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Branch and Bound Computation of the Minimum Norm of a Linear Fractional Transformation over a Structured Set

Robert M'Closkey, Andy Packard, and Jaime Sipila

Abstract— **The minimum norm of a linear fractional transformation (LFT) over a structured set is computed using a branch and bound algorithm. This is a global optimization problem due to the possibility of local minima. Several computationally efficient lower bounds for the minimum norm of the LFT are developed and it is demonstrated that the success of the optimization, as measured by time-to-converge, largely depends on the quality of these bounds.**

Index Terms: branch and bound, fixed-structure synthesis, convex optimization

I. Introduction

Branch and bound algorithms have been proposed for solving a wide variety of global optimization problems that arise in system theory. In [1], [5], [2], both robustness analysis of control systems and controller design are addressed from this perspective. In [7] branch and bound is used to improve the mixed μ upper bound. The present paper is concerned with determining

$$\Phi_{\min}(\mathbf{B}_{\mathbf{\Delta}}) := \min_{\Delta \in \mathbf{B}_{\mathbf{\Delta}}} \bar{\sigma}(F_L(M, \Delta)), \tag{1}$$

where $\mathbf{\Delta}$ is a structured set of real parameters defined as

$$\mathbf{\Delta} = \left\{ \text{diag}\left[\delta_1 I_{r_1}, \ldots, \delta_s I_{r_s}\right] : \delta_i \in \mathbb{R}, \right\},$$

and

$$\mathbf{B}_{\mathbf{\Delta}} = \left\{ \Delta \in \mathbf{\Delta} : \overline{\sigma}(\Delta) \leq 1 \right\},$$

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is the closed unit ball in $\mathbf{\Delta}$. This notation is standard in the structured singular value literature [8]. The matrix M is partitioned as

$$M = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix},$$

and $F_L(\cdot,\cdot)$ is the linear fractional transformation defined as

$$F_L(M, \Delta) = M_{11} + M_{12}\Delta(I - M_{22}\Delta)^{-1}M_{21}.$$

It is assumed that the dimensions of Δ and M are such that the linear fractional transformation is defined.

An application of (1) is in the design of fixed-structure controllers to minimize the closed-loop $\mathcal{H}_\infty$ norm of a linear system. In this case we wish to compute

$$\min_{\Delta \in \mathbf{B}_{\mathbf{\Delta}}} \|F_L(M, \Delta)\|_\infty, \tag{2}$$

where M represents the linear system dynamics and $\mathbf{B}_{\mathbf{\Delta}}$ is a normalized set of control gains or design parameters. Applications of the branch and bound algorithm to this problem, and other problems with different design objectives, may be found in [4], [2]. Our recent paper [9] contains details on computing (2) using the bounds presently developed.

Due to the structured nature of $\mathbf{\Delta}$, (1) may contain multiple local minima. Furthermore, simple examples show that the minimum norm may be achieved at the interior of $\mathbf{B}_{\mathbf{\Delta}}$ and so various schemes which check edges or vertices of the parameter set are not applicable. The main focus of the paper is the development of new lower bounds for (1) which improves the performance of the branch and bound algorithm. We also explore the trade-off between the computation time expended in calculating the bounds versus the quality of the bounds. This trade-off has already been recognized as important in references [1] and

[7]. Section II and the Appendix briefly review the branch and bound algorithm. Section III develops new lower bounds for (1) and Section IV applies these results in some computational experiments.

II. Branch and bound algorithm

The standard branch and bound algorithm is used to estimate the global minimum of $\bar{\sigma}(F_L(M,\Delta)) : \mathbb{R}^s \to \mathbb{R}$ over the s-dimensional "cube" $\mathbf{B}_{\boldsymbol{\Delta}}$ (note that there are s parameters δ_i confined to the interval $[-1,1]$).

For a given sub-cube $\mathcal{Q} \subset \mathbf{B}_{\boldsymbol{\Delta}}$, the algorithm requires upper and lower bounds, denoted $\Phi_{\rm lb}$ and $\Phi_{\rm ub}$, respectively, for

$$\Phi_{\min}(\mathcal{Q}) := \min_{\Delta \in \mathcal{Q}} \bar{\sigma}(F_L(M,\Delta)).$$

The functions $\Phi_{\rm lb}$ and $\Phi_{\rm ub}$ satisfy

$$\Phi_{\rm lb}(\mathcal{Q}) \le \Phi_{\min}(\mathcal{Q}) \le \Phi_{\rm ub}(\mathcal{Q}).$$

At each iteration, a bounding strategy is used with the upper and lower bounds to select the next parameter interval to divide. A continuity condition is imposed to guarantee convergence of the algorithm: let $\operatorname{len}(\mathcal{Q})$ represent the length of the longest edge of $\mathcal{Q}$, then for all $\epsilon > 0$ there exists $\delta > 0$ such that

$$\forall \mathcal{Q} \subset \mathcal{Q}_{\rm init},\ \operatorname{len}(\mathcal{Q}) < \delta \implies \Phi_{\rm ub}(\mathcal{Q}) - \Phi_{\rm lb}(\mathcal{Q}) \le \epsilon. \qquad (3)$$

The branch and bound algorithm is repeated in the Appendix for completeness. Proof of algorithm convergence when the continuity condition is satisfied may be found in [2]. This reference also provides a thorough introduction to the application of branch and bound to control problems.

The performance of the algorithm is strongly influenced by the quality of the bounds, and, since we are attempting to minimize $\bar{\sigma}(F_L(M,\Delta))$, we will demonstrate that a tight *lower* bound is essential for good algorithm performance.

III. Bounds for $\Phi_{\min}(\mathbf{B}_{\boldsymbol{\Delta}})$

This section develops bounds for $\Phi_{\min}(\mathbf{B}_{\boldsymbol{\Delta}})$. If the parameter set is not the unit ball, then an initial scaling and loop shifting may be performed so we need only consider the case when the parameter set is $\mathbf{B}_{\boldsymbol{\Delta}}$. We also assume that $F_L(M,\Delta)$ is well-posed for all $\Delta \in \mathbf{B}_{\boldsymbol{\Delta}}$.

