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# Shape Signifiers for Control of a Low-Order Compressor Model

Rodolphe Sepulchre and Petar Kokotovic

***Abstract*— Rotating stall and surge, two instability mechanisms limiting the performance of aeroengines compressors, are studied on the third-order Moore–Greitzer model. The *skewness* of the compressor characteristic, a single parameter shape signifier, is shown to determine the key qualitative properties of feedback control.**

***Index Terms*— Aeroengine instabilities, compressor characteristics, Moore–Greitzer model.**

## I. INTRODUCTION

In recent years, jet engine compression systems have become a subject of intensive nonlinear dynamics and control studies [1], [3], [5], [7], [8], [11]. These studies were greatly helped by a low-order Moore–Greitzer (MG) model [5], [10] which has served as a guide for conceptualizing different strategies for compressor control [6], [7]. The feasibility of such a control approach was recently demonstrated experimentally [3].

In its simplest form, the MG model consists of three nonlinear differential equations which qualitatively describe the two main compressor instabilities: *rotating stall*, characterized by a region of reduced flow that rotates around the annulus of the compressor, and *surge*, characterized by large axisymmetric oscillations. Surge can damage the compression system and must be prevented. Rotating stall, which causes a major loss of performance, must be either prevented or rapidly removed.

Under manual control the stall-removal process exhibits a hysteresis loop, as shown on two experimental plots in Fig. 1(a) for a single-stage $(N = 1)$ and a three-stage $(N = 3)$ compressor. These plots are taken from Day *et al.* [1]. The critical equilibrium determining the severity of the stall-removal hysteresis is the stall cessation point. In Fig. 1, on the single-stage compressor, this is point $A$ which is located to the left of the peak. In this case the hysteresis is not severe. For the three-stage compressor, this is point $B$, which is located to the right of the peak and the hysteresis is severe. The impact of this qualitative difference on feedback control properties is the main theme of this paper. Within the MG model, our analysis determines which aspects of the compressor's qualitative behavior can be changed when the throttle is used for feedback control. We show that in the MG model the key difference between a mild hysteresis and a severe hysteresis can be deduced from the *skewness* of the compressor characteristic, that is, the difference between the slopes of the characteristic, as illustrated in Fig. 1(b).

In the low-order MG model, we replace the usual cubic parameterization of the compressor characteristic which exhibits only the left-skewness and cannot be adjusted to model a severe hysteresis. With a different form of the characteristic, we capture the critical skewness with a single parameter signifier $\nu$. We show that the left-

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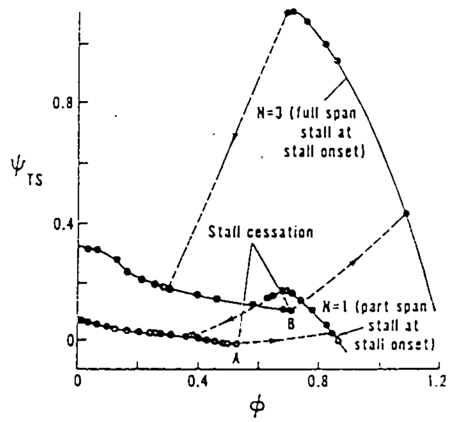
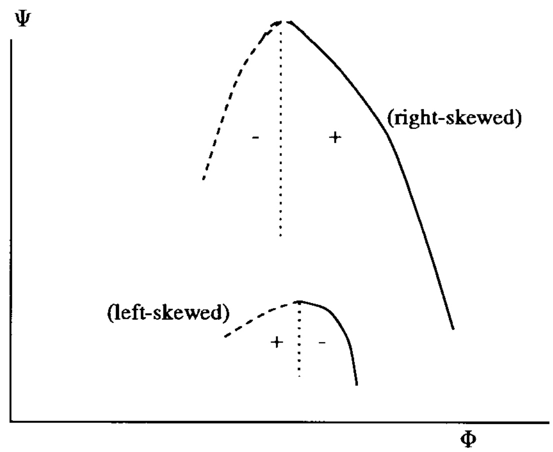



(a)




(b)

Fig. 1. (a) Mild and severe hysteresis in an experimental plot from [1] and (b) left- and right-skewness of the compressor characteristics.

