\gamma^{-1}\boldsymbol{B}^*$$

$$\overline{C}^*\overline{C} = [C_1(\theta) \quad C_2(\theta)]^*(I + \boldsymbol{D}R_\gamma^{-1}\boldsymbol{D}^*)[C_1(\theta) \quad C_2(\theta)]. \qquad (38)$$

Since all these terms on the right are matrices, the H^∞ norm bound condition $\|G(z)\|_\infty < \gamma$ is equivalently transformed to that for $(\overline{A}, \overline{B}, \overline{C})$ by finding these satisfying (38). However, we need yet one more step to transform the original H^∞ control problem into a discrete-time one, because we want the controller (A_d, B_d, C_d, D_d) to remain invariant under this procedure. To this end, let

$$\tilde{P}(z) := \left[\begin{array}{c|cc}\tilde{A} & \tilde{B}_w & \tilde{B}_u\\ \hline \tilde{C}_z & 0 & \tilde{D}_u\\ \tilde{C}_y & 0 & 0\end{array}\right].$$

Note that we can set the direct feedthrough term from w to z to be zero by the form of (29) and (37). Combining this (with state $\tilde{x}$) with the controller (A_d, B_d, C_d, D_d) as in Fig. 1, we have

$$\begin{bmatrix} \tilde{x}_{k+1} \\ x_{d,k+1} \end{bmatrix} = \begin{bmatrix} \tilde{A} + \tilde{B}_u D_d \tilde{C}_y & \tilde{B}_u C_d \\ B_d \tilde{C}_y & A_d \end{bmatrix} \begin{bmatrix} \tilde{x}_k \\ x_{d,k} \end{bmatrix} + \begin{bmatrix} \tilde{B}_w \\ 0 \end{bmatrix} w_k$$
$$z_k = (\tilde{C}_z + \tilde{D}_u D_d \tilde{C}_y)\tilde{x}_k + \tilde{D}_u C_d x_{d,k}. \tag{39}$$

We then have the following theorem.

Theorem 3: Given the sampled feedback system $G(z)$ in Fig. 1 with continuous-time plant (2), choose $(\tilde{A}, \tilde{B}_w, \tilde{B}_u, \tilde{C}_z, \tilde{C}_y, \tilde{D}_u)$ to satisfy

$$\tilde{A} := e^{A_c h} + \boldsymbol{B} R_\gamma^{-1} \boldsymbol{D}^* (C_z e^{A_c \theta})$$
$$\tilde{B}_w \tilde{B}_w^* = \boldsymbol{B}(I - \boldsymbol{D}^* \boldsymbol{D}/\gamma^2)^{-1} \boldsymbol{B}^*$$

$$\tilde{B}_u := \int_0^h e^{A_c(h-\tau)} B_u H(\tau)\, d\tau + \boldsymbol{B} R_\gamma^{-1} \boldsymbol{D}^* C_z \cdot \left\{ \int_0^\theta e^{A_c(\theta-\tau)} B_u H(\tau)\, d\tau + D_u H(\theta) \right\}$$

$$\begin{bmatrix} \tilde{C}_z^* \\ \tilde{D}_u^* \end{bmatrix} [\tilde{C}_z \quad \tilde{D}_u] = \begin{bmatrix} \Phi_1^* \\ \Phi_2^* \end{bmatrix} (I - \boldsymbol{D}\boldsymbol{D}^*/\gamma^2)^{-1} [\Phi_1 \quad \Phi_2]$$
$$\tilde{C}_y := C_y \tag{40}$$

where $\Phi_1(\theta) := C_z e^{A_c \theta}$ and $\Phi_2(\theta) := C_z \int_0^\theta e^{A_c(\theta-\tau)} B_c H(\tau)\, d\tau$. Then the closed-loop system $G_d(z)$ formed with this discrete-time plant with the digital controller (A_d, B_d, C_d, D_d) as in Fig. 1 satisfies $\|G\|_\infty < \gamma$ if and only if $\|G_d\|_\infty < \gamma$.

Proof: Comparing (39) with (29) and (31), we see that (38) can be satisfied if we first take $\tilde{B}_w$ and $\tilde{C}_y$ as above. It follows that $\tilde{A}$ and $\tilde{B}_u$ should satisfy

$$\tilde{A} + \tilde{B}_u D_d C_y = A_{cs} + \boldsymbol{B} R_\gamma^{-1} \boldsymbol{D}^* C_1(\cdot)$$
$$\tilde{B}_u C_d = A_{cd} + \boldsymbol{B} R_\gamma^{-1} \boldsymbol{D}^* C_2(\cdot).$$