In the computational experiments of Section IV, an upper bound is obtained by choosing N random perturbations, $\Delta_i \in \mathbf{B}_{\boldsymbol{\Delta}}$, $i = 1,\ldots,N$ and setting

$$\Phi_{\rm ub}(\mathbf{B}_{\boldsymbol{\Delta}}) = \min\{\bar{\sigma}(F_L(M,\Delta_i)) : i = 1,\ldots,N\} \qquad (4)$$

Other approaches include evaluating $\bar{\sigma}(F_L(M,\Delta))$ at the cube midpoint or corners. Our choice is justified in Section IV.

Several additional quantities are defined to facilitate the development of lower bounds for (1). Let $\mathbf{u}$ and $\mathbf{v}$ be the left and right singular vectors of M_{11} corresponding to its maximum singular value and denote the maximum singular values of M_{ij} as $\bar{\sigma}_{ij}$, $i,j = 1,2$. We may assume that $\bar{\sigma}_{11} \neq 0$, otherwise $\Phi_{\min}(\mathbf{B}_{\boldsymbol{\Delta}}) = 0$ by choosing $\Delta = 0$. Define the following matrices

$$M_0 := \begin{bmatrix} 0 & M_{12} \\ M_{21} & M_{22} \end{bmatrix}$$

$$M_{\mathbf{uv}} := \begin{bmatrix} \sigma_{11} & \mathbf{u}^* M_{12} \\ M_{21}\mathbf{v} & M_{22} \end{bmatrix}$$

$$M_{\mathbf{uv},0} := \begin{bmatrix} 0 & \mathbf{u}^* M_{12} \\ M_{21}\mathbf{v} & M_{22} \end{bmatrix}$$

$$M_{\mathbf{uv}}^{[-1]} := \begin{bmatrix} \frac{1}{\bar{\sigma}_{11}} & -\frac{1}{\bar{\sigma}_{11}}\mathbf{u}^* M_{12} \\ \frac{1}{\bar{\sigma}_{11}} M_{21}\mathbf{v} & M_{22} - \frac{1}{\bar{\sigma}_{11}} M_{21}\mathbf{v}\mathbf{u}^* M_{12} \end{bmatrix}$$

$$M_{\mathbf{uv},0}^{[-1]} := \begin{bmatrix} 0 & -\frac{1}{\bar\sigma_{11}}\mathbf{u}^* M_{12} \\ \frac{1}{\bar\sigma_{11}} M_{21}\mathbf{v} & M_{22} - \frac{1}{\bar\sigma_{11}} M_{21}\mathbf{v}\mathbf{u}^* M_{12} \end{bmatrix}.$$

The "0" in M_0, $M_{\mathbf{uv},0}$ and $M_{\mathbf{uv},0}^{[-1]}$ indicates that the $(1,1)$ sub-matrix is replaced with a matrix of zeros with appropriate dimension.

A series of inequalities that bound (1) from below are stated in

Lemma 1: Assume $I - M_{22}\Delta$ is invertible for all $\Delta \in \mathbf{B}_{\mathbf{\Delta}}$. Assume further

$$\bar\sigma_{11} > \max_{\Delta\in\mathbf{B}_{\mathbf{\Delta}}} \bar\sigma(F_L(M_{\mathbf{uv},0},\Delta)). \tag{5}$$

Under these conditions

$$\begin{aligned}
&\min_{\Delta\in\mathbf{B}_{\mathbf{\Delta}}} \bar\sigma\big(F_L(M,\Delta)\big) \\
&\geq \left[\max_{\Delta\in\mathbf{B}_{\mathbf{\Delta}}} \bar\sigma\left(F_L(M_{\mathbf{uv}}^{[-1]},\Delta)\right)\right]^{-1} &(6)\\
&\geq \left[\frac{1}{\bar\sigma_{11}} + \max_{\Delta\in\mathbf{B}_{\mathbf{\Delta}}} \bar\sigma\left(F_L(M_{\mathbf{uv},0}^{[-1]},\Delta)\right)\right]^{-1} &(7)\\
&\geq \bar\sigma_{11} - \max_{\Delta\in\mathbf{B}_{\mathbf{\Delta}}} \bar\sigma\left(F_L(M_{\mathbf{uv},0},\Delta)\right) &(8)\\
&\geq \bar\sigma_{11} - \max_{\Delta\in\mathbf{B}_{\mathbf{\Delta}}} \bar\sigma\left(F_L(M_0,\Delta)\right) &(9)\\
&\geq \begin{cases} \bar\sigma_{11} - \frac{\bar\sigma_{12}\bar\sigma_{21}}{1-\bar\sigma_{22}} & \text{if } \bar\sigma_{22} < 1 \\ -\infty & \text{if } \bar\sigma_{22} \geq 1 \end{cases}. &(10)
\end{aligned}$$

The expressions in (6), (7), (8) exploit the "directionality" of M_{11} in the LFT by pre- and post-multiplication with its maximum singular vectors. This choice is not necessarily optimal but often gives much better results than (9) or (10).

Remark. Condition (5) is required for well-posedness of (6) and (7). If (5) is not satisfied then (6) and (7) are not defined and in this case (8), (9) and (10) yield useless negative bounds for $\min_{\mathbf{B}_{\mathbf{\Delta}}} \bar\sigma(F_L(M,\Delta))$.

Proof. The first assumption guarantees that $F_L(M,\Delta)$, $F_L(M_0,\Delta)$, $F_L(M_{\mathbf{uv}},\Delta)$ and $F_L(M_{\mathbf{uv},0},\Delta)$ are well-posed for $\Delta \in \mathbf{B}_{\mathbf{\Delta}}$. The second assumption guarantees that

$$F_L\left(\begin{bmatrix} 0 & 1 \\ 1 & -\frac{1}{\bar\sigma_{11}} \end{bmatrix}, F_L(M_{\mathbf{uv},0},\Delta)\right) = F_L\left(\begin{bmatrix} 0 & \mathbf{u}^* M_{12} \\ M_{21}\mathbf{v} & M_{22} - \frac{1}{\bar\sigma_{11}} M_{21}\mathbf{v}\mathbf{u}^* M_{12} \end{bmatrix}, \Delta\right)$$