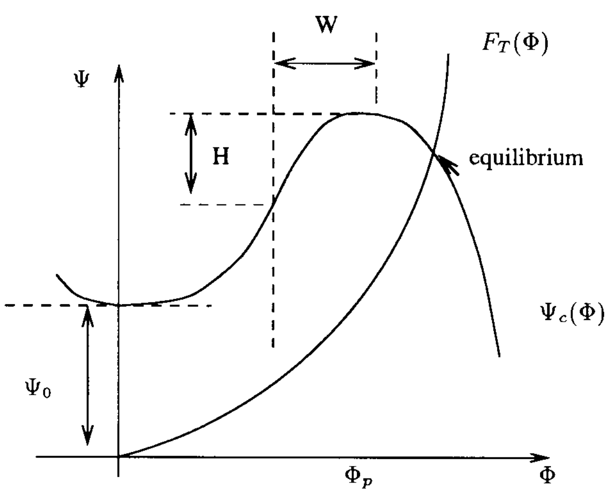



Fig. 2. Compressor and throttle characteristics.

skewness of the characteristic is the cause of a mild hysteresis and the signifier $\nu$ is negative. A severe hysteresis is due to the right-skewness and the signifier $\nu$ is positive.

We begin in Section II with a brief review of the MG model and introduce two shape signifiers: the *slope* signifier, for the control of surge, and the *skewness* signifier, for the control of stall. A two-sine parameterization of the compressor characteristic has resulted in the simplest definitions of these two signifiers. The impact of skewness on the uncontrolled model is analyzed in Section III where we show that the skewness determines the severity of the stall-removal hysteresis: the branch of the stalled equilibria continuously shifts to the right as the skewness parameter $\nu$ is increased from $\nu < 0$ to $\nu > 0$. For $\nu > 0$, the compressor characteristic is right-skewed and the stall cessation equilibrium is to the right of the peak.

The analysis in Section IV shows the impact of skewness on the properties of a family of feedback controllers. This analysis reveals that the minimal feedback information that is required for suppressing the hysteresis changes with the skewness signifier. A specific result is that the use of the stall amplitude as a signal for feedback is necessary for stabilization when $\nu$ is close to zero or positive, and that it can be avoided when $\nu$ is sufficiently negative.

## II. Shape Signifiers for the Moore–Greitzer Model

The dynamics of rotating stall and surge in an axial flow compression system are described by the third-order MG model

$$\dot{\Phi} = \frac{1}{l_c}\left(-\Psi + \frac{1}{2\pi}\int_0^{2\pi} \Psi_c(\Phi + WA\sin\theta)\,d\theta\right) \tag{1}$$

$$\dot{\Psi} = \frac{1}{l_c\beta^2}\left(\Phi - F_T^{-1}(\Psi)\right) \tag{2}$$

$$\dot{A} = \sigma\frac{1}{2\pi}\int_0^{2\pi} \Psi_c(\Phi + WA\sin\theta)\sin\theta d\theta. \tag{3}$$

The meaning of the physical parameters $l_c$, $\sigma$, and $\beta$ is discussed in [3] and [8]. In the following, the integrals in (2) and (3) are denoted by $I_1(\Phi, A)$ and $I_2(\Phi, A)$, respectively.

The variable $A \geq 0$ characterizes the amplitude of the first mode of a nonaxisymmetric disturbance of the flow through the compressor. Under normal operating conditions, the flow is axisymmetric, that is, $A \equiv 0$, and the operating point is located at the intersection of the two static characteristics (Fig. 2): the *compressor characteristic* $\Psi = \Psi_c(\Phi)$, relating the pressure rise $\Psi$ to the mass flow $\Phi$, and the *throttle characteristic* $\Psi = F_T(\Phi)$, relating the pressure loss across the throttle to the mass flow.

The throttle characteristic in Fig. 2 is typically parabolic $F_T = (1/\gamma^2)\Phi_T^2$ where $\gamma$ is proportional to the throttle area. In this paper, $\gamma$ will be the control variable. Typically control actuators are bleed valves and a more realistic choice would be to replace $\gamma$ by $\gamma + u$ and to define $u$ as the control variable. However, this would not alter the conclusions of this paper. By decreasing $\gamma$, the mass flow is reduced and the pressure rise is increased until a maximum, hereafter called the "peak." Beyond this peak, the equilibrium is unstable. Moore and Greitzer postulated an $S$ shape characteristic as on Fig. 2, which is somewhat hypothetical because the part with positive slope cannot be measured experimentally.

The most commonly used parameterization of the $S$-shape compressor characteristic $\Psi_c(\Phi)$ has been the cubic

$$\Psi_c(\Phi) = \Psi_0 + H\left(1 + \frac{3}{2}\left(\frac{\Phi - W}{W}\right) - \frac{1}{2}\left(\frac{\Phi - W}{W}\right)^3\right) \tag{4}$$

where $W$ and $H$ are, respectively, the semi-width and the semi-height of the characteristic.

