Robustness Analysis of Nonlinear Feedback Systems: An Input–Output Approach

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***Abstract*—This paper presents an approach to robustness analysis for nonlinear feedback systems. We pursue a notion of model uncertainty based on the closeness of input–output trajectories which is not tied to a particular uncertainty representation, such as additive, parametric, structured, etc. The basic viewpoint is to regard systems as operators on signal spaces. We present two versions of a global theory where stability is captured by induced norms or by gain functions. We also develop local approaches (over bounded signal sets) and give a treatment for systems with potential for finite-time escape. We compute the relevant stability margin for several examples and demonstrate robustness of stability for some specific perturbations, e.g., small-time delays. We also present examples of nonlinear control systems which have zero robustness margin and are destabilized by arbitrarily small gap perturbations. The paper considers the case where uncertainty is present in the controller as well as the plant and the generalization of the approach to the case where uncertainty occurs in several subsystems in an arbitrary interconnection.**

***Index Terms*—Gap metric, nonlinear systems, robust control.**

NOTATION

$\mathcal{L}_p^r[0,\infty)$	Lebesgue p-space of r-vector valued functions on $[0,\infty)$, with norm $\|\cdot\|_p$.
$C^r[0,\infty)$	Continuous and bounded r-vector valued functions on $[0,\infty)$, with sup norm $\|\cdot\|_\infty$.
$\boldsymbol{T}_\tau$	Truncation operator: for $f(t), 0 \le t < \infty$, $\boldsymbol{T}_\tau f(t) = f(t)$ on $[0,\tau]$, and zero otherwise. The same notation will be used for vector valued functions.
$\mathcal{L}_{p,e}^r[0,\infty)$	$:= \{f(t)\colon\ 0 \le t < \infty \text{ s.t. } \boldsymbol{T}_\tau f(t) \in \mathcal{L}_p^r[0,\infty) \text{ for all } \tau > 0\}$.
$\mathcal{C}_e^r[0,\infty)$	$:= \{f(t)\colon 0 \le t < \infty \text{ s.t. } \boldsymbol{T}_\tau f(t) \in \mathcal{C}^r[0,\infty)$ for all $\tau > 0\}$.
$\|f(t)\|_\tau$	$:= \|\boldsymbol{T}_\tau f(t)\|$, where $f(t)$ is in either $\mathcal{L}_{p,e}^r[0,\infty)$ or $\mathcal{C}_e^r[0,\infty)$, and $\|\cdot\|$ denotes the norm of the relevant normed space.

I. INTRODUCTION

ONE OF the basic properties of stable feedback loops is that they tolerate uncertainties which are sufficiently small in an appropriate sense. Moreover, stable loops have the potential to reduce the effects of uncertainty, if designed appropriately. Uncertainties may be small and yet have complex structure, e.g., infinite dimensional, time-varying, hysteresis, etc., or may even defy a concrete realization in the sense of a dynamical system. Our aim in this paper is to present an input–output approach to uncertainty for nonlinear systems which has the potential to include such a variety of perturbations to the nominal model.

In the context of linear theory, it is well established that the appropriate topology for considering questions of robustness is that induced by the gap metric (the graph topology). Namely, perturbations which are small in the gap are precisely those which give small closed-loop errors in a feedback loop. In contrast, other models of uncertainty have restrictions; e.g., additive uncertainty does not allow a stable and an unstable model to be compared, and parametric uncertainty does not allow changes in model order, small time delays, etc. Accordingly, in this paper we seek a suitable generalization of the gap metric approach to robustness for nonlinear systems.

We consider a system to be defined by its graph, namely the collection of its input–output trajectories. We consider two systems to be close if their graphs are close according to some measure. We will see by theory and example that such a way of comparing systems allows the variety of uncertainty mentioned above. One of the main results of the paper says that robustness to small perturbations of the graph requires that a certain disturbance-to-error mapping has bounded signal amplification. Disturbances need to be injected at both the input and output of the plant and the responses found after and before the respective summing junctions. This mapping is a (nonlinear) parallel projection operator, and the inverse of its gain is the stability margin for plant uncertainty. This result was first presented in the context of nonlinear systems in [14] and generalizes a corresponding result from the linear case [12], [8], [9]. The initial insight for the work of this paper came from the geometrical techniques of [8] and [9]. However, the ideas have connections with a number of works in the literature of nonlinear control. We mention particularly the use of sector conditions for stability in [38], [39], and [27]. Other contributions related to graph representations and the complementarity of graphs as a condition for nonlinear feedback stability include [15], [33], [30], and [23].

There are two basic tools which feature in the approach of this paper for nonlinear systems. The first is a summation operator for the characterization of stability. Any solution of the feedback equations requires that an element of the graph of the plant is added to an element of the graph of the controller to equal the external disturbances. To find the response to arbitrary disturbances, this operation must be invertible, and for stability it must be bounded in a suitable sense. The second

Manuscript received April 12, 1996; revised March 10, 1997. Recommended by Associate Editor, J. Shamma. This work was supported in part by the NSF, AFOSR, EPSRC, and NATO.

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Publisher Item Identifier S 0018-9286(97)06594-X.

tool is the use of a mapping from the system graph onto a perturbed graph and the use of the distance from the identity of this mapping as a measure of distance between systems. Although the assumption of the existence of such a mapping appears to be strong at first, it will be shown that if two systems are stabilizable then there exists such a mapping, and if the closed-loop responses are close for some common controller then the mapping is close to the identity. Moreover, we will be able to construct such mappings explicitly in examples, such as when time delays are introduced.

We now outline the contents of the paper. In Section II, we introduce the summation and parallel projection operators. In Section III, we present a generalization of the gap metric for nonlinear systems defined on extended spaces. Theorem 1 provides the main paradigm of the theory: if the gap between the plant and perturbation is less than the inverse of the norm of a certain parallel projection, then the loop remains stable. The theorem is applied in Example 1 to assess tolerance to time delays when an integrator with input saturation is stabilized by unity feedback. Theorem 2 studies the nonlinear gap topology, and in particular, the relationship between closed-loop norm convergence and convergence in the gap metric. Metric properties of the gap are investigated. An alternative distance measure is presented for which the corresponding main robustness result (Theorem 3) has the circle criterion as a corollary (Example 3). In Section IV, we give a version of the theory for the case where there is a known bound on the norm of potential disturbances. In this case, the main theorem requires the gap and parallel projection norm to be evaluated on a bounded domain (Theorem 4). This result is pertinent in the case where stability, for the nominal or perturbed systems, cannot be guaranteed globally. The theorem is applied to an unstable system with saturation (Example 4). In Section V, we deal with systems with potential for finite-time escape. A modification of the gap and the norm of the parallel projection are again evaluated, on suitably bounded domains, and then compared to assess robustness of stability (Theorem 5). The theorem is applied to an unstable system with quadratic nonlinearity (Example 5). In Section VI, we give a version of our robustness theory for gain-function stability (Theorem 6). The theorem is applied to a system with cubic nonlinearity (Example 6). A brief discussion of hysteresis perturbations is given. Section VII presents two examples of control systems with zero robustness margin: a Nussbaum universal adaptive controller and a parameter adaptive controller. In each case it is shown that these may be destabilized by a perturbation which is infinitesimally small in the gap. The two final sections present generalizations of the robustness theory to the case where uncertainty is present in both the plant and controller (Section VIII) or in several elements in an arbitrary feedback interconnection (Section IX).

II. Preliminaries on Feedback Stabilization

In this paper we assume that the plant and compensator are causal mappings $\boldsymbol{P}\colon \mathcal{U} \to \mathcal{Y}$ and $\boldsymbol{C}\colon \mathcal{Y} \to \mathcal{U}$ which satisfy $\boldsymbol{P}0 = 0$ and $\boldsymbol{C}0 = 0$, where $\mathcal{U}$ and $\mathcal{Y}$ are appropriate signal spaces. We define a signal space to be an extended space or a Banach space of time functions with support on $[0,\infty)$, e.g., $\mathcal{L}^r_{p,e}[0,\infty), \mathcal{C}^r_e[0,\infty)$, or $\mathcal{L}^r_p[0,\infty)$. A departure from this takes place in Section V, where we address systems with potential for finite-time escape. There we allow for the possibility that signals are defined only over a finite window in time.

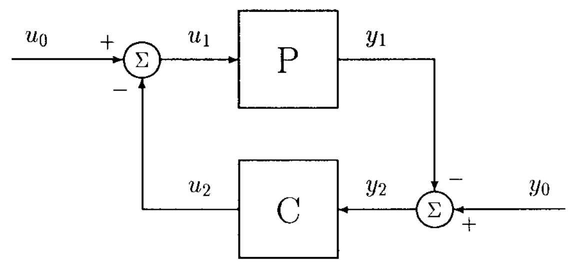

Fig. 1. Standard feedback configuration.

Consider the feedback configuration of Fig. 1 where the signals u_i $(i \in \{0,1,2\})$ belong to $\mathcal{U}$ and y_i $(i \in \{0,1,2\})$ belong to $\mathcal{Y}$. Under mild, physically motivated conditions on $\boldsymbol{P}$ and $\boldsymbol{C}$ (e.g., the product of the instantaneous gains is less than one [37], [4]; see also [2]), the feedback configuration can be guaranteed to be well-posed. Namely, for any $(u_0, y_0) \in \mathcal{U} \times \mathcal{Y} =: \mathcal{W}$, there exist unique signals $u_1, u_2 \in \mathcal{U}$ and $y_1, y_2 \in \mathcal{Y}$ such that the following feedback equations hold:

$$\begin{aligned} u_0 &= u_1 + u_2 \\ y_0 &= y_1 + y_2 \\ y_1 &= \boldsymbol{P} u_1 \\ u_2 &= \boldsymbol{C} y_2 \end{aligned}$$

and moreover

$$\boldsymbol{H}_{\mathrm{P,C}}\colon \mathcal{W} \to \mathcal{W} \times \mathcal{W}\colon \begin{pmatrix} u_0 \\ y_0 \end{pmatrix} \mapsto \left(\begin{pmatrix} u_1 \\ y_1 \end{pmatrix}, \begin{pmatrix} u_2 \\ y_2 \end{pmatrix} \right)$$

is causal. Throughout the paper, well-posedness of the feedback configuration will always be assumed for the nominal as well as for all perturbed systems (though a weaker version will be assumed in Section V). Thus, given that the feedback equations have a solution (e.g., over extended spaces), feedback stability is the requirement that $\boldsymbol{H}_{\mathrm{P,C}}$ is stable, i.e., bounded in a suitable sense.

In the subsequent sections we will consider several alternative notions of stability. These are defined here. Let $\mathcal{X}_i$ $(i = 1, 2)$ be signal spaces or subsets of such spaces. A causal operator $\boldsymbol{F}\colon \mathcal{X}_1 \to \mathcal{X}_2$ is said to be *stable* if $\boldsymbol{F}0 = 0$ and

$$\|\boldsymbol{F}|_{\mathcal{X}_1}\| := \sup\left\{ \frac{\|\boldsymbol{F}x\|_\tau}{\|x\|_\tau}\colon x \in \mathcal{X}_1, \tau > 0, \|x\|_\tau \neq 0 \right\} < \infty.$$

A causal operator $\boldsymbol{F}\colon \mathcal{X}_1 \to \mathcal{X}_2$ is said to be *incrementally stable* if $\boldsymbol{F}0 = 0$ and

$$\|\boldsymbol{F}|_{\mathcal{X}_1}\|_\Delta := \sup\left\{ \frac{\|\boldsymbol{F}x_1 - \boldsymbol{F}x_2\|_\tau}{\|x_1 - x_2\|_\tau}\colon x_1, x_2 \in \mathcal{X}_1, \tau > 0, \|x_1 - x_2\|_\tau \neq 0 \right\} < \infty.$$

It is standard to check that the norm $\|\cdot\|$ and the incremental gain $\|\cdot\|_\Delta$ satisfy the usual triangle and submultiplicative inequalities. We define the gain function $g[\cdot](\cdot)$ of a causal $\boldsymbol{F}$ via

$$g[\boldsymbol{F}](\alpha) := \sup\{\|\boldsymbol{F}x\|_\tau : x \in \mathcal{X}_1, \tau > 0, \|x\|_\tau \le \alpha\}.$$

The operator $\boldsymbol{F}$ is said to be *gain-function (gf)-stable* if $g[\boldsymbol{F}](\alpha)$ remains finite for all $\alpha > 0$.

It is fruitful to think of a system in terms of its graph instead of as a mapping, i.e., as the set of all possible input–output pairs which are compatible with the description of the system. Formally, the *graph* of $\boldsymbol{P}$ is defined as

$$\mathcal{G}_P := \left\{ \begin{pmatrix} u \\ \boldsymbol{P}u \end{pmatrix} : u \in \mathcal{U}, \boldsymbol{P}u \in \mathcal{Y} \right\} \subset \mathcal{W}.$$

In case $\boldsymbol{P}$ is defined for all $u \in \mathcal{U}$ (e.g., when $\boldsymbol{P}$ is an operator on extended spaces), the condition $\boldsymbol{P}u \in \mathcal{Y}$ is redundant. However in general, e.g., in case $\mathcal{U}, \mathcal{Y}$ are Banach spaces, the requirement that $\boldsymbol{P}u \in \mathcal{Y}$ may restrict the inputs to a proper subset of $\mathcal{U}$. We adopt the convention that the elements of the graph are ordered according to the decomposition of the ambient space $\mathcal{W} = \mathcal{U} \times \mathcal{Y}$. Thus, we define the graph (sometimes called the inverse graph) of $\boldsymbol{C}$ as

$$\mathcal{G}_{\mathrm{C}} := \left\{ \begin{pmatrix} \boldsymbol{C}y \\ y \end{pmatrix} : \boldsymbol{C}y \in \mathcal{U}, y \in \mathcal{Y} \right\} \subset \mathcal{W}.$$

In order to study stability of the feedback system in Fig. 1, a convenient device is the *summation operator* defined on the cartesian product of the two graphs $\mathcal{M} := \mathcal{G}_P$ and $\mathcal{N} := \mathcal{G}_C$ as

$$\boldsymbol{\Sigma}_{\mathcal{M},\mathcal{N}} : \mathcal{M} \times \mathcal{N} \to \mathcal{W} : (m, n) \mapsto m + n.$$

Under the well-posedness assumption, $\boldsymbol{\Sigma}_{\mathcal{M},\mathcal{N}}$ has an inverse on the whole of $\mathcal{W}$, and moreover

$$\boldsymbol{\Sigma}_{\mathcal{M},\mathcal{N}}^{-1} = \boldsymbol{H}_{\mathrm{P,C}}.$$

Thus (induced norm, incremental, or gf-) stability of the feedback system is equivalent to the same notion being imposed on $\boldsymbol{\Sigma}_{\mathcal{M},\mathcal{N}}^{-1}$.

In order to study robustness of feedback stability, the following pair of operators plays a central role [8], [9], [14]. Define

$$\boldsymbol{\Pi}_{\mathcal{M}//\mathcal{N}} := \boldsymbol{\Pi}_1 \boldsymbol{H}_{\mathrm{P,C}} \quad \text{and} \quad \boldsymbol{\Pi}_{\mathcal{N}//\mathcal{M}} := \boldsymbol{\Pi}_2 \boldsymbol{H}_{\mathrm{P,C}}$$

where $\boldsymbol{\Pi}_i : \mathcal{W} \times \mathcal{W} \to \mathcal{W}$ denotes the natural projection onto the ith component $(i = 1, 2)$ of $\mathcal{W} \times \mathcal{W}$. These operators represent the mappings, in Fig. 1, from the external disturbances to the input and output of $\boldsymbol{P}$ and the output and input of $\boldsymbol{C}$, respectively. Each of these operators is a parallel projection. The relevant definition, given in [5], is that an operator $\boldsymbol{\Pi} : \mathcal{W} \to \mathcal{W}$ is a (nonlinear) *parallel projection* if for any $w_1, w_2 \in \mathcal{W}$

$$\boldsymbol{\Pi}(\boldsymbol{\Pi} w_1 + (\boldsymbol{I} - \boldsymbol{\Pi}) w_2) = \boldsymbol{\Pi} w_1. \tag{1}$$

We summarize several interesting properties of parallel projection operators which will be used below. First $\boldsymbol{H}_{\mathrm{P,C}} = (\boldsymbol{\Pi}_{\mathcal{M}//\mathcal{N}}, \boldsymbol{\Pi}_{\mathcal{N}//\mathcal{M}})$ and $\boldsymbol{\Pi}_{\mathcal{M}//\mathcal{N}} + \boldsymbol{\Pi}_{\mathcal{N}//\mathcal{M}} = \boldsymbol{I}$. Therefore the stability/causality of one parallel projection implies the stability/causality of the other, and hence of $\boldsymbol{H}_{\mathrm{P,C}}$. Clearly, the stability/causality of $\boldsymbol{H}_{\mathrm{P,C}}$ implies the same property for the two parallel projections. Further, $\boldsymbol{\Pi}_{\mathcal{M}//\mathcal{N}}$ (respectively, $\boldsymbol{\Pi}_{\mathcal{N}//\mathcal{M}}$) is the identity operator on $\mathcal{M}$ (respectively, $\mathcal{N}$) so each has norm greater than or equal to one. Finally, a parallel projection $\boldsymbol{\Pi} : \mathcal{W} \to \mathcal{W}$ always induces a coordinatization of $\mathcal{W}$ in the following sense: any $w \in \mathcal{W}$ has a unique additive decomposition $w = m + n$, where $m \in \boldsymbol{\Pi}\mathcal{W}$ and $n \in (\boldsymbol{I} - \boldsymbol{\Pi})\mathcal{W}$.

III. Global Robustness

In this section we deal with robustness of global stability of feedback systems, in the sense that the induced norm of the input-to-error mapping $\boldsymbol{H}_{\mathrm{P,C}}$ is finite and remains finite for suitable perturbations of the nominal plant $\boldsymbol{P}$. To quantify allowed perturbations we introduce a distance measure which is a generalization of the gap metric to nonlinear systems on extended or Banach signal spaces. For this measure we prove that feedback stability is preserved for perturbations which are smaller than the inverse of the norm of the parallel projection onto the graph of the plant. Next we prove a result which shows a close connection between norm convergence of the closed-loop operators and convergence in the distance measure. We then investigate the metric properties of the measure. Finally, we study a related alternative distance measure and give a direct proof of the corresponding main robustness theorem. The circle criterion is shown to be a corollary.

A. Robust Stability Margin

Let $\mathcal{X}, \mathcal{Y} \subseteq \mathcal{W}$ where $\mathcal{W}$ is a signal space. The following definition represents a generalization of a metric given in [20]:

$$\vec{\delta}(\mathcal{X}, \mathcal{Y}) := \begin{cases} \inf\{\|(\boldsymbol{\Phi} - \boldsymbol{I})|_{\mathcal{X}}\| : \boldsymbol{\Phi} \text{ is a causal,} \\ \qquad \text{bijective map from } \mathcal{X} \text{ to } \mathcal{Y} \text{ with} \\ \qquad \boldsymbol{\Phi}0 = 0\}, \\ \infty \text{ if no such operator } \boldsymbol{\Phi} \text{ exists,} \end{cases}$$
$$\delta(\mathcal{X}, \mathcal{Y}) := \max\{\vec{\delta}(\mathcal{X}, \mathcal{Y}), \vec{\delta}(\mathcal{Y}, \mathcal{X})\}.$$

The theorem below generalizes a standard result from linear robust control. More specifically, if $\boldsymbol{P}, \boldsymbol{P}_1, \boldsymbol{C}$ are linear systems and $\mathcal{U}, \mathcal{Y}$ are Hilbert spaces, $\delta(\cdot, \cdot)$ specializes to the usual gap metric (see Proposition 5 in the Appendix). Theorem 1 then becomes the sufficiency part of [12, Th. 5] and [8, Th. 3]. The tightness of (2) is not examined here. However, it is believed that a suitable necessity construction could be carried out for appropriate classes of plants (cf. [29]). Here we will be content to call $\|\boldsymbol{\Pi}_{\mathcal{M}//\mathcal{N}}\|^{-1}$ the robust stability margin.

Theorem 1: Consider the feedback system in Fig. 1. Denote $\mathcal{M} := \mathcal{G}_{\mathrm{P}}, \mathcal{N} := \mathcal{G}_{\mathrm{C}}$, and let $\boldsymbol{H}_{\mathrm{P,C}}$ be stable. If a system $\boldsymbol{P}_1$, with $\mathcal{M}_1 := \mathcal{G}_{\mathrm{P}_1}$, is such that

$$\vec{\delta}(\mathcal{M}, \mathcal{M}_1) < \|\boldsymbol{\Pi}_{\mathcal{M}//\mathcal{N}}\|^{-1} \tag{2}$$

then $\boldsymbol{H}_{\mathrm{P}_1,\mathrm{C}}$ is stable and

$$\|\boldsymbol{\Pi}_{\mathcal{M}_1//\mathcal{N}}\| \le \|\boldsymbol{\Pi}_{\mathcal{M}//\mathcal{N}}\| \frac{1 + \vec{\delta}(\mathcal{M}, \mathcal{M}_1)}{1 - \|\boldsymbol{\Pi}_{\mathcal{M}//\mathcal{N}}\| \vec{\delta}(\mathcal{M}, \mathcal{M}_1)}.$$

The proof of the theorem uses the following simple lemma.

Lemma 1: Let $\mathcal{X} \subseteq \mathcal{W}$ where $\mathcal{W}$ is a signal space, and $\boldsymbol{A} : \mathcal{X} \to \mathcal{W}$ with $\alpha := \|\boldsymbol{A}\| < 1$. Suppose $w = (\boldsymbol{I} + \boldsymbol{A})x$. Then $\|x - w\|_\tau \le \|w\|_\tau \alpha/(1 - \alpha)$ and $\|x\|_\tau \le \|w\|_\tau/(1 - \alpha)$.

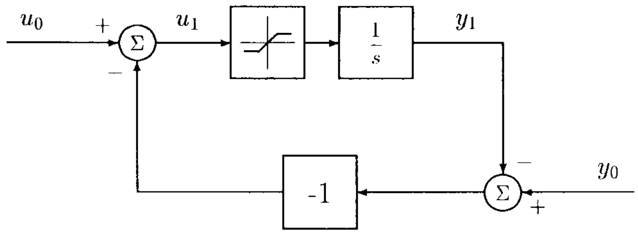

Fig. 2. Integrator with saturation.

