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Constrained stabilization problems for linear plants ☆

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Abstract

Linear systems with magnitude and rate constraints on both the state and control variables are considered. For such systems, semi-global and global constrained stabilization problems are formulated when state feedback controllers are used. Necessary and sufficient conditions for the solvability of the formulated problems are developed. Moreover, design methodologies for such constrained stabilization problems are presented. An important aspect of our development here is a taxonomy of constraints to show clearly for what type of constraints what can or cannot be achieved. © 2002 Elsevier Science Ltd. All rights reserved.

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1. Introduction

While much progress has been made in the development of multivariable linear control theory, its applicability is often restricted by physical limitations. A primary reason for this is that most practical control problems are dominated by *constraints*. Valves can only be operated between fully open and fully closed positions, pumps and compressors have finite throughput capacities, and a tank can only hold a certain volume. These *constraints* are typically ignored in linear multivariable control theory. Ignoring such constraints can be detrimental to the stability and performance of control systems. A classical example for the detrimental effect of neglecting *constraints* is the Chernobyl unit four nuclear power plant disaster in 1986.

Two most commonly encountered *constraints* in control engineering are

- Constraints which pertain to actuators and are typically magnitude or rate constraints.
- Constraints which pertain to states and are typically magnitude constraints on some part of the states.

Actuator constraints have been studied extensively in optimal control in the 1960s and also methods such as describing functions have been developed around that time. But with the development of state space methods in the 1970s and 1980s, there was only a very limited research effort in the effects of actuator saturation. During the 1990s we have witnessed a renewed and intense research activity in the area of control of linear plants with saturating actuators. The early phase of this research renewal in the early 1990s was described in Bernstein and Michel (1995) while Saberi and Stoorvogel (1999) captures the recent research activities in this area.

State constraints are a major concern in many plants. Nearly every application imposes constraints on state as well as control variables. We observe that dynamic models of physical systems are often nonlinear. Linear approximations of such nonlinear systems are obviously valid only in certain constraint regions of state and

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control spaces. In process control, state and control constraints arise from economic necessity of operating the plants near the boundaries of feasible regions. In connection with safety issues, state and control constraints are a major concern in many plants. In certain possibly hazardous systems, such as a nuclear power plant, safety limits on some variables are often imposed. The violations of such predetermined safety measures may cause system malfunction or even damage. This implies that magnitude constraints or bounds on states must be taken as integral parts of any control system design.

However, state constraints, unlike actuator constraints, have not received much attention from a structural point of view. There have been some efforts to deal with state and input constraints utilizing the concept of positive invariant sets. A recent paper (Blanchini, 1999) gives a good overview of these efforts. The available tools presented in this line of work, however, are computationally very demanding and yield highly complex controllers. Model predictive control, which is a popular design technique for industrial processes (Camacho & Bordons, 1998; Maciejowski, 2002), also has been used to deal with constraints on states as well as inputs (Mayne, Rawlings, Rao, & Scokaert, 2000). However, this technique is intrinsically computationally intensive and therefore, not suitable for systems with fast dynamics. Secondly, it is fundamentally a numerical tool and gives only limited insight in the structural properties and effects of constraints on a system.

The focus of this paper is on stabilization of linear systems with state and/or input magnitude constraints. Our primary emphasis is on identifying the structural properties of linear plants under which the so-called constrained semi-global and global stabilization problems are solvable. Whenever the required structural properties are satisfied, design methodologies for constrained semi-global and global output regulation follow from the constructive methods of proving the obtained results. These aspects of our work distinguish us from other works dealing with state and input constraints. One can view our work in the same spirit as the pioneering work of Sontag and Sussmann (1990) which deals with input constraints only.

The paper is organized as follows. Section 2 considers some preliminaries and formal formulations of various problems including the constrained semi-global and global stabilization problems. Section 3 considers a taxonomy of constraints. Let us expand on this. Constraints on the plant reflect in our model in an output of the plant which we label as *constraint output* which is required to lie in a prescribed constraint set. It turns out that the mapping from the input to the constraint output vector or more specifically its structural properties play dominant roles. Based on their impact for control purposes, we categorize these structural properties in two directions that have a profound impact on what can or cannot be achieved. One direction of categorization is based on the right invertibility or lack of it of the mapping from the input to the constraint output vector. This direction of categorization delineates the constraints into two mutually exclusive categories, (1) right invertible constraints representing the case when the mapping from the input to the constraint output vector is right invertible, and (2) non-right invertible constraints representing the case when the mapping from the input to the constraint output vector is not right invertible. Another direction of categorization is based on so-called constraint invariant zeros of the plant, i.e. the invariant zeros of the mapping from the input to the constraint output vector. Like right-invertibility, this direction of categorization delineates the constraints into two main mutually exclusive categories, (1) at most weakly non-minimum phase constraints representing the case when the constraint invariant zeros are in the closed left-half complex plane, and (2) strongly non-minimum phase constraints representing the case when one or more of the constraint invariant zeros are in the open right-half complex plane. The above categorization of constraints dictates the taxonomy of constraints and paves the architecture of our development as given in Section 4 which presents our main results. In fact, the taxonomy of constraints becomes vivid in Section 4 and displays clearly how each category of constraints plays a role in constrained stabilization problems. Among many features discussed in Section 4, we would like to emphasize here the following two important and fundamental features:

- Neither the constrained semi-global nor the constrained global stabilization problem is solvable whenever the constraints are strongly non-minimum phase.
- There exists a perceptible demarkation line between the right and non-right invertible constraints. In particular, the solvability conditions for the constrained semi-global and global stabilization problems do not depend on the shape of the constraint set(s) for right invertible constraints whereas for non-right invertible constraints they indeed do so.

In this paper we consider state feedback controllers. The measurement feedback requires the study of some subtle additional issues that goes beyond the scope of this paper. We let $\mathbb{C}$, $\mathbb{C}^+$, $\mathbb{C}^-$ and $\mathbb{C}^0$ denote, respectively, the entire complex plane, the open right-half complex plane, the open left-half complex plane, and the imaginary axis.

2. Problem formulation

Consider a linear system:

$$\Sigma\colon \begin{cases} \dot{x} = Ax + Bu, \\ z = C_z x + D_z u, \end{cases} \tag{1}$$

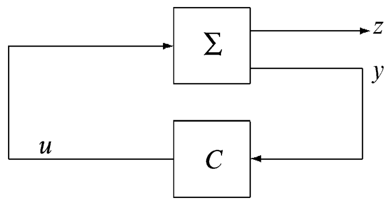

Fig. 1. Closed-loop system subject to constrained output.

where $x \in \mathbb{R}^n, u \in \mathbb{R}^m, z \in \mathbb{R}^p$ are, respectively, state, input and constraint output (see Fig. 1). Without loss of generality we assume $(C_z \;\; D_z)$ is surjective.

In this paper, for a system Σ, constrained stabilization problems via state feedback are considered. Namely, for two a priori given sets $\mathscr{S} \subset \mathbb{R}^p$ and $\mathscr{T} \subset \mathbb{R}^p$ which we refer to as constraint sets, we are interested in stabilization of the plant Σ subject to the requirement that the constraint output z remains in the set $\mathscr{S}$ for all $t \geqslant 0$ while the derivative of the constraint output $\dot{z}$ remains in the set $\mathscr{T}$ for all $t > 0$. In fact, we will only require that the constraints be satisfied for all $t > 0$ and ignore the constraints at time 0. This avoids technicalities due to possible rate limits on the input which require a smooth transition of the input for $t \leqslant 0$ and our to be designed controller which is active for $t > 0$.

In general, the constraint sets $\mathscr{S}$ and $\mathscr{T}$ discussed in most literature are bounded and convex. In this paper, we slightly extend this class of constraint sets and we make the following fundamental assumption on the nature of these constraint sets:

Assumption 1. *The following conditions on $\mathscr{S}$ and $\mathscr{T}$ are satisfied*:

(1) *The sets $\mathscr{S}$ and $\mathscr{T}$ are closed, convex and contain 0 as an interior point.*
(2) *$\mathscr{S} \cap \mathscr{T}$ is bounded.*
(3) *We have $C_z^T D_z = 0$ and*

$$\mathscr{S} = (\mathscr{S} \cap \operatorname{im} C_z) + (\mathscr{S} \cap \operatorname{im} D_z),$$

$$\mathscr{T} = (\mathscr{T} \cap \operatorname{im} C_z) + (\mathscr{T} \cap \operatorname{im} D_z).$$

Remark 2. We observe that $\operatorname{im} C_z$ reflects the state constraints while $\operatorname{im} D_z$ reflects the input constraints. Therefore the decomposition of $\mathscr{S}$ and $\mathscr{T}$ as required in (3) only implies that we have constraints on states and/or inputs. If $C_z = 0$, the problems studied in this paper are referred to as *input-constrained stabilization problems* while if $D_z = 0$, the problems are referred to as *state-constrained stabilization problems*.

Next, we observe that the initial state of the system must obviously be restricted since we cannot satisfy the constraints if the initial state of the system is arbitrary. For this reason, we define an admissible set of initial conditions. It is straightforward to show that if the initial state does not belong to this set, then we can never satisfy our constraint requirements.

Definition 3. Let system (1) and constraint sets $\mathscr{S}$ and $\mathscr{T}$ be given. We define

$$\mathscr{V}(\mathscr{S},\mathscr{T}) := \{x_0 \in \mathbb{R}^n \mid \exists\, u_0 \text{ such that } C_z x_0 + D_z u_0 \in \mathscr{S} \text{ and } C_z(Ax_0 + Bu_0) \in \mathscr{T}\}$$

as the *admissible set of initial conditions*.

Remark 4. In the derivative at time 0 we might expect a term $D_z\dot{u}(0)$ since the derivative of the input affects the derivative of the output z. However, we can omit this term because part (3) of Assumption 1 implies $C_z(Ax_0 + Bu_0) + D_z\dot{u}(0) \in \mathscr{T}$ if and only if $C_z(Ax_0 + Bu_0) \in \mathscr{T}$ and $D_z\dot{u}(0) \in \mathscr{T}$. However at time 0 we do not impose rate constraints. Due to continuity we still need to have $C_z(Ax_0 + Bu_0) \in \mathscr{T}$ but we do not need to impose a condition on the derivative of u since that need not be continuous.