This characteristic, shown in Fig. 3 (dashed, left-hand side), served for the bifurcation analysis in [8] and the bifurcation-softening control in [7]. More general parameterizations were employed in [9], including a concatenation of four polynomials, shown in Fig. 3 (dashed, right-hand side), which was able to reproduce experimentally observed rapid transitions from nonstalled equilibria close to the peak to fully developed stall equilibria. A significant contribution of [9] is the link it established between the shape of the compressor

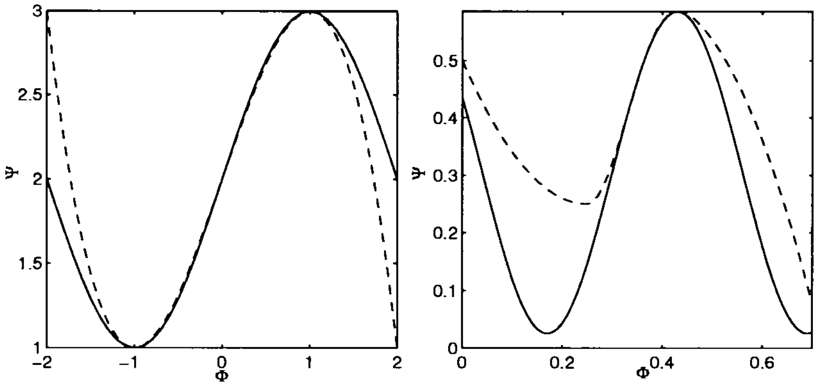



Fig. 3. Dashed: Two examples of characteristics from the literature. Solid: fitting using one-sine curve.

characteristic and compressor qualitative behaviors not captured by the cubic parameterization. In this paper, we will see that a cubic parameterization is indeed insufficient to model a severe hysteresis within the third-order model. For our qualitative analysis the concatenated parameterization is impractical because of a large number of parameters.

In our search for one or two simple "signifiers" which will characterize the qualitative behavior of the MG-model under throttle feedback, we started with another simple $S$-shape curve:

$$\Psi_c(\Phi) = \Psi_0 + H \sin\left(\frac{\pi}{2}\frac{\Phi - W}{W}\right). \tag{5}$$

This "one-sine" parameterization makes use of the same parameters as the cubic (4). For surge studies an important shape signifier is the *maximum slope* of the characteristic, $\pi H/2W$ for the one-sine curve and $3H/2W$ for the cubic curve.

This shape signifier is actually the only information needed for pure "surge control," that is, under the assumption of an axisymmetric flow $(A \equiv 0)$. The second-order surge model can be stabilized by adding damping, that is, by employing feedback of the form

$$u = \frac{1}{\sqrt{\Psi}}(\gamma_{\text{nom}} - \gamma_D \dot{\Phi}). \tag{6}$$

The resulting closed-loop system

$$l_c\ddot{\Phi} + \left(\frac{1}{4l_c\beta^2}\gamma_D - \frac{d\Psi_c}{d\Phi}(\Phi)\right)\dot{\Phi} + \frac{1}{4l_c\beta^2}(\Phi - \gamma_{\text{nom}}) = 0 \tag{7}$$

clearly shows that the maximal slope of the characteristic is the only shape signifier needed in this analysis. It is easily verified that the condition

$$\gamma_D > \beta^2 l_c \frac{d\Psi_c}{d\Phi}\bigg|\max \tag{8}$$

implies global asymptotic stability of the equilibrium $\Phi = \gamma_{\text{nom}}$.

When both surge and stall have to be controlled, the shape of the compressor characteristic in the neighborhood of the peak becomes important. In Fig. 3(a) and (b), the one-sine curve (solid) is symmetric with respect to the peak. The cubic characteristic (4) in Fig. 3(a) (dashed) is "skewed to the left," while the concatenated polynomial in Fig. 3(b) (dashed) is "skewed to the right." To capture this difference in shape, that is, the *skewness of the characteristic with respect to the peak axis* $\Phi = \Phi_p$, we need a second shape signifier.

The simplest way to introduce skewness is with the two-sine parameterization

$$\Psi_c(\Phi) = \Psi_0 + C\left(\sin\left(\omega\frac{\Phi - W}{W}\right) + \nu \sin\left(\frac{\omega}{k}\frac{\Phi - W}{W}\right)\right) \tag{9}$$

where $\nu = 0$ corresponds to the unskewed case. With the choice $k \approx 2^{\text{sign}\,\nu}$ and $\omega$ chosen to adjust the frequency, the skewness signifier is parameter $\nu$: a negative $\nu$ indicates left-skewness while a positive $\nu$ indicates right-skewness.