According to the forms of A_{cd} and $C_2(\cdot)$ in (5), the forms for $\tilde{A}$ and $\tilde{B}_u$ readily follow. Finally, the condition on $\overline{C}$ in (38) is satisfied if

$$\begin{bmatrix} C_1^* \\ C_2^* \end{bmatrix} M [C_1 \quad C_2] = \begin{bmatrix} (\tilde{C}_z + \tilde{D}_u D_d C_y)^* \\ (\tilde{D}_u C_d)^* \end{bmatrix} [\tilde{C}_z + \tilde{D}_u D_d C_y \quad \tilde{D}_u C_d]$$

where $M = I + \boldsymbol{D} R_\gamma^{-1} \boldsymbol{D}^* = (I - \boldsymbol{D}\boldsymbol{D}^*/\gamma^2)^{-1}$. This is easily seen to be equivalent to the requirement given in (40) above. □

The same equivalent system has been obtained by [3]. The advantage here is that once (29) is obtained, the problem is quite simply reduced to that of factorization of matrices. Moreover, from Theorem 2 and Lemma 4 it is clear that the H^∞ norms of $G(z)$ and $G_d(z)$ are assumed at the same frequency. This is not so obvious in the other approaches.

V. State-Space Formulas and Examples

To solve the eigenvalue problem (29) we need to evaluate the integrals appearing in (31). When the hold functions are zero-order hold, however, they can be evaluated by taking suitable exponentials of constant matrices (e.g., [3]).

Assume $D_w = 0$, $D_u = 0$ for brevity, and also assume γ is not a singular value of $\boldsymbol{D}$ throughout. Assume also that the hold function $H(\theta)$ is the zero-order hold: $H(\theta) \equiv H$ (constant matrix). Define

$$\Gamma(t) := \exp\left(\begin{bmatrix} -A_c^T & -C_z^T C_z/\gamma \\ B_w B_w^T/\gamma & A_c \end{bmatrix} t \right) = \begin{bmatrix} \Gamma_{11}(t) & \Gamma_{12}(t) \\ \Gamma_{21}(t) & \Gamma_{22}(t) \end{bmatrix}.$$

Then the hypothesis that γ is not a singular value of $\boldsymbol{D}$ holds if and only if $\Gamma_{11}(h)$ is invertible, and then R_γ becomes invertible [35]. As in [3], the operator R_γ^{-1} can be expressed as

$$R_\gamma^{-1} w = \gamma^{-2} w + \gamma^{-3} [B_w^T \quad 0] \Bigg(\Gamma(t) \begin{bmatrix} -\Gamma_{11}^{-1}(h) & 0 \\ 0 & 0 \end{bmatrix} \cdot \int_0^h \Gamma(h-\tau) \begin{bmatrix} 0 \\ B_w/\gamma \end{bmatrix} w(\tau)\, d\tau + \int_0^t \Gamma(t-\tau) \begin{bmatrix} 0 \\ B_w/\gamma \end{bmatrix} w(\tau)\, d\tau \Bigg).$$

Substituting this into (31) will yield the desired state-space formulas. Recall

$$A_{cs} = e^{A_c h} + \int_0^h e^{A_c(h-\tau)} B_u H D_d C_y d\tau$$
$$A_{cd} = \int_0^h e^{A_c(h-\tau)} B_u H C_d\, d\tau$$
$$K(\theta) = e^{A_c \theta} B_w$$
$$W(\theta) = C_z e^{A_c t} B_w$$
$$C_1(\theta) = C_z \left(e^{A_c \theta} + \int_0^\theta e^{A_c(\theta-\tau)} B_u H D_d C_y\, d\tau \right)$$
$$C_2(\theta) = \int_0^\theta C_z e^{A_c(\theta-\tau)} B_u H C_d\, d\tau.$$

As similarly in [3], we obtain the following:

$$\mathcal{E}_{13} = -\gamma^{-1} \Gamma_{21}(h) \Gamma_{11}(h)^{-1}$$
$$\mathcal{E}_{33} = [I + (B_u H D_d C_y)^* \boldsymbol{\Phi}_{11}(h)] \Gamma_{11}(h)^{-1}$$
$$\mathcal{E}_{43} = (B_u H C_d)^* \boldsymbol{\Phi}_{11}(h) \Gamma_{11}(h)^{-1}$$
$$\mathcal{A}_{11} = \Gamma_{22}(h) - \Gamma_{21}(h)\Gamma_{11}(h)^{-1}\Gamma_{12}(h) + [\boldsymbol{\Phi}_{22}(h) - \Gamma_{21}(h)\Gamma_{11}(h)^{-1}\boldsymbol{\Phi}_{12}(h)] \cdot B_u H D_d C_y$$
$$= (\Gamma_{11}(h)^{-1})^* + [\boldsymbol{\Phi}_{22}(h) - \Gamma_{21}(h)\Gamma_{11}(h)^{-1}\boldsymbol{\Phi}_{12}(h)] \cdot B_u H D_d C_y$$
$$\mathcal{A}_{12} = [\boldsymbol{\Phi}_{22}(h) - \Gamma_{21}(h)\Gamma_{11}(h)^{-1}\boldsymbol{\Phi}_{12}(h)] B_u H C_d$$