Proof: Clearly, $\|x-w\|_\tau \leq \alpha\|x\|_\tau$. Then

$$\begin{aligned}\alpha\|x\|_\tau &\geq (1-\alpha)\|x-w\|_\tau + \alpha\|x-w\|_\tau\\ &\geq (1-\alpha)\|x-w\|_\tau + \alpha\|x\|_\tau - \alpha\|w\|_\tau\end{aligned}$$

from which the inequalities follow. □

Proof of Theorem 1: As observed previously, $\|\boldsymbol{\Pi}_{\mathcal{M}\,\mathcal{N}}\| \geq 1$. Since $\vec{\delta}(\mathcal{M},\mathcal{M}_1) < \infty$ there exists a causal bijective mapping $\boldsymbol{\Phi}$ from $\mathcal{M}$ to $\mathcal{M}_1$ such that

$$\alpha := \|(\boldsymbol{\Phi}-\boldsymbol{I})\boldsymbol{\Pi}_{\mathcal{M}//\mathcal{N}}\| \leq \|\boldsymbol{\Phi}-\boldsymbol{I}\| \cdot \|\boldsymbol{\Pi}_{\mathcal{M}//\mathcal{N}}\| < 1. \quad (3)$$

Consider the equation

$$\begin{aligned} w &= (\boldsymbol{I} + (\boldsymbol{\Phi}-\boldsymbol{I})\boldsymbol{\Pi}_{\mathcal{M}//\mathcal{N}})x \quad &(4)\\ &= (\boldsymbol{\Pi}_{\mathcal{N}//\mathcal{M}} + \boldsymbol{\Phi}\boldsymbol{\Pi}_{\mathcal{M}//\mathcal{N}})x. \quad &(5)\end{aligned}$$

We claim that this equation has a solution for any $w \in \mathcal{W}$. To see this, note that $\boldsymbol{\Sigma}^{-1}_{\mathcal{M}_1,\mathcal{N}}w = (m_1, n)$ for some $m_1 \in \mathcal{M}$ and $n \in \mathcal{N}$ because of the well-posedness assumption. Next, $\boldsymbol{\Phi}m = m_1$ has a solution for some $m \in \mathcal{M}$ since $\boldsymbol{\Phi}$ is surjective. It can now be seen that $x = m + n$ is a solution of (5). Since $m_1 = \boldsymbol{\Pi}_{\mathcal{M}_1//\mathcal{N}}w$ and $m = \boldsymbol{\Pi}_{\mathcal{M}//\mathcal{N}}x$, then

$$\boldsymbol{\Pi}_{\mathcal{M}_1//\mathcal{N}}w = \boldsymbol{\Phi}\boldsymbol{\Pi}_{\mathcal{M}//\mathcal{N}}x.$$

At the same time, from (4) and Lemma 1, $\|x\|_\tau \leq \|w\|_\tau/(1-\alpha)$. Thus

$$\begin{aligned}\|\boldsymbol{\Pi}_{\mathcal{M}_1//\mathcal{N}}\| &\leq \|\boldsymbol{\Pi}_{\mathcal{M}//\mathcal{N}}\| \cdot \frac{\|\boldsymbol{\Phi}\|}{1-\alpha}\\ &\leq \|\boldsymbol{\Pi}_{\mathcal{M}//\mathcal{N}}\| \cdot \frac{1+\|\boldsymbol{\Phi}-\boldsymbol{I}\|}{1-\|\boldsymbol{\Pi}_{\mathcal{M}//\mathcal{N}}\| \cdot \|\boldsymbol{\Phi}-\boldsymbol{I}\|}.\end{aligned}$$

Since this is true for all such $\boldsymbol{\Phi}$, the result follows on taking the infimum over $\boldsymbol{\Phi}$. □

Example 1 (Integrator with Saturation): Consider the feedback configuration of Fig. 2. We will use Theorem 1 to show that the feedback loop remains stable in the presence of sufficiently small time delays in the plant. The nominal plant $\boldsymbol{P}$ is defined by

$$\begin{aligned}\dot{x}(t) &= \mathrm{sat}(u_1(t)), \qquad x(0) = 0\\ y_1(t) &= x(t)\end{aligned}$$

where $\mathrm{sat}(u_1) = u_1$ when $|u_1| \leq 1$ and is equal to $\mathrm{sign}(u_1)$ when $|u_1| > 1$. We take $\mathcal{U} = \mathcal{Y} = \mathcal{L}_{\infty,e}$ and choose the feedback controller to be $\boldsymbol{C} = -1$. The instantaneous gains of $\boldsymbol{P}$ and $\boldsymbol{C}$ are zero and one, respectively, so the loop is well-posed. The feedback equations reduce to

$$\dot{x} = \mathrm{sat}(v_0 - x), \qquad x(0) = 0 \quad (6)$$

where $v_0 = u_0 + y_0$. We first calculate the stability margin $\|\boldsymbol{\Pi}_{\mathcal{M}//\mathcal{N}}\|^{-1}$ where

$$\boldsymbol{\Pi}_{\mathcal{M}//\mathcal{N}}\colon \begin{pmatrix} u_0 \\ y_0 \end{pmatrix} \longmapsto \begin{pmatrix} u_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} v_0 - x \\ x \end{pmatrix}.$$

For any $v_0, u_0 = y_0 = v_0/2$ gives $\binom{u_1}{y_1} = \binom{v_0-x}{x}$ with the smallest input norm. Thus

$$\begin{aligned}\|\boldsymbol{\Pi}_{\mathcal{M}//\mathcal{N}}\| &= 2\left\| v_0 \mapsto \begin{pmatrix} v_0 - x \\ x \end{pmatrix}\right\|\\ &= 2 \times \max\{\|v_0 \mapsto (v_0 - x)\|, \|v_0 \mapsto x\|\} \quad (7)\end{aligned}$$

for the mapping defined by (6). We now claim that $\|v_0 \mapsto x\| = 1$. To see this, consider any interval $[0,T]$ and suppose that $x(t)$ achieves a maximum which is positive at $t_0 \in [0,T]$. Then, for any $\epsilon > 0$, there exists $0 < t_1 < t_0$ such that $x(t_1) > x(t_0) - \epsilon$ and $\dot{x}(t_1) > 0$. The latter means that $v_0(t_1) > x(t_1)$, hence $v_0(t_1) > x(t_0) - \epsilon$ for any ϵ. A similar argument applies for the minimum of $x(t)$, so $\|v_0\|_T \geq \|x\|_T$. But $v_0(t) = 1$ for all t gives $x(t) = 1 - e^{-t}$, hence $\|v_0 \mapsto x\| = 1$. Next observe that $\|v_0 \mapsto (v_0 - x)\| \leq 1 + \|v_0 \mapsto x\| = 2$. In fact, this upper bound can be approached arbitrarily closely by the input $v_0(t) = 1$ for $0 \leq t < T$ and $v_0 = -1$ for $t \geq T$, since then $(v_0 - x)(T) = -2 + e^{-T}$. Therefore, from (7), we conclude that $\|\boldsymbol{\Pi}_{\mathcal{M}//\mathcal{N}}\| = 4$.

We will now calculate the gap between $\boldsymbol{P}$ and a perturbation $\boldsymbol{P}_1$ in order to apply Theorem 1. Assume that the perturbed plant $\boldsymbol{P}_1$ is described by

$$\dot{x}(t) = \mathrm{sat}(u_1(t-h)), \qquad x(0) = 0$$

and $u_0(t) = y_0(t) = 0$ for $t < 0$, which means that $u_1(t) = 0$ for $t < 0$. Define a mapping $\boldsymbol{\Phi}\colon \mathcal{M} \to \mathcal{M}_1$ by

$$\boldsymbol{\Phi}\begin{pmatrix} u_1(t) \\ x(t) \end{pmatrix} = \begin{pmatrix} u_1(t) \\ x(t-h) \end{pmatrix}.$$

Then

$$\begin{aligned}|x(t) - x(t-h)| &\leq \sup_{\tau\in[t-h,t]} |\dot{x}(\tau))| \cdot h\\ &\leq \sup_{\tau\in[t-h,t]} |u_1(\tau)| \cdot h.\end{aligned}$$

Hence

$$\|\boldsymbol{I} - \boldsymbol{\Phi}\| = \sup_{\tau, \|u_1\|_\tau \neq 0} \frac{\|x(t) - x(t-h)\|_\tau}{\max\{\|u_1\|_\tau, \|x\|_\tau\}} \leq h$$

which means that $\vec{\delta}(\mathcal{M},\mathcal{M}_1) \leq h$. In fact, the gap is equal to this bound (for $h \leq 1$). To see this, take $u_1(t) = 1$ on $[0,h]$, for which $(\boldsymbol{P}u_1)(t) = t$ on $[0,h]$. Since $\boldsymbol{T}_h w = \binom{*}{0}$ for any $w \in \mathcal{M}_1$ we have

$$\vec{\delta}(\mathcal{M},\mathcal{M}_1) \geq \frac{\|\boldsymbol{P}u_1\|_h}{\max\{\|u_1\|_h, \|\boldsymbol{P}u_1\|_h\}} = \frac{h}{\max\{1,h\}}.$$

Theorem 1 now asserts that $\boldsymbol{H}_{\mathrm{P,C}}$ will be stable if $\vec{\delta}(\mathcal{M},\mathcal{M}_1) < \|\boldsymbol{\Pi}_{\mathcal{M}//\mathcal{N}}\|^{-1}$, which predicts that the perturbed system will remain stable for all values of the time delay $h < 1/4$. It should be noted that this prediction of delay margin is conservative and can be improved (e.g., by use of the scaling $1/s \to k/s$ and $\boldsymbol{C} \to -1/k$). □

B. Nonlinear Gap Topology

In [40, Th. 1] it was shown, for the case of linear systems over Hilbert spaces, that open-loop uncertainties which correspond to small closed-loop errors are precisely those that are small in the gap. The following theorem is an attempt to generalize this result to the nonlinear case. In particular, it shows the equivalence in the case that the nominal closed-loop is incrementally bounded. In general, it shows that closed-loop convergence in norm implies convergence in the gap. Conversely, any stabilizing controller for a given plant will stabilize some neighborhood of the plant. However, closeness in the gap does not necessarily imply closeness of the closed-loop operators, as shown in Example 2. For this to be the case some form of continuity of the nominal closed-loop operator is necessary.

Theorem 2: Let $\boldsymbol{H}_{\mathrm{P,C}}$ be stable and consider a sequence of plants $\boldsymbol{P}_i$ for $i = 1, 2, \cdots$. Define $\mathcal{M} := \mathcal{G}_{\mathrm{P}}, \mathcal{M}_i := \mathcal{G}_{\mathrm{P}_i}$, and $\mathcal{N} := \mathcal{G}_{\mathrm{C}}$. Then the following statements hold.

1) If $\boldsymbol{H}_{\mathrm{P}_i,\mathrm{C}}$ is stable for all sufficiently large i and $\|\boldsymbol{H}_{\mathrm{P,C}} - \boldsymbol{H}_{\mathrm{P}_i,\mathrm{C}}\| \to 0$ as $i \to \infty$, then $\delta(\mathcal{M}, \mathcal{M}_i) \to 0$.

2) If $\delta(\mathcal{M}, \mathcal{M}_i) \to 0$ as $i \to \infty$, then $\boldsymbol{H}_{\mathrm{P}_i,\mathrm{C}}$ is stable for all sufficiently large i. Furthermore:

   a) if for some $\tau > 0, \boldsymbol{T}_\tau \boldsymbol{H}_{\mathrm{P,C}} \boldsymbol{T}_\tau$ is continuous, then $\|(\boldsymbol{H}_{\mathrm{P,C}} - \boldsymbol{H}_{\mathrm{P}_i,\mathrm{C}})w\|_\tau \to 0$ as $i \to \infty$ for any $w \in \mathcal{W}$.

   b) if $\boldsymbol{H}_{\mathrm{P,C}}$ is incrementally stable, then $\|\boldsymbol{H}_{\mathrm{P,C}} - \boldsymbol{H}_{\mathrm{P}_i,\mathrm{C}}\| \to 0$ as $i \to \infty$.

Proof of 1): Define

$$\boldsymbol{\Phi}_i := \boldsymbol{I} - \boldsymbol{\Pi}_{\mathcal{M}//\mathcal{N}} + \boldsymbol{\Pi}_{\mathcal{M}_i//\mathcal{N}}.$$

Since $\|\boldsymbol{H}_{\mathrm{P,C}} - \boldsymbol{H}_{\mathrm{P}_i,\mathrm{C}}\| \to 0$, it follows that $\|\boldsymbol{\Phi}_i - \boldsymbol{I}\| \to 0$. We now show that $\boldsymbol{\Phi}_i$ maps $\mathcal{M}$ bijectively onto $\mathcal{M}_i$.

Let $m \in \mathcal{M}$. Then

$$\begin{aligned}\boldsymbol{\Phi}_i m &= m - \boldsymbol{\Pi}_{\mathcal{M}//\mathcal{N}} m + \boldsymbol{\Pi}_{\mathcal{M}_i//\mathcal{N}} m\\ &= \boldsymbol{\Pi}_{\mathcal{M}_i//\mathcal{N}} m \in \mathcal{M}_i.\end{aligned}$$

Thus $\boldsymbol{\Phi}_i$ maps $\mathcal{M}$ into $\mathcal{M}_i$. Further, $m = \boldsymbol{\Phi}_i m + n$ for $n = \boldsymbol{\Pi}_{\mathcal{N}//\mathcal{M}_i} m \in \mathcal{N}$, and this is unique among such additive decompositions into a sum of elements in $\mathcal{M}_i$ and $\mathcal{N}$ [cf. (1)]. Now consider the operator $\boldsymbol{\Sigma}_{\mathcal{M},-\mathcal{N}}$, which has a well-defined (not necessarily bounded) inverse by the assumption of well-posedness. We observe that $\boldsymbol{\Pi}_{\mathcal{M}//-\mathcal{N}}(\boldsymbol{\Phi}_i m) = m$, since we can write $\boldsymbol{\Phi}_i m = m + (-n)$. Thus $\boldsymbol{\Pi}_{\mathcal{M}//-\mathcal{N}} \boldsymbol{\Phi}_i|_{\mathcal{M}} = \boldsymbol{I}_{\mathcal{M}}$ which shows that $\boldsymbol{\Phi}_i$ is a 1–1 map from $\mathcal{M}$ into $\mathcal{M}_i$. By a similar argument we get $\boldsymbol{\Phi}_i \boldsymbol{\Pi}_{\mathcal{M}//-\mathcal{N}}|_{\mathcal{M}_i} = \boldsymbol{I}_{\mathcal{M}_i}$. Therefore $\boldsymbol{\Phi}_i$ is a 1–1 map from $\mathcal{M}$ *onto* $\mathcal{M}_i$. This means that $\vec{\delta}(\mathcal{M}, \mathcal{M}_i) \to 0$.

Using the operators $\boldsymbol{\Psi}_i := \boldsymbol{I} + \boldsymbol{\Pi}_{\mathcal{M}//\mathcal{N}} - \boldsymbol{\Pi}_{\mathcal{M}_i//\mathcal{N}}$, we can show analogously that $\vec{\delta}(\mathcal{M}_i, \mathcal{M}) \to 0$.

2): Suppose $\delta(\mathcal{M}, \mathcal{M}_i) \to 0$. Then there exists $\boldsymbol{\Phi}_i$ mapping $\mathcal{M}$ bijectively onto $\mathcal{M}_i$ so that $\|(\boldsymbol{\Phi}_i - \boldsymbol{I})|_{\mathcal{M}}\| \to 0$. It therefore follows from Theorem 1 that $\boldsymbol{H}_{\mathrm{P}_i,\mathrm{C}}$ is stable for sufficiently large i. Now note, as in the proof of Theorem 1, that $w = (\boldsymbol{I} + (\boldsymbol{\Phi}_i - \boldsymbol{I})\boldsymbol{\Pi}_{\mathcal{M}//\mathcal{N}})x_i$ has a solution $x_i \in \mathcal{W}$ for all i and $\boldsymbol{\Pi}_{\mathcal{M}_i//\mathcal{N}} w = \boldsymbol{\Phi}_i \boldsymbol{\Pi}_{\mathcal{M}//\mathcal{N}} x_i$. Let $\alpha_i := \|(\boldsymbol{\Phi}_i - \boldsymbol{I})\boldsymbol{\Pi}_{\mathcal{M}//\mathcal{N}}\|$ and assume for convenience that $\alpha_i < 1$ for all i. Then $\|w - x_i\|_\tau \leq \|w\|_\tau \alpha_i/(1-\alpha_i)$ and $\|x_i\|_\tau \leq \|w\|_\tau/(1-\alpha_i)$ from Lemma 1. Now consider the following identity:

$$\begin{aligned}&(\boldsymbol{\Pi}_{\mathcal{M}//\mathcal{N}} - \boldsymbol{\Pi}_{\mathcal{M}_i//\mathcal{N}})w\\ &\quad = (\boldsymbol{I} - \boldsymbol{\Phi}_i)\boldsymbol{\Pi}_{\mathcal{M}//\mathcal{N}} x_i + \boldsymbol{\Pi}_{\mathcal{M}//\mathcal{N}} w - \boldsymbol{\Pi}_{\mathcal{M}//\mathcal{N}} x_i. \qquad (8)\end{aligned}$$

Part a) now follows from the continuity of $\boldsymbol{T}_\tau \boldsymbol{\Pi}_{\mathcal{M}//\mathcal{N}} = \boldsymbol{T}_\tau \boldsymbol{\Pi}_{\mathcal{M}//\mathcal{N}} \boldsymbol{T}_\tau$, the uniform boundedness of x_i, and the fact that $\alpha_i \to 0$ as $i \to \infty$. Assuming incremental boundedness, we have from (8)

$$\|\boldsymbol{\Pi}_{\mathcal{M}//\mathcal{N}} - \boldsymbol{\Pi}_{\mathcal{M}_i//\mathcal{N}}\| \leq \frac{\alpha_i}{1-\alpha_i}(1 + \|\boldsymbol{\Pi}_{\mathcal{M}//\mathcal{N}}\|_\Delta)$$

from which Part b) follows. □

Remark: Theorem 2 was first given in [14] for stability defined in the sense of incremental gain boundedness. We point out that the proof of [14, Th. 2(b) ⇒ (a)] contains an error. Namely, the summation operator cannot be premultiplied in the fourth displayed equation of [14, p. 93] because of incompatible domain. However, the result is still valid and can be proved exactly as Theorem 2-2)-b) of the present paper. □

Example 2 (Discontinuity of Closed-Loop Operators): In the feedback configuration of Fig. 1, let $\boldsymbol{P} = 0$, $\boldsymbol{P}_i\ (i = 1, 2, \cdots)$ be defined as scalar multiplication by $1/i$ and $\boldsymbol{C}$ be a relay with dead zone and a unit time delay in series

$$\boldsymbol{C}(u_2)(t) = y_2(t) = \begin{cases} 1 & \text{for} \quad u_2(t-1) \;\geq\; 1,\\ 0 & \text{for} \quad |u_2(t-1)| \;<\; 1,\\ -1 & \text{for} \quad u_2(t-1) \;\leq\; -1.\end{cases}$$

The feedback systems are well-posed due to the time delay. Let $\mathcal{U} = \mathcal{Y} = \mathcal{L}_{\infty,e}$. For the constant input $w := \binom{u_0}{y_0} \equiv \binom{\epsilon}{1}$, with $t \geq 0$ and $0 < \epsilon < 1$

$$(\boldsymbol{\Pi}_{\mathcal{M}//\mathcal{N}} w)(t) = \begin{pmatrix}\epsilon - 1\\ 0\end{pmatrix}, \qquad \text{for} \quad t \geq 1$$

while

$$(\boldsymbol{\Pi}_{\mathcal{M}_i//\mathcal{N}} w)(t) = \begin{pmatrix}\epsilon\\ \epsilon/i\end{pmatrix}, \qquad \text{for all} \quad t.$$

It follows that $\|(\boldsymbol{H}_{\mathrm{P,C}} - \boldsymbol{H}_{\mathrm{P}_i,\mathrm{C}})w\|_\infty \geq 1$ for all i while $\delta(\mathcal{M}, \mathcal{M}_i) \to 0$. This behavior is due to the fact that the nominal closed-loop operator is discontinuous. □

C. Metric Properties of the Gap

We now investigate the metric properties of $\delta(\cdot,\cdot)$. In fact, we show that a suitable scaling of $\delta(\cdot,\cdot)$, defined by $d(\cdot,\cdot) := \log(1 + \delta(\cdot,\cdot))$, is a metric under certain natural assumptions imposed on its arguments (cf. [20]). We begin with the triangle inequality.

Proposition 1: Let $\mathcal{X}, \mathcal{Y}, \mathcal{Z} \subset \mathcal{W}$ where $\mathcal{W}$ is a signal space. Then

$$d(\mathcal{X}, \mathcal{Z}) \leq d(\mathcal{X}, \mathcal{Y}) + d(\mathcal{Y}, \mathcal{Z}).$$

Proof: If either of $d(\mathcal{X},\mathcal{Y}), d(\mathcal{Y},\mathcal{Z})$ is ∞, then the statement is obvious. So assume that both $d(\mathcal{X},\mathcal{Y}), d(\mathcal{Y},\mathcal{Z})$ are less than ∞ and, consequently, that there exist 1–1 mappings $\boldsymbol{\Phi}_i$ $(i=1,2)$ for which $\boldsymbol{\Phi}_1\mathcal{X}=\mathcal{Y}$ and $\boldsymbol{\Phi}_2\mathcal{Y}=\mathcal{Z}$. Then $\boldsymbol{\Phi}=\boldsymbol{\Phi}_2\boldsymbol{\Phi}_1$ is 1–1 and maps $\mathcal{X}$ onto $\mathcal{Z}$. From the identity

$$\boldsymbol{\Phi}_2\boldsymbol{\Phi}_1-\boldsymbol{I}=(\boldsymbol{\Phi}_2-\boldsymbol{I})\boldsymbol{\Phi}_1+\boldsymbol{\Phi}_1-\boldsymbol{I}$$

we obtain

$$\begin{aligned}&\|(\boldsymbol{\Phi}_2\boldsymbol{\Phi}_1-\boldsymbol{I})|_{\mathcal{X}}\|\\&\quad\le\|(\boldsymbol{\Phi}_2-\boldsymbol{I})|_{\mathcal{Y}}\|\cdot\|\boldsymbol{\Phi}_1|_{\mathcal{X}}\|+\|(\boldsymbol{\Phi}_1-\boldsymbol{I})|_{\mathcal{X}}\|\\&\quad\le\|(\boldsymbol{\Phi}_2-\boldsymbol{I})|_{\mathcal{Y}}\|(\|(\boldsymbol{\Phi}_1-\boldsymbol{I})|_{\mathcal{X}}\|+1)+\|(\boldsymbol{\Phi}_1-\boldsymbol{I})|_{\mathcal{X}}\|.\end{aligned}$$

This shows that

$$\begin{aligned}&\log(1+\|(\boldsymbol{\Phi}_2\boldsymbol{\Phi}_1-\boldsymbol{I})|_{\mathcal{X}}\|)\\&\quad\le\log(1+\|(\boldsymbol{\Phi}_1-\boldsymbol{I})|_{\mathcal{X}}\|)+\log(1+\|(\boldsymbol{\Phi}_2-\boldsymbol{I})|_{\mathcal{Y}}\|)\end{aligned}$$

and completes the proof. □

In general $\delta(\mathcal{X},\mathcal{Y})=0$ does not imply $\mathcal{X}=\mathcal{Y}$. However, this is the case for the following type of subsets. We say that $\mathcal{X}\subset\mathcal{W}$ is *locally and asymptotically complete* (l.a.c.) if, for all $\tau>0, \boldsymbol{T}_\tau\mathcal{X}$ is a closed subset of $\boldsymbol{T}_\tau\mathcal{W}$, and moreover $\mathcal{X}$ has the property that it contains any $w\in\mathcal{W}$ such that $\boldsymbol{T}_\tau w\in\boldsymbol{T}_\tau\mathcal{X}$ for all τ. This type of subset is motivated by the following proposition.