The skewness signifier $\nu$ will play a key role in determining the effectiveness of the throttle (or bleed valve) control in counteracting stall and surge. When $\nu$ is negative and $|\nu|$ is not small (left-skewness), the throttle control can be effective. This is not so in the case of right-skewness, that is, when $\nu$ is positive.

The values of $\omega$, $C$, and $\nu$ which fit a particular compressor characteristic can be obtained from the formulas

$$\nu = \frac{k\cos(\omega)}{\cos\left(\frac{\omega}{k}\right)}, \qquad C = \frac{H}{\sin(\omega) + \nu\sin\left(\frac{\omega}{k}\right)}. \tag{10}$$

The two constraints express the matching conditions $\Psi_c(\Phi_p) = \Psi_0 + H$ and $(d\Psi_c/d\Phi)(\Phi_p) = 0$.

As an example, the two-sine parameterization is used to match the same dashed curves shown in Fig. 3. The result with $\nu \approx -1$ is shown in Fig. 4(a) and with $\nu \approx +0.1$ in Fig. 4(b). In both cases, the skewness of the characteristic is captured.

The fact that the two-sine parameterization of the cubic curve (4) results in a negative $\nu$ is not accidental. It is because the skewness around the peak is governed by the change of curvature, that is, by the *the third derivative of* $\Psi_c$. For the cubic parameterization (4), the third derivative is a negative constant, which necessarily yields a *left-skewness*, and therefore a shape signifier $\nu < 0$.

A convenience of polynomial parameterizations is that the integrals $I_1$ and $I_2$ can be analytically evaluated. Fortunately, this convenience is not lost with one- and two-sine parameterizations. For the one-sine approximation, we obtain

$$\begin{aligned} I_1(\Phi, A) &= \Psi_0 + \sin\left(\omega\frac{\Phi - W}{W}\right)\frac{1}{2\pi}\int_0^{2\pi}\cos(\omega A\sin\theta)\,d\theta \\ &= \Psi_0 + (\Psi_c(\Phi) - \Psi_0)J_0(\omega A) \end{aligned} \tag{11}$$

and

$$\begin{aligned} I_2(\Phi, A) &= \sin\left(\omega\frac{\Phi - W}{W}\right)\frac{1}{2\pi}\int_0^{2\pi}\sin(\omega A\sin\theta)\sin\theta\,d\theta \\ &= \frac{W}{\omega}\frac{d\Psi_c}{d\Phi}(\Phi)J_1(\omega A) \end{aligned} \tag{12}$$

where $J_0(s)$ and $J_1(s)$ are the two first Bessel functions of the first kind (in particular, $J_0' = -J_1$). For the two-sine parameterization, one can use the same formulas for each term.

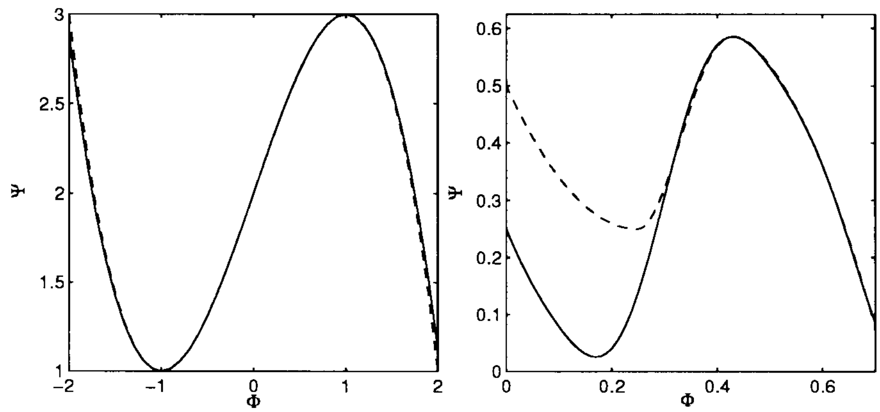



Fig. 4. (a) Left-skewed and (b) right-skewed characteristics. Dashed: polynomials. Solid: two-sine curve with (a) $\nu < 0$ and (b) $\nu > 0$.