is well-posed for $\Delta \in \mathbf{B}_{\mathbf{\Delta}}$. Thus $[I - (M_{22} - \frac{1}{\bar\sigma_{11}} M_{21}\mathbf{v}\mathbf{u}^* M_{12})\Delta]$ is invertible for all $\Delta \in \mathbf{B}_{\mathbf{\Delta}}$ and so $F_L(M_{\mathbf{uv}}^{[-1]},\Delta)$ and $F_L(M_{\mathbf{uv},0}^{[-1]},\Delta)$ are well-posed. For any unit vectors $\mathbf{u}$ and $\mathbf{v}$, $\bar\sigma(F_L(M,\Delta)) \geq |\mathbf{u}^* F_L(M,\Delta)\mathbf{v}|$, so with the particular choice of $\mathbf{u}$ and $\mathbf{v}$ as the singular vectors corresponding to $\bar\sigma(M_{11})$:

$$\begin{aligned}
&\min_{\Delta\in\mathbf{B}_{\mathbf{\Delta}}} \bar\sigma(F_L(M,\Delta)) \\
&\geq \min_{\Delta\in\mathbf{B}_{\mathbf{\Delta}}} |\mathbf{u}^* F_L(M,\Delta)\mathbf{v}| \\
&= \min_{\Delta\in\mathbf{B}_{\mathbf{\Delta}}} |F_L(M_{\mathbf{uv}},\Delta)| \\
&= \min_{\Delta\in\mathbf{B}_{\mathbf{\Delta}}} \left[\left|F_L(M_{\mathbf{uv}}^{[-1]},\Delta)\right|\right]^{-1} \\
&= \left[\max_{\Delta\in\mathbf{B}_{\mathbf{\Delta}}} \left|F_L(M_{\mathbf{uv}}^{[-1]},\Delta)\right|\right]^{-1} \\
&= \left[\max_{\Delta\in\mathbf{B}_{\mathbf{\Delta}}} \left|\frac{1}{\bar\sigma_{11}} + F_L(M_{\mathbf{uv},0}^{[-1]},\Delta)\right|\right]^{-1} \\
&\geq \left[\frac{1}{\bar\sigma_{11}} + \max_{\Delta\in\mathbf{B}_{\mathbf{\Delta}}} \left|F_L(M_{\mathbf{uv},0}^{[-1]},\Delta)\right|\right]^{-1},
\end{aligned}$$

where we have used the fact that $F_L(M_{\mathbf{uv}},\Delta) = (F_L(M_{\mathbf{uv}}^{[-1]},\Delta))^{-1}$ (see [10]). These inequalities establish (6) $\geq$ (7).

The following is derived from $F_L(M_{\mathbf{uv}}^{[-1]},\Delta)$:

$$\frac{1}{\frac{1}{\bar\sigma_{11}} + F_L(M_{\mathbf{uv},0}^{[-1]},\Delta)} = \bar\sigma_{11} + F_L(M_{\mathbf{uv},0},\Delta).$$

This relation may be rearranged to

$$\frac{\bar\sigma_{11} F_L(M_{\mathbf{uv},0}^{[-1]},\Delta)}{\frac{1}{\bar\sigma_{11}} + F_L(M_{\mathbf{uv},0}^{[-1]},\Delta)} = -F_L(M_{\mathbf{uv},0},\Delta).$$

This last expression is used to show the remaining inequalities.

Suppose that $\max_{\Delta\in\mathbf{B}_{\mathbf{\Delta}}}\left|F_L(M_{\mathbf{uv},0}^{[-1]},\Delta)\right|$ is achieved at Δ_0, then

$$\begin{aligned}
&\left[\tfrac{1}{\bar\sigma_{11}}+\max\nolimits_{\Delta\in\mathbf{B}_{\mathbf{\Delta}}}\left|F_L(M_{\mathbf{uv},0}^{[-1]},\Delta)\right|\right]^{-1}\\
&\qquad=\left[\frac{1}{\bar\sigma_{11}}+\left|F_L(M_{\mathbf{uv},0}^{[-1]},\Delta_0)\right|\right]^{-1}\\
&\qquad=\bar\sigma_{11}-\frac{\bar\sigma_{11}|F_L(M_{\mathbf{uv},0}^{[-1]},\Delta_0)|}{\frac{1}{\bar\sigma_{11}}+|F_L(M_{\mathbf{uv},0}^{[-1]},\Delta_0)|}\\
&\qquad\geq\bar\sigma_{11}-\left|\frac{\bar\sigma_{11}F_L(M_{\mathbf{uv},0}^{[-1]},\Delta_0)}{\frac{1}{\bar\sigma_{11}}+F_L(M_{\mathbf{uv},0}^{[-1]},\Delta_0)}\right|\\
&\qquad=\bar\sigma_{11}-|F_L(M_{\mathbf{uv},0},\Delta_0)|\\
&\qquad\geq\bar\sigma_{11}-\max_{\Delta\in\mathbf{B}_{\mathbf{\Delta}}}|F_L(M_{\mathbf{uv},0},\Delta)|\\
&\qquad\geq\bar\sigma_{11}-\max_{\Delta\in\mathbf{B}_{\mathbf{\Delta}}}\bar\sigma(F_L(M_0,\Delta)).
\end{aligned}$$

Thus, (7) $\geq$ (8) $\geq$ (9) are established. The last inequality, namely (9) $\geq$ (10), follows from

$$\begin{aligned}
\max_{\Delta\in\mathbf{B}_{\mathbf{\Delta}}}\bar\sigma(F_L(M_0,\Delta))&\leq\max_{\substack{\Delta\in\mathbb{C}^{q\times q}\\ \|\Delta\|\leq 1}}\bar\sigma(F_L(M_0,\Delta))\\
&=\frac{\bar\sigma_{12}\bar\sigma_{21}}{1-\bar\sigma_{22}},
\end{aligned}$$

when $\bar\sigma_{22}<1$. ■

The expressions (6) through (9) each contain a term of the form

$$\max_{\Delta\in\mathbf{B}_{\mathbf{\Delta}}}\bar\sigma(F_L(\tilde M,\Delta)),\tag{11}$$

where $\tilde M$ is an appropriate matrix. Lower bounds for (6)–(9) are developed by replacing (11) with its *upper* bound computed using the structure singular value theory (see [8]). Toward this end, define two sets of scaling matrices that commute with $\mathbf{\Delta}$

$$\begin{aligned}
\mathbb{D}&=\left\{\text{diag }[D_1,\ldots,D_S]:\ D_i\in\mathbb{C}^{r_i\times r_i},D_i=D_i^*>0\right\}\\
\mathbb{G}&=\left\{\text{diag }[G_1,\ldots,G_S]:\ G_i\in\mathbb{C}^{r_i\times r_i},G_i=G_i^*\right\}.
\end{aligned}$$