$$\begin{aligned}
\mathcal{A}_{31} =& \gamma\Gamma_{11}(h)^{-1}\Gamma_{12}(h) - \gamma(B_uHD_dC_y)^* \\
&\cdot [\boldsymbol{\Omega}_{12}(h) - \boldsymbol{\Phi}_{11}(h)\Gamma_{11}(h)^{-1}\boldsymbol{\Phi}_{12}(h)]B_uHD_dC_y \\
&+ \gamma\Gamma_{11}(h)^{-1}\boldsymbol{\Phi}_{12}(h)B_uHD_dC_y \\
&+ (\gamma\Gamma_{11}(h)^{-1}\boldsymbol{\Phi}_{12}(h)B_uHD_dC_y)^* \\
\mathcal{A}_{32} =& \gamma\Gamma_{11}(h)^{-1}\boldsymbol{\Phi}_{12}(h)B_uHC_d - \gamma(B_uHD_dC_y)^* \\
&\cdot [\boldsymbol{\Omega}_{12}(h) - \boldsymbol{\Phi}_{11}(h)\Gamma_{11}(h)^{-1}\boldsymbol{\Phi}_{12}(h)]B_uHC_d \\
\mathcal{A}_{41} =& \mathcal{A}_{32}^* \\
\mathcal{A}_{42} =& -\gamma(B_uHC_d)^*[\boldsymbol{\Omega}_{12}(h) - \boldsymbol{\Phi}_{11}(h)\Gamma_{11}(h)^{-1}\boldsymbol{\Phi}_{12}(h)] \\
&\cdot B_uHC_d \quad (41)
\end{aligned}$$

where

$$\boldsymbol{\Phi}(t) := \int_0^t \Gamma(\tau)\,d\tau$$

$$\boldsymbol{\Omega}(t) := \int_0^t \left(\int_0^\theta \Gamma(\tau)\,d\tau\right) d\theta$$

can also be evaluated by taking suitable exponentials (cf. [3]). For example

$$\boldsymbol{\Phi}(t) = [I \quad 0] \exp\left(\begin{bmatrix} F & I \\ 0 & 0 \end{bmatrix} t\right) \begin{bmatrix} 0 \\ I \end{bmatrix}$$

where $F = \begin{bmatrix} -A_c^T & -C_z^TC_z/\gamma \\ B_wB_w^T/\gamma & A_c \end{bmatrix}$. To obtain $\Omega(t)$, we use this formula again. Actually, more compact formulas are given in [15].

We now give two examples.

Example 2: Let us compute the frequency response of the continuous-time plant

$$G(s) = \frac{1}{s+1}$$

in the sense defined here. Observe that since the H^∞-norm is equal to the L^2-induced norm in the time domain, it should give precisely the same value as in the continuous-time case, which is one, irrespective of the sampling period h.

Let

$$A_c = -1, \qquad B_w = C_z = 1, \qquad \beta = \gamma^{-1},$$

$$\alpha = \sqrt{|\beta^2 - 1|}.$$

Then $\Gamma(h)$ can be computed as shown at the bottom of the page, depending on $\gamma < 1$ or $\gamma > 1$. When $\gamma = 1$, they agree and are equal to

$$\begin{bmatrix} 1+h & -h \\ h & 1-h \end{bmatrix}.$$

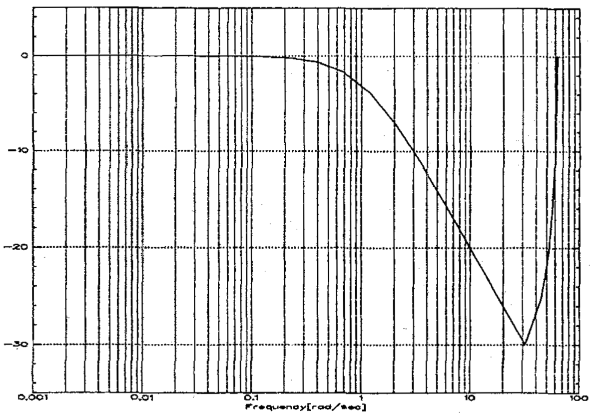

Fig. 3. Lifted frequency response for $1/(s+1)$.