Remark 5. Consider the case when there are no rate constraints, that is when $\mathscr{T} = \mathbb{R}^p$. Then, in view of Assumption 1, the admissible set of initial conditions $\mathscr{V}(\mathscr{S}, \mathbb{R}^p)$ can be rewritten as

$$\mathscr{V}(\mathscr{S}, \mathbb{R}^p) := \{x_0 \in \mathbb{R}^n \mid C_z x_0 \in \mathscr{S}\}.$$

We formulate our problems either in global or in semi-global setting. In the global setting we consider arbitrary initial conditions in the set of admissible initial conditions. In a semi-global setting we assume that the initial conditions are in some arbitrary compact set contained in the interior of the set of admissible initial conditions.

Problem 6. *Let the system* (1) *along with constraint sets $\mathscr{S} \subset \mathbb{R}^p$ and $\mathscr{T} \subset \mathbb{R}^p$ be given. The global constrained stabilization via state feedback is to find, if possible, a state feedback* (*possibly nonlinear and time-varying*) *$u(t) = f(x(t), t)$ such that the following conditions hold*:

(1) *The equilibrium point $x = 0$ of the closed-loop system is asymptotically stable with $\mathscr{V}(\mathscr{S},\mathscr{T})$ contained in its basin of attraction.*
(2) *For any $x_0 \in \mathscr{V}(\mathscr{S},\mathscr{T})$, we have $z(t) \in \mathscr{S}$ for all $t \geqslant 0$ and $\dot{z}(t) \in \mathscr{T}$ for all $t > 0$.*

Problem 7. *Consider the system* (1) *along with constraint sets $\mathscr{S} \subset \mathbb{R}^p$ and $\mathscr{T} \subset \mathbb{R}^p$. The semi-global constrained stabilization via state feedback is to find, for any a priori given compact set $\mathscr{W}$ contained in the interior of $\mathscr{V}(\mathscr{S},\mathscr{T})$, if possible, a state feedback* (*possibly nonlinear and time-varying*) *$u(t) = f(x(t), t)$ such that* *the following conditions hold*:

(1) *The equilibrium point $x = 0$ of the closed-loop system is asymptotically stable with $\mathscr{W}$ contained in its basin of attraction.*
(2) *For any $x_0 \in \mathscr{W}$, we have $z(t) \in \mathscr{S}$ for all $t \geqslant 0$ and $\dot{z}(t) \in \mathscr{T}$ for all $t > 0$.*

3. Taxonomy of constraints

Consider the system Σ given by (1) and characterized by the quadruple (A, B, C_z, D_z). It turns out that certain structural properties of this system play dominant roles in the study of constrained semi-global and global stabilization. Specifically, the right invertibility, the location of invariant zeros, and the order of infinite zeros determine what can or cannot be achieved. This section is devoted to a taxonomy of constraints in the context of stabilization.

The first category in the taxonomy of constraints is based on whether the system Σ is right invertible or not.

Definition 8. The constraints are said to be

- *right invertible constraints* if the system Σ is right invertible.
- *non-right invertible constraints* if the system Σ is non-right invertible.

The second category in the taxonomy of constraints is based on the location of the invariant zeros of the system Σ. Because of its importance, we specifically label the invariant zeros of the system Σ as the *constraint invariant zeros* of the plant.

Definition 9. The invariant zeros of the system Σ are called the *constraint invariant zeros* of the plant associated with the constraint output z.

Definition 10. The constraints are said to be

- *minimum phase constraints* if all the constraint invariant zeros are in $\mathbb{C}^-$.
- *weakly minimum phase constraints* if all the constraint invariant zeros are in $\mathbb{C}^- \cup \mathbb{C}^0$ with the restriction that at least one such constraint invariant zero is in $\mathbb{C}^0$ and any such constraint invariant zero in $\mathbb{C}^0$ is simple.
- *weakly non-minimum phase constraints* if all the constraint invariant zeros are in $\mathbb{C}^- \cup \mathbb{C}^0$ and at least one constraint invariant zero in $\mathbb{C}^0$ is not simple.
- *at most weakly non-minimum phase constraints* if all the constraint invariant zeros are in $\mathbb{C}^- \cup \mathbb{C}^0$.
- *strongly non-minimum phase constraints* if one or more of the constraint invariant zeros are in $\mathbb{C}^+$.

The third categorization is based on the order of the infinite zeros of the system Σ. Because of its importance, we specifically label the infinite zeros of the system Σ as the constraint infinite zeros of the plant.

Definition 11. The infinite zeros of the subsystem Σ are called the *constraint infinite zeros* of the plant associated with the constrained output z.

Definition 12. The constraints are said to be *type one constraints* if the order of all constraint infinite zeros is less than or equal to one.

4. Main results for semi-global and global stabilization

In this section we study in detail the constrained semi-global and global stabilization problems utilizing state feedback. We divide our development into two subsections, one for right invertible constraints and the other for non-right invertible constraints. The rational for such a division lies in the fundamental way the study of either semi-global or global stabilization problem differs for these two categories of constraints. In fact, there exists a clear demarkation line between the right and non-right invertible constraints. For instance, as will be evident soon, *the solvability conditions for the constrained semi-global and global stabilization problems do not depend on the shape of the constraint set for right invertible constraints whereas for non-right invertible constraints they indeed do so*.

4.1. Right invertible constraints

In this section we provide necessary and sufficient conditions for the solvability of Problems 6 and 7 whenever the constraints are right invertible. It is worth noting here that the right invertible constraints include as a special case amplitude and rate constraints on actuators. Whenever we have constraints only on the control variable u, we have $z = u$ implying that $C_z = 0$ and $D_z = I_m$. In other words, since Σ characterized by $(A, B, 0, I_m)$ can easily be verified to be right invertible, we note that the amplitude and rate constraints on actuators are indeed right invertible constraints.

We first consider the case of utilizing state feedback controllers. We have the following theorem concerned with Problem 6.

Theorem 13. *Consider the plant Σ as given by* (1) *and constraint sets $\mathscr{S}$ and $\mathscr{T}$ that satisfy Assumption* 1. *Assume that the set $\mathscr{S}$ is bounded. Assume also that the constraints are right invertible. Then the global constrained stabilization problem via state feedback as defined in Problem* 6 *is solvable if and only if the* *following conditions hold*:

(1) (A,B) *is stabilizable.*
(2) *The constraints are at most weakly non-minimum phase.*
(3) *The constraints are of type one.*

Remark 14. Consider the case when $\mathscr{S}$ is not bounded but $\mathscr{S} \cap \mathscr{T}$ is bounded. For this case, the conditions (1) and (2) are still necessary. On the other hand, the condition (3) will then be sufficient but not necessary. However, what is necessary is that the system characterized by (A,B,C_z,D_z) has no infinite zeros of order greater than two. Obtaining the necessary and sufficient conditions for this case depends on the precise shape of the sets $\mathscr{S}$ and $\mathscr{T}$.

We have the following theorem concerned with Problem 7.

Theorem 15. *Consider the plant Σ as given by* (1) *and constraint sets $\mathscr{S}$ and $\mathscr{T}$ that satisfy Assumption* 1. *Assume that the constraints are right invertible. Then the semi-global constrained stabilization problem via state feedback as defined in Problem* 7 *is solvable if and only if the following conditions hold*:

(1) (A,B) *is stabilizable.*
(2) *The constraints are at most weakly non-minimum phase.*

Remark 16. Consider the case when we have constraints only on actuator amplitude and rate, i.e. let $C_z = 0$. In other words, a subset of the input channels is subject to amplitude and rate constraints. Then, it is straightforward to show that the constraint invariant zeros of Σ (i.e. the invariant zeros of the system Σ characterized by $(A,B,0,D_z)$) coincide with a subset of the eigenvalues of A. This observation implies that the requirement of at most weakly non-minimum phase constraints in Theorems 13 and 15 is equivalent to requiring that a particular subset of eigenvalues of A lies in the closed left-half plane. Obviously, such a condition is always satisfied if we are dealing with asymptotically null controllable systems with bounded controls. It is interesting to consider a special case which corresponds to $C_z = 0$ and $D_z = I_m$, that is all the input channels are subject to amplitude and rate constraints. In this case, the constraint invariant zeros of Σ coincide with all the eigenvalues of A. Therefore, the requirement of at most weakly non-minimum phase constraints in Theorems 13 and 15 is equivalent to requiring that the given system be asymptotically null controllable with bounded controls. Moreover the system $(A,B,0,I_m)$ has no infinite zeros of order greater than 1 and hence the condition (3) of Theorem 13 is automatically satisfied.

For systems with only input saturation, it is known that global stabilization requires, in general, nonlinear feedback while semi-global stabilization can be achieved whenever it can be done by utilizing simply linear time-invariant feedback laws. Therefore, a question that arises naturally is whether an analogous result is valid under a broad framework of constraints as formulated here. The following theorem answers this question:

Theorem 17. *Consider the plant Σ as given by* (1) *and constraint sets $\mathscr{S}$ and $\mathscr{T}$ that satisfy Assumption* 1. *Assume that the constraints are right invertible. Then the following hold*:

(1) *Under the condition that* $\operatorname{im} C_z \subset \mathscr{T}$ (*i.e. no rate constraints on states*), *if a semi-global constrained stabilization problem via state feedback as defined in Problem* 7 *is solvable*, *then it is also solvable via a linear time-invariant state feedback law.*
(2) *If* $\operatorname{im} C_z \not\subset \mathscr{T}$ (*i.e. rate constraints on states are present*), *whenever a semi-global constrained stabilization problem via state feedback as defined in Problem* 7 *is solvable*, *in general it might not be solvable via a linear time-invariant state feedback law. That is, there exist a plant Σ as given by* (1) *and constraint sets $\mathscr{S}$ and $\mathscr{T} \not\supset \operatorname{im} C_z$ that satisfy Assumption* 1 *for which the semi-global stabilization problem is solvable via a nonlinear feedback law but for which there exists no linear feedback law that solves the problem.*

Remark 18. As will become clear from the proof, the possibility of not having a linear time-invariant feedback law solving a semi-global constrained stabilization problem (whenever the solvability conditions as given by Theorem 15 are satisfied) arises if the given plant has a peculiar infinite zero structure. More specifically, it arises if the given plant has infinite zeros of order 1 along with higher order infinite zeros. If the infinite zeros of order 1 are not present but infinite zeros of order greater than 1 are present, one can utilize linear time-invariant feedback laws to solve the posed problem. In other words, in most practical cases one will be able to use linear time-invariant feedback laws.

Theorems 13 and 15 tell us that the global and semi-global constrained stabilization problems are solvable only for a system Σ which has at most weakly non-minimum phase constraints. Suppose that the given system Σ has strictly non-minimum phase constraints. In this case, one cannot enlarge the domain of attraction arbitrarily, and the domain of attraction is intrinsically restricted. Our next goal is to obtain a maximally achievable domain of attraction in an absolute sense (i.e. for those initial conditions outside of such a set the given system cannot be stabilized).