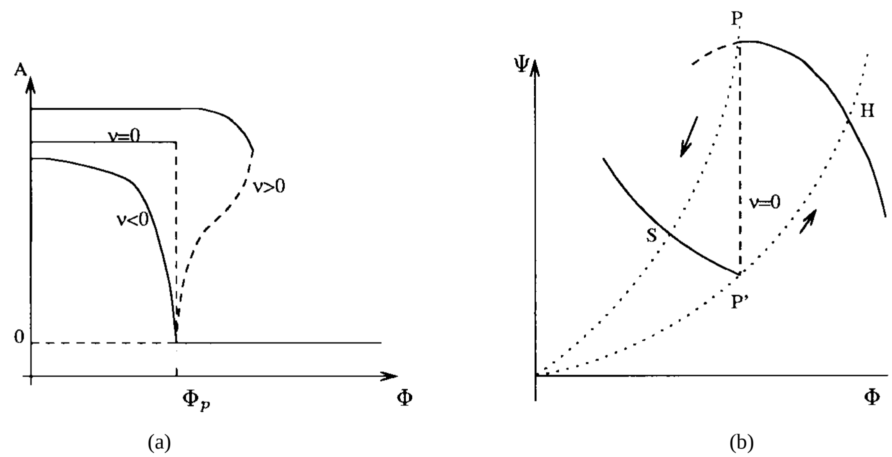



Fig. 5. (a) $(A, \Phi)$ bifurcation diagram and (b) $(\Psi, \Phi)$ equilibria diagram (unskewed case).

## III. Impact of Skewness on the Uncontrolled Model

We now study how the steady-state behavior of the uncontrolled MG model (2) and (3) depends on the skewness. Because (3) does not depend on $\Psi$, we can first study the equilibria of $\dot{A} = F_1(A, \Phi) = 0$ treating $\Phi$ as a parameter. In Fig. 5(a), the stable and the unstable equilibria are, respectively, the solid and the dashed sections of the curves. A bifurcation occurs at $A = 0$, $\Phi = \Phi_p$, which is supercritical for $\nu < 0$ and subcritical for $\nu > 0$. In all three cases, the no-stall equilibria $A = 0$, $\Phi < \Phi_p$ are unstable.

Although $\Phi$ is not a parameter but a state, the bifurcation diagram in Fig 5(a) is important from a control point of view: suppose that a given controller succeeds in tracking a set value $\Phi$. Then if $\Phi > \Phi_p$, the corresponding no-stall equilibrium $A = 0$ is stable, while for $\Phi$ below $\Phi_p$, a stable stall equilibrium $(A > 0)$ will appear. If $\nu < 0$, the stall amplitude of the stable equilibrium will increase smoothly as a function of $\Phi$. However, when $\nu > 0$, a "jump" will occur from a no stall situation $(A = 0)$ to a fully developed stall $(A = A_{\max})$.

We now turn our attention to the equilibria of the whole MG model. For a given solution $(\Phi, A)$ of $F_1(A, \Phi) = 0$, the corresponding pressure rise $\Psi$ is obtained by solving the equation $\dot{\Phi} = 0$, which yields $\Psi = F_2(A, \Phi)$. The corresponding throttle value is $\gamma = \Phi/\sqrt{\Psi}$.

For the no-stall case $A = 0$, $J_0(0) = 1$. Hence $\dot{\Phi} = 0$ reduces to $\Psi = \Psi_c(\Phi)$, i.e., the no-stall equilibria are on the compressor characteristic. These equilibria are unstable left of the peak. For the stall case $(A > 0)$, the projection of the stall equilibria to the plane $(\Psi, \Phi)$ is a curve shown in Fig. 5(b) for the symmetric case $(\nu = 0)$. The vertical segment $PP'$ in Fig. 5(b) corresponds to the vertical segment in Fig. 5(a). The arc $SP'$ in Fig. 5(b) corresponds to the horizontal stall branch in Fig. 5(a).

### A. Skewness and Hysteresis

To the right of the peak $P$, that is, for $\Phi > \Phi_p$, the no-stall equilibria are stable. As the throttle slowly closes beyond $P$, the occurrence of stall is represented by the jump from $P$ to $S$. In Fig. 5(a), this corresponds to a jump from the lower segment $A = 0$ to the higher segment. If the throttle is slowly reopened in order to recover the no-stall regime, the stable stall equilibrium moves along the arc $SP'$ toward $P'$. At $P'$, the stall is suddenly extinguished and the operating point jumps to the stable no-stall equilibrium $H$. This operating condition is still undesirable because of the reduced pressure rise. When the throttle begins to close to recover a desired operating point on the compressor characteristics, an attempt to increase the pressure rise beyond the peak would cause a new jump to $S$. This completes the cycle $S \rightarrow P' \rightarrow H \rightarrow P \rightarrow S$ which is the stall-removal hysteresis. Its reduction was one of the main accomplishments of the feedback control described and experimentally validated in [3].