Proposition 2: Suppose $\boldsymbol{P}$: $\mathcal{U}\to\mathcal{Y}$ is causal and $\boldsymbol{T}_\tau\boldsymbol{P}\boldsymbol{T}_\tau$ is defined and continuous on $\mathcal{U}$ for all $\tau>0$. Then $\mathcal{G}_{\rm P}$ is an l.a.c. subset of $\mathcal{W}=\mathcal{U}\times\mathcal{Y}$.

We remark that, in case $\boldsymbol{P}$ is not defined on the whole of $\mathcal{U}$ (e.g., when $\mathcal{U},\mathcal{Y}$ are Banach spaces), the property that $\boldsymbol{T}_\tau\boldsymbol{P}\boldsymbol{T}_\tau$ is defined for every $u\in\mathcal{U}$ is called causal extendibility. This property requires that for any $u\in\mathcal{U}$ and any τ, there exists u' with $\boldsymbol{P}u'\in\mathcal{Y}$ such that $\boldsymbol{T}_\tau(u-u')=0$. This concept has been studied for linear shift-invariant systems in [13].

Proof: Fix a value $\tau>0$ and consider a sequence $w_i=\binom{u_i}{y_i}\in\mathcal{G}_{\rm P}$ $(i=1,2,\cdots)$ such that $\boldsymbol{T}_\tau w_i$ is a Cauchy sequence. Let

$$\hat{w}=\begin{pmatrix}\hat{u}\\\hat{y}\end{pmatrix}=\lim_{i\to\infty}\boldsymbol{T}_\tau\begin{pmatrix}u_i\\y_i\end{pmatrix}.$$

Since $\boldsymbol{T}_\tau\boldsymbol{P}\boldsymbol{T}_\tau$ is continuous then $\boldsymbol{T}_\tau\boldsymbol{P}\boldsymbol{T}_\tau u_i\to\boldsymbol{T}_\tau\boldsymbol{P}\boldsymbol{T}_\tau\hat{u}$. But $\boldsymbol{T}_\tau\boldsymbol{P}\boldsymbol{T}_\tau u_i=\boldsymbol{T}_\tau\boldsymbol{P}u_i=\boldsymbol{T}_\tau y_i$. Hence $\boldsymbol{T}_\tau\boldsymbol{P}\boldsymbol{T}_\tau\hat{u}=\hat{y}$, which means that

$$\hat{w}=\boldsymbol{T}_\tau\begin{pmatrix}\boldsymbol{T}_\tau\hat{u}\\\boldsymbol{P}\boldsymbol{T}_\tau\hat{u}\end{pmatrix}\in\boldsymbol{T}_\tau\mathcal{G}_{\rm P}.$$

This proves that $\boldsymbol{T}_\tau\mathcal{G}_{\rm P}$ is closed.

Now consider any $w=\binom{u}{y}\in\mathcal{W}$ such that $\boldsymbol{T}_\tau w\in\boldsymbol{T}_\tau\mathcal{G}_{\rm P}$ for all $\tau>0$. Thus, for any τ, there exists a $v\in\mathcal{U}$ such that $\boldsymbol{T}_\tau y=\boldsymbol{T}_\tau\boldsymbol{P}(\boldsymbol{T}_\tau u+(\boldsymbol{I}-\boldsymbol{T}_\tau)v)$. Hence, by causality (replacing $\boldsymbol{T}_\tau\boldsymbol{P}$ by $\boldsymbol{T}_\tau\boldsymbol{P}\boldsymbol{T}_\tau$ in the last equation), $\boldsymbol{T}_\tau y=\boldsymbol{T}_\tau\boldsymbol{P}u$ for all τ. This implies that $y=\boldsymbol{P}u$ and so $w\in\mathcal{G}_{\rm P}$. □

Proposition 3: Let $\mathcal{X},\mathcal{Y}\subset\mathcal{W}$ be l.a.c. subsets. Then, $\delta(\mathcal{X},\mathcal{Y})=0$ implies that $\mathcal{X}=\mathcal{Y}$.

Proof: Assuming that $\delta(\mathcal{X},\mathcal{Y})=0$, for any $1>\epsilon>0$ there exists a bijective mapping $\boldsymbol{\Phi}_\epsilon$: $\mathcal{X}\to\mathcal{Y}$ such that $\|(\boldsymbol{\Phi}_\epsilon-\boldsymbol{I})|_{\mathcal{X}}\|<\epsilon$. For any given $y\in\mathcal{Y}$ there exists a family $x_\epsilon\in\mathcal{X}$ such that $\boldsymbol{\Phi}_\epsilon x_\epsilon=y$. Moreover, for any $\tau>0, \|y-x_\epsilon\|_\tau<\|y\|_\tau\epsilon/(1-\epsilon)$, as in Lemma 1. Hence we can find a sequence $x_\epsilon\in\mathcal{X}$ such that, for any τ, $\boldsymbol{T}_\tau x_\epsilon\to\boldsymbol{T}_\tau y$. Since $\boldsymbol{T}_\tau\mathcal{X}$ is closed, it follows that $\boldsymbol{T}_\tau y\in\boldsymbol{T}_\tau\mathcal{X}$. Since $\mathcal{X}$ is l.a.c., it now follows that $y\in\mathcal{X}$, hence $\mathcal{Y}\subseteq\mathcal{X}$. Conversely, given any $x\in\mathcal{X}$, the family $y_\epsilon=\boldsymbol{\Phi}_\epsilon x\in\mathcal{Y}$ satisfies $\|y_\epsilon-x\|_\tau=\|(\boldsymbol{\Phi}_\epsilon-\boldsymbol{I})x\|_\tau<\epsilon\|x\|_\tau$. Again, using the fact that $\mathcal{Y}$ is l.a.c., it follows that $x\in\mathcal{Y}$, hence $\mathcal{X}\subseteq\mathcal{Y}$. □

Corollary: $d(\cdot,\cdot)$ is a metric on l.a.c. subsets of $\mathcal{W}$. □

D. An Alternative Distance Measure

It is possible to make Theorem 1 stronger by altering the definition of induced norm used on $\boldsymbol{\Phi}-\boldsymbol{I}$. Let $\boldsymbol{F}$ be a causal operator on $\mathcal{X}\subset\mathcal{W}$ with $\boldsymbol{F}0=0$. Define the *ls-gain*

$$\|\boldsymbol{F}|_{\mathcal{X}_1}\|_{\rm l.s.}:=\limsup_{\tau>0}\ \sup_{\substack{x\in\mathcal{X}_1\\\|x\|_\tau\neq0}}\frac{\|\boldsymbol{F}x\|_\tau}{\|x\|_\tau}$$

and the *Banach gain*

$$\|\boldsymbol{F}|_{\mathcal{X}_1}\|_B:=\sup_{\substack{x\in\mathcal{X}_1\\x\neq0}}\ \limsup_{\tau>0}\frac{\|\boldsymbol{F}x\|_\tau}{\|x\|_\tau}.$$

In general $\|\boldsymbol{F}|_{\mathcal{X}_1}\|_B\le\|\boldsymbol{F}|_{\mathcal{X}_1}\|_{\rm l.s.}\le\|\boldsymbol{F}|_{\mathcal{X}_1}\|$. If $\mathcal{X}_1$ satisfies the property that $\boldsymbol{T}_\tau x\in\boldsymbol{T}_\tau\mathcal{X}_1$ implies that $\boldsymbol{T}_\tau x\in\mathcal{X}_1$ (truncation invariance), then it can be seen that $\|\boldsymbol{F}|_{\mathcal{X}_1}\|_B=\|\boldsymbol{F}|_{\mathcal{X}_1}\|_{\rm l.s.}=\|\boldsymbol{F}|_{\mathcal{X}_1}\|$. If $\boldsymbol{F}$ and $\mathcal{X}_1$ are shift-invariant, then $\|\boldsymbol{F}|_{\mathcal{X}_1}\|_{\rm l.s.}=\|\boldsymbol{F}|_{\mathcal{X}_1}\|$. However, in general, the three gains may be different. If $\mathcal{W}$ is a Banach space, then $\|\cdot\|_B$ coincides with the usual induced norm. Now it is possible to rework Lemma 1 and the proof of Theorem 1 using $\|\cdot\|_B$ or $\|\cdot\|_{\rm l.s.}$. The changes are that the inequalities hold only for *sufficiently large* τ. At the final step of the proof of the theorem one obtains bounds on $\|\boldsymbol{\Pi}_{\mathcal{M}_1//\mathcal{N}}\|_B$ or $\|\boldsymbol{\Pi}_{\mathcal{M}_1//\mathcal{N}}\|_{\rm l.s.}$, respectively. However, these are the same as $\|\boldsymbol{\Pi}_{\mathcal{M}_1//\mathcal{N}}\|$ because of the truncation invariance of the space $\mathcal{W}$. Thus one can use the Banach or ls-gain in the definition of $\vec{\delta}(\mathcal{M},\mathcal{M}_1)$, but otherwise the statement of Theorem 1 is unchanged. We remark that Theorem 2–2) and Proposition 3 do not hold with such strengthening.

Further consideration of the proof of Theorem 1 indicates that the conditions imposed on $\boldsymbol{\Phi}$ in the definition of $\vec{\delta}(\mathcal{X},\mathcal{Y})$ can be relaxed while allowing the basic robust stability theorem to remain valid. The essential requirement on $\boldsymbol{\Phi}$ is that the mapping is surjective. The theorem still holds even if $\boldsymbol{\Phi}$ fails to be injective, a single-valued map, or defined on the whole of $\mathcal{M}$.

Below we give an alternative distance measure which is motivated by the above observations and a direct derivation of the corresponding robustness theorem. A discussion on the relationship of this measure with the gap metric for linear operators on Hilbert space is given in the Appendix.

Let $\mathcal{X}, \mathcal{Y} \subset \mathcal{W}$, where $\mathcal{W}$ is a signal space. We define the following:

$$\vec{\delta}_0(\mathcal{X}, \mathcal{Y}) := \limsup_{\tau > 0} \sup_{\substack{y \in \mathcal{Y} \\ \|y\|_\tau \neq 0}} \inf_{\substack{x \in \mathcal{X} \\ \|x\|_\tau \neq 0}} \frac{\|y - x\|_\tau}{\|x\|_\tau}.$$

Theorem 3: Theorem 1 holds with $\vec{\delta}(\mathcal{M}, \mathcal{M}_1)$ replaced by $\vec{\delta}_0(\mathcal{M}, \mathcal{M}_1)$.

Proof: Suppose $\|\boldsymbol{\Pi}_{\mathcal{M}//\mathcal{N}}\| = \gamma$ and $\vec{\delta}_0(\mathcal{M}, \mathcal{M}_1) < \alpha$ with $\alpha\gamma < 1$. Take any $w \in \mathcal{W}$ and write $w = m_1 + n$, where $m_1 \in \mathcal{M}_1$ and $n \in \mathcal{N}$. Such a decomposition exists and is unique because of the well-posedness assumption of the perturbed feedback loop. By definition, we can find $m \in \mathcal{M}$, possibly depending on τ, such that $\|m_1 - m\|_\tau \leq \alpha \|m\|_\tau$ for all sufficiently large $\tau > 0$. (If $\|m_1\|_\tau = 0$, then we can choose $\|m\|_\tau = 0$.) For the nominal feedback system, we must have $\boldsymbol{\Pi}_{\mathcal{M}//\mathcal{N}}\ (m + n) = m$ so that $\|m\|_\tau \leq \gamma \|m + n\|_\tau$ for all $\tau > 0$. Now note that

$$\begin{aligned} \|m_1 + n\|_\tau &= \|m_1 - m + m + n\|_\tau \\ &\geq -\|m_1 - m\|_\tau + \|m + n\|_\tau \\ &\geq -\alpha \|m\|_\tau + \frac{1}{\gamma}\|m\|_\tau \\ &= \frac{1 - \alpha\gamma}{\gamma}\|m\|_\tau \end{aligned}$$

for sufficiently large τ. Further, $\|m_1\|_\tau \leq \|m_1 - m\|_\tau + \|m\|_\tau \leq (1 + \alpha)\|m\|_\tau$ for sufficiently large τ. Thus

$$\|m_1\|_\tau \leq \frac{(1 + \alpha)\gamma}{1 - \alpha\gamma}\|m_1 + n\|_\tau$$

for sufficiently large τ. This gives the required bound on $\|\boldsymbol{\Pi}_{\mathcal{M}_1//\mathcal{N}}\|_{\text{l.s.}}$ which is equal to $\|\boldsymbol{\Pi}_{\mathcal{M}_1//\mathcal{N}}\|$. □

Example 3 (Circle Criterion): Here we show that the standard circle criterion is a corollary of Theorem 3. Let $\mathcal{U} = \mathcal{L}_2, \mathcal{Y} = \mathcal{L}_2, \boldsymbol{P} = N$, where N is a memoryless nonlinearity satisfying $0 < k_1 \leq N(u_1)/u_1 \leq k_2$ for any real u_1 and $\boldsymbol{C}$ is a linear shift invariant system with transfer function $-g(s)$. Suppose that $g(j\omega)$ does not penetrate the disc with diameter $[-k_1^{-1}, -k_2^{-1}]$, encircles it the correct number of times for closed-loop stability, and the loop is well-posed (e.g., $g(s)$ is strictly proper). We will show that the feedback system is stable.

We take as a nominal $\boldsymbol{P}_0$ the linear gain

$$k_c = \left(k_1 k_2 + \sqrt{(1 + k_1^2)(1 + k_2^2)} - 1\right)/(k_1 + k_2).$$

This is chosen so that the line with slope k_c bisects the angle between the two lines with slopes k_1 and k_2. It is straightforward to show that

$$\vec{\delta}_0(k_1, k_c) = \vec{\delta}_0(k_c, k_2) = \sin(\phi) =: \gamma$$

where 2ϕ is the angle between the two lines with slopes k_1 and k_2. We now show that $\vec{\delta}_0(k_c, N) \leq \gamma$. It suffices to show that for any point on the graph of N we can select a point x on the nominal graph so that $\|y - x\| \leq \gamma \|x\|$. This can always be done by selecting x (on the line with slope k_c) so that $y - x$ is orthogonal to y (see Fig. 3). Then $\|y - x\|/\|x\| \leq \sin(\phi) = \gamma$.

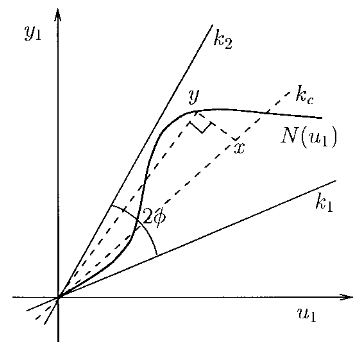

Fig. 3. Sector bounded nonlinearity.

Now consider the (stereographic) projection of the disc with diameter $[-k_1^{-1}, -k_2^{-1}]$, and of the locus $g(j\omega)$, onto the Riemann sphere of unit diameter placed above the origin of the complex plane. It can be seen directly (see also [35] and [36]) that $\alpha := \|\boldsymbol{\Pi}_{\mathcal{M}//\mathcal{N}}\|^{-1}$ (where $\mathcal{M} = \mathcal{G}_{\mathrm{P}_0}$ and $\mathcal{N} = \mathcal{G}_{\mathrm{C}}$) is the smallest chordal distance of $g(j\omega)$ to $-1/k_c$. As long as $g(j\omega)$ avoids the circle with diameter $[-k_1^{-1}, -k_2^{-1}]$, it follows that $\alpha > \gamma$. (Note that γ is the chordal radius of the projection of the disc from the projected center $-1/k_c$.) This proves the assertion on the stability of the feedback system by Theorem 3. □

IV. Stability and Robustness on Bounded Sets

It is often the case that a feedback system cannot have a bounded response outside a restricted set of disturbance signals. Such an example is an unstable nominal plant with input saturation. In this section we present a local version of the robustness theory, and introduce a suitable modification of $\vec{\delta}(\cdot, \cdot)$, for such a class of disturbances. The main result (Theorem 4) states that, as long as the plant perturbation is less than a certain robustness margin, bounded operation for the perturbed feedback system can be guaranteed over a suitably restricted set of disturbance signals. The predicted size of the allowable disturbance set for the perturbed system decreases with the magnitude of the perturbation from the nominal plant.

Let $\boldsymbol{P}, \boldsymbol{P}_1$ be causal operators mapping $\mathcal{U} \to \mathcal{Y}, \boldsymbol{C}$ a causal operator from $\mathcal{Y} \to \mathcal{U}$, and as before, denote $\mathcal{M} := \mathcal{G}_{\mathrm{P}}, \mathcal{M}_1 := \mathcal{G}_{\mathrm{P}_1}$, and $\mathcal{N} := \mathcal{G}_{\mathrm{C}}$. Let $\mathcal{S}_r$ be the open ball of radius r, i.e., $\mathcal{S}_r \subset \mathcal{W}$ is defined by

$$\mathcal{S}_r := \{w \in \mathcal{W}\colon \sup_\tau \|w\|_\tau < r\}$$

and define

$$\vec{\delta}_{\mathcal{S}_r}(\mathcal{M}, \mathcal{M}_1) := \begin{cases} \inf\{\|(\boldsymbol{\Phi} - \boldsymbol{I})|_{\mathcal{M}\cap\mathcal{S}_r}\|\colon \boldsymbol{\Phi} \text{ is causal,} \\ \quad \text{maps } \mathcal{M} \cap \mathcal{S}_r \text{ into } \mathcal{M}_1 \text{ with } \boldsymbol{\Phi}0 \\ \quad = 0, \text{ and is such that } \boldsymbol{T}_\tau(\boldsymbol{\Phi} - \boldsymbol{I})\boldsymbol{T}_\tau \\ \quad \text{is compact for all } \tau > 0\}, \\ \infty \text{ if no such operator } \boldsymbol{\Phi} \text{ exists.} \end{cases}$$

Theorem 4: Let $\boldsymbol{H}_{\mathrm{P,C}}$ be bounded on $\mathcal{S}_r$ with

$$\|\boldsymbol{\Pi}_{\mathcal{M}//\mathcal{N}}|_{\mathcal{S}_r}\| = \alpha$$

and let $\boldsymbol{P}_1$ be such that $\vec{\delta}_{\mathcal{S}_{\alpha r}}(\mathcal{M}, \mathcal{M}_1) = \gamma$ with $\gamma < \alpha^{-1}$.

Then $\boldsymbol{H}_{\mathrm{P}_1,\mathrm{C}}$ is bounded on $\mathcal{S}_{r(1-\alpha\gamma)}$ with

$$\|\boldsymbol{\Pi}_{\mathcal{M}_1//\mathcal{N}}|_{\mathcal{S}_{r(1-\alpha\gamma)}}\| \le \frac{\alpha(1+\gamma)}{1-\alpha\gamma}. \tag{9}$$

The proof of the theorem uses the following basic lemma.

Lemma 2: Let $\mathcal{X}$ be a Banach space, $\mathcal{S}_r$ the open ball of radius r in $\mathcal{X}$, and consider a mapping $\boldsymbol{A}$: $\mathcal{S}_r \to \mathcal{X}$. Suppose $\boldsymbol{A}$ is a continuous compact mapping such that $\zeta := \|\boldsymbol{A}|_{\mathcal{S}_r}\| < 1$. Then the equation $(\boldsymbol{I}+\boldsymbol{A})x = w$ has a solution $x \in \mathcal{S}_r$ for any given $w \in \mathcal{S}_{(1-\zeta)r}$, and moreover

$$\|x\| \le \frac{1}{1-\zeta}\|w\|.$$

Proof: Define the homotopy $\boldsymbol{B}_\lambda := \boldsymbol{I} + \lambda\boldsymbol{A}$: $\mathcal{S}_r \to \mathcal{X}$ for $\lambda \in [0,1]$. Since $\boldsymbol{B}_0$ is the identity operator and $w \in \mathcal{S}_r$, the Leray–Schauder degree of w relative to the set $\mathcal{S}_r$ and the map $\boldsymbol{B}_0$ (see [18, Th. 4.3.1]) is degree$(\boldsymbol{B}_0, \mathcal{S}_r, w) = 1$. Furthermore, since $\|\lambda\boldsymbol{A}\| \le \zeta$ and $w \in \mathcal{S}_{(1-\zeta)r}$, it follows that $w \notin \boldsymbol{B}_\lambda(\partial\mathcal{S}_r)$ for any $\lambda \in [0,1]$, where $\partial\mathcal{S}_r$ denotes the boundary of $\mathcal{S}_r$. Thus, we have degree$(\boldsymbol{B}_\lambda, \mathcal{S}_r, w) = 1$ for all $\lambda \in [0,1]$ (see [18, Th. 4.3.4]). Consequently, there exists an $x \in \mathcal{S}_r$ such that $\boldsymbol{B}_1 x = (\boldsymbol{I}+\boldsymbol{A})x = w$ (see [18, Th. 4.3.2]), and the required bound follows immediately. □

Proof of Theorem 4: Since $\vec{\delta}_{\mathcal{S}_{\alpha r}}(\mathcal{M}, \mathcal{M}_1) = \gamma$, then for any $\gamma_1 > \gamma$ there exists a causal map $\boldsymbol{\Phi}$: $\mathcal{M} \cap \mathcal{S}_{\alpha r} \to \mathcal{M}_1$ with $\|(\boldsymbol{I}-\boldsymbol{\Phi})|_{\mathcal{M}\cap\mathcal{S}_{\alpha r}}\| \le \gamma_1$ and $\boldsymbol{T}_\tau(\boldsymbol{\Phi}-\boldsymbol{I})\boldsymbol{T}_\tau$ compact for all $\tau > 0$. We choose such a $\boldsymbol{\Phi}$ for which $\gamma_1 < \alpha^{-1}$. The operator $\boldsymbol{A} := (\boldsymbol{\Phi}-\boldsymbol{I})\boldsymbol{\Pi}_{\mathcal{M}//\mathcal{N}}$ satisfies the conditions of Lemma 2 (with $\zeta \le \alpha\gamma_1$). Hence, the equation

$$w = (\boldsymbol{I} + (\boldsymbol{\Phi}-\boldsymbol{I})\boldsymbol{\Pi}_{\mathcal{M}//\mathcal{N}})x$$

has a solution $x \in \mathcal{S}_r$ for any $w \in \mathcal{S}_{r(1-\alpha\gamma_1)}$, and moreover

$$\|x\|_\tau \le \frac{\|w\|_\tau}{1-\alpha\gamma_1}.$$

Since $w = \boldsymbol{\Pi}_{\mathcal{N}//\mathcal{M}}x + \boldsymbol{\Phi}\boldsymbol{\Pi}_{\mathcal{M}//\mathcal{N}}x$, where $\boldsymbol{\Pi}_{\mathcal{N}//\mathcal{M}}x \in \mathcal{N}$ and $\boldsymbol{\Phi}\boldsymbol{\Pi}_{\mathcal{M}//\mathcal{N}}x \in \mathcal{M}_1$, and the perturbed system is well-posed, then

$$\boldsymbol{\Pi}_{\mathcal{M}_1//\mathcal{N}}w = \boldsymbol{\Phi}\boldsymbol{\Pi}_{\mathcal{M}//\mathcal{N}}x.$$

Hence

$$\begin{aligned}\|\boldsymbol{\Pi}_{\mathcal{M}_1//\mathcal{N}}w\|_\tau &\le \|\boldsymbol{\Phi}|_{\mathcal{M}\cap\mathcal{S}_{\alpha r}}\| \cdot \|\boldsymbol{\Pi}_{\mathcal{M}//\mathcal{N}}|_{\mathcal{S}_r}\| \cdot \|x\|_\tau \\ &\le (1+\gamma_1)\alpha\frac{\|w\|_\tau}{1-\alpha\gamma_1}.\end{aligned}$$

The above holds for any $\gamma_1 > \gamma$. Therefore (9) holds true. □

We include here a useful proposition which shows that linear integral operators are compact when restricted to a finite interval. In particular, linear systems with strictly proper transfer functions define such operators. This fact will be used in the example below which illustrates Theorem 4.