The presentation of this result and also the proofs of all major theorems rely on *one* specific decomposition of the system. This decomposition is nothing else than the decomposition related to the special coordinate basis (scb) as presented in Sannuti and Saberi (1987) and Saberi and Sannuti (1990). As is well known by now, the scb of a system displays clearly both the finite and infinite zero structures of a given system. Also, the familiar properties of linear systems such as stabilizability, controllability, detectability, observability, left and right invertibility, can easily be ascertained from the scb. An expanded form of scb plays a crucial role in the context of proofs as well as in controller design, and it will be presented later on. However, for the presentation of the results related to non-minimum phase constraints, a compact form of scb suffices, and is given below.

Utilizing the state and input space coordinates of scb for the quadruple (A,B,C_z,D_z) together with a preliminary state feedback $u = Fx + \Gamma_u\tilde{u}$, one can rewrite the general system Σ given by (1) as

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} A_{11} & 0 \\ 0 & A_{22} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ B_2 \end{pmatrix} \tilde{u} + \begin{pmatrix} K_1 \\ K_2 \end{pmatrix} z, \tag{2a}$$

$$z = (C_{z,1} \;\; C_{z,2}) \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \tilde{D}_z\tilde{u}. \tag{2b}$$

Here, $x_1 \in \mathbb{R}^{n_1}$ and $x_2 \in \mathbb{R}^{n_2}$ with $n_1 + n_2 = n$. We emphasize that many submatrices given in (2) have definite structure which we judiciously point out as the need arises. At this time we point out that the subsystem $(A_{22}, B_2, C_{z,2}, \tilde{D}_z)$ is strongly controllable and has no invariant zeros (see Sannuti & Saberi, 1987; Saberi & Sannuti, 1990). For the case of right invertible constraints, there is additional structure; in particular $C_{z,1} = 0$.

In the case of right-invertible constraints, the eigenvalues of A_{11} are equal to the invariant zeros of the system Σ. In the case of non-right invertible constraints (i.e when $C_{z,1} \neq 0$), the unobservable eigenvalues of the pair $(C_{z,1}, A_{11})$ are precisely the invariant zeros of the system Σ.

When we have right-invertible constraints, we extract from (2) two subsystems. The first subsystem is given by

$$\Sigma_s : \dot{x}_1 = A_{11}x_1 + K_1 z, \quad x_1 \in \mathbb{R}^{n_1}. \tag{3}$$

The above subsystem Σ_s represents the zero dynamics of the system Σ characterized by the quadruple (A,B,C_z,D_z).

The second subsystem extracted from (2) is given by

$$\begin{aligned} \dot{x}_2 &= A_{22}x_2 + B_2\tilde{u} + K_2 z, \quad x_2 \in \mathbb{R}^{n_2}, \\ z &= C_{z,2}x_2 + \tilde{D}_z\tilde{u}. \end{aligned} \tag{4}$$

We define the following admissible set of initial conditions for the above system:

$$\begin{aligned} \mathscr{V}_2(\mathscr{S},\mathscr{T}) = \{x_{2,0} \in \mathbb{R}^{n_2} \mid \exists u_0 \text{ such that } z_0 \in \mathscr{S} \\ \text{and } C_{z,2}(A_{22}x_{2,0} + B_2u_0 + K_2z_0) \in \mathscr{T}\}, \end{aligned} \tag{5}$$

where $z_0 = C_{z,2}x_{2,0} + \tilde{D}_z u_0$.

Before stating our results for the case of non-minimum phase constraints, we need to recall the following definition of the region of asymptotic null-controllability subject to input constraints:

Definition 19. Consider the system:

$$\bar{\Sigma} : \dot{\bar{x}} = \bar{A}\bar{x} + \bar{B}\bar{u}, \quad \bar{x} \in \mathbb{R}^n, \quad \bar{u} \in \mathbb{R}^m. \tag{6}$$

For any $\bar{\mathscr{S}} \subset \mathbb{R}^m$ and $\bar{\mathscr{T}} \subset \mathbb{R}^m$, the *region of asymptotic null-controllability subject to the input constraint sets* $\bar{\mathscr{S}}$ *and* $\bar{\mathscr{T}}$ is the set of initial conditions defined by

$$\begin{aligned} \mathscr{R}_c(\bar{\Sigma}, \bar{\mathscr{S}}, \bar{\mathscr{T}}) = \Big\{ \bar{x}_0 \in \mathbb{R}^n \mid \exists \bar{u} \text{ such that } \bar{u}(t) \in \bar{\mathscr{S}} \text{ and} \\ \dot{\bar{u}}(t) \in \bar{\mathscr{T}} \text{ for all } t > 0 \text{ and} \\ \lim_{t\to\infty} \bar{x}(t) = 0 \text{ where } \bar{x}(0) = \bar{x}_0 \Big\}. \end{aligned}$$

Viewing z as the input to the subsystem Σ_s given in (3) with $z(t) \in \mathscr{S}$ and $\dot{z}(t) \in \mathscr{T}$ for all $t > 0$, and in accordance with the above definition, we next denote by $\mathscr{R}_c(\Sigma_s, \mathscr{S}, \mathscr{T})$ the region of asymptotic null-controllability of system Σ_s subject to input constraint sets $\mathscr{S}$ and $\mathscr{T}$.

We have the following theorem:

Theorem 20. *Consider the plant Σ as given by* (1) *and constraint sets $\mathscr{S}$ and $\mathscr{T}$ that satisfy Assumption* 1. *Assume that the constraints are right invertible. Also, for a given stabilizing controller $u = f(x)$, denote its domain of attraction as $\mathscr{R}_A^f(\Sigma)$. Then we have*:

$$\mathscr{R}_A^f(\Sigma) \subseteq \mathscr{R}_c(\Sigma_s, \mathscr{S}, \mathscr{T}) \times \mathscr{V}_2(\mathscr{S}, \mathscr{T}). \tag{7}$$

Under the additional constraint that $\mathscr{T} = \mathbb{R}^p$ (i.e. no rate constraint is present), for any compact set $\mathscr{R}$ satisfying $\mathscr{R} \subset \rho\mathscr{R}_c(\Sigma_s, \mathscr{S}, \mathbb{R}^p)$ for some $\rho < 1$, we can find a stabilizing controller for the complete system Σ with domain of attraction containing $\mathscr{R} \times \mathscr{V}_2(\mathscr{S}, \mathbb{R}^p)$ and for all initial conditions in $\mathscr{R} \times \mathscr{V}_2(\mathscr{S}, \mathbb{R}^p)$ satisfies the constraint $z(t) \in \mathscr{S}$ for all $t > 0$.

Remark 21. We emphasize that the second part of the above theorem does not hold if we have rate constraints (i.e. if $\mathscr{T} \neq \mathbb{R}^p$).

4.1.1. Proofs and construction of controllers

Both for construction of controllers and the proof of the results in the previous section, we use the decomposition in the two subsystems (3) and (4) as defined in the previous section. We observe clearly that we can control the first subsystem (3) only through z. Also, from the scb decomposition it follows that the second subsystem characterized by the quadruple $(A_{22}, B_2, C_{z,2}, \tilde{D}_z)$ has no finite invariant zeros and is right-invertible. This implies that we can guarantee by suitable choice of $\tilde{u}$ that z is arbitrary close to any desired signal. Therefore, we basically design a controller in two phases:

- First design a desired feedback for the first subsystem (3) using z as the (constrained) input signal such that the first subsystem exhibits a desired closed-loop behavior.
- Secondly, design a feedback for the second subsystem with state x_2, input u and output z such that
  (1) the output z is close to the desired feedback for the first subsystem,
  (2) the output satisfies the constraints,
  (3) the state x_2 of the second subsystem exhibits a desirable behavior.

All feedback designs in this paper are constructed in accordance with this two-phase design.

We need to discuss what kind of initial conditions can be considered for the first subsystem (3). In fact, since we have no state constraints on this subsystem, we can have arbitrary initial conditions for it. Hence, we consider arbitrary initial conditions in $\mathscr{W}_1 = \mathbb{R}^{n_1}$ in the global case while in the semi-global case we consider initial conditions in some arbitrary compact set $\mathscr{W}_1$.

Similarly, the initial conditions for the second subsystem must be in some set $\mathscr{W}_2$. In the global case, we have $\mathscr{W}_2 = \mathscr{V}_2(\mathscr{S}, \mathscr{T})$ using the definition in (5) while in the semi-global case, we have that $\mathscr{W}_2$ is an arbitrary compact set contained in the interior of $\mathscr{V}_2(\mathscr{S}, \mathscr{T})$. Due to space limitation we have omitted the proof of Theorem 17. The interested reader can obtain the proof from Saberi, Han, and Stoorvogel (2001).

4.1.1.1. Proof of Theorem 13.

We first show that the conditions of Theorem 13 are necessary. The necessity of condition (1) of Theorem 13 is trivial. The necessity of condition (2) of Theorem 13 is a consequence of the results obtained in Shi, Saberi, and Stoorvogel (2000) and the following lemma which is a direct consequence of the decomposition given in (3) and (4),

Lemma 22. *Let the system* (1) *and constraint sets* $\mathscr{S}$ *and* $\mathscr{T}$ *be given. There exists a state feedback that solves the global constrained stabilization problem for the system* (1) *only if the system* (3) *is globally stabilizable by a constrained state feedback* $z = f(x_1)$, *i.e.*

(1) *The equilibrium point* $x_1 = 0$ *of the closed-loop system is globally asymptotically stable.*
(2) *For any initial condition,* $z(t) \in \mathscr{S}$ *and* $\dot{z}(t) \in \mathscr{T}$ *for all* $t > 0$.

In order to complete the proof that the conditions of Theorem 13 are necessary we need to prove the following lemma which states that condition (3) is also a necessary condition for solvability of Problem 6.