The desire to model the hysteresis in the form in which it has been observed experimentally motivated us to introduce the two-sine parameterization (9). This parameterization can describe a crucial aspect of the stall-removal hysteresis which cannot be modeled by the cubic parameterization (4). The two-sine parameterization exhibits the *stable stall equilibria to the right of the peak, that is for* $\Phi > \Phi_p$.

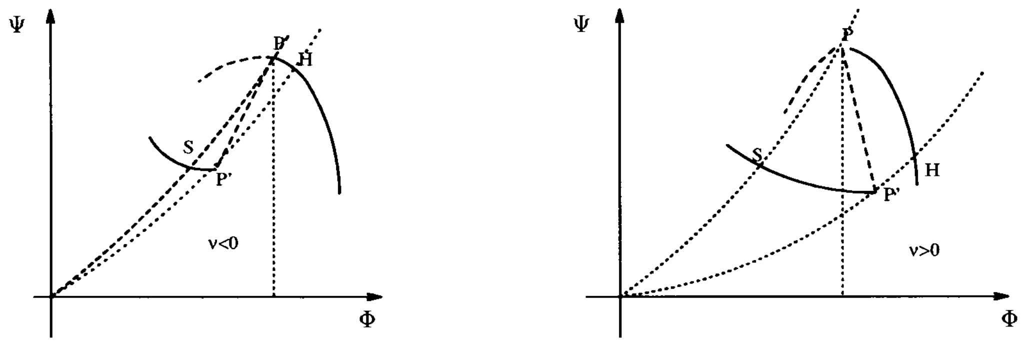



Fig. 6. $(\Psi, \Phi)$ equilibria diagrams with left- and right-skewness.

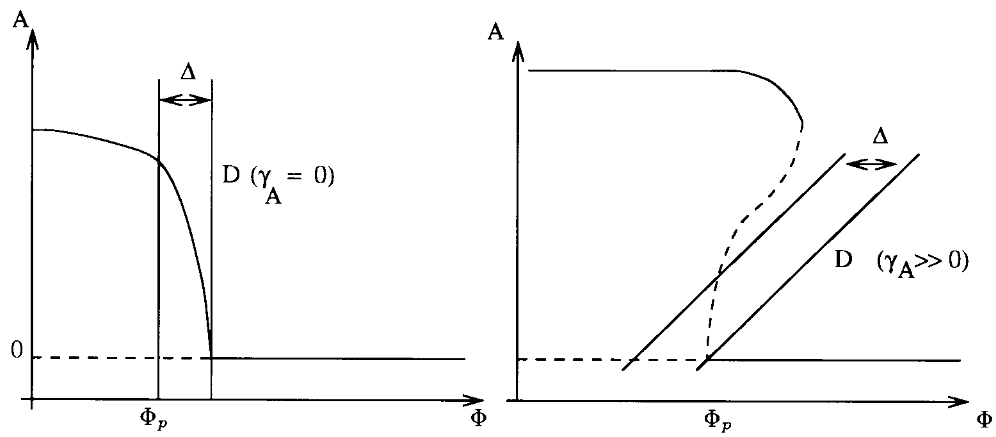



Fig. 7. Closed-loop equilibria with left- and right-skewness.

It is the experimentally observed existence of these equilibria that dramatically increases the severity of the compressor hysteresis. The two-sine parameterization reveals that these equilibria are caused by the right-skewness, that is, when the signifier $\nu$ is positive. The characteristics $\Psi_c$ with $\nu < 0$ and $\nu > 0$ shown in Fig. 6 give rise to the two fundamentally different arcs of the stable stall equilibria. Repeating our discussion for the two situations depicted in Fig. 6, we can easily see that in the case $\nu > 0$ the hysteresis is much larger and the stall persists over a wider range of the throttle openings. It is of major practical importance that this crucial phenomenon is determined by a single skewness signifier $\nu$.

Within the three-dimensional MG model, the skewness of the compressor characteristic is thus the shape signifier which determines if the stall cessation point $P'$ is to the left or to the right of the peak. As a consequence, different types of hysteresis are obtained just by varying the parameter $\nu$. In a *higher-order* MG model a cubic characteristic may be able to model different types of hysteresis. However, the skewness captures this important compressor feature in a low-order model.

## IV. Impact of Skewness on Feedback Control

### A. Structural Limitations of Throttle Control

Our task is now to investigate to what extent the undesirable steady-state behavior of the uncontrolled compression system can be altered by throttle control using state feedback. The first and foremost limitation is that throttle control cannot create new equilibria. In other words, a desired operating point has to be selected at an equilibrium of the uncontrolled system taking into account that the throttle control variable $\gamma$ can only determine the steady-state value of $\Phi/\sqrt{\Psi}$. The corresponding equilibria are then imposed by the equations $\dot{A} = 0$ and $\dot{\Phi} = 0$ which are independent of $\gamma$.