Define

$$\begin{aligned}
\gamma_0:=\inf\Big\{\gamma>0:\inf_{\tilde D,\tilde G}\bar\sigma&\left[j\tilde G(I+\tilde G^2)^{-\frac12}\right.\\
&\left.+(I+\tilde G^2)^{-\frac12}\tilde D_L\tilde M\tilde D_R\right]<1\Big\},
\end{aligned}\tag{12}$$

where

$$\tilde D_L=\begin{bmatrix}\frac{1}{\sqrt{\gamma}}&0\\0&D\end{bmatrix},\quad \tilde D_R=\begin{bmatrix}\frac{1}{\sqrt{\gamma}}&0\\0&D^{-1}\end{bmatrix},\quad D\in\mathbb{D}$$

$$\tilde G=\begin{bmatrix}0&0\\0&G\end{bmatrix},\quad G\in\mathbb{G},\quad j=\sqrt{-1}.$$

With γ_0 so defined we have

$$\max_{\Delta\in\mathbf{B}_{\mathbf{\Delta}}}\bar\sigma(F_L(\tilde M,\Delta))\leq\gamma_0.$$

For notational clarity, an additional subscript is used in γ_0 to indicate the equation from which it was derived, i.e., $\gamma_{0,(6)}\geq\max\bar\sigma(F_L(M_{\mathbf{uv}}^{[-1]},\Delta))$, $\gamma_{0,(7)}\geq\max\bar\sigma(F_L(M_{\mathbf{uv},0}^{[-1]},\Delta))$, etc. Thus, $\gamma_{0,(6)}$ is computed with $\tilde M=M_{\mathbf{uv}}^{[-1]}$, $\gamma_{0,(7)}$ with $\tilde M=M_{\mathbf{uv},0}^{[-1]}$, $\gamma_{0,(8)}$ with $\tilde M=M_{\mathbf{uv},0}$, and $\gamma_{0,(9)}$ with $\tilde M=M_0$. The following lower bounds are computed for (6) through (9)

$$\begin{aligned}
&\left[\max_{\Delta\in\mathbf{B}_{\mathbf{\Delta}}}\bar\sigma\Big(F_L(M_{\mathbf{uv}}^{[-1]},\Delta)\Big)\right]^{-1}\geq\\
&\qquad\beta_{(6)}:=\begin{cases}\frac{1}{\gamma_{0,(6)}}&\text{(12) feasible}\\ 0&\text{(12) infeasible}\end{cases},
\end{aligned}\tag{13}$$

$$\left[\frac{1}{\bar\sigma_{11}}+\max_{\Delta\in\mathbf{B}_{\mathbf{\Delta}}}\bar\sigma\Big(F_L(M_{\mathbf{uv},0}^{[-1]},\Delta)\Big)\right]^{-1}\geq$$

$$\beta_{(7)} := \begin{cases} \left[\frac{1}{\bar{\sigma}_{11}} + \gamma_{0,(7)}\right]^{-1} & (12) \text{ feasible} \\ 0 & (12) \text{ infeasible} \end{cases}, \tag{14}$$

$$\bar{\sigma}_{11} - \max_{\Delta \in \mathbf{B}_{\boldsymbol{\Delta}}} \bar{\sigma}\left(F_L(M_{\mathbf{uv},0}, \Delta)\right) \geq$$

$$\beta_{(8)} := \begin{cases} \max\{\bar{\sigma}_{11} - \gamma_{0,(8)}, 0\} & (12) \text{ feasible} \\ 0 & (12) \text{ infeasible} \end{cases}, \tag{15}$$

$$\bar{\sigma}_{11} - \max_{\Delta \in \mathbf{B}_{\boldsymbol{\Delta}}} \bar{\sigma}\left(F_L(M_0, \Delta)\right) \geq$$

$$\beta_{(9)} := \begin{cases} \bar{\sigma}_{11} - \gamma_{0,(9)} & (12) \text{ feasible} \\ -\infty & (12) \text{ infeasible} \end{cases}, \tag{16}$$

$$\beta_{(10)} := \begin{cases} \bar{\sigma}_{11} - \frac{\bar{\sigma}_{12}\bar{\sigma}_{21}}{1-\bar{\sigma}_{22}} & \text{if } \bar{\sigma}_{22} < 1 \\ -\infty & \text{if } \bar{\sigma}_{22} \geq 1 \end{cases}. \tag{17}$$

Interestingly, $\beta_{(6)}$ to $\beta_{(10)}$ satisfy the same ordering as (6) to (10). This is stated in the following lemma.

Lemma 2: Compute $\beta_{(6)}$ through $\beta_{(10)}$ according to (13) through (17). Then

$$\beta_{(6)} \geq \beta_{(7)} \geq \beta_{(8)} \geq \beta_{(9)} \geq \beta_{(10)}.$$

Any of these bounds may be chosen as Φ_{lb} for use in the branch and bound algorithm. These lower bounds and the upper bound (4) satisfy the continuity condition (3). The proof of this fact is in the Appendix.

Before proceeding with the proof of Lemma 2, recall that the *Redheffer star product* [10] of two matrices P and T is

$$\mathcal{S}(P,T) := \begin{bmatrix} F_L(P, T_{11}) & P_{12}(I - T_{11}P_{22})^{-1}T_{12} \\ T_{21}(I - P_{22}T_{11})^{-1}P_{21} & F_U(T, P_{22}) \end{bmatrix},$$

where P and T are compatibly partitioned as

$$P = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} \qquad T = \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix}.$$

The star product is well-posed when $[I - P_{22}T_{11}]$ is invertible. Furthermore if $\bar{\sigma}(P) \leq 1$ and $\bar{\sigma}(T) \leq 1$ then $\bar{\sigma}(\mathcal{S}(P,T)) \leq 1$ if the connection is well-posed. These properties will be used in the following proof.