Lemma 23. *Consider the plant* Σ *as given by* (1). *Let the assumptions of Theorem* 13 *be satisfied. Then the global constrained stabilization problem as defined in Problem* 6 *is solvable only if the system* (A, B, C_z, D_z) *has no infinite zeros of order greater than one.*

Proof. First note that, since the system is right invertible, having no infinite zeros of order greater than one is equivalent to $(C_zB\ D_z)$ being surjective. Therefore, if the system has infinite zeros of order greater than one, then there exists a vector $c \neq 0$ such that $c^{\mathrm{T}} D_z = 0$ and $c^{\mathrm{T}} C_z B = 0$. Moreover, since $\mathscr{T}$ contains zero in its interior, we can guarantee that $c \in \mathscr{T}$. Let $z_0 \in \mathscr{S}$ be such that $\langle z, c\rangle \leqslant \langle z_0, c\rangle$ for all $z \in \mathscr{S}$. Since $\mathscr{S}$ is convex as well as compact such a z_0 always exists. Next, because (A, B, C_z, D_z) is right invertible there exist initial condition x_0 and input function $\bar{u}$ such that the output z satisfies $z(0) = z_0$ and $\dot{z}(0) = c$, i.e.

$$z_0 = C_z x_0 + D_z \bar{u}(0) \in \mathscr{S},$$

$$c = C_z(Ax_0 + B\bar{u}(0)) + D_z \dot{\bar{u}}(0) \in \mathscr{T}.$$

Clearly $x_0 \in \mathscr{V}(\mathscr{S}, \mathscr{T})$. But if we start at time 0 in x_0, then we have for any input signal u

$$\langle c, z(0)\rangle = \langle c, C_z x_0\rangle = \langle c, z_0\rangle,$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle c, z(t)\rangle|_{t=0} = \langle c, C_z A x_0\rangle = \langle c, c\rangle > 0.$$

Therefore $\langle c, z(t)\rangle > \langle c, z_0\rangle$ for small $t > 0$ and for any input u. By definition of z_0 this implies $z(t) \notin \mathscr{S}$ for small $t > 0$ and for any input u. Therefore, there exist initial conditions in $\mathscr{V}(\mathscr{S}, \mathscr{T})$ which cannot be stabilized without violating our constraints which yields the required contradiction. □

This establishes the necessity of our conditions. The next step is to prove sufficiency by explicitly designing a suitable feedback. Before we do so, in view of scb, we need to recall a finer structure of (2), namely

$$C_{z,2} = \begin{pmatrix} C_1 \\ 0 \end{pmatrix}, \quad \tilde{D}_z = \begin{pmatrix} 0 & 0 & 0 \\ D_1 & 0 & 0 \end{pmatrix}, \tag{8}$$

$$B_2 = (\,0 \quad \tilde{B}_2 \quad \tilde{B}_3\,)$$

with D_1 invertible. Since we have no infinite zeros of order greater than 1, we have the additional structure that $C_1 \tilde{B}_2$ is invertible. Also, we decompose $\tilde{u}$ and z

to be compatible with the above: into z_1, z_2 and u_1, u_2, u_3 respectively. We impose next a more stringent bound on the rate by choosing $\tilde{\mathscr{T}} \subset \mathscr{T}$ such that $\mathscr{S}$ and $\tilde{\mathscr{T}}$ satisfy Assumption 1 while $\tilde{\mathscr{T}}$ is bounded. This is done to utilize an upper bound on the rate which is needed in our proof. We observe that the assumptions on the sets $\mathscr{S}$ and $\tilde{\mathscr{T}}$ guarantee that we can decompose the sets $\mathscr{S}$ and $\tilde{\mathscr{T}}$ compatible with the decomposition of z:

$$\mathscr{S} = \mathscr{S}_1 \times \mathscr{S}_2, \quad \text{and} \quad \tilde{\mathscr{T}} = \tilde{\mathscr{T}}_1 \times \tilde{\mathscr{T}}_2 \tag{9}$$

such that $z \in \mathscr{S}$ if and only if $z_1 \in \mathscr{S}_1$ and $z_2 \in \mathscr{S}_2$. The decomposition of $\tilde{\mathscr{T}}$ is similar.

In our design for the first subsystem, we choose an input z which satisfies the constraints such that $z(t) \in (1-\rho)\mathscr{S}$ and $\dot{z}(t) \in (1-\rho)\tilde{\mathscr{T}}$, where the parameter $\rho \in (0,1)$. By choosing ρ close to 1, we have limited control effort for the first subsystem but in our design for $\tilde{u}$ to track the desired output z, we have more flexibility. Conversely, choosing ρ small gives us more control effort for the first subsystem but in our design for $\tilde{u}$ we need to track the desired output z quite accurately. Based on these arguments, we fix the parameter $\rho \in (0,1)$. Next we choose a $\delta > 0$ such that $\delta \bar{\mathscr{S}}_1 \subset \rho \tilde{\mathscr{T}}$, where

$$\bar{\mathscr{S}}_1 = \{ z_1 - z_2 \mid z_1 \in \mathscr{S}_1,\ z_2 \in \mathscr{S}_1 \}. \tag{10}$$

Obviously, such a δ always exists. We now focus our design for the first subsystem (3) while viewing z as an input variable. At first we let $z = z_0 + v$ and rewrite subsystem (3) as

$$\dot{x}_1 = A_{11}x_1 + K_1 z_0 + K_1 v. \tag{11}$$

We note that the conditions of the theorem imply that all the eigenvalues of A_{11} are in the closed left-half plane. Next, we would like to construct a state feedback law $z_0 = f(x_1)$ such that it satisfies the constraints $z_0(t) \in (1-\rho)\mathscr{S}$ and $\dot{z}_0(t) \in (1-\rho)\tilde{\mathscr{T}}$ for all $t > 0$ while rendering the zero equilibrium point of the closed-loop system of (11) and $z_0 = f(x_1)$ globally attractive (i.e. $x_1(t) \to 0$ as $t \to \infty$) in the presence of signals v satisfying:

$$\|v(t)\| \leqslant M e^{-\delta t} \tag{12}$$

for some $M > 0$. Moreover, the feedback law $z_0 = f(x_1)$ should render the zero equilibrium point of the closed-loop system when $v = 0$ locally exponentially stable. Such a nonlinear feedback law $z_0 = f(x_1)$ can be obtained from Shi et al. (2000). Note that although Shi et al. (2000) considers only the case $v = 0$, the needed extension to this more general case is quite straightforward.

Next we consider the second subsystem, namely (4). The main design objective is to find a suitable input $\tilde{u}$ to the second subsystem (4) such that for any initial condition of the first subsystem (4) in $\mathscr{W}_1$ and for any initial condition of the second subsystem (3) we have

$$\|z(t) - f(x_1(t))\| \leqslant M e^{-\delta t} \tag{13}$$

for all $t > 0$ and for any function f satisfying:

$$f(x_1(t)) \in (1-\rho)\mathscr{S}, \quad \frac{d}{dt} f(x_1(t)) \in (1-\rho)\tilde{\mathscr{T}}$$

for all $t > 0$. Obviously, we must also guarantee that $z(t) \in \mathscr{S}$ and $\dot{z}(t) \in \mathscr{T}$ for all $t > 0$.

To proceed, let us next partition $f(x_1)$ to be compatible with the partitioning of z:

$$f(x_1) = \begin{pmatrix} f_1(x_1) \\ f_2(x_1) \end{pmatrix}.$$

We are now ready to construct the required feedback laws for $\tilde{u}$. Our objective in designing it is to guarantee that $v := z - f(x_1)$ satisfies (12) while z satisfies the constraints. Knowing the properties of scb, it can be shown that one can choose $\tilde{u}_3 = Fx_2$ such that the system (2) with inputs $\tilde{u}_1$ and $\tilde{u}_2$ and output z is invertible and moreover the additional invariant zeros introduced by the feedback $\tilde{u}_3 = Fx_2$ are placed in a desired location in the open left-half plane. With this choice of $\tilde{u}_3$, we obtain:

$$z_1 = C_1 x_2, \quad z_2 = D_1 \tilde{u}_1,$$

$$\dot{z}_1 = C_1(A_{22}x_2 + \tilde{B}_3 F x_2 + K_2 z) + C_1 \tilde{B}_2 \tilde{u}_2.$$

Then choose the feedback laws,

$$\tilde{u}_1(t) = D_1^{-1} f_2(x_1(t))$$

$$\begin{aligned}\tilde{u}_2(t) = (C_1\tilde{B}_2)^{-1} \Big(& -C_1 A_{22} x_2(t) + \tilde{B}_3 F x_2(t) \\ & + K_2 z(t) - \delta(z_1(t) - f_1(x_1(t))) \\ & + (1 - e^{-\delta t}) \frac{d}{dt} f_1(x_1(t)) \Big).\end{aligned}$$

We emphasize that the above feedback laws are time-varying nonlinear state feedback laws. These feedback laws guarantee that $z_2 = f_2(x_1)$ and that $z_1(t) \to f_1(x_1(t))$ as $t \to \infty$ for all initial conditions in the set of admissible set of initial conditions $\mathscr{V}(\mathscr{S}, \mathscr{T})$. We show next that z_1 and z_2 with the above feedback laws satisfy all the constraints. We observe first that $f(x_1) \in \mathscr{S}$ and $(d/dt) f(x_1) \in \tilde{\mathscr{T}}$ which guarantees that $z_2 = f_2(x_1) \in \mathscr{S}_2$ and $\dot{z}_2 = (d/dt) f_2(x_1) \in \tilde{\mathscr{T}}_2$. This implies that z_2 satisfies all the constraints. We focus next on showing that z_1 satisfies all the constraints, i.e. $z_1(t) \in \mathscr{S}_1$ and $\dot{z}_1(t) \in \tilde{\mathscr{T}}_1$ for all $t > 0$. We have:

$$\dot{z}_1(t) = -\delta(z_1(t) - f_1(x_1(t))) + (1 - e^{-\delta t}) \frac{d}{dt} f_1(x_1(t)). \tag{14}$$

Integrating this equation we obtain:

$$z_1(t) = e^{-\delta t} z_1(0) + (1 - e^{-\delta t}) f_1(x_1(t)). \tag{15}$$

Since $f_1(x_1) \in \mathscr{S}_1$ and $z_1(0) \in \mathscr{S}_1$, we find, using the convexity of $\mathscr{S}_1$, that $z_1(t) \in \mathscr{S}_1$. We still need to guarantee that z_1 satisfies our rate constraints. We again use (14). Since $(d/dt) f_1(x_1(t)) \in (1-\rho)\tilde{\mathscr{T}}_1$, we obviously have that

$$(1 - e^{-\delta t}) \frac{d}{dt} f_1(x_1(t)) \in (1-\rho)\tilde{\mathscr{T}}_1.$$

Moreover

$\delta(z_1(t) - f_1(x_1(t))) \in \delta\bar{\mathscr{S}}_1 \subset \rho\tilde{\mathscr{T}}_1.$

This guarantees that $\dot{z}_1(t) \in \tilde{\mathscr{T}}_1$ as required. Next, note that $v_1(t) = z_1(t) - f_1(x_1(t))$ satisfies according to (15):

$v_1(t) = \mathrm{e}^{-\delta t} z_1(0) - \mathrm{e}^{-\delta t} f_1(x_1(t)) \in \mathrm{e}^{-\delta t}(\mathscr{S}_1 - \tilde{\mathscr{T}}_1)$

for any $t > 0$ and hence the error signal $v = z - f(x_1)$ satisfies:

$\|v(t)\| \leqslant M\mathrm{e}^{-\delta t}$

for all $t > 0$, where M is some positive constant. This immediately shows that, for all initial conditions in $\mathscr{V}(\mathscr{S},\mathscr{T})$, $x_1(t) \to 0$ as $t \to \infty$ which in turn guarantees that $z(t) \to 0$ as $t \to \infty$. Moreover, it is straightforward to show that $x_2(t) \to 0$ as $t \to \infty$ for all initial conditions in $\mathscr{V}(\mathscr{S},\mathscr{T})$. Thus, we conclude that the zero equilibrium point of the closed-loop system is attractive for all initial conditions in $\mathscr{V}(\mathscr{S},\mathscr{T})$. Finally, it is obvious that the zero equilibrium point of the closed-loop system is locally asymptotically stable. This concludes our proof.