Among the equilibria of the uncontrolled model, the most desirable operating point is at the peak because it corresponds to the maximum pressure rise and to a no-stall situation ($A = 0$). We will therefore examine if this equilibrium can be stabilized by a control law of the form

$$u = \frac{1}{\sqrt{\Psi}}(\Phi_c - \gamma_P(\Phi - \Phi_c) - \gamma_D\dot{\Phi} + \gamma_A A). \qquad (13)$$

The set-point $\Phi_c$ determines the location of the closed-loop equilibrium $(A, \Phi, \dot{\Phi}) = (0, \Phi_c, 0)$, which corresponds to the peak by selecting $\Phi_c = \Phi_p$. The roots of

$$\gamma_A A_e - (1 + \gamma_P)(\Phi_e - \Phi_c) = 0 \qquad (14)$$

determine whether this is the only closed-loop equilibrium or not. With the help of Fig. 7, these roots are easily visualized as the intersections of the line $D \equiv A_e = [(1+\gamma_P)/\gamma_A](\Phi_e - \Phi_c)$ with the equilibria curves from Fig. 5. In the case of left-skewness, the choice $\Phi_c = \Phi_p$ results in the unique equilibrium $(0, \Phi_p, 0)$ because the slope of $D$ is positive. This unique equilibrium is already achieved even with $\gamma_A = \gamma_P = 0$, that is, without any feedback of the mass flow or of the stall amplitude. (Note that for $\gamma_A = 0$, the line $D$ is vertical. In the case of right-skewness, the closed-loop equilibrium is unique only if the slope of $D$ is not too large, that is if $\gamma_A \gg \gamma_P$.)

Because of modeling imperfections, the peak location is uncertain and the set-point $\Phi_c$ will never be equal to $\Phi_p$. Instead we will have $\Phi_c = \Phi_p + \Delta$, where $\Delta$ represents uncertainty. For the determination of the closed-loop equilibrium, a constant $\Delta$ causes a horizontal shift of the line $D$. For $\Delta > 0$, the closed-loop equilibrium will be shifted to the right and will be a stable operating point on the compressor characteristic. However, for $\Delta < 0$, the line $D$ creates at least two new equilibria. The two equilibria near the peak are unstable for the uncontrolled compressor. The main task of feedback control is to stabilize one of the two equilibria. Then a small uncertainty $\Delta < 0$ will cause only a small shift of the closed-loop equilibrium, rather than a large jump observed without control. This explains why it is crucial not only to examine the stabilization of the peak with (13) and $\Phi_c = \Phi_p$, but also to analyze whether the same control law will result in stable equilibria near the peak when $\Phi_c = \Phi_p + \Delta$.

A simple calculation not presented here shows that for the unstable no-stall equilibria ($A = 0$, $\Phi < \Phi_p$), one of the unstable eigenvalues of the linearized system is uncontrollable and, hence, none of these equilibria can be stabilized by smooth feedback. We therefore concentrate on the stabilization of the stall equilibria.

### B. Local Stabilization

With the understanding that the uncertainty $\Delta$ will determine the actual location of the stall equilibrium ($A_e$, $\Phi_e$ $\Psi_e$), we rewrite the control law (13) in the form

$$u = \frac{1}{\sqrt{\Psi}}(\Phi_e - \gamma_P(\Phi - \Phi_e) - \gamma_D\dot{\Phi} + \gamma_A(A - A_e). \tag{15}$$

Introducing the error coordinates $(a, \phi, \dot{\phi}) := (A - A_e, \Phi - \Phi_e, \dot{\Phi})$, the linearized model of the compression system (3) under the feedback control (15) can be written in the form

$$\begin{aligned}\dot{a} &= \sigma(I_{2a}a + I_{2\phi}\phi)\\ \ddot{\phi} &= \left(\frac{\gamma_A}{4l_c^2\beta^2} + \sigma I_{2\phi}I_{2a}\right)a + \left(I_{2\phi}^2 - \frac{1+\gamma_P}{4l_c^2\beta^2}\right)\phi\\ &\quad + \left(I_{1\phi} - \frac{\gamma_D}{4l_c^2\beta^2}\right)\dot{\phi}.\end{aligned} \tag{16}$$