Proof of Lemma 2

Proof of $\beta_{(6)} \geq \beta_{(7)}$. Suppose that the computation of $\gamma_{0,(7)}$ is feasible and $\beta_{(7)}$ is computed according to (14) (if the computation of $\gamma_{0,(7)}$ is not feasible then $\beta_{(6)} \geq \beta_{(7)}$ is trivially satisfied). Then there exist $G \in \mathbb{G}$ and $D \in \mathbb{D}$ such that $\bar{\sigma}(T) \leq 1$, where

$$T = j\begin{bmatrix} 0 & 0 \\ 0 & G \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & (I+G^2)^{-\frac{1}{2}} \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & (I+G^2)^{-\frac{1}{2}} \end{bmatrix}\begin{bmatrix} \frac{1}{\sqrt{\gamma_{0,(7)}}} & 0 \\ 0 & D \end{bmatrix} M_{\mathbf{uv},0}^{[-1]} \begin{bmatrix} \frac{1}{\sqrt{\gamma_{0,(7)}}} & 0 \\ 0 & D^{-1} \end{bmatrix}.$$

Now, with the same D and G consider the matrix

$$W = j\begin{bmatrix} 0 & 0 \\ 0 & G \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & (I+G^2)^{-\frac{1}{2}} \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & (I+G^2)^{-\frac{1}{2}} \end{bmatrix} *$$

$$\begin{bmatrix} \frac{1}{\sqrt{\gamma_{0,(7)}\left(1+\frac{1}{\gamma_{0,(7)}\bar{\sigma}_{11}}\right)}} & 0 \\ 0 & D \end{bmatrix} M_{\mathbf{uv}}^{[-1]} \begin{bmatrix} \frac{1}{\sqrt{\gamma_{0,(7)}\left(1+\frac{1}{\gamma_{0,(7)}\bar{\sigma}_{11}}\right)}} & 0 \\ 0 & D^{-1} \end{bmatrix}.$$

With some effort, W may be manipulated into the following representation

$$W = \mathcal{S}(P,T) \quad \text{where } P = \begin{bmatrix} \frac{1}{\bar{\sigma}_{11}\gamma_{0,(7)}+1} & \sqrt{\frac{\bar{\sigma}_{11}\gamma_{0,(7)}}{\bar{\sigma}_{11}\gamma_{0,(7)}+1}} \\ \sqrt{\frac{\bar{\sigma}_{11}\gamma_{0,(7)}}{\bar{\sigma}_{11}\gamma_{0,(7)}+1}} & 0 \end{bmatrix}.$$

The star product is well-posed since $P_{22} = 0$. By hypothesis

$\bar\sigma(T) \le 1$, and it may be confirmed that

$$\bar\sigma(P) \equiv 1 \quad \forall \bar\sigma_{11} > 0,\ \gamma_{0,(7)} > 0.$$

Thus, $\bar\sigma(W) = \bar\sigma(\mathcal{S}(P,T)) \le 1$. In light of the definition of $\beta_{(6)}$, the "performance" scaling employed in W,

$$\gamma_{0,(7)}\left(1 + \frac{1}{\gamma_{0,(7)}\bar\sigma_{11}}\right),$$

is an *upper* bound for the best value given by $\gamma_{0,(6)}$ in (13). Hence,

$$\beta_{(6)} = \frac{1}{\gamma_{0,(6)}} \ge \frac{1}{\gamma_{0,(7)}\left(1 + \frac{1}{\gamma_{0,(7)}\bar\sigma_{11}}\right)} = \frac{1}{\gamma_{0,(7)} + \frac{1}{\bar\sigma_{11}}} = \beta_{(7)}.$$

Proof of $\beta_{(7)} \ge \beta_{(8)}$. Let $\gamma_{0,(8)}$ and $\beta_{(8)}$ be computed as in (15). Note that $\beta_{(8)} = 0$ if either $\gamma_{0,(8)}$ is not defined or $\bar\sigma_{11} \le \gamma_{0,(8)}$. In these cases $\beta_{(7)} \ge \beta_{(8)}$ is trivially satisfied. Thus we may assume $\bar\sigma_{11} > \gamma_{0,(8)} > 0$ without loss of generality. Now let $D \in \mathbb{D}$ and $G \in \mathbb{G}$ be the associated scales such that $\bar\sigma(T) \le 1$ where

$$T = j\begin{bmatrix} 0 & 0 \\ 0 & G \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & (I+G^2)^{-\frac{1}{2}} \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & (I+G^2)^{-\frac{1}{2}} \end{bmatrix}\begin{bmatrix} \frac{1}{\sqrt{\gamma_{0,(8)}}} & 0 \\ 0 & D \end{bmatrix} M_{\mathbf{uv},0} \begin{bmatrix} \frac{1}{\sqrt{\gamma_{0,(8)}}} & 0 \\ 0 & D^{-1} \end{bmatrix}.$$

Keeping D and G fixed, consider the scaled matrix

$$W = j\begin{bmatrix} 0 & 0 \\ 0 & G \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & (I+G^2)^{-\frac{1}{2}} \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & (I+G^2)^{-\frac{1}{2}} \end{bmatrix} * \begin{bmatrix} -\bar\sigma_{11}\sqrt{\frac{1-\frac{\gamma_{0,(8)}}{\bar\sigma_{11}}}{\gamma_{0,(8)}}} & 0 \\ 0 & D \end{bmatrix} M_{\mathbf{uv},0}^{[-1]} \begin{bmatrix} \bar\sigma_{11}\sqrt{\frac{1-\frac{\gamma_{0,(8)}}{\bar\sigma_{11}}}{\gamma_{0,(8)}}} & 0 \\ 0 & D^{-1} \end{bmatrix},$$

where we have used the fact that $1 > \gamma_{0,(8)}/\bar\sigma_{11} > 0$. The star product may be used to represent W as

$$W = \mathcal{S}(P,T), \quad \text{where } P = \begin{bmatrix} 0 & \sqrt{1 - \frac{\gamma_{0,(8)}}{\bar\sigma_{11}}} \\ \sqrt{1 - \frac{\gamma_{0,(8)}}{\bar\sigma_{11}}} & -\frac{\gamma_{0,(8)}}{\bar\sigma_{11}} \end{bmatrix}.$$

The star product is well-posed because $\bar\sigma(T_{11}) \le 1$ and $\bar\sigma(P_{22}) = \frac{\gamma_{0,(8)}}{\bar\sigma_{11}} < 1$. A calculation reveals that

$$\bar\sigma(P) \equiv 1 \quad \forall\, 1 > \frac{\gamma_{0,(8)}}{\bar\sigma_{11}} > 0,$$

and since $\bar\sigma(T) \le 1$ we conclude that $\bar\sigma(W) \le 1$.