Proposition 4: A linear operator defined by $y(t) = \int_0^T g(t,\tau)u(\tau)d\tau$ is compact when restricted to $\mathcal{L}_\infty[0,T]$ if $g(t,\tau) \in \mathcal{L}_\infty([0,T]\times[0,T])$.

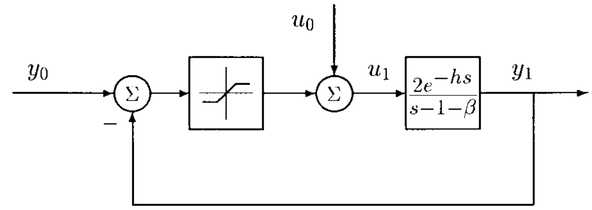

Fig. 4. Unstable plant with saturation.

Proof: On the rectangle $[0,T]\times[0,T]$, $g(t,\tau)$ can be uniformly approximated by simple functions, i.e., functions which have finite range. Since measurable sets can be approximated arbitrarily closely in measure by rectangles, the integral operator can be approximated in norm by replacing $g(t,\tau)$ with functions of the form $\Sigma_{i=1}^n f_i(t)k_i(\tau)$, where each of the functions f_i, k_i is a scalar times the characteristic function of an interval. But such approximations are finite rank operators and hence compact, so the original operator is compact. □

Example 4 (Robust Stabilization of an Unstable Linear System with Saturation, Over a Bounded Set of Disturbances): Consider the feedback configuration of Fig. 4, where $\boldsymbol{P}_1$ is described by

$$\dot{x}(t) = (1+\beta)x(t) + 2u_1(t-h), \qquad y_1(t) = x(t)$$

with $x(0) = 0, u_1(t) = 0$ for $t < 0$, and the nominal plant $\boldsymbol{P}$ has $\beta = h = 0$. We wish to illustrate Theorem 4 by finding bounds on the parameters β and h for which stability of the feedback system can be guaranteed. We take $\mathcal{U} = \mathcal{Y} = \mathcal{L}_\infty$.

We begin by finding the $\mathcal{L}_\infty$-induced gain of the mapping $\binom{u_0}{y_0} \mapsto \binom{u_1}{y_1}$, for the nominal model ($\beta = h = 0$). We first compute the gain of the mapping $\binom{u_0}{y_0} \mapsto y_1 = x$, for u_0, y_0 constrained so that $|u_0| \le b_1$ and $|y_0| \le b_2$. The nominal closed loop is described by the equations

$$\begin{aligned}\dot{x}(t) &= x(t) + 2u_0(t) + 2\,\mathrm{sat}\,(y_0(t) - x(t)) \\ &=: f(x(t), u_0(t), y_0(t))\end{aligned}$$

(with f plotted in Fig. 5 as a function of x). We first note that if $y_0 \ge 1$, then $f(x,u_0,y_0) > 0$ for all $x \ge 0$ if $u_0 > 0$, which means that x can become unbounded for constant, bounded disturbances. We therefore need to assume that $b_2 < 1$. We next note that if $2u_0 + y_0 > 1$, then $f(x,u_0,y_0) > 0$ for all $x \ge 0$, and once again x can become unbounded. We therefore assume that $2b_1 + b_2 \le 1$. We now claim that $x_{\max} = 2(b_1 + b_2)$ is the infimum of $x > 0$ such that $f(x,u_0,y_0)$ is nonpositive for all $|u_0| \le b_1, |y_0| \le b_2$. To see this, note that the two rightmost x-axis intercepts of f are $2(u_0 + y_0)$ and $2 - 2u_0$ and $2(u_0+y_0) \le x_{\max} \le 2 - 2u_0$. Thus

$$\begin{aligned}\sup\{x(t)&: \dot{x}(t) = f(x(t),u_0(t),y_0(t)),\ x(0) = 0, \\ &|u_0| \le b_1,\ |y_0| \le b_2\}\end{aligned}$$

cannot exceed $x_{\max}$. A similar argument for the range of negative values shows that $x(t) \ge -x_{\max}$. In fact, by examining the form of f in Fig. 5, it can be seen that the upper bound is tight for $u_0(t) = b_1, y_0(t) = b_2$ (constants). Let us now specialize to the case where $b_1 = b_2 = r/3$ with $r \le 1$. Then

$$\sup\left\{\|x(t)\|_\infty: \left\|\begin{pmatrix}u_0\\y_0\end{pmatrix}\right\| \le \frac{r}{3}\right\} = \frac{4r}{3}.$$

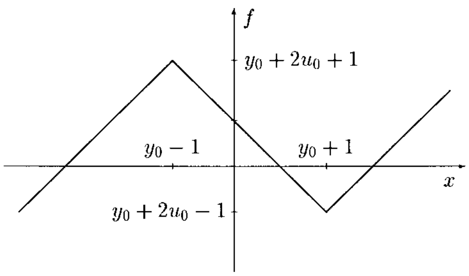

Fig. 5. $f(x, u_0, y_0)$ as a function of x.

Hence

$$\left\| \left(\begin{pmatrix} u_0 \\ y_0 \end{pmatrix} \mapsto x \right) \Big|_{\mathcal{S}_{1/3}} \right\| = 4.$$

Next we estimate the gain of $\binom{u_0}{y_0} \mapsto u_1$ subject to $|u_0| \leq 1/3, |y_0| \leq 1/3$. Since $u_1 = u_0 + \mathrm{sat}(y_0 - x)$, then

$$\begin{aligned} |u_1| &\leq |u_0| + |y_0| + |x| \\ &\leq \frac{r}{3} + \frac{r}{3} + \frac{4r}{3} = 2r \end{aligned}$$

providing $|u_0| \leq r/3, |y_0| \leq r/3$. In fact, this upper bound can be achieved for $r \leq 3/5$ (for which the saturation remains inactive) as follows. Take $u_0 = y_0 = r/3$ on a sufficiently long interval so that $x(T)$ is close to $4r/3$, then apply $u_0(T) = y_0(T) = -r/3$. Thus the mapping $\binom{u_0}{y_0} \mapsto u_1$ has gain six on $\mathcal{S}_{1/3}$ which means that

$$\left\| \left(\begin{pmatrix} u_0 \\ y_0 \end{pmatrix} \mapsto \begin{pmatrix} u_1 \\ y_1 \end{pmatrix} \right) \Big|_{\mathcal{S}_{1/3}} \right\| = 6.$$

We now consider a perturbation of the parameters $h \geq 0$ and β from their nominal values ($\beta = h = 0$) and apply the above theory to assess robustness of stability. To this end we obtain an estimate of the gap between $\boldsymbol{P}$ and $\boldsymbol{P}_1$. It can be shown (e.g., see [34, p. 234]) that the respective graphs are given by $\mathcal{G}_{\mathrm{P}} = G\mathcal{L}_\infty$ and $\mathcal{G}_{\mathrm{P}_1} = G_1\mathcal{L}_\infty$ where

$$G = \begin{pmatrix} M \\ N \end{pmatrix} := \begin{pmatrix} \dfrac{s-1}{s+1} \\ \dfrac{2}{s+1} \end{pmatrix}$$

and

$$G_1 = \begin{pmatrix} M_1 \\ N_1 \end{pmatrix} := \begin{pmatrix} \dfrac{s-1-\beta}{s+1} \\ \dfrac{2e^{-hs}}{s+1} \end{pmatrix}.$$

Let $(V, U) := (1, 1)$ (which satisfies $(V, U)\binom{M}{N} = 1$), and define the mapping

$$\boldsymbol{\Phi} := \begin{pmatrix} M_1 \\ N_1 \end{pmatrix} (V, U)$$

which takes $\mathcal{G}_{\mathrm{P}}$ onto $\mathcal{G}_{\mathrm{P}_1}$. Then

$$\begin{aligned} (\boldsymbol{I} - \boldsymbol{\Phi})|_{\mathcal{G}_{\mathrm{P}}} &= \begin{pmatrix} M \\ N \end{pmatrix} (V, U) - \begin{pmatrix} M_1 \\ N_1 \end{pmatrix} (V, U) \\ &= \left(\begin{pmatrix} M \\ N \end{pmatrix} - \begin{pmatrix} M_1 \\ N_1 \end{pmatrix} \right) (1, 1) \\ &= \begin{pmatrix} \dfrac{\beta}{s+1} \\ \dfrac{2}{s+1}(1 - e^{-hs}) \end{pmatrix} (1, 1). \end{aligned}$$

We now estimate the $\mathcal{L}_\infty$-induced norm of $(\boldsymbol{I} - \boldsymbol{\Phi})|_{\mathcal{G}_{\mathrm{P}}}$. It holds that

$$\begin{aligned} \|(\boldsymbol{I} - \boldsymbol{\Phi})|_{\mathcal{G}_{\mathrm{P}}}\| &\leq \|(\boldsymbol{I} - \boldsymbol{\Phi})|_{\mathcal{W}}\| \\ &= \|(\boldsymbol{I} - \boldsymbol{\Phi})|_{\mathcal{W}}\|_B \\ &= 2 \left\| \begin{pmatrix} \dfrac{\beta}{s+1} \\ \dfrac{2}{s+1}(1 - e^{-hs}) \end{pmatrix} \right\| \\ &= 2 \max \left\{ \left\| \frac{\beta}{s+1} \right\|, \left\| \frac{2}{s+1}(1 - e^{-hs}) \right\| \right\}. \end{aligned}$$

But $\|\beta/(s+1)\| = \|\beta e^{-t}\|_{\mathcal{L}_1} = \beta$, and $\|2(1 - e^{-hs})/(s+1)\| = 2(\int_0^h e^{-t}\, dt + \int_h^\infty (1 - e^{-h}) e^{h-t}\, dt) = 4(1 - e^{-h})$, being in both cases the $\mathcal{L}_1$-norm of the relevant convolution kernel. It follows that

$$\gamma := \|(\boldsymbol{I} - \boldsymbol{\Phi})|_{\mathcal{G}_{\mathrm{P}}}\| \leq \max\{2\beta, 8(1 - e^{-h})\}.$$

From Proposition 4, it follows that this choice of a $\boldsymbol{\Phi}$ satisfies the compactness requirement in the definition of $\vec{\delta}_{\mathcal{S}_r}(\cdot, \cdot)$. We finally apply Theorem 4 for $r = 1/3$. Since the norm of the nominal parallel projection is $\alpha = 6$, it follows that the perturbed system is stable on the restricted set $\mathcal{S}_{(1/3)(1-6\gamma)}$ provided that $\gamma \leq \max\{2\beta, 8(1 - e^{-h})\} < 1/6$. $\square$

V. Robustness for Systems with Potential for Finite-Time Escape

In this section we are motivated by the need to provide a robustness theory for nonlinear systems which allows for the possibility of a finite-time escape. An example of such a system is given by

$$\begin{aligned} \dot{x}(t) &= x^2(t) + u_1(t), \qquad \text{with} \quad x(0) = 0 \\ y_1(t) &= x(t). \end{aligned}$$

We first remark that global robust stabilization is impossible for such a system. To see this, consider an arbitrarily small time delay at the plant input. The feedback equations are

$$\begin{aligned} \dot{x}(t) &= x^2(t) + u_0(t-h) - u_2(t-h) \\ u_2(t) &= \boldsymbol{C}(y_0(t) - x(t)). \end{aligned} \tag{10}$$

If $y_0(t) = 0$ and $u_0(t) = d$ for $t \in [0, h]$, then $x(t) = 0$ on the same interval. This means that $u_2(t) = 0$ on $[0, h]$ for any causal controller $\boldsymbol{C}$. Then, on the interval $[h, 2h]$ the state evolves according to $\dot{x}(t) = x^2(t) + d$ with a solution of the form $x(t) = \sqrt{d}\, \tan(\sqrt{d}(t-h))$. Clearly if $d > (\pi/2h)^2$, the state escapes to infinity before $t = 2h$. Thus, no (causal) controller can prevent finite-time escape in the presence of small time delays *unless* the disturbances are subject to some fixed bound. Below we will discuss a way to extend the theory of Section IV to deal with such systems. The main new element is the fact that such systems cannot be viewed as operators on signal spaces in the way which has been assumed so far.

The basic idea of the summation operator carries over even if some responses escape to infinity in finite time. To see this, note that the solution of the feedback equations requires that for any external disturbance $w = \binom{u_0}{y_0} \in \mathcal{W}$, the components of the feedback system respond accordingly by producing responses $m = \binom{u_1}{y_1}$ and $n = \binom{u_2}{y_2}$ so that $w = m + n$. However, in the present context, some of these signals may be defined only over a finite interval $[0, T)$ and escape to infinity at $t = T$. Below we will impose a well-posedness assumption on the feedback system which requires that this decomposition is unique on the interval over which all the signals are defined. In this case, the summation operator still plays its normal role, and its inverse, being the map from the external signal w to the feedback signals (m, n), is a well-defined map with the provision that (m, n) may not necessarily belong to $\mathcal{W} \times \mathcal{W}$. Thus, the difference from the earlier situation is that the graphs of the plant and the controller belong to a space which includes signals that are defined only on finite intervals $[0, T)$. We will study the case where the nominal feedback system is bounded on some bounded set $\mathcal{S}_r \subset \mathcal{W}$, i.e., $\boldsymbol{\Sigma}_{\mathcal{M},\mathcal{N}}^{-1}\mathcal{S}_r \subset \mathcal{W} \times \mathcal{W}$ with $\|\boldsymbol{\Sigma}_{\mathcal{M},\mathcal{N}}^{-1}|_{\mathcal{S}_r}\| < \infty$, and we will give conditions for a perturbed system to be stable in the same sense.

Formally, a system $\boldsymbol{P}$ is a collection of input–ouput pairs with the provision that these may be defined only on a finite interval $[0, T)$. We will assume for all systems considered that finite-time escape behavior cannot occur instantaneously. Thus, in this section, the following replaces our standing well-posedness assumption.

Assumption 1: All feedback systems considered, together with their perturbations, satisfy the property that for each $w = \binom{u_0}{y_0} \in \mathcal{W}$ there exists a unique pair (m, n) with $m = \binom{u_1}{y_1} \in \mathcal{G}_{\mathrm{P}}$ and $n = \binom{u_2}{y_2} \in \mathcal{G}_{\mathrm{C}}$ such that $w = m + n$ over a maximal interval $[0, T)$ with $0 < T \leq \infty$. Moreover, the mappings $w \mapsto m, n$ are causal on $[0, T)$. Finally, if T is finite, then $\|m\|_\tau \to \infty$ as τ tends to T from below. □

For this type of system the conclusions of Theorem 4 carry over without modification as we now show.

Theorem 5: Let $\boldsymbol{P}, \boldsymbol{P}_1$, and $\boldsymbol{C}$ satisfy Assumption 1. Denote $\mathcal{M} = \mathcal{G}_{\mathrm{P}}, \mathcal{M}_1 = \mathcal{G}_{\mathrm{P}_1}, \mathcal{N} = \mathcal{G}_{\mathrm{C}}$. Let $\boldsymbol{H}_{\mathrm{P,C}}$ be bounded on $\mathcal{S}_r$ with

$$\|\boldsymbol{\Pi}_{\mathcal{M}_1//\mathcal{N}}|_{\mathcal{S}_r}\| = \alpha.$$

Suppose there exists a mapping $\boldsymbol{\Phi}$: $\mathcal{M} \cap \mathcal{S}_{\alpha r} \to \mathcal{M}_1 \cap \mathcal{W}$ such that $\|(\boldsymbol{\Phi} - \boldsymbol{I})|_{\mathcal{M}\cap\mathcal{S}_{\alpha r}}\| = \gamma_1 < \alpha^{-1}$, and $\boldsymbol{A} = (\boldsymbol{\Phi} - \boldsymbol{I})\boldsymbol{\Pi}_{\mathcal{M}//\mathcal{N}}$ satisfies the conditions of Lemma 2. Then $\boldsymbol{H}_{\mathrm{P}_1,\mathrm{C}}$ is bounded on $\mathcal{S}_{(1-\alpha\gamma_1)r}$ with

$$\|\boldsymbol{\Pi}_{\mathcal{M}_1//\mathcal{N}}|_{\mathcal{S}_{(1-\alpha\gamma_1)r}}\| \leq \frac{(1+\gamma_1)\alpha}{1-\alpha\gamma_1}.$$

Proof: As in the proof of Theorem 4, for any $w \in \mathcal{S}_{(1-\alpha\gamma_1)r}$ there exists a solution $x \in \mathcal{W}$ to the equation

$$w = (\boldsymbol{I} + (\boldsymbol{\Phi} - \boldsymbol{I})\boldsymbol{\Pi}_{\mathcal{M}//\mathcal{N}})x. \tag{11}$$

Moreover, for any $\tau > 0, \|x\|_\tau \leq \|w\|_\tau/(1-\alpha\gamma_1) \leq r$. Note that $\boldsymbol{\Phi}\boldsymbol{\Pi}_{\mathcal{M}//\mathcal{N}}x \in \mathcal{M}_1$. By Assumption 1, there exists a maximal $T > 0$ for which $\boldsymbol{T}_\tau\boldsymbol{\Sigma}_{\mathcal{M}_1,\mathcal{N}}^{-1}w$ is bounded for all $\tau < T$. Over such an interval

$$\boldsymbol{T}_\tau\boldsymbol{\Pi}_{\mathcal{M}_1//\mathcal{N}}w = \boldsymbol{T}_\tau\boldsymbol{\Phi}\boldsymbol{\Pi}_{\mathcal{M}//\mathcal{N}}x.$$

Hence

$$\begin{aligned}\|\boldsymbol{\Pi}_{\mathcal{M}_1//\mathcal{N}}w\|_\tau &\leq \|\boldsymbol{\Phi}|_{\mathcal{M}\cap\mathcal{S}_{\alpha r}}\| \cdot \|\boldsymbol{\Pi}_{\mathcal{M}//\mathcal{N}}|_{\mathcal{S}_r}\| \cdot \|x\|_\tau \\ &\leq \frac{(1+\gamma_1)\alpha}{1-\alpha\gamma_1}\|w\|_\tau \leq (1+\gamma_1)\alpha r.\end{aligned} \tag{12}$$

Note that (12) provides a uniform bound on the response of the perturbed system for any $\tau < T$. This fact prevents the possibility of finite-time escape since the perturbed system satisfies Assumption 1. Hence $T = \infty$. □

Example 5 (Robustness of Stability for a System with Quadratic Nonlinearity): We consider the feedback interconnection of Fig. 1 with $\boldsymbol{P}$ defined by

$$\begin{aligned}\dot{x}(t) &= x^2(t) + u_1(t), \qquad x(0) = 0 \\ y_1(t) &= x(t)\end{aligned}$$

and the controller $\boldsymbol{C}$ by

$$u_2(t) = y_2^2(t) - ky_2(t).$$

(This controller aims to cancel the quadratic term and replace it with a stable linear term.) We will take $\mathcal{U} = \mathcal{Y} = \mathcal{L}_\infty[0, \infty)$. The closed-loop system evolves according to

$$\dot{x}(t) = (2y_0(t) - k)x(t) + u_0(t) + ky_0(t) - y_0^2(t)$$

and satisfies Assumption 1. We will study the robustness of this system to a time delay at the plant input, using Theorem 5.

Claim 1:

$$\begin{aligned}&\|\boldsymbol{\Pi}_{\mathcal{M}//\mathcal{N}}|_{\mathcal{S}_r}\| \\ &\quad= \max\left\{\frac{1+k-r}{k-2r},\right. \\ &\qquad\qquad \left. 1 + \frac{(1+2k-3r)(r+k^2-3r^2)}{(k-2r)^2}\right\} \\ &\quad=: f(r,k).\end{aligned} \tag{13}$$

Proof of Claim 1: Recall that $\boldsymbol{\Pi}_{\mathcal{M}//\mathcal{N}}$ is the mapping from $\binom{u_0}{y_0} \mapsto \binom{u_1}{x}$. We will first consider the bounds

$$\|u_0\|_\infty \leq \beta, \qquad \|y_0\|_\infty \leq \eta.$$

To guarantee boundedness of the induced norm it is necessary that $\eta < k/2$. Otherwise $u_0 \equiv 0$, and $y_0 \equiv k/2$ gives $\dot{x} = k^2/4$ which diverges. We claim that

$$\sup_{\|u_0\|_\infty \leq \beta, \|y_0\|_\infty \leq \eta} \|x\|_\infty = \sup_{|u_0| \leq \beta, |y_0| \leq \eta} |A(u_0, y_0)| =: A_{\max} \tag{14}$$

where $A(u_0, y_0) = (u_0 + ky_0 - y_0^2)/(k - 2y_0)$. This can be seen by noting that $x(t)\dot{x}(t) < 0$ when $|2y_0 - k| \cdot |x(t)| > |u_0 +$

$ky_0 - y_0^2|$. Thus $|x(t)|$ is decreasing whenever $|x(t)|$ exceeds $A_{\max}$. But $x(t) \to A(u_0, y_0)$ as $t \to \infty$ for u_0, y_0 identically constant.