According to (41), the characteristic equation $\det(\lambda\mathcal{E} - \mathcal{A}) = 0$ becomes

$$\det\begin{bmatrix} \lambda - \Gamma_{11}^{-1} & -\lambda\gamma^{-1}\Gamma_{21}\Gamma_{11}^{-1} \\ -\gamma\Gamma_{11}^{-1}\Gamma_{12} & \lambda\Gamma_{11}^{-1} - 1 \end{bmatrix} =$$
$$\Gamma_{11}^{-1}(\lambda^2 + \Gamma_{11}^{-1}(-\Gamma_{21}\Gamma_{12} - \Gamma_{11}^2 - 1)\lambda + 1) = 0.$$

In view of the identity

$$\Gamma_{22} - \Gamma_{21}\Gamma_{11}^{-1}\Gamma_{12} = \Gamma_{11}^{-1}$$

we see that the coefficient of λ is

$$-\Gamma_{11} - \Gamma_{22} = \begin{cases} -2\cos\alpha h & \gamma < 1 \\ -2 & \gamma = 1 \\ -2\cosh\alpha h & \gamma > 1. \end{cases}$$

Since $|2\cos\alpha h| \le 1$ and $|2\cosh\alpha h| > 1$, it is easy to see that $\det(\lambda\mathcal{E} - \mathcal{A}) = 0$ admits a solution of modulus one if and only if $\gamma \le 1$. The largest γ that can be assumed among them is one, equal to the H^∞-norm of $1/(s+1)$ in the continuous-time sense. The frequency where this norm is attained is $\omega = 0$.

To compute the frequency response, we must solve

$$e^{2j\omega h} - 2\left(\cos\frac{\sqrt{1-\gamma^2}}{\gamma}h\right)e^{j\omega h} + 1 = 0$$

for γ at each ω. For γ not being a singular value of $\boldsymbol{D}$, this is easily solved as

$$\gamma = \begin{cases} \frac{1}{\sqrt{1+\omega^2}}, & \omega \le \pi/h \\ \frac{1}{\sqrt{1+(2\pi/h-\omega)^2}}, & \omega > \pi/h. \end{cases}$$

Observe that this is precisely equal to the continuous-time counterpart for $\omega \le \pi/h$. This can also be seen from the Bode plot in Fig. 3 for the case $h = 0.1$.

$$\Gamma(h) = \begin{cases} \begin{bmatrix} \cos\alpha h + \frac{1}{\alpha}\sin\alpha h & -\frac{\beta}{\alpha}\sin\alpha h \\ \frac{\beta}{\alpha}\sin\alpha h & \cos\alpha h - \frac{1}{\alpha}\sin\alpha h \end{bmatrix} & \gamma < 1 \\ \begin{bmatrix} \cosh\alpha h + \frac{1}{\alpha}\sin\alpha h & -\frac{\beta}{\alpha}\sinh\alpha h \\ \frac{\beta}{\alpha}\sinh\alpha h & \cosh\alpha h - \frac{1}{\alpha}\sinh\alpha h \end{bmatrix} & \gamma > 1 \end{cases}$$

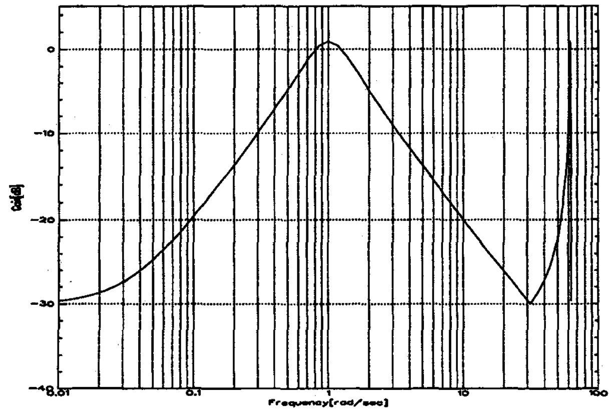

Fig. 4. A closed-loop case.

See also [34] for a second-order example where aliasing effect clearly appears as a very high peak at a low frequency.

Example 3: We now give a hybrid closed-loop case of Fig. 1 in which the continuous-time plant and the discrete-time controller are specified by

$$z = \frac{1}{s+1}(w+u), \qquad y = z$$
$$u(\theta) = -\frac{h}{z-1}Sy, \qquad h = 0.1 \text{ (s)}.$$

The hold function is the zero-order hold. The controller is the discretization of $1/s$. The closed-loop stability is guaranteed for small enough h. In Fig. 4, we show the frequency response of the closed-loop system from w to z. It is interesting to observe that in this case the highest gain is actually larger than 0 dB which is the gain of the corresponding continuous-time gain. This computation is done by implementing (41) to Xmath.

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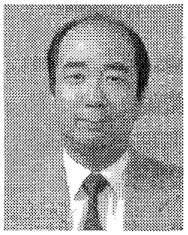

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