Let $\gamma_{0,(7)}$ be computed in the definition of $\beta_{(7)}$. From the definition of W with its associated performance scale the following holds,

$$\gamma_{0,(7)} \le \frac{\gamma_{0,(8)}}{\bar\sigma_{11}^2\left(1 - \frac{\gamma_{0,(8)}}{\bar\sigma_{11}}\right)}.$$

Thus,

$$\begin{aligned}
\beta_{(7)} &= \frac{1}{\frac{1}{\bar\sigma_{11}} + \gamma_{0,(7)}} \\
&\ge \frac{1}{\frac{1}{\bar\sigma_{11}} + \frac{\gamma_{0,(8)}}{\bar\sigma_{11}^2(1-\frac{\gamma_{0,(8)}}{\bar\sigma_{11}})}} \\
&= \bar\sigma_{11} - \bar\sigma_{11}\frac{\frac{\bar\sigma_{11}\gamma_{0,(8)}}{\bar\sigma_{11}^2 - \bar\sigma_{11}\gamma_{0,(8)}}}{1 + \frac{\bar\sigma_{11}\gamma_{0,(8)}}{\bar\sigma_{11}^2 - \bar\sigma_{11}\gamma_{0,(8)}}} \\
&= \bar\sigma_{11} - \gamma_{0,(8)} \\
&= \beta_{(8)}.
\end{aligned}$$

Proof of $\beta_{(8)} \ge \beta_{(9)}$. Compute $\beta_{(9)}$ and its corresponding $\gamma_{0,(9)}$. Without loss of generality we may assume that the computation of $\gamma_{0,(9)} > 0$ is feasible. Then there exist $D \in \mathbb{D}$ and $G \in \mathbb{G}$ such that $\bar\sigma(W) \le 1$ where

$$W = j\begin{bmatrix} 0 & 0 \\ 0 & G \end{bmatrix}\begin{bmatrix} I & 0 \\ 0 & (I+G^2)^{-\frac{1}{2}} \end{bmatrix}$$

$$+\begin{bmatrix} I & 0 \\ 0 & (I+G^2)^{-\frac{1}{2}} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{\gamma_{0,(9)}}} I & 0 \\ 0 & D \end{bmatrix} M_0 \begin{bmatrix} \frac{1}{\sqrt{\gamma_{0,(9)}}} I & 0 \\ 0 & D^{-1} \end{bmatrix}.$$

Thus,

$$\bar{\sigma}\left(j \begin{bmatrix} 0 & 0 \\ 0 & G \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & (I+G^2)^{-\frac{1}{2}} \end{bmatrix} \right.$$
$$\left. + \begin{bmatrix} 1 & 0 \\ 0 & (I+G^2)^{-\frac{1}{2}} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{\gamma_{0,(9)}}} & 0 \\ 0 & D \end{bmatrix} M_{\mathbf{uv},0} \begin{bmatrix} \frac{1}{\sqrt{\gamma_{0,(9)}}} & 0 \\ 0 & D^{-1} \end{bmatrix} \right)$$
$$= \bar{\sigma}\left(\begin{bmatrix} \mathbf{u}^* & 0 \\ 0 & I \end{bmatrix} W \begin{bmatrix} \mathbf{v} & 0 \\ 0 & I \end{bmatrix} \right)$$
$$\leq \bar{\sigma}\left(\begin{bmatrix} \mathbf{u}^* & 0 \\ 0 & I \end{bmatrix} \right) \bar{\sigma}(W) \bar{\sigma}\left(\begin{bmatrix} \mathbf{v} & 0 \\ 0 & I \end{bmatrix} \right)$$
$$\leq 1.$$

From the definition of $\beta_{(8)}$ and $\gamma_{0,(8)}$ we have $\gamma_{0,(8)} \leq \gamma_{0,(9)}$, so $\beta_{(8)} \geq \beta_{(9)}$.

Proof of $\beta_{(9)} \geq \beta_{(10)}$. Assume $\bar{\sigma}_{22} < 1$ otherwise the inequality is trivially satisfied. The calculation of $\gamma_{0,(9)}$ is feasible in this case. Now define

$$\hat{\gamma}_0 := \inf \left\{ \gamma > 0 : \inf_{\tilde{D}} \bar{\sigma}\left(\tilde{D} M_0 \tilde{D}^{-1} \right) < 1 \right\},$$

where

$$\tilde{D} := \begin{bmatrix} \frac{1}{\sqrt{\gamma}} I & 0 \\ 0 & dI \end{bmatrix}, \quad d > 0.$$

Note that $\hat{\gamma}_0 > 0$ since $\bar{\sigma}_{22} < 1$. Also note that $\hat{\gamma}_0 \geq \gamma_{0,(9)}$ since there are fewer degrees of freedom in the computation of $\hat{\gamma}_0$ than in the computation of $\gamma_{0,(9)}$. The $\tilde{D}$ scaling in the computation of $\hat{\gamma}_0$ assumes a full complex block perturbation and since the structured singular value is equal to its upper bound in this case (see [8]), the following equality holds

$$\hat{\gamma}_0 = \max_{\substack{\Delta \in \mathbb{C}^{q \times q} \\ \bar{\sigma}(\Delta) \leq 1}} \bar{\sigma}(F_L(M_0, \Delta)).$$

Singular value inequalities may be used to show

$$\bar{\sigma}(F_L(M_0, \Delta)) \leq \frac{\bar{\sigma}_{12}\bar{\sigma}_{21}}{1 - \bar{\sigma}_{22}} \quad \forall\, \Delta \in \mathbb{C}^{q \times q}, \bar{\sigma}(\Delta) \leq 1.$$

Collecting these results

$$\gamma_{0,(9)} \leq \hat{\gamma}_0 \leq \frac{\bar{\sigma}_{12}\bar{\sigma}_{21}}{1 - \bar{\sigma}_{22}}.$$

This establishes $\beta_{(9)} \geq \beta_{(10)}$. ■

IV. Computational results

Computation of γ_0, and hence $\Phi_{\mathrm{lb}}(\mathbf{B}_{\boldsymbol{\Delta}})$ via (13) to (16), may be formulated as the minimization of a linear functional subject to a linear matrix inequality constraint and is efficiently computed using commercially available software packages. The simulations in this paper implemented the branch and bound algorithm and lower bounds using Matlab's LMI Control Toolbox [6].