Remark 24. If we have only amplitude constraints (i.e. $\mathscr{T} = \mathbb{R}^p$), then it is clear that the time-invariant feedback

$$\tilde{u}_1(t) = D_1^{-1} f_2(x_1(t)),$$

$$\tilde{u}_2(t) = (C_1\tilde{B}_2)^{-1}\left(-C_1A_{22}x_2(t) + \tilde{B}_3Fx_2(t) + K_2z(t) - \delta\left(z_1(t) - f_1(x_1(t)) + \frac{\mathrm{d}}{\mathrm{d}t} f_1(x_1(t))\right)\right),$$

$$\tilde{u}_3(t) = Fx(t)$$

achieves stability and respects our constraints for a sufficiently large δ. Obviously, a large δ is not possible whenever rate-constraints exist.

4.1.1.2. Proof of Theorem 15.

Again, in view of the decomposition given in (3) and (4), the following lemma is obvious:

Lemma 25. *Let the system* (1) *and constraint sets* $\mathscr{S}$ *and* $\mathscr{T}$ *be given. There exists a state feedback that solves the semi-global constrained stabilization problem for the system* (1) *only if the system* (3) *is semi-globally stabilizable by a constrained state feedback, i.e. for any compact set* $\mathscr{W}_1$, *there exists a state feedback* $z = f(x_1)$ *such that*

(1) *The equilibrium point* $x_1 = 0$ *of the closed-loop system is asymptotically stable with* $\mathscr{W}_1$ *contained in its basin of attraction.*
(2) *For any* $x_1(0) \in \mathscr{W}_1$, $z(t) \in \mathscr{S}$ *and* $\dot{z}(t) \in \mathscr{T}$ *for all* $t > 0$.

Note that (3) is semi-globally stabilizable by a constrained state feedback only if A_{11} has all its eigenvalues in the closed left-half plane. Secondly, if the system (A, B, C_z, D_z) is right invertible then the eigenvalues of A_{11} are the invariant zeros of the system (A, B, C_z, D_z) and hence the above proves necessity of Theorem 15. Remains to prove sufficiency of the conditions in Theorem 15. We can hence assume that from now on A_{11} has all its eigenvalues in the closed left-half plane. We prove sufficiency by an explicit design of a suitable controller. We begin with the design of a state feedback controller as needed for Theorem 15.

The basic philosophy of our controller design as before is as follows. We first design a suitable stabilizing controller $z = f_1(x_1)$ for the subsystem (3). Next we consider the subsystem (4). We need to design an input u such that the output z tracks the desired feedback for the first subsystem while avoiding constraint violation and while guaranteeing stability of the second subsystem.

In our global design, we needed to guarantee stability for all initial conditions for the first subsystem in $\mathbb{R}^{n_1}$ and for all initial conditions of the second subsystem in $\mathscr{V}_2(\mathscr{S},\mathscr{T})$. For our semi-global design, we need to guarantee stability for all initial conditions for the first subsystem in some compact subset $\mathscr{W}_1 \subset \mathbb{R}^{n_1}$ and for all initial conditions of the second subsystem in a compact subset $\mathscr{W}_2$, where $\mathscr{W}_2$ is such that there exists $\zeta \in (0, 0.5)$ such that $\mathscr{W}_2 \subset (1 - 2\zeta)\mathscr{V}_2(\mathscr{S},\mathscr{T})$. Our design has a larger domain of attraction for smaller ζ but at the expense of the need for a higher gain in the second subsystem. Obviously, there exists a compact set $\tilde{\mathscr{S}} \subset \mathscr{S}$ such that $\mathscr{W}_2 \subset (1 - \zeta)\mathscr{V}_2(\tilde{\mathscr{S}},\mathscr{T})$. We assume that $\tilde{\mathscr{S}}$ and $\mathscr{T}$ still satisfy assumption 1. We actually design a feedback such that $z(t) \in \tilde{\mathscr{S}}$ for all $t > 0$. Because we are proving sufficiency, this restriction is without loss of generality and enables a simplification in the proof.

Assumption 1 enables us to decompose $\tilde{\mathscr{S}}$ and $\mathscr{T}$:

$$\tilde{\mathscr{S}} \cap \operatorname{im} C_z = \tilde{\mathscr{S}}_1, \quad \tilde{\mathscr{S}} \cap \operatorname{im} D_z = \tilde{\mathscr{S}}_2,$$

$$\mathscr{T} \cap \operatorname{im} C_z = \mathscr{T}_1, \quad \mathscr{T} \cap \operatorname{im} D_z = \mathscr{T}_2,$$

Step 1 (*Controller design for the zero-dynamics*): We now focus our design for the first subsystem (3) while viewing z as an input variable. As before, at first we let $z = z_0 + v$ and rewrite subsystem (3) as

$$\dot{x}_1 = A_{11}x_1 + K_1z_0 + K_1v. \tag{16}$$

We note again that the conditions of the theorem imply that all the eigenvalues of A_{11} are in the closed left-half plane. As will become transparent in the design for our second subsystem, we need to choose δ_2 such that:

$$\delta_2\bar{\mathscr{S}}_1 \subset \frac{\zeta}{3}\mathscr{T}_1, \tag{17}$$

where as before

$$\bar{\mathscr{S}}_1 = \{z_1 - z_2 \mid z_1 \in \tilde{\mathscr{S}}_1,\ z_2 \in \tilde{\mathscr{S}}_1\}. \tag{18}$$

Clearly, such a δ_2 exists since $\bar{\mathscr{S}}_1$ is bounded.

Our objective is to design a stabilizing feedback $z_0 = f(x_1)$ such that the equilibrium point of the closed-loop system of (16) and $z_0 = f(x_1)$ with $v = 0$ is asymptotically stable. Moreover, for all v satisfying the bound

$$\|v(t)\| \leqslant M e^{-\delta_2 t} \tag{19}$$

for all $t > 0$, and for all initial conditions in some arbitrarily large but compact subset $\mathscr{W}_1 \subset \mathbb{R}^{n_1}$, we satisfy $x_1(t) \to 0$ as $t \to \infty$. Finally, we need to guarantee also that

$$z_0(t) \in (1-\zeta)\tilde{\mathscr{S}}_1, \quad \dot{z}_0(t) \in \frac{\zeta\delta_2}{2e^{-1}}\tilde{\mathscr{S}}_1 \cap \frac{\zeta}{3}\mathscr{T}_1 \tag{20}$$

for all $t > 0$. Again, it will become clear in our design for our second subsystem why it is desirable to guarantee that z_0 satisfies these bounds. One can design such a suitable feedback law $f(x_1)$ as a linear state feedback law as described below. For further details we refer to Stoorvogel and Saberi (1999).

Let P_μ be the solution of the continuous-time algebraic Riccati equation:

$$A_{11}^{\mathrm{T}} P_\mu + P_\mu A_{11} - P_\mu K_1 K_1^{\mathrm{T}} P_\mu + \mu^2 I = 0.$$

It is well known that $\lim_{\mu\to 0} P_\mu = 0$. It is also shown for any compact subset $\mathscr{W}_1$, there exists a μ^* such that for all $\mu \in (0, \mu^*]$:

(1) $z_0 = -K_1^{\mathrm{T}} P_\mu x_1$ is a stabilizing controller for system (16) with $\mathscr{W}_1$ contained in the domain of attraction.
(2) (20) is satisfied for all t and for all v satisfying (19).

Hence we can choose $f(x_1) = -K_1^{\mathrm{T}} P_\mu x_1$ for some $\mu \in (0, \mu^*]$ to obtain a suitable feedback for this first subsystem. In the rest of the proof, we assume in fact that f is a linear function of the state as presented above.

Step 2 (*Controller design for the second subsystem*): Our next design objective is to find a suitable input $\tilde{u}$ to the second subsystem (4) such that (19) is satisfied where

$$v(t) = z(t) - f(x_1(t))$$

for all $t > 0$ and where $z_0(t) = f(x_1(t))$ satisfies (20). Obviously, we must guarantee that also $z(t) \in \tilde{\mathscr{S}}$ for all $t \geqslant 0$ and $\dot{z}(t) \in \mathscr{T}$ for all $t > 0$ while assuring the stability of the resulting closed-loop system with the desired domain of attraction.

At this stage, in order to proceed with our design, we need to reveal from scb certain finer structure the matrices $A_{22}, A_{21}, \tilde{B}_2, \tilde{B}_3$, and C_1. Indeed we have $A_{22} = \tilde{A}_{22} + \tilde{B}_2 G$ for some compatible matrices G. Moreover, we have

$$\tilde{A}_{22} = \begin{pmatrix} A_1 & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & A_s & 0 \\ 0 & \cdots & 0 & A_c \end{pmatrix}, \quad \tilde{B}_2 = \begin{pmatrix} B_{2,1} & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & B_{2,s} & 0 \\ 0 & \cdots & 0 & 0 \end{pmatrix},$$

$$\tilde{B}_3 = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ B_c \end{pmatrix}, \quad C_1 = \begin{pmatrix} C_{11} & 0 & \cdots & 0 & 0 \\ 0 & \ddots & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & 0 & 0 \\ 0 & \cdots & 0 & C_{1s} & 0 \end{pmatrix},$$

and for $i = 1, 2, \ldots, s$:

$$A_i = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & 1 \\ 0 & \cdots & 0 & 0 & 0 \end{pmatrix} \in \mathbb{R}^{k_i \times k_i}, \quad B_{2,i} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 0 \\ 1 \end{pmatrix},$$

$$C_{1i} = (1 \quad 0 \quad \cdots \quad 0).$$

Here, A_c and B_c are matrices of appropriate dimension such that the pair (A_c, B_c) is controllable.