We next note that the maximum of $|A(u_0, y_0)|$ will be achieved for $u_0 = \pm\beta$, so that $A_{\max} = \max\{A_+, A_-\}$ where $A_\pm = \max_{|y_0|\le\eta} |A(\pm\beta, y_0)|$. Computing the partial derivative

$$\frac{\partial A}{\partial y_0} = \frac{4u_0 + (k-2y_0)^2 + k^2}{2(k-2y_0)^2}$$

shows A_+ achieves its maximum at an extreme point $y_0 = \pm\eta$, while A_- achieves its maximum either at an extreme point or at a possible turning point if $\partial A/\partial y_0 = 0$ in the interval $-\eta \le y_0 \le \eta$. Considering first the extreme points, it is clear that $|A(\beta, -\eta)| < |A(-\beta, -\eta)|$ and $|A(-\beta, \eta)| < |A(\beta, \eta)|$. Furthermore, it is straightforward to check that $|A(-\beta, -\eta)| < |A(\beta, \eta)|$. Considering the sign of $\partial A/\partial y_0$ at the extreme points, a turning point of A_- exists in the interval $-\eta \le y_0 \le \eta$, providing

$$\frac{(k-2\eta)^2 + k^2}{4} < \beta < \frac{(k+2\eta)^2 + k^2}{4} \tag{15}$$

in which case it occurs at $y_0 = (k - \sqrt{4\beta - k^2})/2$ with a corresponding value of $|A(-\beta, y_0)| = \sqrt{4\beta - k^2}/2$. From (15)

$$\sqrt{4\beta - k^2}/2 < \eta + k/2 \tag{16}$$

$$A(\beta, \eta) = \frac{\beta + k\eta - \eta^2}{k - 2\eta} > \frac{k^2}{2(k-2\eta)}. \tag{17}$$

It is easy to check that the upper bound in (16) is smaller than the lower bound in (17). We have therefore shown that $A_{\max} = A(\beta, \eta)$.

We now turn to the mapping $\binom{u_0}{y_0} \mapsto u_1$. From the expression $u_1 = u_0 - (y_0 - x)^2 + k(y_0 - x)$ it is easy to see that

$$\sup\{\|u_1\|_\infty\colon \|u_0\|_\infty \le \beta, \|y_0\|_\infty \le \eta\} \le B_{\max} \tag{18}$$

where $B_{\max} := \beta + (\eta + A_{\max})^2 + k(\eta + A_{\max})$. In fact, equality is achieved in (18), since the disturbances $u_0 = \beta$ on $[0, T), u_0(T) = -\beta, y_0 = \eta$ on $[0, T), y_0(T) = -\eta$ make $x(T)$ as close as desired to $A_{\max}$ and $|u_1(T)|$ as close as desired to $B_{\max}$ for large enough T. We will now specialize to the case $\beta = \eta = r < k/2$. Then

$$\begin{aligned}\frac{1}{r}\sup&\left\{\left\|\boldsymbol{\Pi}_{\mathcal{M}//\mathcal{N}}\begin{pmatrix}u_0\\ y_0\end{pmatrix}\right\|_\infty : \left\|\begin{pmatrix}u_0\\ y_0\end{pmatrix}\right\|_\infty \le r\right\}\\ &= \frac{1}{r}\max\{A_{\max}, B_{\max}\}\\ &= f(r,k).\end{aligned}$$

It can be checked that each of the terms in (13) is a monotonically increasing function of r for fixed $k > 2r$ (for the second expression, each of the numerator factors divided by $(k - 2r)$ is separately increasing). Therefore, we conclude that $f(r,k)$ is equal to $\|\boldsymbol{\Pi}_{\mathcal{M}//\mathcal{N}}|\mathcal{S}_r\|$. □

It is interesting to consider the behavior of $\|\boldsymbol{\Pi}_{\mathcal{M}//\mathcal{N}}|\mathcal{S}_r\|$ for *fixed* r and variable k. As $k \downarrow 2r$ and as $k \uparrow \infty, f(r,k) \to \infty$. Generally there is an intermediate value of k at which $f(r,k)$ achieves a minimum, $k_{\min}$ (see Fig. 6). A plot of $k_{\min}$ and $f(r, k_{\min})$ versus r is shown in Fig. 7 (where the values for $k_{\min}$ are given on the left and the values for $f(r, k_{\min}(r))$ on the right). Experience from linear control design indicates that the parallel projection norm should not exceed about four or five for good robustness properties. From Fig. 7 it is necessary that $r \le 0.0784$ in order that $\|\boldsymbol{\Pi}_{\mathcal{M}//\mathcal{N}}|\mathcal{S}_r\| \le 5$, assuming that $k = k_{\min}$. This suggests that good performance of this control system requires a rather tight constraint on the disturbance signal magnitudes.

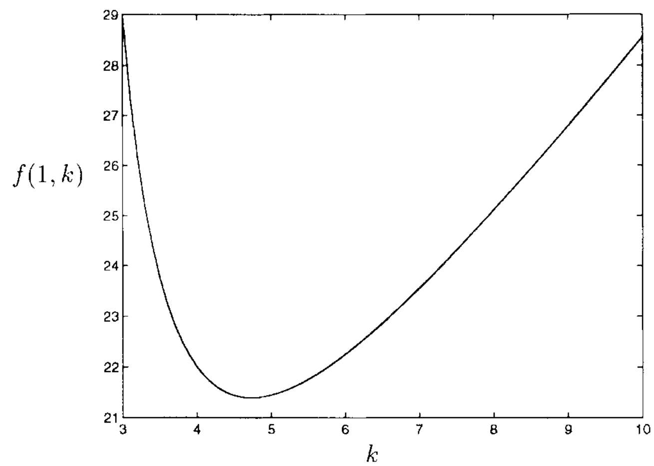

Fig. 6. Plot of $\|\boldsymbol{\Pi}_{\mathcal{M}//\mathcal{N}|\mathcal{S}_1}\| = f(1,k)$ versus k.

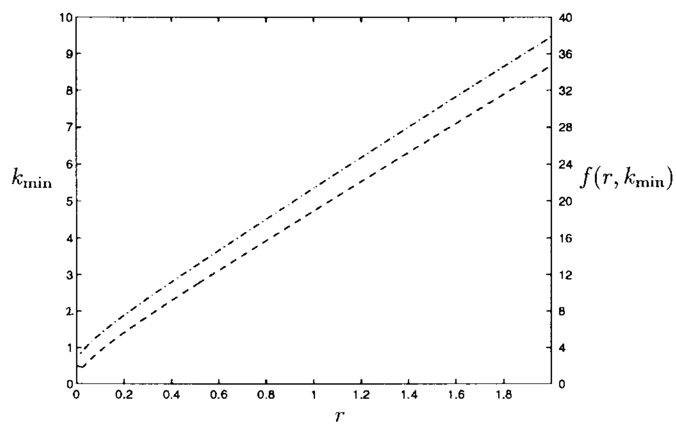

Fig. 7. $k_{\min}$ (dashed) and $f(r, k_{\min})$ (dashed-dotted) versus r.

We will now consider the effect of time delays on the control system, and we will use Theorem 5 to give a bound on the delay which will not destabilize the loop. Suppose $\boldsymbol{P}_1$ is defined by the equations

$$\begin{aligned}\dot{x}(t) &= x^2(t) + u_1(t), \qquad x(0) = 0\\ y_1(t) &= x(t-h).\end{aligned}$$

(Since $\boldsymbol{P}$ is shift-invariant, it is equivalent to place the delay at u_1 or y_1.) We first remark that Assumption 1 is satisfied because the closed-loop system is defined through a differential equation. Next we consider the mapping $\boldsymbol{\Phi}\colon \mathcal{G}_{\mathrm{P}} \to \mathcal{G}_{\mathrm{P}_1}$ defined by

$$\boldsymbol{\Phi}\begin{pmatrix}u_1(t)\\ x(t)\end{pmatrix} = \begin{pmatrix}u_1(t)\\ x(t-h)\end{pmatrix}.$$

We will check below that $\boldsymbol{A} := (\boldsymbol{\Phi} - \boldsymbol{I})\boldsymbol{\Pi}_{\mathcal{M}//\mathcal{N}}$ satisfies the conditions of Lemma 2. We will now proceed to bound the norm of $\boldsymbol{A}$.

Claim 2:

$$\|(\boldsymbol{\Phi} - \boldsymbol{I})\boldsymbol{\Pi}_{\mathcal{M}//\mathcal{N}}|_{\mathcal{S}_r}\| \leq \frac{2h(k(1+k) - 2r^2)}{k - 2r} =: g(r,k).$$

Proof of Claim 2: Note that

$$\left\|\boldsymbol{A}\begin{pmatrix} u_0 \\ y_0 \end{pmatrix}\right\|_\infty = \|x(t) - x(t-h)\|_\infty \leq h\|\dot{x}(t)\|_\infty.$$

Since

$$\dot{x} = x^2 + u_1 = (2y_0 - k)x + u_0 + ky_0 - y_0^2$$

then

$$\sup_{\substack{\|u_0\|_\infty \leq \beta \\ \|y_0\|_\infty \leq \eta}} \|\dot{x}\|_\infty \leq C_{\max} \tag{19}$$

where

$$C_{\max} := \beta + k\eta + \eta^2 + (2\eta + k) \cdot \sup_{\substack{\|u_0\|_\infty \leq \beta \\ \|y_0\|_\infty \leq \eta}} \|x\|_\infty.$$

In fact, equality is achieved in (19), since the disturbances $u_0 = \beta$ on $[0,T), u_0(T) = -\beta, y_0 = \eta$ on $[0,T), y_0(T) = -\eta$ make $|\dot{x}(T)|$ as close as desired to $C_{\max}$ for large enough T. Once again we can set $\beta = \eta = r$ to find

$$\frac{C_{\max}}{r} = \frac{2(k(1+k) - 2r^2)}{k - 2r}$$

which is a monotonically increasing function of r for $r < k/2$. Therefore, we have the bound $\|\boldsymbol{A}|_{\mathcal{S}_r}\| \leq g(r,k)$. □

Considering $g(r,k)$ for fixed r and variable k, we note that as $k \downarrow 2r$ and as $k \uparrow \infty$, $g(r,k) \to \infty$. A minimum of $g(r,k)$ is achieved at $k_1(r) := 2r + \sqrt{2r^2 + 2r}$, which may be close to, but is not identical to, $k_{\min}$ calculated previously. This gives

$$g(r,k_1) = 2h\{4r + 1 + 2\sqrt{2r^2 + 2r}\}.$$

To guarantee stability of the perturbed system (with $k = k_1$) on some bounded set we need $g(r,k_1) < 1$. Note that the perturbed loop will only be guaranteed to be bounded on a set $\mathcal{S}_{r(1-g(r,k_1))}$.

Suppose we wish to determine a bound h_0 such that the nominal loop will tolerate any time delay $h \leq h_0$ for a suitable k in the presence of external disturbances $\|u_0\|_\infty \leq 1, \|y_0\|_\infty \leq 1$. We require, for $h = h_0$, $r(1 - g(r,k_1)) = 1$ where we select $k = k_1(r)$. Solving for h_0 gives

$$h_0 = \frac{r - 1}{2r\{4r + 1 + 2\sqrt{2r^2 + 2r}\}}$$

which is maximized for $r = 2.147$ giving $h_0 = 0.0158$. The corresponding value for k is $k_1(r) = 7.969$. Under such conditions we can compute the following bound for the perturbed loop:

$$\begin{aligned}\|\boldsymbol{\Pi}_{\mathcal{M}_1//\mathcal{N}}|_{\mathcal{S}_1}\| &\leq \|\boldsymbol{\Phi}\boldsymbol{\Pi}_{\mathcal{M}//\mathcal{N}}|_{\mathcal{S}_r}\| \frac{1}{1 - g(r,k_1(r))} \\ &\leq (\|(\boldsymbol{\Phi} - \boldsymbol{I})\boldsymbol{\Pi}_{\mathcal{M}//\mathcal{N}}|_{\mathcal{S}_r}\| + \|\boldsymbol{\Pi}_{\mathcal{M}//\mathcal{N}}|_{\mathcal{S}_r}\|)r \\ &\leq (0.534 + 41.273) \times 2.147 = 89.752.\end{aligned}$$

It remains only to verify that $\boldsymbol{A} = (\boldsymbol{\Phi} - \boldsymbol{I})\boldsymbol{\Pi}_{\mathcal{M}//\mathcal{N}}$ satisfies the conditions of Lemma 2. Note that the first component of $\boldsymbol{A}$ is zero, while the second component equals $x(t) - x(t-h)$. It follows from the fact that $x(t)$ is the output of an integral operator that $\boldsymbol{A}$ is causal and continuous for $u_0, y_0 \in \mathcal{L}_\infty[0,\infty)$, with $\|u_0\|_\infty, \|y_0\|_\infty \leq r$. We will now show that the mapping $\binom{u_0}{y_0} \mapsto x$ is (sequentially) compact on $[0,T]$. This mapping is determined by

$$\dot{x}(t) = -a(t)x(t) + b(t) \tag{20}$$

where $a(t) = k - 2y_0(t)$, and $b(t) = u_0(t) + ky_0(t) - y_0(t)^2$ both belong to $\mathcal{L}_\infty$. We need to show that, for any bounded sequences $a_i(t), b_i(t)$ in $\mathcal{L}_\infty[0,T]$, the corresponding sequence $x_i(t)$ has a convergent subsequence. Note that (20) has a solution

$$x(t) = \int_0^t \exp\left\{-\int_\tau^t a(\sigma)\, d\sigma\right\} b(\tau)\, d\tau.$$

From Proposition 4 we know that $\int_\tau^t a(\sigma)d\sigma$ defines a compact operator from $\mathcal{L}_\infty[0,T]$ to $\mathcal{L}_\infty([0,T] \times [0,T])$. Hence $\int_\tau^t a_i(\sigma)\, d\sigma$ has a subsequence which converges uniformly on $[0,T] \times [0,T]$, and so the same is true of $\exp\{-\int_\tau^t a(\sigma)\, d\sigma\}$. Thus we can select a subsequence so that

$$x_i(t) = \int_0^T (g(t,\tau) + e_i(t,\tau))b_i(\tau)\, d\tau$$

where $g(t,\tau) \in \mathcal{L}_\infty([0,T] \times [0,T])$ and $|e_i(t,\tau)| \to 0$ uniformly on $[0,T] \times [0,T]$. By Proposition 4 we can select another subsequence so that

$$\int_0^T g(t,\tau)b_i(\tau)\, d\tau$$

converges, in which case $x_i(t)$ converges to the same limit. The conclusion now follows by noting that $\boldsymbol{A}$ is the sum of two compact operators, which is therefore compact. □

VI. Robustness Analysis Using Gain Functions

In the present section we will develop a version of the robustness results of Section III which makes use of a "gain function" to quantify the size of closed-loop operators and the mismatch between the nominal and perturbed plant. In the first result we show how the gain function of the parallel projection must relate to that of a mapping from the nominal to the perturbed system graph for stability to be preserved. In the second result we show that the existence of such a mapping is a consequence of simultaneous stabilizability, and we investigate how the mismatch between the graphs relates to gain functions of the closed-loop errors.

We recall the standard notation $\mathcal{K}_\infty$ for the set of functions g: $[0,\infty) \to [0,\infty)$ which are continuous, strictly increasing, and satisfy $g(0)=0$ and $g(\infty)=\infty$. As usual we denote $\mathcal{M} := \mathcal{G}_{\rm P}, \mathcal{M}_1 := \mathcal{G}_{{\rm P}_1}$, and $\mathcal{N} := \mathcal{G}_{\rm C}$.

Theorem 6: Let $\boldsymbol{H}_{\rm P,C}$ be gf-stable. If there exists a map $\boldsymbol{\Phi}$ from a subset $\mathcal{D} \subseteq \mathcal{M}$ onto $\mathcal{M}_1$, and if there exists a function $\epsilon(\cdot) \in \mathcal{K}_\infty$ such that

$$g[\boldsymbol{I}-\boldsymbol{\Phi}] \circ g[\boldsymbol{\Pi}_{\mathcal{M}//\mathcal{N}}](\alpha) \le (1+\epsilon)^{-1}(\alpha)$$

for all $\alpha \ge 0$, then $\boldsymbol{H}_{{\rm P}_1,{\rm C}}$ is gf-stable and

$$g[\boldsymbol{\Pi}_{\mathcal{M}_1//\mathcal{N}}](\alpha) \le g[\boldsymbol{\Phi}] \circ g[\boldsymbol{\Pi}_{\mathcal{M}//\mathcal{N}}] \circ (1+\epsilon^{-1})(\alpha). \tag{21}$$

We remark that, as pointed out in Section III-D, $\boldsymbol{\Phi}$ is allowed to be a multivalued map, e.g., defined as the inverse relation of a map $\boldsymbol{\Psi}$: $\mathcal{M}_1 \to \mathcal{D} \subseteq \mathcal{M}$ selected so as to control the size of $\boldsymbol{I}-\boldsymbol{\Psi}^{-1}$; however, the theorem will be worked out in the simpler form. It was suggested in [10] that a result along the lines of Theorem 6 may be obtained by applying the Mareels–Hill small gain lemma [19] to the basic framework set out in [14]. Subsequently, Teel developed an independent approach to such a result using $\mathcal{K}_\infty$-functions [31].

Proof of Theorem 6: It is a standing assumption that $\boldsymbol{\Sigma}_{\mathcal{M}_1,\mathcal{N}}$ has a well-defined inverse. This, together with the fact that $\boldsymbol{\Phi}$ is surjective, implies as in the proof of Theorem 1 that, for any $w \in \mathcal{W}$, there is an $x \in \mathcal{W}$ such that

$$w = (\boldsymbol{I} + (\boldsymbol{\Phi}-\boldsymbol{I})\boldsymbol{\Pi}_{\mathcal{M}//\mathcal{N}})x \tag{22}$$

and moreover, that

$$\boldsymbol{\Pi}_{\mathcal{M}_1//\mathcal{N}} w = \boldsymbol{\Phi}\boldsymbol{\Pi}_{\mathcal{M}//\mathcal{N}} x. \tag{23}$$

From (22) it follows that for any τ

$$\begin{aligned}\|x\|_\tau &\le \|w\|_\tau + \|(\boldsymbol{\Phi}-\boldsymbol{I})\boldsymbol{\Pi}_{\mathcal{M}//\mathcal{N}}x\|_\tau\\ &\le \|w\|_\tau + g[\boldsymbol{\Phi}-\boldsymbol{I}] \circ g[\boldsymbol{\Pi}_{\mathcal{M}//\mathcal{N}}](\|x\|_\tau)\\ &\le \|w\|_\tau + (1+\epsilon)^{-1}(\|x\|_\tau).\end{aligned}$$

Hence

$$\begin{aligned}\|w\|_\tau &\ge (1-(1+\epsilon)^{-1})(\|x\|_\tau)\\ &= \epsilon \circ (1+\epsilon)^{-1}(\|x\|_\tau).\end{aligned}$$

Therefore

$$\begin{aligned}\|x\|_\tau &\le (1+\epsilon)\circ\epsilon^{-1}(\|w\|_\tau)\\ &= (1+\epsilon^{-1})(\|w\|_\tau).\end{aligned}$$

From (23) it now follows that $\boldsymbol{\Pi}_{\mathcal{M}_1//\mathcal{N}}$ is gf-stable and is such that (21) holds. □

Theorem 7: Let $\boldsymbol{H}_{\rm P,C}$ be gf-stable. The following hold.

1) Let $\boldsymbol{H}_{{\rm P}_1,{\rm C}}$ be gf-stable. Then, there exists a causal bijective mapping $\boldsymbol{\Phi}$: $\mathcal{M} \to \mathcal{M}_1$ such that $g[\boldsymbol{\Phi}-\boldsymbol{I}](\alpha) \le g[\boldsymbol{\Pi}_{\mathcal{M}_1//\mathcal{N}} - \boldsymbol{\Pi}_{\mathcal{M}//\mathcal{N}}](\alpha)$ for all $\alpha \ge 0$.

2) Suppose that $g[\boldsymbol{\Pi}_{\mathcal{M}//\mathcal{N}}] = \gamma_0$ and that there exists a mapping $\boldsymbol{\Phi}$ from a subset $\mathcal{D} \subseteq \mathcal{M}$ onto $\mathcal{M}_1$ for which $g[\boldsymbol{\Phi}-\boldsymbol{I}] = \gamma_1$. If

$$\sup_{\alpha>0} \frac{1}{\alpha}(\gamma_1 \circ \gamma_0)(\alpha) = \lambda < 1 \tag{24}$$

then $\boldsymbol{\Pi}_{\mathcal{M}_1//\mathcal{N}}$ is gf-stable and

$$\begin{aligned}&g[\boldsymbol{\Pi}_{\mathcal{M}_1//\mathcal{N}} - \boldsymbol{\Pi}_{\mathcal{M}//\mathcal{N}}](\alpha)\\ &\quad\le \frac{\lambda}{1-\lambda}\alpha + g_\Delta[\boldsymbol{\Pi}_{\mathcal{M}//\mathcal{N}}]\left(\alpha, \frac{\lambda}{1-\lambda}\right)\end{aligned} \tag{25}$$

where

$$g_\Delta[\boldsymbol{F}](\alpha,\beta) := \sup_{\tau>0, \|x_1\|_\tau \le \alpha, \|x_1-x_2\|_\tau \le \beta} \|\boldsymbol{F}x_1 - \boldsymbol{F}x_2\|_\tau.$$

Proof of 1): This result follows in the same way as the proof of Theorem 2-1) using the choice

$$\boldsymbol{\Phi} = \boldsymbol{I} - (\boldsymbol{\Pi}_{\mathcal{M}//\mathcal{N}} - \boldsymbol{\Pi}_{\mathcal{M}_1//\mathcal{N}}).$$

2): Since $\lambda < 1$, Theorem 6 implies that $\boldsymbol{H}_{{\rm P}_1,{\rm C}}$ is gf-stable. As in the proof of Theorem 1, for any $w \in \mathcal{W}$, there exists an $x \in \mathcal{W}$ such that (22) holds. From Lemma 1, $\|w-x\|_\tau \le \|w\|_\tau \lambda/(1-\lambda)$ and $\|x\|_\tau \le \|w\|_\tau/(1-\lambda)$. From (23)

$$\begin{aligned}&(\boldsymbol{\Pi}_{\mathcal{M}//\mathcal{N}} - \boldsymbol{\Pi}_{\mathcal{M}_1//\mathcal{N}})w\\ &\quad= (\boldsymbol{I}-\boldsymbol{\Phi})\boldsymbol{\Pi}_{\mathcal{M}//\mathcal{N}}x + \boldsymbol{\Pi}_{\mathcal{M}//\mathcal{N}}w - \boldsymbol{\Pi}_{\mathcal{M}//\mathcal{N}}x\end{aligned} \tag{26}$$

which gives the required result. □

Theorem 7 provides the analog of Theorem 2 in the context of gain functions. In particular, 1) captures the essence of the idea: "convergence of closed-loop norms implies convergence of the gap to zero." More precisely, if a plant $\boldsymbol{P}$ and a sequence of plants $\boldsymbol{P}_i$ $(i=1,2,\cdots)$ are all stabilized by a compensator $\boldsymbol{C}$, and the gain function of the difference of the corresponding closed-loop operators tends to zero, then there are mappings between $\mathcal{G}_{\rm P}$ and $\mathcal{G}_{{\rm P}_i}$ which tend to the identity operator in the same sense. The converse idea is expressed in 2). Namely, if there are mappings from $\mathcal{G}_{\rm P}$ onto $\mathcal{G}_{{\rm P}_i}$ which tend to the identity in such a way that the gain function of the difference (of these mappings from the identity) acting on the gain function of the nominal closed-loop tends to zero, then $\boldsymbol{C}$ stabilizes the $\boldsymbol{P}_i$'s, and the closed-loop errors tend to zero, providing the nominal closed loop has a bounded incremental gain function.