The constants $I_{1\phi}$, $I_{2a}$, and $I_{2\phi}$ are the partial derivatives of $I_1$ and $I_2$ evaluated at the equilibrium. We simplify the expressions below by neglecting $0(A^4)$-terms. Evaluating the constants $I_{1a}$, $I_{2a}$, and $I_{1\phi}$ for the parameterization (9), we obtain the expressions

$$\begin{aligned}I_{2\phi} &= I_{1a} = \frac{d^2\Psi_c}{d\phi^2}(\Phi_e)\frac{A_e}{2} + 0(A^3)\\ I_{1\phi} &= I_{2a} = \frac{1}{8}\frac{d^3\Psi_c}{d\Phi^3}(\Phi_e) = c\,\nu A_e^2\end{aligned} \tag{17}$$

where $c = c(\Phi_e) = (C/8k)\omega^3 \cos[(\omega/W)\Phi_e](-1 + 1/k^2)$. While $I_{2\phi} = I_{1a}$, the error in $I_{1\phi} = I_{2a}$ is $0(A^4)$.

The stability conditions for the linearized system (16) are strongly affected by the skewness signifier $\nu$. A first condition yields

$$-c\frac{1+\gamma_P}{l_c^2}\nu A^2 + \gamma_A\left|\frac{d^2\Psi_c}{d\phi^2}(\Phi_e)\right|A > 0. \tag{18}$$

If $\nu < 0$, each term of (18) is positive and the inequality holds even for $\gamma_A = \gamma_P = 0$, that is, without any feedback of the mass flow or of the stall amplitude. If $\nu \geq 0$, the first term becomes negative. In this case, a feedback of the stall amplitude is *necessary* for stabilization and its gain must satisfy $\gamma_A > \{c\nu/[|\ (d^2\Psi/d\phi^2)(\Phi_e)\ |]\}A$.

A second stability condition yields

$$1 + \gamma_P - \nu c A^2\gamma_D > \beta^2\left(\frac{d^2\Psi_c}{d\phi^2}(\Phi_e)A\right)^2. \tag{19}$$

For large values of the parameter $\beta$ and as $A$ increases, the right-hand side rapidly becomes larger than one. As a consequence, it is not possible to stabilize equilibria on the entire stall branch with a stall amplitude feedback alone ($\gamma_P = \gamma_D = 0$). This has been previously observed by Eveker *et al.* [3] who proposed to incorporate in the feedback law a $\dot{\Phi}$-term, that is, $\gamma_D > 0$. However, it is apparent from (19) that with $\gamma_D > 0$ and $\gamma_P = 0$, stabilization can be achieved only if $\nu < 0$. If $\nu \geq 0$, a feedback of the mass flow is necessary for stabilization of the whole branch of unstable stall equilibria.

The third stability condition is

$$\begin{aligned}\frac{\gamma_D}{\beta^2}\left(\frac{1+\gamma_P}{\beta^2}\right) - c\nu A^2\left(\frac{1+\gamma_P}{\beta^2} + \frac{\gamma_D^2}{\beta^4}\right)&\\ - \frac{d^2\Psi_c}{d\Phi^2}\frac{\gamma_D}{\beta^2}A^2 \geq \left|\frac{d^2\Psi_c}{d\Phi^2}\right|\gamma_A A.&\end{aligned} \tag{20}$$

When $\gamma_D = 0$, the right-hand side is positive and of order $A$, while the left-hand side is of order $A^2$. This shows the need for a nonzero gain $\gamma_D$ for $A$ small, that is, near the peak.

The minimal requirements for feedback stabilization of the unstable stall equilibria of the MG model can be summarized as follows: In the case of left-skewness $(\nu < 0)$, the minimal requirement for feedback stabilization is a term $-\gamma_D\dot{\Phi}$ which provides damping. In the case of right-skewness ($\nu > 0$), a full state feedback is *necessary* to achieve stabilization, while $\gamma_A > \{c\nu/[|\ (d^2\Psi/d\phi^2)(\Phi_e)\ |]\}A$ and (20) further constrain the gains $\gamma_D$, $\gamma_A$, and $\gamma_P$.

## V. Conclusions

This paper has addressed the throttle control of rotating stall and surge in a low-order compressor model. We have shown that the *skewness* of the compressor characteristic, captured by a single parameter $\nu$, is a key shape signifier for throttle control. The use of other types of actuations [2], [4], such as air injection, may be less sensitive to this shape signifier. A cubic characteristic inherently leads to left-skewness ($\nu < 0$), while real compressors might exhibit right-skewness ($\nu > 0$). We have shown that a variation of $\nu$ has important consequences on the open-loop hysteresis observed without control and on its potential suppression by feedback control. In the case of left-skewness, the open-loop hysteresis can be suppressed by a $\dot{\Phi}$-feedback. In the case of right-skewness, the open-loop hysteresis is larger and the control requires full state feedback.

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