In view of the above and in view of (8), the system of equations given in (4) can be rewritten as

$$\dot{x}_2 = \tilde{A}_{22} x_2 + \tilde{B}_2(\tilde{u}_2 + G x_2) + \tilde{B}_3 \tilde{u}_3 + K_2 z,$$

$$z = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} C_1 x_2 \\ D_1 \tilde{u}_1 \end{pmatrix} \tag{21}$$

with D_1 invertible. Next, we partition K_2, x_2, z_1, $\tilde{u}_2$, and $f(x_1)$ in conformity with the partitioning of matrices $\tilde{A}_{22}$, $\tilde{B}_2$, etc. We have

$$K_2 = \begin{pmatrix} K_{2,1} \\ \vdots \\ K_{2,s} \\ K_{2,c} \end{pmatrix}, \quad x_2 = \begin{pmatrix} x_{2,1} \\ x_{2,2} \\ \vdots \\ x_{2,s} \\ x_c \end{pmatrix}, \quad z_1 = \begin{pmatrix} z_{1,1} \\ z_{1,2} \\ \vdots \\ z_{1,s} \end{pmatrix},$$

$$\tilde{u}_2 = \begin{pmatrix} u_{2,1} \\ u_{2,2} \\ \vdots \\ u_{2,s} \end{pmatrix}, \quad f(x_1) = \begin{pmatrix} f_1(x_1) \\ f_2(x_1) \end{pmatrix},$$

$$f_1(x_1) = \begin{pmatrix} f_{1,1}(x_1) \\ f_{1,2}(x_1) \\ \vdots \\ f_{1,s}(x_1) \end{pmatrix}, \quad G = \begin{pmatrix} G_1 \\ \vdots \\ G_s \end{pmatrix}.$$

We are now ready to design $\tilde{u}$. We will first focus on $\tilde{u}_1$ and $\tilde{u}_3$. We choose $\tilde{u}_1 = D_1^{-1} f_2(x_1)$. To get $\tilde{u}_3$, we choose first a matrix F_c such that the eigenvalues of $A_c + B_c F_c$ are all at desired locations in the open left half plane. Such a selection of F_c is possible since (A_c, B_c) is controllable. We then choose $\tilde{u}_3 = F_c x_c$.

It remains to choose $\tilde{u}_2$. To do so, let us study the system (21). By substituting for $\tilde{u}_1$ and $\tilde{u}_3$ as chosen, we can rewrite (21) as

$$\begin{aligned}
&\dot{x}_{2,i} = A_i x_{2,i} + B_{2,i}(u_{2,i} + G_i x_2) + K_{2,i} z,\\
&z_{1,i} = C_{1i} x_{2,i}\\
&\text{for } i = 1, 2, \ldots, s, \text{ and}\\
&\dot{x}_c = (A_c + B_c F_c) x_c + K_{2,c} z.
\end{aligned} \tag{22}$$

Now our objective in designing $\tilde{u}_2$ is to guarantee that $z_{1,i} - f_{1,i}(x_1)$ converges to zero exponentially while making sure that the state constraints are satisfied. To proceed further, we define functions $m_{i,j}(x)$, $i = 1, \ldots, s$, $j = 1, \ldots, k_i + 1$ as follows: $m_{i,1}(x) := f_{1,i}(x_1)$, and for $i = 1, \ldots, s$, $j = 2, \ldots, k_i + 1$

$$m_{i,j}(x) := -K_{2,i,j-1} z - \delta_j (x_{2,i,j-1} - m_{i,j-1}(x)) + \frac{\mathrm{d}}{\mathrm{d}t} m_{i,j-1}(x),$$

where the parameter δ_j is such that δ_2 is as chosen before, and $\delta_3 > 0, \ldots, \delta_{k_i+1} > 0$ are to be chosen subsequently. We would like to point out that $m_{i,j}(x)$ as defined above are linear functions of x. We define next certain variables, $\varepsilon_{i,j}$, $i = 1, \ldots, s$, $j = 2, \ldots, k_i$ such that

$$\varepsilon_{i,j}(t) := x_{2,i,j} - m_{i,j}(x).$$

We are now ready to choose the components of $\tilde{u}_2$, namely $u_{2,i}$, $i = 1, \ldots, s$. If $k_i > 1$, we choose

$$u_{2,i} = -G_i x_2 + m_{i,k_i+1}(x). \tag{23}$$

If $k_i = 1$ for some i, say $i = \alpha$, then $u_{2,\alpha}$ is chosen as

$$u_{2,\alpha} = -G_\alpha x_2 + m_{\alpha,2}(x) + \varepsilon_{\alpha,2}(t), \tag{24}$$

where $\varepsilon_{\alpha,2}(t)$ is to be chosen soon. We note that if $k_i \neq 1$ for any $i = 1, \ldots, s$, obviously the system (22) with the choice of $u_{2,i}$ as chosen in (23) is exponentially stable. We define next,

$$\varepsilon_2(t) := \begin{pmatrix} \varepsilon_{1,2}(t) \\ \vdots \\ \varepsilon_{s,2}(t) \end{pmatrix}.$$

Let us next focus on the behavior of the constraints under the feedback laws chosen above. We observe that z_2 satisfies the constraints due to the choice of $\tilde{u}_1$. Hence we focus on z_1. We have

$$\begin{aligned}
z_1 = C_1 x_2(t) &= \mathrm{e}^{-\delta_2 t} C_1 x_2(0) + f_1(x_1(t))\\
&\quad - \mathrm{e}^{-\delta_2 t} f_1(x_1(0)) + \int_0^t \mathrm{e}^{-\delta_2(t-\tau)} \varepsilon_2(\tau)\,\mathrm{d}\tau\\
&= \mathrm{e}^{-\delta_2 t} C_1 x_2(0) + (1 - \mathrm{e}^{-\delta_2 t}) f_1(x_1(t))\\
&\quad + \mathrm{e}^{-\delta_2 t} \int_0^t \frac{\mathrm{d}}{\mathrm{d}t} f_1(x_1(\tau))\,\mathrm{d}\tau\\
&\quad + \int_0^t \mathrm{e}^{-\delta_2(t-\tau)} \varepsilon_2(\tau)\,\mathrm{d}\tau.
\end{aligned}$$

Since $z_0 = f_1(x_1)$ satisfies (20) we get:

$$\mathrm{e}^{-\delta_2 t} \int_0^t \frac{\mathrm{d}}{\mathrm{d}t} f_1(x_1(\tau))\,\mathrm{d}\tau \in \frac{\zeta}{2} \tilde{\mathscr{S}}_1.$$

Moreover, $C_1 x_2(0) \in (1-\zeta)\tilde{\mathscr{S}}_1$ and $f_1(x_1(t)) \in (1-\zeta)\tilde{\mathscr{S}}_1$. Therefore, if we guarantee that

$$\int_0^t \mathrm{e}^{-\delta_2(t-\tau)} \varepsilon_2(\tau)\,\mathrm{d}\tau \in \frac{\zeta}{2} \tilde{\mathscr{S}}_1, \tag{25}$$

then we obtain $C_1 x_2(t) \in \tilde{\mathscr{S}}_1$ as required.

Next, we need to consider the rate constraint on z_1.

$$\begin{aligned}
\dot{z}_1 = \frac{\mathrm{d}}{\mathrm{d}t} C_1 x_2 &= -\delta_2 (C_1 x_2 - f_1(x_1))\\
&\quad + \frac{\mathrm{d}}{\mathrm{d}t} f_1(x_1) + \varepsilon_2(t),
\end{aligned} \tag{26}$$

We know from (20) that $(\mathrm{d}/\mathrm{d}t) f_1(x_1) \in (\zeta/3)\mathscr{T}_1$. Since δ_2 satisfies (17) combined with the fact that $C_1 x_2 - f_1(x_1) \in \bar{\mathscr{S}}_1$ we find that $\delta_2(C_1 x_2 - f_1(x_1)) \in (\zeta/3)\mathscr{T}_1$. Hence, we obtain that $(\mathrm{d}/\mathrm{d}t) C_1 x_2 \in \mathscr{T}_1$, if we guarantee that:

$$\varepsilon_2(t) \in \left(1 - \frac{2\zeta}{3}\right) \mathscr{T}_1. \tag{27}$$

Therefore, if we can guarantee (25) and (27), then we satisfy our constraints. We still need to show that the difference between z and $f(x_1)$, which is equal to the disturbance v in the first subsystem, satisfies (19). We have

$$v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix},$$

where $v_2 = 0$ and rewriting (26) we get

$$\dot{v}_1(t) = -\delta_2 v_1(t) + \varepsilon_2(t)$$

with $v_1(0) \in \bar{\mathscr{S}}_1$. Therefore, v satisfies (19) if ε_2 satisfies:

$$\int_0^t \mathrm{e}^{\delta_2 \tau} \varepsilon_2(\tau)\,\mathrm{d}\tau \in \tilde{\mathscr{S}}_1. \tag{28}$$

We will next consider how we can guarantee that $\varepsilon_2(t)$ satisfies (25), (27) and (28). It is easy to see that these three conditions are satisfied if for all $t > 0$ we have:

$$\varepsilon_2(t) \in \mathrm{e}^{-\delta_3 t} \left(1 - \frac{2\zeta}{3}\right) \mathscr{T}_1 \tag{29}$$

for δ_3 large enough.

For ease of notation, we define

$$\mathscr{A} = \{ i \mid i = 1, \ldots, s,\ k_i = 1 \},$$

$$\mathscr{A}^c = \{ i \mid i = 1, \ldots, s,\ k_i > 1 \}.$$

With $i \in \mathscr{A}^c$, we have

$$\varepsilon_{i,2}(t) = \mathrm{e}^{-\delta_3 t}\varepsilon_{i,2}(0) + \int_0^t \mathrm{e}^{-\delta_3(t-\tau)}\varepsilon_{i,3}(\tau)\,\mathrm{d}\tau$$

with $\varepsilon_{i,3}(t) = 0$ if $k_i = 2$. For $i \in \mathscr{A}$, we can obtain arbitrary $\varepsilon_{i,2}$ by choosing $u_{2,i}$ as in (24). Assume for $i \in \mathscr{A}$ we choose:

$$\varepsilon_{i,2}(t) = \mathrm{e}^{-\delta_3 t}\varepsilon_{i,2}(0) \tag{30}$$

with $\varepsilon_{i,2}(0)$ still to be chosen. If we guarantee:

(1) $\varepsilon_2(0) \in (1-\zeta)\mathscr{T}_1$,
(2) $\int_0^t \mathrm{e}^{\delta_3 t}\varepsilon_{i,3}(\tau)\,\mathrm{d}\tau$ small enough for i with $k_i \geqslant 3$,

then (29) is satisfied. We will see later how, by choosing $\delta_4, \delta_5, \ldots$, we can guarantee item (2). Let us next consider item (1). We have

$$\varepsilon_2(0) = \dot{z}_1(0) + \delta_2(z_1(0) - f_1(x_1(0))) - \frac{\mathrm{d}}{\mathrm{d}t} f_1(x_1(0)).$$

We have:

$$\delta_2(z_1(0) - f_1(x_1(0))) \in \delta_2\bar{\mathscr{S}}_1 \subset \frac{\zeta}{3}\mathscr{T}_1,$$

$$\frac{\mathrm{d}}{\mathrm{d}t} f_1(x_1(0)) \in \frac{\zeta}{3}\mathscr{T}_1$$

and hence as soon as we guarantee that $\dot{z}_1(0) \in (1-2\zeta)\mathscr{T}_1$ we know that item (1) is satisfied.