The number of decision variables required in the computation of $\beta_{(6)}$ through $\beta_{(9)}$ is constant, regardless of the choice of $\tilde{M}$. The calculation of $\beta_{(10)}$, however, is computationally inexpensive. Thus it is informative to compare the gap between these bounds and their computation times. Figure 1 shows the averaged (and normalized) values of $\beta_{(6)}$ to $\beta_{(10)}$ for 1000 random M matrices. The matrices are scaled to ensure $(I - M_{22}\Delta)^{-1}$ exists for all $\Delta \in \mathbf{B}_{\boldsymbol{\Delta}}$. The dimensions of the partitions of M in these simulations are

$$M_{11} \in \mathbb{C}^{4\times 4} \quad M_{12} \in \mathbb{C}^{4\times 8} \quad M_{21} \in \mathbb{C}^{8\times 4} \quad M_{22} \in \mathbb{C}^{8\times 8}. \quad (18)$$

The perturbation set contains six real parameters:

$$\boldsymbol{\Delta} = \{\mathrm{diag}[\delta_1, \delta_2, \delta_3, \delta_4, \delta_5 I_2, \delta_6 I_2] : \delta_i \in \mathbb{R}, |\delta_i| \leq 1\}. \quad (19)$$

The average values of $\beta_{(7)}$ to $\beta_{(10)}$ are normalized with respect to the average value of $\beta_{(6)}$. In other words, for a given M, the bounds are computed, then divided by the value of $\beta_{(6)}$, and finally averaged across all simulations with the 1000 random matrices. For example, the chart shows that on average $\beta_{(8)}$ yields a lower bound that is approximately half that of $\beta_{(6)}$. This gives a quantitative assessment of the gap between the lower bounds. A bound whose computation is infeasible or yields a negative number is taken to be zero for purposes of the bar chart. In the majority of random cases used to compile the chart, $\bar{\sigma}(M_{22}) > 1$ so $\beta_{(10)}$ is zero. This lowers the average value of $\beta_{(10)}$ to essentially zero. This comparison is slightly unfair to $\beta_{(10)}$ since in the branch and bound search $\beta_{(10)}$ approaches the other bounds from below and the gap between the bounds is reduced as the algorithm progresses. The situation represented in Figure 1, however, reflects the performance of the bounds at the beginning of the branch and bound search.

The average computation times of the bounds are also shown in Figure 1. As in the case of the bounds, the computation times are normalized with respect to the average execution time for $\beta_{(6)}$. The figure demonstrates that $\beta_{(6)}$ is the tightest lower bound and requires slightly less time to compute than $\beta_{(7)}$ through $\beta_{(9)}$. The figure also shows that $\beta_{(10)}$ requires much less computation than the other bounds but also gives a very poor lower bound.

Figure 1 also shows that there is no incentive to use $\beta_{(7)}$, $\beta_{(8)}$, or $\beta_{(9)}$ for Φ_{lb} but a comparison between the performance of the branch and bound routine using $\beta_{(6)}$ or $\beta_{(10)}$ for Φ_{lb} should be informative and will reveal the trade-off between a tight (but computationally expensive) lower bound and a crude (but computationally cheap) lower bound.

Figure 2 summarizes the results of 1000 simulations of the branch and bound algorithm with $\Phi_{\mathrm{lb}} = \beta_{(6)}$ and Φ_{ub} given by (4). In the computation of Φ_{ub}, N is chosen to be 40. The objective is to spend roughly equal time refining the upper bound as it takes to compute the lower bound. A typical choice for Φ_{ub} is to calculate $\bar{\sigma}(F_L(M, \Delta))$ at the mid-point of the parameter set but by replacing this choice with (4) more effort is spent on computing the upper bound. Our approach is heuristic but we have noted modest savings in total computation time. The random M's in the branch and bound simulations have the dimensions given in (18) and (19). The search is terminated when the global lower bound is greater than 90% of the global upper bound. In other words, $\Phi_{\min}$ is computed to within 10%. The computation time and total number of iterations are displayed. These figures reveal that the branch and bound performs efficiently on most problems when $\Phi_{\mathrm{lb}} = \beta_{(6)}$. There are, however, some exceptional cases for which the optimization requires much longer computation time. The simulations were performed on a 400 MHz PC.

In contrast to the results with $\beta_{(6)}$, the performance of the branch and bound algorithm is not satisfactory when $\Phi_{\mathrm{lb}} = \beta_{(10)}$. A number of simulations were performed and all required more than *12 hours* of computation for convergence. Given these excessive simulation times it was not possible to compile statistics in the spirit of Figure 2 for this case. Even though the computation time for $\beta_{(10)}$ is more than an order of magnitude smaller that the computation time for $\beta_{(6)}$, the large number of iterations required for convergence prohibits the use of the algorithm for problems with more that a few parameters. The bound $\beta_{(6)}$, though, gives an overall savings in computation time even though the bound itself takes longer to compute than $\beta_{(10)}$.

V. Conclusion

Branch and bound is a useful tool for solving certain control analysis and design problems. The algorithm performance, however, depends on the quality of the bounds. This paper considered the computation of $\min_{\Delta \in \mathbf{B}_\Delta} F_L(M, \Delta)$. Four new lower bounds, that may be calculated via convex optimization, were developed for this problem. Simulation results revealed that tight bounds, though computationally expensive, are essential for good algorithm performance and lead to a large savings in total computation time.