Below we present two examples. The first example (Example 6) shows a gf-stable feedback system and, using Theorem 6, gives a bound on a possible time delay that can be tolerated in the feedback loop and for which gf-stability can be guaranteed for the perturbed system. The second example (Example 7) illustrates the usefulness of equation (26) to estimate closed-loop errors. This example does not impose the "small gain" condition (24), so (25) is not directly applicable.

Example 6 (A gf-Stable System with Cubic Nonlinearity): In the feedback configuration of Fig. 1, define $\boldsymbol{P}$ by

$$\begin{aligned}\dot{x}(t) &= -x(t)^3 + u_1(t), \qquad x(0)=0\\ y_1(t) &= x(t)\end{aligned} \tag{27}$$

and $\boldsymbol{C}$ by $u_2(t) = -y_2(t)$. The closed loop is then given by

$$\dot{x}(t) = -x(t)^3 - x(t) + v_0(t) \tag{28}$$

where $v_0 = u_0 + y_0$. We take $\mathcal{U} = \mathcal{Y} = \mathcal{L}_{\infty,e}$. We will first calculate the gain function of the parallel projection $g[\boldsymbol{\Pi}_{\mathcal{M}//\mathcal{N}}](\alpha)$ where $\boldsymbol{\Pi}_{\mathcal{M}//\mathcal{N}}$ is the mapping $\binom{u_0}{y_0} \mapsto \binom{v_0-x}{x}$.

Note that maximizing over the set $\|\binom{u_0}{y_0}\|_\infty \le \alpha$ is equivalent to maximizing over $\|v_0\|_\infty \le 2\alpha$. We first consider the mapping $v_0 \mapsto x$ defined by (28). Clearly

$$\sup_{\|v_0\|_\infty \le 2\alpha} \|x\|_\infty \le \inf\{|x|: x\dot{x} < 0 \text{ in (28) for all } |v_0| \le 2\alpha\} = f(2\alpha) \tag{29}$$

where $f(2\alpha)$ is the unique real root of the equation $x^3 + x = 2\alpha$. In fact, the bound in (29) is tight since $f(2\alpha)$ can be approached arbitrarily by choosing $v_0 = 2\alpha$. We next observe that

$$\sup_{\|v_0\|_\infty \le 2\alpha} \|v_0 - x\|_\infty \le 2\alpha + f(2\alpha)$$

which again is tight, since we can set $v_0 = 2\alpha$ on $[0, T)$ and $v_0(T) = -2\alpha$. We thus obtain

$$g[\boldsymbol{\Pi}_{\mathcal{M}//\mathcal{N}}](\alpha) = 2\alpha + f(2\alpha).$$

We now consider the effect of a small time delay, namely a perturbed plant $\boldsymbol{P}_1$

$$\begin{aligned}\dot{x}(t) &= -x(t)^3 + u_1(t), \qquad x(0) = 0\\ y_1(t) &= x(t-h).\end{aligned}$$

Consider a mapping $\boldsymbol{\Phi}$: $\mathcal{M} \to \mathcal{M}_1$ defined by $\boldsymbol{\Phi}\binom{u_1(t)}{x(t)} = \binom{u_1(t)}{x(t-h)}$. Note that, in (27), given any $u_1 \in \mathcal{L}_{\infty,e}$, $\|x\|_\tau \le \|u_1\|_\tau^{1/3}$ and so $\|\dot{x}\|_\tau \le \|x\|_\tau^3 + \|u_1\|_\tau \le 2\|u_1\|_\tau$. Since $|x(t) - x(t-h)| \le h\|\dot{x}\|_\tau$ for $t \le \tau$, we get

$$\begin{aligned} g[\boldsymbol{I} - \boldsymbol{\Phi}](\alpha) &= \sup\left\{\left\|(\boldsymbol{I} - \boldsymbol{\Phi})\begin{pmatrix}u_1\\x\end{pmatrix}\right\|_\tau : \left\|\begin{pmatrix}u_1\\x\end{pmatrix}\right\|_\tau \le \alpha\right\}\\ &\le \sup\left\{\left\|(\boldsymbol{I} - \boldsymbol{\Phi})\begin{pmatrix}u_1\\x\end{pmatrix}\right\|_\tau : \|u_1\|_\tau \le \alpha\right\}\\ &\le h\sup\{\|\dot{x}\|_\tau: \|u_1\|_\tau \le \alpha\} \le 2\alpha h. \end{aligned} \tag{30}$$

Note, for $\alpha \ge 1$, all the above inequalities hold with equality. Also

$$\begin{aligned} g[\boldsymbol{I} - \boldsymbol{\Phi}](\alpha) &= h\sup\left\{\|x\|_\tau^3 + \|u_1\|_\tau: \left\|\begin{pmatrix}u_1\\x\end{pmatrix}\right\|_\tau \le \alpha\right\}\\ &\le h(\alpha^3 + \alpha) \end{aligned} \tag{31}$$

which, with a choice of $u_1 = \alpha^3$ on $[0, T)$ and $u_1(T) = -\alpha$, can be approached arbitrarily closely.

Fig. 8 shows the gain function of the parallel projection, which we denote by $F(\alpha)$. Fig. 9 shows the upper bound on the distance to the identity for the map $\boldsymbol{\Phi}$ computed using (30) and (31), for $h = 0.1$. This upper bound we denote by $G(\alpha)$. In order to guarantee gf-stability for the perturbed system using Theorem 6, it is required that the composition of the two functions is bounded away from the identity map in the sense of the theorem. For this to be the case, because of the shape of the two functions, it turns out that we only need to check that $G(F(\alpha)) < 1$ for the value of α for which $F(\alpha) = 1$. This value is $\alpha_0 = 0.2733$. It follows that h should not exceed a maximal value of $\alpha_0/2 = 0.1366$. Fig. 10 shows, for $h = 0.12$, the composition function $G(F(\alpha))$ (solid line) compared to the line of slope 1 (dotted line). □

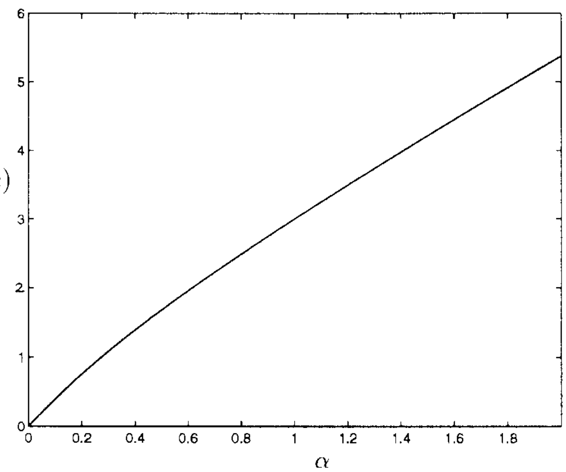

Fig. 8. $F(\alpha) := g[\boldsymbol{\Pi}_{\mathcal{M}//\mathcal{N}}](\alpha)$ versus α.

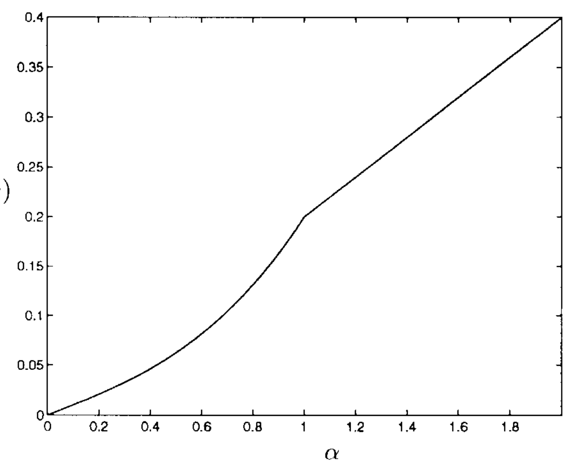

Fig. 9. $G(\alpha) := g[\boldsymbol{I} - \boldsymbol{\Phi}](\alpha)$ versus α for $h = 0.1$.

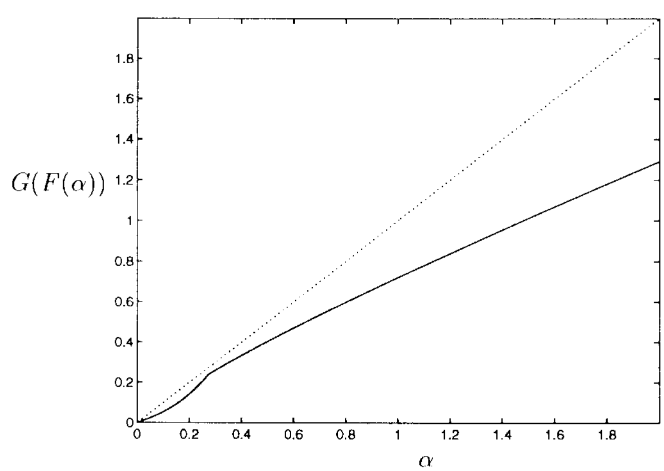

Fig. 10. $G(F(\alpha))$ (solid line) versus α for $h = 0.12$.

Example 7 (A System with Hysteresis): We consider perturbations of hysteresis type on a feedback loop as shown in Fig. 11. The hysteresis is modeled by the (simple) *hysteron*, $\boldsymbol{\Upsilon}_\epsilon$, shown in Fig. 12, as defined in [17], also called the ordinary play. Its behavior on continuous functions can be physically realized using the piston and cylinder arrangement

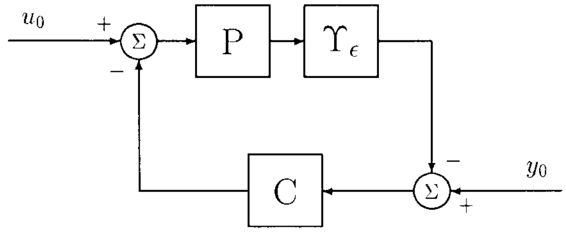

Fig. 11. Feedback system with hysteresis perturbation.

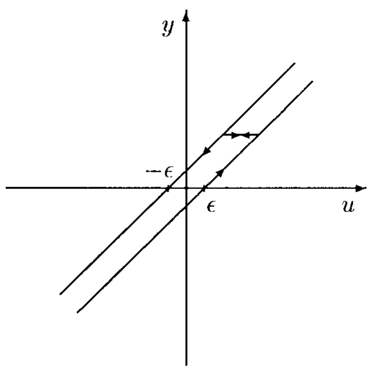

Fig. 12. Hysteron.

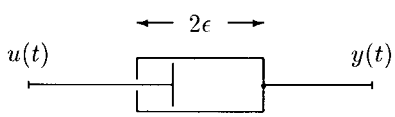

Fig. 13. Piston and cylinder.

of Fig. 13. In case the piston is at the extreme right-hand position and $u(t)$ is increasing, or at the extreme left-hand position and $u(t)$ is decreasing, then $y(t) - u(t)$ remains constant. Otherwise, $y(t)$ remains constant. It is typical to initialize $u(t)$ and $y(t)$ to zero at $t = 0$ and the piston to the midpoint position. The definition of $\boldsymbol{\Upsilon}_\epsilon$ on continuous, piecewise monotone inputs $u(t)$ is illustrated in Fig. 12.

We consider the graphs of $\boldsymbol{P}, \boldsymbol{P}_1$ to be

$$\mathcal{M} = \begin{pmatrix} \boldsymbol{I} \\ \boldsymbol{P} \end{pmatrix} \mathcal{C}_e^\infty[0, \infty), \qquad \mathcal{M}_1 = \begin{pmatrix} \boldsymbol{I} \\ \boldsymbol{\Upsilon}_\epsilon \boldsymbol{P} \end{pmatrix} \mathcal{C}_e^\infty[0, \infty).$$

Observe that there is a natural bijective mapping $\boldsymbol{\Phi}_\epsilon$ from $\mathcal{M}$ to $\mathcal{M}_1$ which is the identity on the input component and satisfies

$$\|(\boldsymbol{I} - \boldsymbol{\Phi}_\epsilon)x\|_\infty = \|(\boldsymbol{I} - \boldsymbol{\Upsilon}_\epsilon)\boldsymbol{P}x\|_\infty \le \epsilon$$

for any $x \in \mathcal{M}$. From (26) we get

$$\|(\boldsymbol{\Pi}_{\mathcal{M}//\mathcal{N}} - \boldsymbol{\Pi}_{\mathcal{M}_1//\mathcal{N}})w\|_\infty \le \epsilon(1 + \|\boldsymbol{\Pi}_{\mathcal{M}//\mathcal{N}}\|_\Delta)$$

since $w = (\boldsymbol{I} + (\boldsymbol{\Phi}_\epsilon - \boldsymbol{I})\boldsymbol{\Pi}_{\mathcal{M}//\mathcal{N}})x$ implies $\|w - x\|_\infty < \epsilon$. We remark that this bound can also be obtained by modeling the hysteresis nonlinearity by the identity map plus an external source for a disturbance of magnitude no larger than ϵ. □

VII. Examples of Nonrobust Controllers

The examples of this section have zero robustness margin, and the feedback loops are destabilized by arbitrarily small gap perturbations of the plant.

Example 8 (Nussbaum Universal Controller): In the feedback loop of Fig. 1 define $\boldsymbol{P}$ by

$$\begin{aligned} \dot{x}(t) &= ax(t) + bu_1(t) \\ y_1(t) &= x(t) \end{aligned}$$

where $b \neq 0$, but otherwise a and b are unknown. Let $\boldsymbol{C}$ be defined by [1, p. 291], [24]

$$\begin{aligned} \dot{\theta}(t) &= y_2^2(t) \\ u_2(t) &= y_2(t)\theta^2(t)\cos\theta(t). \end{aligned}$$

In the case where $u_0 \equiv 0$ and $y_0 \equiv 0$ (so $u_1 \equiv -u_2$ and $y_1 \equiv -y_2$), the controller regulates the output $y_1(t)$ to zero asymptotically for an arbitrary initial condition $x(0) = x_0$. However, we will show that, if $u_0 \neq 0$ and $|u_0| \le \epsilon$ for an arbitrarily small $\epsilon > 0$, then $x(t)$ can become unbounded with $x(0) = 0$. Thus, any form of the robustness margin as considered in this paper should be assigned the value zero. We will further show by construction that an arbitrarily small gap perturbation of the plant from the assumed model class can lead to instability (even of the autonomous system).

We assume that $\mathcal{U} = \mathcal{Y} = \mathcal{L}_\infty$ and take $x(0) = \theta(0) = 0$. Consider a nominal plant with $a = 0, b = 1$, and let $u_0 \equiv \epsilon > 0$, $y_0 \equiv 0$. The feedback equations reduce to

$$\dot{x} = x\theta^2\cos\theta + \epsilon \tag{32}$$

$$\dot{\theta} = x^2. \tag{33}$$

First, we claim that as $t \to \infty$, $\theta(t) \to \infty$. To see this assume the contrary, i.e., that $\theta(t)$ remains bounded. Since $\dot{\theta} \ge 0$ [from (33)], $\theta(t) \to \theta_0 < \infty$. Then (33) implies that $x(t) \to 0$. At the same time (32) implies that $\dot{x}(t) \to \epsilon \neq 0$. This is a contradiction. Therefore, $\theta(t) \to \infty$. Second, we claim that $x(t)$ grows unbounded as well. It is easy to see that the gain $\theta^2\cos\theta$ in (32) exceeds any value. For instance, in the interval $[\theta(t_1), \theta(t_2)]$, where $\theta(t_1) = 2k\pi - (\pi/4)$, $\theta(t_2) = 2k\pi + (\pi/4)$, and t_1, t_2 are chosen accordingly, the feedback gain is

$$\theta^2(t)\cos\theta(t) \ge M_k := \frac{\sqrt{2}}{2}\left(2k\pi - \frac{\pi}{4}\right)^2.$$

It is clear that M_k exceeds any value for a suitable choice of the integer $k > 0$. Over such an interval

$$\frac{dx}{d\theta} = \frac{x\theta^2\cos\theta + \epsilon}{x^2} \ge \frac{M_k}{x}.$$

Integration over $[t_1, t_2]$ gives

$$\begin{aligned} |x(t_2)| &\ge \sqrt{2M_k(\theta(t_2) - \theta(t_1)) + x^2(t_1)} \\ &\ge \sqrt{2M_k(\theta(t_2) - \theta(t_1))} = \sqrt{M_k\pi}. \end{aligned}$$

Thus, $|x(t)|$ exceeds any bound as claimed. We conclude that the relevant parallel projection operator is unbounded over any bounded set, no matter how small.

We now consider a perturbation of the nominal plant by introducing a first-order lag with transfer function $M/(s + M)$ $(M > 1)$ in series with the plant. We claim that this perturbation is arbitrarily small in the gap for M sufficiently large. This can be seen as follows. The graphs of $\boldsymbol{P}$ and $\boldsymbol{P}_1$ are

$$\mathcal{G}_{\mathrm{P}} = \begin{pmatrix} \dfrac{s}{s+1} \\ \dfrac{1}{s+1} \end{pmatrix} \mathcal{L}_\infty =: G\mathcal{L}_\infty$$

$$\mathcal{G}_{\mathrm{P}_1} = \begin{pmatrix} \dfrac{s}{s+1} \\ \dfrac{M}{(s+1)(s+M)} \end{pmatrix} \mathcal{L}_\infty.$$

Consider a mapping $\boldsymbol{\Phi}$ from $\mathcal{G}_{\mathrm{P}}$ onto $\mathcal{G}_{\mathrm{P}_1}$, which is the identity on the input component. Then, for $v \in \mathcal{L}_\infty$

$$(\boldsymbol{I} - \boldsymbol{\Phi}) \begin{pmatrix} \dfrac{s}{s+1} \\ \dfrac{1}{s+1} \end{pmatrix} v = \begin{pmatrix} 0 \\ \dfrac{s}{(s+1)(s+M)} \end{pmatrix} v.$$

Since (1, 1) is a left inverse of G, we see that $\|\boldsymbol{T}_\tau v\|_\infty \le 2\|\boldsymbol{T}_\tau G v\|_\infty$. Thus

$$\begin{aligned} \frac{\left\| \boldsymbol{T}_\tau \dfrac{s}{(s+1)(s+M)} v \right\|_\infty}{\|\boldsymbol{T}_\tau G v\|_\infty} &\le \frac{2\left\| \boldsymbol{T}_\tau \dfrac{s}{(s+1)(s+M)} v \right\|_\infty}{\|\boldsymbol{T}_\tau v\|_\infty} \\ &\le \frac{2}{M-1} \|Me^{-Mt} - e^{-t}\|_{\mathcal{L}_1} \\ &\le \frac{4}{M-1} \end{aligned}$$

for any $v \in \mathcal{L}_\infty$. Thus $\|(\boldsymbol{I} - \boldsymbol{\Phi})|_{\mathcal{G}_{\mathrm{P}}}\| \le 4/(M-1)$ which proves the assertion.

We now examine the behavior of the perturbed closed-loop system. We continue to assume the nominal parameters $a = 0, b = 1$. The autonomous feedback system evolves according to

$$\frac{dx}{dt} = y\theta^2 \cos\theta \tag{34}$$

$$\frac{dy}{dt} = M(x-y) \tag{35}$$

$$\frac{d\theta}{dt} = y^2. \tag{36}$$

A typical response of the system is shown in Fig. 14, where $x(0) = y(0) = 10, \theta(0) = 0$, and $M = 10$. A similar diverging solution can be observed as M is varied.

To better understand the form of the solution we can view x and y as functions of θ and eliminate t from (34)–(36) to obtain

$$\frac{dx}{d\theta} = \frac{\theta^2 \cos\theta}{y} \tag{37}$$

$$\frac{dy}{d\theta} = \frac{M(x-y)}{y^2}. \tag{38}$$

The numerical solution of x and y versus θ, for $x(0) = y(0) = 10$ and $M = 10$, is shown in Fig. 15. Numerical analysis of the data suggests dominant terms in x of $\theta^{5/4}\sin\theta$ and in y of $\theta^{3/4}$. Further numerical analysis suggests the following asymptotic expansion for $x(\theta)$ and $y(\theta)$:

$$x(\theta) = \theta^{5/4}(a_1 \sin(\theta) + a_2) + \theta^{3/4}(a_6 \sin(\theta) + a_7) + \cdots \tag{39}$$

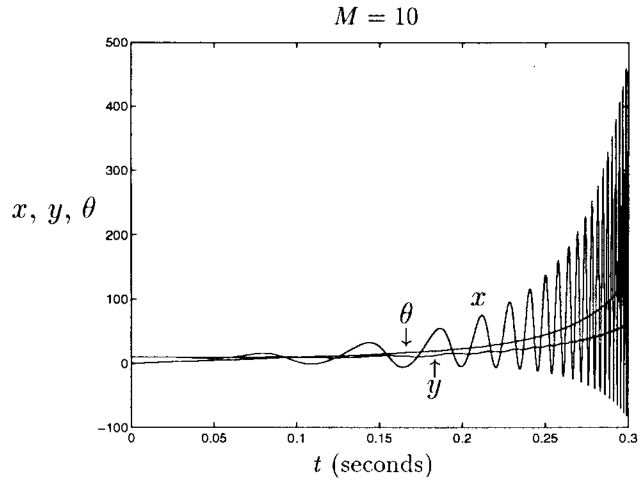

Fig. 14. Universal controller with perturbed plant.