However, because $x_2(0) \in (1-2\zeta)\mathscr{V}_2(\mathscr{S},\mathscr{T})$ there exists $u(0)$ such that $\dot{z}_1(0) \in (1-2\zeta)\mathscr{T}_1$. It is easily verified that the only components of $u(0)$ which affect $\dot{z}_1(0)$ are exactly the $u_{2,i}(0)$ with $i \in \mathscr{A}$ which, according to (24), is equivalent to choosing $\varepsilon_{i,2}(0)$ with $i \in \mathscr{A}$ appropriately.

This yields a system with the desired properties but the feedback is partially determined in open loop due to our choice in (30) and therefore not acceptable. Choose instead for each t, $\varepsilon_{i,2}(t)$ with $i \in \mathscr{A}$ to minimize the following criterion:

$$\min\{\gamma \mid \varepsilon_2(t) \in \gamma\mathscr{T}_1\}. \tag{31}$$

Note that the existence of $\varepsilon_{i,2}(t)$ with $i \in \mathscr{A}$ that minimize this criterion is a consequence of the fact that the set $\mathscr{T}_1$ is bounded. Clearly, the optimal $\varepsilon_{i,2}(t)$ with $i \in \mathscr{A}$ becomes a function of $\varepsilon_{i,2}(t)$ with $i \in \mathscr{A}^c$. But for $i \in \mathscr{A}^c$, the $\varepsilon_{i,2}(t)$ are a function of the state and hence the $\varepsilon_{i,2}(t)$ with $i \in \mathscr{A}$ are determined according to a state feedback. Note that (30) is a suboptimal choice for the optimization in (31) yielding (29) and therefore, we have that

$$\gamma \leqslant \mathrm{e}^{-\delta_3 t}\left(1 - \frac{2\zeta}{3}\right)$$

and hence the choice for $\varepsilon_{i,2}(t)$ according to the optimization is a state feedback which also satisfies (29). Note that in general the dependence of $\varepsilon_{i,2}$ with $i \in \mathscr{A}$ on the $\varepsilon_{i,2}$ with $i \in \mathscr{A}^c$ is nonlinear. There are a few instances where we can guarantee a linear feedback. Clearly, if either the set $\mathscr{A}$ or the set $\mathscr{A}^c$ is empty, then this mapping is automatically linear since either its domain or its range is 0-dimensional. Moreover, if $\operatorname{im} C_z \subset \mathscr{T}$, then $\mathscr{T}_1$ is equal to the whole space and we get an optimal value $\gamma = 0$ by choosing $\varepsilon_{i,2}(t) = 0$ for $i \in \mathscr{A}$ which clearly also yields a linear feedback.

Finally, we still need to choose $\delta_4, \delta_5, \ldots$. We note that we have the following structure when $k_i > 2$.

$$\dot{\varepsilon}_{i,j} = -\delta_j\varepsilon_{i,j} + \varepsilon_{i,j+1} \quad \text{for } j = 1,\ldots,k_i - 1,$$
$$\dot{\varepsilon}_{i,j} = -\delta_j\varepsilon_{i,j} \quad \text{for } j = k_i.$$

From the above structure, it should be obvious that we can make the $\varepsilon_{i,j}$ small by a suitable design of the δ_j. We have to make sure that

$$\int_0^t \mathrm{e}^{\delta_3 t}\varepsilon_{i,3}(\tau)\,\mathrm{d}\tau$$

is small enough for those i with $k_i \geqslant 3$. By making δ_4 large enough, we can make this arbitrarily small provided that

$$\int_0^t \mathrm{e}^{\delta_4 t}\varepsilon_{i,4}(\tau)\,\mathrm{d}\tau$$

is small enough. If $k_i = 4$ this is actually equal to zero and otherwise we can use a similar argument to make this small enough by choosing δ_5 large enough. In this way, we can recursively determine $\delta_4, \delta_5, \ldots, \delta_{k_i+1}$.

Finally, note that all the $\varepsilon_{i,j}(t)$ converge to zero exponentially and therefore for $i = 2$ this implies that $\varepsilon_{i,2}(t)$ converges to zero exponentially and hence the difference between $x_{i,1}$ and $f_{1,i}(x_1)$ converges to zero exponentially. Since $f_1(x_1)$ also converges to zero exponentially we find that $x_{i,1}$ converges to zero exponentially. This also implies that z converges to zero exponentially. Finally it implies that $m_{i,2}(x)$ converges to zero exponentially. Similarly, since $\varepsilon_{i,3}(t)$ converges to zero exponentially we have that the difference between $x_{i,1}$ and $m_{i,2}(x)$ converges to zero exponentially. Hence, if $k_i > 1$, $m_{i,2}(x)$ converges to zero exponentially and we find that $x_{i,2}$ converges to zero exponentially. As before, this also implies that $m_{i,3}(x_1)$ converges to zero exponentially. Continuing with this recursive argument, we find that all states converge to zero exponentially and therefore, the constructed feedback has the desired attractivity as well as stability.

4.1.1.3. *Proof of Theorem 20.*

We first note that any stabilizing controller satisfies (7). Consider an initial condition $x(0) = (x_1(0), x_2(0))$ in the domain of attraction $\mathscr{R}_A^f(\Sigma)$. We note that by construction of $\mathscr{V}_2$ if $x_2(0) \notin \mathscr{V}_2(\mathscr{S},\mathscr{T})$ then we cannot even satisfy the constraints at time 0. On the other hand, the first subsystem Σ_s is controlled through z and has initial condition $x_1(0)$. Since $x(0)$ is in the domain of attraction, we know $x(t) \to 0$ as $t \to \infty$ but then also in the system Σ_s we have that the state $x_1(t)$ converges to zero as $t \to \infty$ while z satisfies the constraints. This implies by definition that $x_1(0) \in \mathscr{R}_c(\Sigma_s,\mathscr{S},\mathscr{T})$.

In order to prove the second part of Theorem 20, we need the following lemma which is a direct consequence of results from van Moll (1999).

Lemma 26. *Given a linear system*:

$$\bar{\Sigma}:\ \dot{\bar{x}} = \bar{A}\bar{x} + \bar{B}\bar{u} + d, \quad \bar{x} \in \mathbb{R}^n,\ \bar{u} \in \mathbb{R}^m$$

and a convex set $\bar{\mathscr{S}}$ which contains 0 *in the interior. Assume there exists a feedback $\bar{u} = g(\bar{x})$ and a set $\mathscr{R}^g_A(\bar{\Sigma})$ such that in the absence of d, 0 is the equilibrium point of the closed-loop system which is asymptotically stable and its domain of attraction contains $\mathscr{R}^g_A(\bar{\Sigma})$. Moreover, for all initial conditions in the set $\mathscr{R}^g_A(\bar{\Sigma})$ and in the absence of d (i.e. $d = 0$) we have that $\bar{u}(t) \in \bar{\mathscr{S}}$.*

Then we can find for any compact set $\mathscr{R}$ satisfying $\mathscr{R} \subset \zeta\mathscr{R}^g_A(\bar{\Sigma})$ for some $\zeta < 1$, a feedback $\bar{u} = h(\bar{x})$ such that h is continuous and for any M, there exists an r large enough such that for any d satisfying:

$$\|d(t)\| \leqslant Me^{-rt}$$

and for all initial conditions in the set $\mathscr{R}$ we have that $\bar{u}(t) \in \bar{\mathscr{S}}$ and all the trajectories $\bar{x}(t)$ go to zero exponentially as $t \to \infty$.

The above lemma yields the proof of Theorem 20 immediately by utilizing the same construction as given in the proof of Theorem 15 but using $z_0 = h(x_1)$ as constructed in the above lemma instead of $z = f(x_1)$. We should note that since we have no rate constraints we can choose $\delta_2 = r$ (with r as given in the above lemma) since the feedback as constructed in the above lemma requires a rapid enough decay in the disturbance signal d (which equals v as used in our proof of Theorem 15).

4.2. Non-right invertible constraints

In this section, we consider non-right invertible constraints. In the case of right invertible constraints, as we have seen earlier, if a stabilization problem is solvable for one pair of constraint sets $\mathscr{S}$ and $\mathscr{T}$, the same stabilization problem is solvable for any pair of constraint sets $\mathscr{S}$ and $\mathscr{T}$ irrespective of their shape as long as they satisfy Assumption 1. In general this is not so for non-right invertible constraints. This adds a layer of complexity and renders the case of non-right invertible constraints profoundly different from the case of right invertible constraints.

In what follows, we first provide a set of necessary conditions for the solvability of the posed stabilization problems. This set of conditions do not depend on the shape of the sets $\mathscr{S}$ and $\mathscr{T}$, but essentially show the complexity involved with non-right invertible constraints. Next, we present an example showing that the necessary and sufficient conditions for the solvability of the posed stabilization problems in general invariably depend on the shape (i.e. on the specific features) of the sets $\mathscr{S}$ and $\mathscr{T}$. We then pursue the task of identifying a class of non-right invertible constraints for which the necessary and sufficient conditions for the solvability of the posed stabilization problems can be formulated without involving any specific features of the sets $\mathscr{S}$ and $\mathscr{T}$ as long as they satisfy Assumption 1.

As we shall see shortly for both global and semi-global stabilizations, the condition given in the previous section for the right invertible case remains necessary for the case of non-right invertible constraints. However, these conditions are no longer sufficient. In fact we show that an intricate set of additional conditions are required in case of non-right invertible constraints.