Appendix

The standard branch and bound algorithm and notation is taken from [2] and is reproduced below:

$k = 0$

$\mathcal{L}_0 = \{\mathbf{B}_\Delta\};$

$L_0 = \Phi_{\mathrm{lb}}(\mathbf{B}_\Delta);$

$U_0 = \Phi_{\mathrm{ub}}(\mathbf{B}_\Delta);$

while $U_k - L_k > \epsilon, \{$

choose $\mathcal{Q} \in \mathcal{L}_k$ *such that* $\Phi_{\mathrm{lb}} = L_k;$

split $\mathcal{Q}$ *along its longest edge into* $\mathcal{Q}_I$ *and* $\mathcal{Q}_{II};$

$\mathcal{L}_{k+1} := (\mathcal{L}_k - \{\mathcal{Q}\}) \cup \{\mathcal{Q}_I, \mathcal{Q}_{II}\};$

$L_{k+1} := \min_{\mathcal{Q} \in \mathcal{L}_{k+1}} \Phi_{\mathrm{lb}}(\mathcal{Q});$

$U_{k+1} := \min_{\mathcal{Q} \in \mathcal{L}_{k+1}} \Phi_{\mathrm{ub}}(\mathcal{Q});$

$k = k + 1;$

$\}$

The iteration index is denoted by k, the list of cubes by $\mathcal{L}_k$, the lower bound by L_k and the upper bound by U_k for $\Phi_{\min}(\mathbf{B}_\Delta)$ at the end of k iterations. The cube with the lowest lower bound is split along its longest edge. This strategy seems to work well in most cases although worst-case combinatoric behavior is possible. See [2] for proofs of convergence of the algorithm.

Proof of continuity condition (3). Let $\alpha \mathbf{B}_{\Delta_0}$ denote a ball of radius $\alpha > 0$ centered at $\Delta_0 \in \mathbf{B}_\Delta$. The uniform continuity condition can be proven by showing that for any $\epsilon > 0$ there exists an α such that

$$|\Phi_{\mathrm{ub}}(\alpha \mathbf{B}_{\Delta_0}) - \Phi_{\mathrm{lb}}(\alpha \mathbf{B}_{\Delta_0})| < \epsilon \quad \forall \Delta_0 \in \mathbf{B}_\Delta. \tag{20}$$

The minimum norm of the LFT of M over the set $\alpha \mathbf{B}_{\Delta_0}$ is

$$\Phi_{\min}(\alpha \mathbf{B}_{\Delta_0}) = \min_{\substack{\Delta \in \boldsymbol{\Delta} \\ \bar{\sigma}(\Delta) \le \alpha}} F_L(\bar{M}, \Delta),$$

where

$$\bar{M} = \begin{bmatrix} \bar{M}_{11} & \bar{M}_{12} \\ \bar{M}_{21} & \bar{M}_{22} \end{bmatrix}$$

$$= \begin{bmatrix} M_{11} + M_{12}\Delta_0(I - M_{22}\Delta_0)^{-1}M_{21} & M_{12}(I - \Delta_0 M_{22})^{-1}) \\ (I - M_{22}\Delta_0)^{-1}M_{21} & (I - M_{22}\Delta_0)^{-1}M_{22}. \end{bmatrix}$$

Singular values inequalities can be used to show that

$$\Phi_{\mathrm{ub}}(\alpha \mathbf{B}_{\Delta_0}) \le \bar{\sigma}(\bar{M}_{11}) + \alpha \frac{\bar{\sigma}(\bar{M}_{12})\bar{\sigma}(\bar{M}_{21})}{1 - \alpha \bar{\sigma}(\bar{M}_{22})}.$$

Furthermore, if Φ_{lb} is chosen as any of the bounds (13) to (17), then for α sufficiently small

$$\Phi_{\mathrm{lb}}(\alpha \mathbf{B}_{\Delta_0}) \ge \bar{\sigma}(\bar{M}_{11}) - \alpha \frac{\bar{\sigma}(\bar{M}_{12})\bar{\sigma}(\bar{M}_{21})}{1 - \alpha \bar{\sigma}(\bar{M}_{22})}. \tag{21}$$

The right hand side of (21) is merely $\beta_{(10)}$ (the lowest of the lower bounds) computed for a ball of size α instead of the unit ball $\mathbf{B}_{\boldsymbol{\Delta}}$. The gap between the bounds is

$$|\Phi_{\mathrm{ub}}(\alpha\mathbf{B}_{\boldsymbol{\Delta_0}}) - \Phi_{\mathrm{lb}}(\alpha\mathbf{B}_{\boldsymbol{\Delta_0}})| \leq 2\alpha\frac{\bar{\sigma}(\bar{M}_{12})\bar{\sigma}(\bar{M}_{21})}{1-\alpha\bar{\sigma}(\bar{M}_{22})}.$$

Now define

$$\begin{aligned}
\Gamma_{12} &:= \max_{\Delta_0\in\mathbf{B}_{\boldsymbol{\Delta}}} \bar{\sigma}(\bar{M}_{12}) = \max_{\Delta\in\mathbf{B}_{\boldsymbol{\Delta}}} \bar{\sigma}\left(M_{12}(I-\Delta M_{22})^{-1})\right) \\
\Gamma_{21} &:= \max_{\Delta_0\in\mathbf{B}_{\boldsymbol{\Delta}}} \bar{\sigma}(\bar{M}_{21}) = \max_{\Delta\in\mathbf{B}_{\boldsymbol{\Delta}}} \bar{\sigma}\left((I-M_{22}\Delta)^{-1}M_{21}\right) \\
\Gamma_{22} &:= \max_{\Delta_0\in\mathbf{B}_{\boldsymbol{\Delta}}} \bar{\sigma}(\bar{M}_{22}) = \max_{\Delta\in\mathbf{B}_{\boldsymbol{\Delta}}} \bar{\sigma}\left((I-M_{22}\Delta)^{-1}M_{22}\right).
\end{aligned}$$

The Γ_{ij} are well defined since $\mathbf{B}_{\boldsymbol{\Delta}}$ is compact and $I - M_{22}\Delta$ is invertible for all $\Delta \in \mathbf{B}_{\boldsymbol{\Delta}}$. Thus, it is possible to choose $\alpha > 0$ such that (20) is satisfied since

$$|\Phi_{\mathrm{ub}}(\alpha\mathbf{B}_{\boldsymbol{\Delta_0}}) - \Phi_{\mathrm{lb}}(\alpha\mathbf{B}_{\boldsymbol{\Delta_0}})| \leq 2\alpha\frac{\Gamma_{12}\Gamma_{21}}{1-\alpha\Gamma_{22}}$$

independent of $\Delta_0 \in \mathbf{B}_{\boldsymbol{\Delta}}$. ■

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