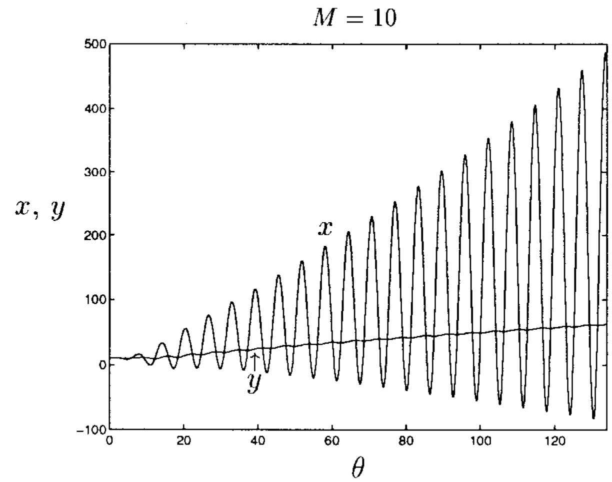

Fig. 15. Universal controller with perturbed plant.

$$y(\theta) = \theta^{3/4} a_3 + \theta^{1/4} a_5 + \theta^{-1/4}(a_4 \cos(\theta) + a_8) + \cdots \tag{40}$$

where the fractional powers of θ on the right-hand side (RHS) decrease by one-half between subsequent terms of the series. Similar series are common in the solution of certain types of linear time-varying differential equations (e.g., see [6]). This is an instance of a nonlinear differential equation whose solution appears to have a similar asymptotic expansion. We have no formal proof that the solution to (37) and (38) can be represented by (39) and (40). However, the following considerations support this hypothesis. Substitution of (39) and (40) into (37) and (38) gives the following algebraic equations for the coefficients a_1 to a_4 of the dominant powers of θ in the drift and the periodic components:

$$a_1 = \left(\frac{15}{8M^2}\right)^{1/8}, \qquad a_2 = \frac{3}{4Ma_1^3}$$

$$a_3 = \frac{1}{a_1}, \qquad a_4 = -Ma_1^3.$$

Interestingly enough, these values are independent of initial conditions and agree with the results of numerical integration using a variety of initial conditions. The form of the series justifies the observed growth of x and y and moreover suggests, since $\dot{\theta} = y^2(t) = O(\theta^{6/4})$, that the solution of (34)–(36) diverges in finite time. The prediction for finite-time escape appears again to be corroborated by the simulations. We remark that the independence of the leading terms from initial conditions suggests that this type of behavior (which occurs also in Example 9) indicates the presence of some kind of "explosive attractor." □

Example 9 (A Parameter Adaptive Controller): In the feedback configuration of Fig. 1 define $\boldsymbol{P}$ by

$$\begin{aligned}\dot{x}(t) &= ax(t) + bu_1(t)\\ y_1(t) &= x(t)\end{aligned}$$

where $b \neq 0$ and $\operatorname{sign}(b)$ is known, but otherwise a and b are unknown. Let $\boldsymbol{C}$ be defined by

$$\begin{aligned}\dot{\theta}(t) &= \gamma y_2(t)^2\\ u_2(t) &= -\theta(t)y_2(t)\end{aligned}$$

where γ is a constant chosen so that $\gamma b > 0$. In the case where $u_0 \equiv 0$ and $y_0 \equiv 0$ (so $u_1 \equiv -u_2$ and $y_1 \equiv -y_2$), the system is globally stable, since the Lyapunov function $V(x, \theta) := x^2 + \alpha\ (\theta - \hat{\theta})^2$ has $\dot{V} \equiv 0$ for $\alpha = b/\gamma$ and $\hat{\theta} = a/b$. This controller can be obtained from a model reference adaptive scheme (e.g., see, [1, p. 127]) with a zero reference signal.

Now consider a nominal plant with $a = 0, b = 1$, and take $\gamma = 1$. We assume that $\mathcal{U} = \mathcal{Y} = \mathcal{L}_\infty$ and take $x(0) = \theta(0) = 0$. If $u_0 \equiv \epsilon > 0$ and $y_0 \equiv 0$, the feedback equations reduce to

$$\begin{aligned}\dot{x} &= \epsilon - \theta x\\ \dot{\theta} &= x^2.\end{aligned}$$

The same reasoning as Example 8 shows that $\theta(t) \to \infty$ as $t \to \infty$. It can also be seen that $x(t) \to 0$ as $t \to \infty$. (To see this, notice that $x(t) > 0$ for $t > 0$ since $x(t) = 0 \Rightarrow \dot{x}(t) = \epsilon > 0$. If $x(t) < \epsilon/\theta(t)$), then $x(t)$ is increasing. But $\epsilon/\theta(t)$ is decreasing to zero, so eventually there will be a t_0 such that $x(t_0) = \epsilon/\theta(t_0)$ and thereafter $\dot{x}(t)$ remains negative. Now if $x(t) \geq \eta > 0$ for sufficiently large t, then eventually $\dot{x}(t)$ is less than any negative number, which is a contradiction.) Then, the external disturbances $u_0 \equiv \epsilon$ on $[0, T]$, $y_0 \equiv 0$ on $[0, T)$ and $y_0(T) = \epsilon$ give $u_1(T) = \epsilon + \theta(T)(\epsilon - x(T))$ which can be made arbitrarily large. We conclude that the relevant parallel projection operator is unbounded over any bounded set, no matter how small. Thus, any form of the robustness margin as considered in this paper should be assigned the value zero.

We now consider a small gap perturbation of the nominal plant, namely we introduce an all-pass factor $(M-s)/(M+s)$ in series with the plant. It can be shown as in Example 8 that the gap tends to zero as $M \to \infty$. The autonomous feedback system with perturbed plant now evolves according to

$$\begin{aligned}\dot{y} &= z + \theta y\\ \dot{z} &= -M(z + 2\theta y)\\ \dot{\theta} &= y^2.\end{aligned}$$

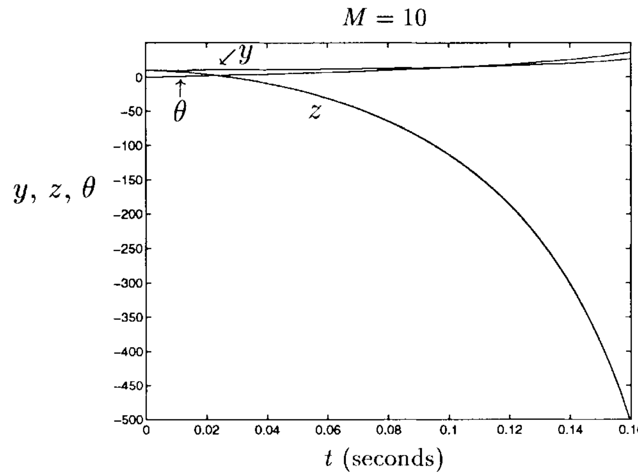

Fig. 16. Parameter adaptive controller with perturbed plant.

A typical response of the system is shown in Fig. 16 where $y(0) = z(0) = 10$, $\theta(0) = 0$, and $M = 10$.

Again we eliminate t to obtain

$$\frac{dy}{d\theta} = \frac{z + \theta y}{y^2} \tag{41}$$

$$\frac{dz}{d\theta} = -\frac{M(z + 2\theta y)}{y^2}. \tag{42}$$

The diverging solution appears to be governed by a series expansion with leading terms

$$\begin{aligned}y(\theta) &= a_1\theta + a_0 + \theta^{-1}(a_{-1,2}(\log\ \theta)^2\\ &\quad + a_{-1,1}\ \log\ \theta + a_{-1,0})\cdots\\ z(\theta) &= b_1\theta + b\ \log\ \theta + b_0 + \cdots.\end{aligned}$$

Substitution of the series into (41) and (42) gives

$$\begin{aligned}a_1^2 &= 1, \qquad a_0 = b_1 = -2Ma_1\\ a_{-1,2} &= -M^2 a_1, \qquad b = -2M^2 a_1\end{aligned}$$

which are independent of initial condition apart from the sign. These values agree with the results of simulation with a variety of initial conditions. Since $\dot{\theta} = y^2 = O(\theta^2)$, the finite time escape is expected (and corroborated by simulation). We remark that the infinite gain of certain closed-loop operators, and the consequent lack of robustness, of parameter adaptive controllers was first pointed out in [25]. □

VIII. Combined Plant and Controller Uncertainty

In this section we discuss how the results of Section III extend to the case where uncertainty occurs in both $\boldsymbol{P}$ and $\boldsymbol{C}$.

Let $\boldsymbol{P}$ and $\boldsymbol{P}_1$ be causal operators from $\mathcal{U} \to \mathcal{Y}$, and let $\boldsymbol{C}$ and $\boldsymbol{C}_1$ be causal operators from $\mathcal{Y} \to \mathcal{U}$. Denote their respective graphs by $\mathcal{M} := \mathcal{G}_{\mathrm{P}}, \mathcal{M}_1 := \mathcal{G}_{\mathrm{P}_1}, \mathcal{N} := \mathcal{G}_{\mathrm{C}}$, and $\mathcal{N}_1 := \mathcal{G}_{\mathrm{C}_1}$.

Theorem 8: Let $\boldsymbol{H}_{\mathrm{P,C}}$ be stable. If

$$\vec{\delta}(\mathcal{M}, \mathcal{M}_1)\|\boldsymbol{\Pi}_{\mathcal{M}//\mathcal{N}}\| + \vec{\delta}(\mathcal{N}, \mathcal{N}_1)\|\boldsymbol{\Pi}_{\mathcal{N}//\mathcal{M}}\| < 1 \tag{43}$$

then $\boldsymbol{H}_{\mathrm{P}_1,\mathrm{C}_1}$ is stable and

$$\|\boldsymbol{\Pi}_{\mathcal{M}_1//\mathcal{N}_1}\| \leq \frac{\|\boldsymbol{\Pi}_{\mathcal{M}//\mathcal{N}}\|(1+\vec{\delta}(\mathcal{M},\mathcal{M}_1))}{1-\vec{\delta}(\mathcal{M},\mathcal{M}_1)\|\boldsymbol{\Pi}_{\mathcal{M}//\mathcal{N}}\|-\vec{\delta}(\mathcal{N},\mathcal{N}_1)\|\boldsymbol{\Pi}_{\mathcal{N}//\mathcal{M}}\|}.$$

An analogous bound holds for $\boldsymbol{\Pi}_{\mathcal{N}_1//\mathcal{M}_1}$.

Proof: It is sufficient to assume from (43) the weaker requirement that there exist mappings $\boldsymbol{\Phi}_{\mathcal{M}}$ from a subset $\mathcal{D}_{\mathcal{M}} \subseteq \mathcal{M}$ onto $\mathcal{M}_1$ and $\boldsymbol{\Phi}_{\mathcal{N}}$ from a subset $\mathcal{D}_{\mathcal{N}} \subseteq \mathcal{N}$ onto $\mathcal{N}_1$, such that

$$\begin{aligned}\|(\boldsymbol{\Phi}_{\mathcal{M}}-\boldsymbol{I})\boldsymbol{\Pi}_{\mathcal{M}//\mathcal{N}}+(\boldsymbol{\Phi}_{\mathcal{N}}-\boldsymbol{I})\boldsymbol{\Pi}_{\mathcal{N}//\mathcal{M}}\| &\leq \|\boldsymbol{\Phi}_{\mathcal{M}}-\boldsymbol{I}\|\cdot\|\boldsymbol{\Pi}_{\mathcal{M}//\mathcal{N}}\| \\ &\quad + \|\boldsymbol{\Phi}_{\mathcal{N}}-\boldsymbol{I}\|\cdot\|\boldsymbol{\Pi}_{\mathcal{N}//\mathcal{M}}\| < 1.\end{aligned}$$

Consider the equation

$$\begin{aligned}w &= (\boldsymbol{I}+(\boldsymbol{\Phi}_{\mathcal{M}}-\boldsymbol{I})\boldsymbol{\Pi}_{\mathcal{M}//\mathcal{N}}+(\boldsymbol{\Phi}_{\mathcal{N}}-\boldsymbol{I})\boldsymbol{\Pi}_{\mathcal{N}//\mathcal{M}})x \\ &= (\boldsymbol{\Phi}_{\mathcal{M}}\boldsymbol{\Pi}_{\mathcal{M}//\mathcal{N}}+\boldsymbol{\Phi}_{\mathcal{N}}\boldsymbol{\Pi}_{\mathcal{N}//\mathcal{M}})x.\end{aligned}$$

Since $\boldsymbol{\Phi}_{\mathcal{M}}$ (respectively, $\boldsymbol{\Phi}_{\mathcal{N}}$) is surjective onto $\mathcal{M}_1$ (respectively, $\mathcal{N}_1$), for any $w \in \mathcal{W}$ we can find a solution $x \in \mathcal{W}$. In particular, if $m_1 = \boldsymbol{\Pi}_{\mathcal{M}_1//\mathcal{N}_1}w$ and $n_1 = \boldsymbol{\Pi}_{\mathcal{N}_1//\mathcal{M}_1}w$ then $x = m + n$ where $m \in \mathcal{D}_{\mathcal{M}}, n \in \mathcal{D}_{\mathcal{N}}$ are chosen so that $m_1 = \boldsymbol{\Phi}_{\mathcal{M}}m$ and $n_1 = \boldsymbol{\Phi}_{\mathcal{N}}n$. It follows that

$$\|x\|_\tau \leq \frac{\|w\|_\tau}{1-\|\boldsymbol{\Phi}_{\mathcal{M}}-\boldsymbol{I}\|\cdot\|\boldsymbol{\Pi}_{\mathcal{M}//\mathcal{N}}\|-\|\boldsymbol{\Phi}_{\mathcal{N}}-\boldsymbol{I}\|\cdot\|\boldsymbol{\Pi}_{\mathcal{N}//\mathcal{M}}\|}.$$

Finally, since $\boldsymbol{\Pi}_{\mathcal{M}_1//\mathcal{N}_1}w = \boldsymbol{\Phi}_{\mathcal{M}}\boldsymbol{\Pi}_{\mathcal{M}//\mathcal{N}}x$, the required bound, and hence the stability of $\boldsymbol{H}_{\mathrm{P}_1,\mathrm{C}_1}$, now follow. □

Theorem 9: Let $\boldsymbol{H}_{\mathrm{P},\mathrm{C}}$ be stable. The following hold.

1) Let $\boldsymbol{H}_{\mathrm{P}_1,\mathrm{C}_1}$ be stable. Then there exist causal bijective mappings $\boldsymbol{\Phi}_{\mathcal{M}}$ from $\mathcal{M}$ onto $\mathcal{M}_1$ and $\boldsymbol{\Phi}_{\mathcal{N}}$ from $\mathcal{N}$ onto $\mathcal{N}_1$ such that

$$\begin{aligned}\|\boldsymbol{\Phi}_{\mathcal{M}}-\boldsymbol{I}\| &\leq \|\boldsymbol{\Pi}_{\mathcal{M}//\mathcal{N}}-\boldsymbol{\Pi}_{\mathcal{M}_1//\mathcal{N}_1}\| \\ \|\boldsymbol{\Phi}_{\mathcal{N}}-\boldsymbol{I}\| &\leq \|\boldsymbol{\Pi}_{\mathcal{N}//\mathcal{M}}-\boldsymbol{\Pi}_{\mathcal{N}_1//\mathcal{M}_1}\|.\end{aligned}$$

2) Suppose that there exist mappings $\boldsymbol{\Phi}_{\mathcal{M}}$ from a subset $\mathcal{D}_{\mathcal{M}} \subseteq \mathcal{M}$ onto $\mathcal{M}_1$ and $\boldsymbol{\Phi}_{\mathcal{N}}$ from a subset $\mathcal{D}_{\mathcal{N}} \subseteq \mathcal{N}$ onto $\mathcal{N}_1$ such that $\alpha < 1$ where

$$\begin{aligned}\alpha := \|(\boldsymbol{\Phi}_{\mathcal{M}}-\boldsymbol{I})|_{\mathcal{D}_{\mathcal{M}}}\|\cdot\|\boldsymbol{\Pi}_{\mathcal{M}//\mathcal{N}}\| \\ + \|(\boldsymbol{\Phi}_{\mathcal{N}}-\boldsymbol{I})|_{\mathcal{D}_{\mathcal{N}}}\|\cdot\|\boldsymbol{\Pi}_{\mathcal{N}//\mathcal{M}}\|.\end{aligned}$$

Then

$$\begin{aligned}\|\boldsymbol{\Pi}_{\mathcal{M}//\mathcal{N}}-\boldsymbol{\Pi}_{\mathcal{M}_1//\mathcal{N}_1}\| &\leq \frac{1}{1-\alpha}\|(\boldsymbol{\Phi}_{\mathcal{M}}-\boldsymbol{I})|_{\mathcal{D}_{\mathcal{M}}}\|\cdot\|\boldsymbol{\Pi}_{\mathcal{M}//\mathcal{N}}\| \\ &\quad + \frac{\alpha}{1-\alpha}\|\boldsymbol{\Pi}_{\mathcal{M}//\mathcal{N}}\|_\Delta.\end{aligned}$$

Proof:

1) We define $\boldsymbol{\Phi}_{\mathcal{M}} = \boldsymbol{I} - \boldsymbol{\Pi}_{\mathcal{M}//\mathcal{N}} + \boldsymbol{\Pi}_{\mathcal{M}_1//\mathcal{N}_1}$. As in the proof of Theorem 2-1) $\boldsymbol{\Phi}_{\mathcal{M}}$ maps $\mathcal{M}$ into $\mathcal{M}_1$ and $\boldsymbol{\Phi}_{\mathcal{M}}|_{\mathcal{M}} = \boldsymbol{\Pi}_{\mathcal{M}_1//\mathcal{N}_1}|_{\mathcal{M}}$. Moreover, $\boldsymbol{\Sigma}_{\mathcal{M},-\mathcal{N}_1}$ has a well-defined (not necessarily bounded) inverse by the assumption of well-posedness and so $\boldsymbol{\Pi}_{\mathcal{M}//-\mathcal{N}_1}\boldsymbol{\Pi}_{\mathcal{M}_1//\mathcal{N}_1}|_{\mathcal{M}} = \boldsymbol{I}_{\mathcal{M}}$ and $\boldsymbol{\Pi}_{\mathcal{M}_1//\mathcal{N}_1}\boldsymbol{\Pi}_{\mathcal{M}//-\mathcal{N}_1}|_{\mathcal{M}_1} = \boldsymbol{I}_{\mathcal{M}_1}$ [cf. proof of Theorem 2-1)]. Hence $\boldsymbol{\Phi}_{\mathcal{M}}$ maps $\mathcal{M}$ bijectively onto $\mathcal{M}_1$. Similarly, we can define $\boldsymbol{\Phi}_{\mathcal{N}} = \boldsymbol{I} - \boldsymbol{\Pi}_{\mathcal{N}//\mathcal{M}} + \boldsymbol{\Pi}_{\mathcal{N}_1//\mathcal{M}_1}$ and show it is a bijection from $\mathcal{N}$ to $\mathcal{N}_1$.

2) For any $w \in \mathcal{W}$ we can find an $x \in \mathcal{W}$ such that $(\boldsymbol{I}+(\boldsymbol{\Phi}_{\mathcal{M}}-\boldsymbol{I})\boldsymbol{\Pi}_{\mathcal{M}//\mathcal{N}}+(\boldsymbol{\Phi}_{\mathcal{N}}-\boldsymbol{I})\boldsymbol{\Pi}_{\mathcal{N}//\mathcal{M}})x = w$ as in the proof of Theorem 8. The identity

$$\begin{aligned}(\boldsymbol{\Pi}_{\mathcal{M}//\mathcal{N}}-\boldsymbol{\Pi}_{\mathcal{M}_1//\mathcal{N}_1})w \\ = (\boldsymbol{I}-\boldsymbol{\Phi}_{\mathcal{M}})\boldsymbol{\Pi}_{\mathcal{M}//\mathcal{N}}x+\boldsymbol{\Pi}_{\mathcal{M}//\mathcal{N}}w-\boldsymbol{\Pi}_{\mathcal{M}//\mathcal{N}}x\end{aligned}$$

and Lemma 1 now give the required result. □

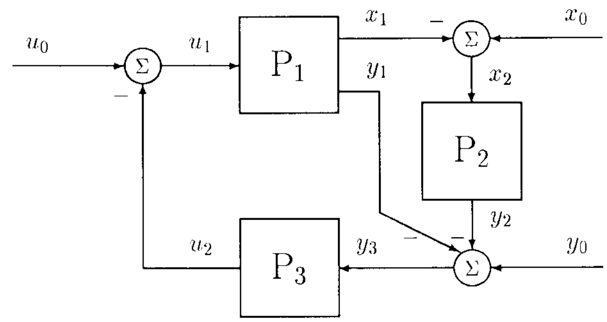

Fig. 17. Feedback interconnection with three subsystems.

IX. General Feedback Configurations

In this section we point out that the framework for robustness analysis, involving graphs and summation operators, applies equally well to more general feedback configurations. We will demonstrate this by way of example.

Consider an interconnection of three systems as shown in Fig. 17. We consider three signal spaces $\mathcal{U}, \mathcal{X}$, and $\mathcal{Y}$ where $u_i \in \mathcal{U}$ $(i = 0,1,2,3)$, etc., and denote $\mathcal{W} = \mathcal{U} \times \mathcal{X} \times \mathcal{Y}$. We denote by $\boldsymbol{\Pi}_i$ $(i = 1,2,3)$ the natural projection from $\mathcal{W}$ onto $\mathcal{U}, \mathcal{X}$, and $\mathcal{Y}$, respectively. We embed the graphs of $\boldsymbol{P}_i$ in $\mathcal{W}$ as follows:

$$\mathcal{G}_{\mathrm{P}_1} = \begin{pmatrix} u_1 \\ \boldsymbol{P}_1 u_1 \end{pmatrix} = \begin{pmatrix} u_1 \\ x_1 \\ y_1 \end{pmatrix}$$

$$\mathcal{G}_{\mathrm{P}_2} = \begin{pmatrix} 0 \\ x_2 \\ \boldsymbol{P}_2 x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ x_2 \\ y_2 \end{pmatrix}$$

$$\mathcal{G}_{\mathrm{P}_3} = \begin{pmatrix} \boldsymbol{P}_3 y_3 \\ 0 \\ y_3 \end{pmatrix} = \begin{pmatrix} u_2 \\ 0 \\ y_3 \end{pmatrix}.$$

Write $\mathcal{M}_i := \mathcal{G}_{\mathrm{P}_i}$ $(i = 1,2,3)$ and define the summation operator

$$\begin{aligned}\boldsymbol{\Sigma}_{\mathcal{M}_1,\mathcal{M}_2,\mathcal{M}_3} &: \mathcal{M}_1 \times \mathcal{M}_2 \times \mathcal{M}_3 \to \mathcal{W} \\ &: (m_1, m_2, m_3) \mapsto m_1 + m_2 + m_3.\end{aligned}$$

We will assume well-posedness of the feedback configuration, which means that $\boldsymbol{\Sigma}_{\mathcal{M}_1,\mathcal{M}_2,\mathcal{M}_3}$ has a well-defined inverse on the whole of $\mathcal{W}$. Stability of the feedback loop requires that $\boldsymbol{\Sigma}^{-1}_{\mathcal{M}_1,\mathcal{M}_2,\mathcal{M}_3}$ is stable.

The mappings

$$\boldsymbol{\Pi}_{(i)} := \boldsymbol{\Pi}_i \boldsymbol{\Sigma}^{-1}_{\mathcal{M}_1,\mathcal{M}_2,\mathcal{M}_3}, \qquad (i = 1,2,3)$$

from the external disturbances onto the graphs of the components, are generalizations of the notion of nonlinear parallel projection to the case where more than two manifolds (in this case *three*) "coordinatize" the space $\mathcal{W}$. It follows easily that

$$\begin{aligned} \boldsymbol{\Pi}_{(1)} + \boldsymbol{\Pi}_{(2)} + \boldsymbol{\Pi}_{(3)} &= \boldsymbol{I} \\ \boldsymbol{\Pi}_{(i)}\boldsymbol{\Pi}_{(i)} &= \boldsymbol{\Pi}_{(i)}, \qquad \text{for } i = 1,2,3 \\ \boldsymbol{\Pi}_{(i)}\boldsymbol{\Pi}_{(j)} &= 0, \qquad \text{for } i \neq j. \end{aligned}$$

In fact, the generalization of (1) which defines such a set of projections is

$$\boldsymbol{\Pi}_{(i)}(\boldsymbol{\Pi}_{(1)}w_1 + \boldsymbol{\Pi}_{(2)}w_2 + \boldsymbol{\Pi}_{(3)}w_3) = \boldsymbol{\Pi}_{(i)}w_i$$

for any $w_1, w_2, w_3 \in \mathcal{W}$.

We now consider perturbed systems $\boldsymbol{P}_1', \boldsymbol{P}_2', \boldsymbol{P}_3'$, acting on the appropriate spaces, with graphs $\mathcal{M}_i' := \mathcal{G}_{\mathrm{P}_i'}$. Accordingly, we define

$$\boldsymbol{\Pi}_{(i)}' := \boldsymbol{\Pi}_i \boldsymbol{\Sigma}^{-1}_{\mathcal{M}_1', \mathcal{M}_2', \mathcal{M}_3'}, \qquad (i = 1,2,3).$$

Theorem 10: Let $\boldsymbol{\Sigma}^{-1}_{\mathcal{M}_1, \mathcal{M}_2, \mathcal{M}_3}$ be stable. If

$$\alpha := \sum_{i=1}^{3} \vec{\delta}(\mathcal{M}_i, \mathcal{M}_i') \|\boldsymbol{\Pi}_{(i)}\| < 1$$

then $\boldsymbol{\Sigma}^{-1}_{\mathcal{M}_1', \mathcal{M}_2', \mathcal{M}_3'}$ is stable and

$$\|\boldsymbol{\Pi}_{(i)}'\| \leq \|\boldsymbol{\Pi}_{(i)}\| \frac{1 + \vec{\delta}(\mathcal{M}_i, \mathcal{M}_i')}{1 - \alpha}.$$

Proof: As in the proof of Theorem 8, it is sufficient to assume the weaker condition that there exist mappings $\boldsymbol{\Phi}_i$ from $\mathcal{D}_i \subseteq \mathcal{M}_i$ onto $\mathcal{M}_i'$ such that

$$\left\| \sum_{i=1}^{3} (\boldsymbol{\Phi}_i - \boldsymbol{I}) \boldsymbol{\Pi}_{(i)} \right\| \leq \sum_{i=1}^{3} \|(\boldsymbol{\Phi}_i - \boldsymbol{I})|_{\mathcal{D}_i}\| \cdot \|\boldsymbol{\Pi}_{(i)}\| < 1.$$

Consider the equation

$$\begin{aligned} w &= \left(\boldsymbol{I} + \sum_{i=1}^{3} (\boldsymbol{\Phi}_i - \boldsymbol{I}) \boldsymbol{\Pi}_{(i)} \right) x \\ &= \left(\sum_{i=1}^{3} \boldsymbol{\Phi}_i \boldsymbol{\Pi}_{(i)} \right) x. \end{aligned}$$

As before, because of the well-posedness assumption of the perturbed system and the surjectivity of the maps $\boldsymbol{\Phi}_i$ $(i = 1,2,3)$, there exists a solution $x \in \mathcal{W}$ of the above equation for any $w \in \mathcal{W}$ (cf. proof of Theorem 8). In fact, the solution is such that $\boldsymbol{\Pi}_{(i)}' w = \boldsymbol{\Phi}_i \boldsymbol{\Pi}_{(i)} x$ for each i. Bounds on each of the closed-loop operator norms $\|\boldsymbol{\Pi}_{(i)}'\|$ $(i = 1,2,3)$ now follow immediately. $\square$

In conclusion, we would like to point out that the main elements of the framework when applied to more general situations (i.e., more than three plants and arbitrary interconnections) are: 1) to introduce additive disturbance signals at each interconnection point which belong to suitable signal spaces; 2) to embed the graph of each operator in the cross product of these spaces; and (3) to consider the generalized parallel projections onto the system graphs. A useful sign convention is to arrange that each disturbance enters the summation junction with a positive sign and all other signals with a negative sign.

X. Concluding Remarks

This paper has developed an input–output framework for robustness analysis of nonlinear systems which is a generalization of the linear gap metric theory. The essence of the approach was shown to be adaptable to a variety of situations, e.g., global, local, possible finite-time escape. The initial insight for the approach was provided by an (abstract) geometric treatment of the linear gap-robustness theory [8], [9]. This work highlighted the role of the parallel projection operator for robustness, and in particular, that the inverse of its norm is the maximum aperture between plant and perturbed plant which can be tolerated if preservation of feedback stability is to be guaranteed. The existence and geometric significance of a nonlinear parallel projection operator, when the elements of the feedback system are nonlinear, was studied in [5]. Suitable generalizations of the gap and variants of the basic robustness result [12, Th. 5], [9, Th. 3] were presented in [11] and in [14], for a concept of differential stability and of incremental gain stability, respectively. The present work builds on [14], which contains two basic results of our theory (analogous to Theorems 1 and 2) for nonlinear systems on Banach spaces. In [14], the condition of incremental gain stability was used, which is quite restrictive in the context of nonlinear systems (e.g., see [3]). This observation, and the experience gained from treating specific examples, motivated the development presented here.

We remark that, in each version of the theory presented in this paper, the nonlinear gain of a parallel projection needs to be computed. The examples presented here were chosen to be tractable with hand calculations. In general, appropriate computational tools are required (see [32], [16], [26], and [7] for some examples of recent work in this area). The problem of computing the nonlinear gap distance measures introduced in this paper is also a topic which requires further investigation. In most cases we were content to define a $\boldsymbol{\Phi}$ which was the identity mapping when restricted to the system's input component. This is usually quite conservative.

Finally, we point out that one of the most useful aspects of the linear gap theory has been in the area of controller synthesis. In particular, minimization of the $\mathcal{H}_\infty$-norm of the parallel projection, suitably weighted, is the basis of the $\mathcal{H}_\infty$ loop-shaping method [21], [22]. The results of this paper indicate that a generalization of this design method can be built around a nonlinear gap robustness theory in an analogous way to the linear case. This consideration highlights the need to find tractable methods to optimize system gains for nonlinear systems.

Appendix
Connection of $\delta(\cdot,\cdot)$ with the Gap Between Linear Systems over Hilbert Spaces

We prove that certain versions of the gap as defined in the present paper, when specialized to the case of linear systems over Hilbert spaces, coincide with the usual gap metric (cf. [12]). Further, we show that in the same case a compactness condition can be imposed on the mapping $\boldsymbol{\Phi}$ without altering the value of the gap. This fact suggests that the conditions imposed in Sections IV and V on $\boldsymbol{\Phi}$, for capturing a suitable notion of distance between systems on bounded sets, are reasonable.

In the proposition below, $\vec{\delta}_0(\cdot,\cdot)$ is the same as the definition in Section III-D except that no truncations are taken, $\vec{\delta}_1(\cdot,\cdot)$ represents the usual directed gap between linear systems over Hilbert spaces (see [12]), $\vec{\delta}_2(\cdot,\cdot)$ represents the specialization of $\vec{\delta}(\cdot,\cdot)$ given in Section III-A but with the Banach gain, and $\vec{\delta}_3(\cdot,\cdot)$ represents a similar definition with the additional compactness assumption (cf. Section IV).

Proposition 5: Let $\boldsymbol{P},\boldsymbol{P}_1$ be linear, shift-invariant, finite-dimensional dynamical systems with strictly proper transfer functions, acting on signals in $\mathcal{L}_2^m[0,\infty) \cong \mathcal{H}_2^m$. Define $\mathcal{W} = \mathcal{H}_2^m \times \mathcal{H}_2^n$ and the graphs

$$\mathcal{M} = \mathcal{G}_{\rm P} = \begin{pmatrix} M \\ N \end{pmatrix}\mathcal{H}_2^m, \qquad \mathcal{M}_1 = \mathcal{G}_{{\rm P}_1} = \begin{pmatrix} M_1 \\ N_1 \end{pmatrix}\mathcal{H}_2^m \subset \mathcal{W}$$

where NM^{-1}, $N_1(M_1)^{-1}$ are normalized coprime factorizations over $\mathcal{H}_\infty$ of the respective transfer functions. Define

$$\vec{\delta}_0(\boldsymbol{P},\boldsymbol{P}_1) = \sup_{\substack{m_1\in\mathcal{M}_1\\ \|m_1\|_2\neq 0}} \inf_{\substack{m\in\mathcal{M}\\ \|m\|_2\neq 0}} \frac{\|m_1 - m\|_2}{\|m\|_2}$$

$$\vec{\delta}_1(\boldsymbol{P},\boldsymbol{P}_1) = \|(\boldsymbol{\Pi}_{\mathcal{M}_1} - \boldsymbol{\Pi}_{\mathcal{M}})\boldsymbol{\Pi}_{\mathcal{M}}\|$$

$$\vec{\delta}_2(\boldsymbol{P},\boldsymbol{P}_1) = \begin{cases} \inf\{\|(\boldsymbol{\Phi} - \boldsymbol{I})|_{\mathcal{M}}\|\colon \boldsymbol{\Phi}\colon \mathcal{M}\to\mathcal{M}_1 \text{ is} \\ \qquad \text{causal, bijective, } \boldsymbol{\Phi}0 = 0\}, \\ \infty \text{ if no such operator } \boldsymbol{\Phi} \text{ exists,} \end{cases}$$

$$\vec{\delta}_3(\boldsymbol{P},\boldsymbol{P}_1) = \begin{cases} \inf\{\|(\boldsymbol{\Phi} - \boldsymbol{I})|_{\mathcal{M}}\|\colon \boldsymbol{\Phi}\colon \mathcal{M}\to\mathcal{M}_1 \text{ is} \\ \qquad \text{causal, causally invertible, } \boldsymbol{\Phi}0 = 0, \\ \qquad \text{with } \boldsymbol{T}_\tau(\boldsymbol{I} - \boldsymbol{\Phi})\boldsymbol{T}_\tau|_{\mathcal{M}} \text{ compact}\}, \\ \infty \text{ if no such operator } \boldsymbol{\Phi} \text{ exists} \end{cases}$$

where $\boldsymbol{\Pi}_{\mathcal{K}}$ denotes the orthogonal projection onto a closed subspace $\mathcal{K}\subset\mathcal{W}$ and $\|\cdot\|$ denotes the usual induced norm on a Hilbert space. Then

$$\vec{\delta}_0(\boldsymbol{P},\boldsymbol{P}_1) = \vec{\delta}_1(\boldsymbol{P}_1,\boldsymbol{P}).$$

(Note the order of $\boldsymbol{P}$ and $\boldsymbol{P}_1$.) Also, if $\delta_1(\boldsymbol{P},\boldsymbol{P}_1) := \max\{\vec{\delta}_1(\boldsymbol{P},\boldsymbol{P}_1), \vec{\delta}_1(\boldsymbol{P}_1,\boldsymbol{P})\} < 1$, then

$$\vec{\delta}_0(\boldsymbol{P},\boldsymbol{P}_1) = \vec{\delta}_1(\boldsymbol{P},\boldsymbol{P}_1) = \vec{\delta}_2(\boldsymbol{P},\boldsymbol{P}_1) = \vec{\delta}_3(\boldsymbol{P},\boldsymbol{P}_1).$$

Proof: First note that

$$\begin{aligned}\vec{\delta}_1(\boldsymbol{P}_1,\boldsymbol{P}) &= \|(\boldsymbol{\Pi}_{\mathcal{M}_1} - \boldsymbol{\Pi}_{\mathcal{M}})\boldsymbol{\Pi}_{\mathcal{M}_1}\| \\ &= \sup_{\substack{m_1\in\mathcal{M}_1\\ \|m_1\|_2\neq 0}} \inf_{\substack{m\in\mathcal{M}\\ \|m\|_2\neq 0}} \frac{\|m_1 - m\|_2}{\|m_1\|_2}.\end{aligned}$$

Now consider $m\in\mathcal{M}$ and $m_1\in\mathcal{M}_1$ fixed and separated by an angle θ. Then

$$\inf_{\lambda\in\mathbb{C}} \frac{\|m_1 - \lambda m\|_2}{\|m_1\|_2} = \sin(\theta) = \inf_{\lambda\in\mathbb{C}} \frac{\|m_1 - \lambda m\|_2}{\|\lambda m\|_2}$$

where the first infimum is achieved by taking $m_1 - \lambda m$ perpendicular to m and the second by taking $m_1 - \lambda m$ perpendicular to m_1. This is already enough to show the first part of the proposition.

Now if $\delta_1(\boldsymbol{P},\boldsymbol{P}_1) < 1$, it is standard that $\delta_1(\boldsymbol{P},\boldsymbol{P}_1) = \vec{\delta}_1(\boldsymbol{P},\boldsymbol{P}_1) = \vec{\delta}_1(\boldsymbol{P}_1,\boldsymbol{P})$ [12, Proposition 3]. Therefore, under this condition, $\vec{\delta}_0(\boldsymbol{P},\boldsymbol{P}_1) = \vec{\delta}_1(\boldsymbol{P},\boldsymbol{P}_1)$.

Consider any (causal) mapping $\boldsymbol{\Phi}\colon \mathcal{M}\to\mathcal{M}_1$. Then

$$\begin{aligned}\vec{\delta}_1(\boldsymbol{P},\boldsymbol{P}_1) &= \sup_{x\in\mathcal{M},\|x\|_2=1} \|(\boldsymbol{\Pi}_{\mathcal{M}_1} - \boldsymbol{I})x\|_2 \\ &\leq \sup_{x\in\mathcal{M},\|x\|_2=1} \|(\boldsymbol{\Phi} - \boldsymbol{I})x\|_2 = \|(\boldsymbol{\Phi} - \boldsymbol{I})|_{\mathcal{M}}\|.\end{aligned}$$

Since this is true for any $\boldsymbol{\Phi}$, then $\vec{\delta}_1(\boldsymbol{P},\boldsymbol{P}_1) \leq \vec{\delta}_2(\boldsymbol{P},\boldsymbol{P}_1)$. Clearly, $\vec{\delta}_2(\boldsymbol{P},\boldsymbol{P}_1) \leq \vec{\delta}_3(\boldsymbol{P},\boldsymbol{P}_1)$.

Now suppose that $\delta_1(\boldsymbol{P},\boldsymbol{P}_1) = \alpha < 1$. Then there exists a $Q\in\mathcal{H}_\infty^{m\times m}$ such that

$$\left\| \begin{pmatrix} M \\ N \end{pmatrix} - \begin{pmatrix} M_1 \\ N_1 \end{pmatrix} Q \right\|_\infty = \alpha$$

with $Q^{-1}\in\mathcal{H}_\infty^{m\times m}$ [28]. Let V, U be matrices over $\mathcal{H}_\infty$ such that $VM + UN = I$. Define

$$\Phi = \begin{pmatrix} M_1 \\ N_1 \end{pmatrix} Q(V \quad U).$$

Multiplication of $\mathcal{H}_2$ signals by Φ is the frequency domain representation (i.e., after taking Fourier transforms) of a causal, shift-invariant mapping on $\mathcal{L}_2$. Allowing the usual abuse of notation, operators considered below as acting on $\mathcal{L}_2$ are written as multiplication by matrices of H_∞ functions, though they are strictly to be interpreted as their time-domain equivalents. Observe that

$$(\Phi - I)|_{\mathcal{M}} = \left\{ \begin{pmatrix} M_1 \\ N_1 \end{pmatrix} Q - \begin{pmatrix} M \\ N \end{pmatrix} \right\} (V \quad U)|_{\mathcal{M}}$$

and

$$(\Phi - I)\begin{pmatrix} M \\ N \end{pmatrix} v = \left\{ \begin{pmatrix} M_1 \\ N_1 \end{pmatrix} Q - \begin{pmatrix} M \\ N \end{pmatrix} \right\} v$$

for $v\in\mathcal{L}_2$. Since $\binom{M}{N}$ is an isometry on $\mathcal{L}_2$, then $\|(\Phi - I)|_{\mathcal{M}}\| = \alpha$. Now let V_1, U_1 be matrices over $\mathcal{H}_\infty$ such that $V_1 M_1 + U_1 N_1 = I$. Then

$$\begin{pmatrix} M \\ N \end{pmatrix} Q^{-1}(V_1 \quad U_1) \cdot \begin{pmatrix} M_1 \\ N_1 \end{pmatrix} Q(V \quad U)|_{\mathcal{M}} = I.$$

Hence Φ has a causal inverse (between $\mathcal{M}$ and $\mathcal{M}_1$). To complete the proof we need to find a Φ so that $\Phi - I$ is compact on finite intervals. This will be achieved if $\Phi - I$ can be made strictly proper (see Proposition 4). Take any $\epsilon > 0$, so that $\alpha + \epsilon < 1$. Without loss of generality, $M(\infty) = M_1(\infty) = I$. Define

$$\hat{Q}(s) := Q(s)\frac{1}{\tau s + 1} + I\frac{\tau s}{\tau s + 1} \tag{44}$$

$$= (Q - I)\frac{1}{\tau s + 1} + I \tag{45}$$

and observe that $\|Q(\infty) - I\| \leq \alpha < 1$. Thus we can find a disc centered at the origin so that $\|Q(s) - I\| \leq \alpha + \epsilon$ outside the disc in the right half-plane. Thus $\hat{Q}(s)^{-1}$ will be bounded outside this disc, from (45). But within the disc $Q(s)^{-1}$ is bounded. Hence, from (44), $\hat{Q}(s)^{-1}$ will be bounded within this disc for sufficiently small τ. This means that $\hat{Q}(s)$ is invertible for sufficiently small $\tau > 0$. Further, inside *any* such disc, in the RHP, $\|Q(s) - \hat{Q}(s)\|$ can be made arbitrarily small by choosing τ small enough, and

$$\left\| \begin{pmatrix} M \\ N \end{pmatrix} - \begin{pmatrix} M_1 \\ N_1 \end{pmatrix} \hat{Q} \right\|$$

can be made less than $\alpha + \epsilon$. On the other hand, we can choose such a disc to be sufficiently large so that

$$\begin{aligned} &\left\| \begin{pmatrix} M \\ N \end{pmatrix} - \begin{pmatrix} M_1 \\ N_1 \end{pmatrix} \hat{Q} \right\| \\ &\quad \le \left\| \begin{pmatrix} M \\ N \end{pmatrix} - \begin{pmatrix} I \\ 0 \end{pmatrix} \hat{Q} \right\| + \left\| \left(\begin{pmatrix} M_1 \\ N_1 \end{pmatrix} - \begin{pmatrix} I \\ 0 \end{pmatrix} \right) \hat{Q} \right\| \\ &\quad \le \left\| \begin{pmatrix} M \\ N \end{pmatrix} - \begin{pmatrix} I \\ 0 \end{pmatrix} \right\| + \left\| \begin{pmatrix} I \\ 0 \end{pmatrix} - \begin{pmatrix} I \\ 0 \end{pmatrix} \hat{Q} \right\| \\ &\qquad + \left\| \begin{pmatrix} M_1 \\ N_1 \end{pmatrix} - \begin{pmatrix} I \\ 0 \end{pmatrix} \right\| \cdot \|\hat{Q}\| \\ &\quad \le \frac{\epsilon}{4} + \left\| (I - Q(s)) \frac{1}{\tau s + 1} \right\| + \frac{\epsilon}{4} \end{aligned}$$

outside that disc, independent of τ. So, once again, if the disc is large enough $I - Q(s)$ can be kept within $\epsilon/2$ of $I - Q(\infty)$ in norm, i.e.,

$$\left\| \begin{pmatrix} M \\ N \end{pmatrix} - \begin{pmatrix} M_1 \\ N_1 \end{pmatrix} \hat{Q} \right\| \le \alpha + \epsilon.$$

Thus we can find a $\hat{Q} \in \mathcal{H}_\infty$ which is invertible so that

$$\left\| \begin{pmatrix} M \\ N \end{pmatrix} - \begin{pmatrix} M_1 \\ N_1 \end{pmatrix} \hat{Q} \right\|_\infty \le \alpha + \epsilon$$

for ϵ arbitrary, and with

$$\begin{pmatrix} M \\ N \end{pmatrix} - \begin{pmatrix} M_1 \\ N_1 \end{pmatrix} \hat{Q}$$

strictly proper, i.e., compact on finite intervals. But the same is now true of $\Phi - I$.

Therefore $\vec{\delta}_3(\boldsymbol{P}, \boldsymbol{P}_1) \le \alpha + \epsilon$, for any ϵ, which means that $\vec{\delta}_3(\boldsymbol{P}, \boldsymbol{P}_1) \le \vec{\delta}_1(\boldsymbol{P}, \boldsymbol{P}_1)$. □

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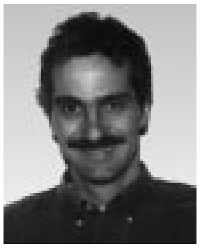

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