Consider the special coordinate basis as given in (2). In order to proceed again we need further structure to be recalled from scb. In order to show this structure, we need to have a transformation in the constrained output space in addition to the earlier transformations in the state and input spaces. There exists a transformation matrix Γ_z such that $\bar{z} = \Gamma_z z$ yields the following decomposition

$$\bar{z} = \begin{pmatrix} z_{11} \\ z_{12} \\ z_2 \end{pmatrix} = \begin{pmatrix} C_{z,12} \\ 0 \\ 0 \end{pmatrix} x_1 + \begin{pmatrix} 0 \\ \bar{C}_1 \\ 0 \end{pmatrix} x_2 + \begin{pmatrix} 0 \\ 0 \\ D_1 \end{pmatrix} \tilde{u}_1. \tag{32}$$

Note that choosing a basis in the output space affects our sets $\mathscr{S}$ and $\mathscr{T}$. Therefore, we obtain new constraint sets $\tilde{\mathscr{S}}$ and $\tilde{\mathscr{T}}$. Since $C_z^{\mathrm{T}} D_z = 0$ we can guarantee that these new constraint sets still satisfy Assumption 1.

Consider our original system in the special coordinate basis as given in (2) together with the extra output transformation in (32). By defining $\tilde{A}_1 = A_{11}$, $\tilde{B}_1 = K_1\Gamma_z^{-1}$, $\tilde{C}_1 = C_{z,12}$, $\tilde{x}_1 = x_1$, $\tilde{v}_1 = \bar{z}$ and $\tilde{z}_1 = z_{11}$, we obtain for $i = 1$ the following system:

$$\tilde{\Sigma}_i:\ \begin{cases} \dot{\tilde{x}}_i = \tilde{A}_i\tilde{x}_i + \tilde{B}_i\tilde{v}_i, \\ \tilde{z}_i = \tilde{C}_i\tilde{x}_i. \end{cases} \tag{33}$$

This is basically the first subsystem as defined in the previous section but still with an output constraint which was absent in the right invertible case. So basically from the system Σ we constructed $\tilde{\Sigma}_1$. In a similar fashion we can construct $\tilde{\Sigma}_2$ from $\tilde{\Sigma}_1$ and so on. However, at each step we should make sure that the matrix $\tilde{B}_i$ has full column rank and the matrix $\tilde{C}_i$ has full row rank to proceed with the next step. This can of course be done without loss of generality. This chain ends if we obtain a subsystem $\tilde{\Sigma}_i$ which is right invertible in the sense that $\tilde{\Sigma}_{i+1}$ satisfies $\tilde{C}_{i+1} = 0$. Another possibility for termination is that after some steps we get $\tilde{B}_i = 0$ which obviously implies that we can end the chain. We know that (it can be shown easily) if the pair (A, B) of the given system Σ is stabilizable, then all the systems $\tilde{\Sigma}_i$ as defined in (33) are stabilizable.

We have the following result for the case of amplitude constraints only.

Theorem 27. *Consider the plant Σ as given by* (1) *with constraint sets $\mathscr{S}$ and $\mathscr{T}=\mathbb{R}^p$ satisfying Assumption* 1. *Let the chain of systems $\tilde{\Sigma}_i$ $(i=1,\dots,s)$ be as described above. Then the semi-global constrained stabilization problem as formulated in Problem* 7 *is solvable only if the following conditions are satisfied*:

(1) (A,B) *is stabilizable.*
(2) *The constraints are at most weakly non-minimum phase.*
(3) *All the systems $\tilde{\Sigma}_i$ $(i=1,\dots,s)$ have at most weakly non-minimum phase constraints.*
(4) *The systems $\tilde{\Sigma}_i$ $(i=1,\dots,s)$ with realization* (33) *satisfy*:

$$\ker \tilde{C}_i \subset \ker \tilde{C}_i\tilde{A}_i. \tag{34}$$

Moreover, the global constrained stabilization problem as formulated in Problem 6 *is solvable only if the above conditions* (1)–(4) *and the following condition are satisfied*:

(5) *The constraints are of type one.*

Proof. The necessity of (1) and (2) are obvious. The condition (5) is also a direct consequence of earlier arguments.

In order to show item (3), consider one of the systems $\tilde{\Sigma}_i$. This system has input constraints and output constraints in the sense that $\tilde{v}_i$ and $\tilde{z}_i$ must both be bounded, i.e. $\tilde{v}_i \in \mathscr{V}_i$ and $\tilde{z}_i \in \mathscr{S}_i$ for some bounded sets $\mathscr{V}_i$ and $\mathscr{S}_i$. Based on earlier theorems, the necessity of condition (3) is then obvious.

To show the necessity of condition (4), we proceed as follows. Consider the system $\tilde{\Sigma}_i$ whose input $\tilde{v}_i$ and output $\tilde{z}_i$ are bounded. Assume $\tilde{z}_i$ is constrained to be in the set $\mathscr{S}_i$. We will prove this implication by contradiction. Assume that there exists a vector $\check{x}$ such that $\tilde{C}_i\check{x}=0$ but $\tilde{C}_i\tilde{A}_i\check{x}\neq 0$. For any $\varepsilon>0$, there exists a vector $\hat{x}$ such that $\tilde{C}_i\hat{x}$ is in the interior of $\mathscr{S}_i$ but $\tilde{C}_i\hat{x}+\varepsilon\tilde{C}_i\tilde{A}_i\check{x}\notin\mathscr{S}_i$. Consider for any scalar λ the initial condition $x_i(0)=\hat{x}+\lambda\check{x}$. It is easily verified that this initial condition is admissible. We have:

$$\dot{z}_1(0)=\tilde{C}_i\tilde{A}_i\hat{x}+\tilde{C}_i\tilde{B}_i\hat{u}_i(0)+\lambda\tilde{C}_i\tilde{A}_i\check{x}. \tag{35}$$

For large enough λ the last term in this derivative will dominate the first two. Recall in that respect that $\hat{x}(0)$ is fixed and we can choose $\hat{u}_i$ but it is constrained to be in a fixed bounded set. However, if we move in the direction $\tilde{C}_i\tilde{A}_i\check{x}$, then we will be outside the set $\mathscr{S}_i$ very quickly when we choose ε sufficiently small combined with the fact that

$$z_1(0)+\varepsilon\tilde{C}_i\tilde{A}_i\check{x}=\tilde{C}_i\hat{x}+\varepsilon\tilde{C}_i\tilde{A}_i\check{x}\notin\mathscr{S}_i.$$

Note that, since the complement of $\mathscr{S}_i$ is open, the small perturbation caused by the first two terms in the derivative of $\dot{z}_1(0)$ in (35) cannot avoid that we will leave $\mathscr{S}_i$ since they only cause a minor perturbation compared to the dominant third term. The initial condition $x_i(0)$ is in the interior of the admissible set of initial conditions for the system $\tilde{\Sigma}_i$ but we cannot avoid constraint violation with this initial condition. This yields a contradiction to the claim that this system was semi-globally stabilizable. Therefore such a $\check{x}$ for which $\tilde{C}_i\check{x}=0$ but $\tilde{C}_i\tilde{A}_i\check{x}\neq 0$ does not exist, and this yields the fourth condition. □

Remark 28. The condition (34) immediately implies that the order of infinite zeros of each subsystem $\tilde{\Sigma}_i$, $i=1,\dots s$, is less than or equal to one.

The following example indicates that the conditions given in Theorem 27 are just necessary conditions and are not sufficient to solve the constrained stabilization problems. Also, this example shows that the solvability conditions for global and semi-global stabilization in the case of non-right invertible constraints (unlike the case of right invertible constraints) in general depend on the *particular* choice of constraint sets $\mathscr{S}$ and $\mathscr{T}$.

Example 29. Consider the following system (Kosut, 1988):

$$\dot{x}_1=x_2,$$

$$\dot{x}_2=-a_1x_1-a_2x_2-b_1u,$$

$$z=(u^{\mathrm{T}},x^{\mathrm{T}})^{\mathrm{T}},$$

where z required to be constrained in hypercubes, and $a_1=3575$, $a_2=333$, $b_1=305\,555$.

Note that the transfer matrix from u to z in this example is non-right invertible. We obtain $\tilde{\Sigma}_1$ and $\tilde{\Sigma}_2$ as:

$$\tilde{\Sigma}_1\colon \begin{cases} \dot{\tilde{x}}_1=\begin{pmatrix}0 & 1\\ -a_1 & -a_2\end{pmatrix}\tilde{x}_1+\begin{pmatrix}0 & 0\\ 0 & -b_1\end{pmatrix}\tilde{v}_1,\\ \tilde{z}_1=\tilde{x}_1,\end{cases}$$

$$\tilde{\Sigma}_2\colon \begin{cases} \dot{\tilde{x}}_2=(0\quad 1)\tilde{v}_2,\\ \tilde{z}_2=\tilde{x}_2.\end{cases}$$

Note that to construct $\tilde{\Sigma}_2$ from $\tilde{\Sigma}_1$ we have removed the redundancy (a column equal to 0) in $\tilde{B}_1$. This example satisfies the necessary conditions in Theorem 27. On the other hand, suppose we require that:

$$u\in[c_1,d_1],\quad x_1\in[c_2,d_2],\quad x_2\in[c_3,d_3], \tag{36}$$

where $0\in(c_i,d_i)$. We have that 0 is an interior point of the constraints set and hence we have $c_2<0$ and $c_3<0$. Therefore, if $x_1(0)=c_2$, $x_2(0)=c_3$, we get that x_1 will leave $[c_2,d_2]$, which shows that global constrained stabilization is not possible. Moreover, since $\dot{x}_2$ is bounded, an initial condition very close to the boundary will still violate the constraint conditions and hence semi-global stabilization is not possible either. Hence, for the given constraint sets (36), we cannot achieve semi-global or global constrained stabilization. However, it is trivial to show that there exist other constraint sets for x (for instance ellipsoidal sets) such that we can achieve semi-global or global constrained stabilization. This implies that the solvability conditions depend on the particular choice of the constraint sets unlike in the case of right invertible constraints.

We pursue next the task of identifying a class of non-right invertible constraints for which the necessary and sufficient conditions for the solvability of the posed stabilization problems can be formulated without involving any specific features of the sets $\mathscr{S}$ and $\mathscr{T}$ as long as they satisfy Assumption 1.

To proceed we first consider the system $\tilde{\Sigma}_1$ as given in (33) for $i = 1$. First note that using the structure from scb, we can decompose this system as follows:

$$\begin{pmatrix} \dot{\tilde{x}}_1^1 \\ \dot{\tilde{x}}_1^2 \end{pmatrix} = \begin{pmatrix} \tilde{A}_1^1 & \tilde{A}_1^2 \\ 0 & \tilde{A}_1^3 \end{pmatrix} \begin{pmatrix} \tilde{x}_1^1 \\ \tilde{x}_1^2 \end{pmatrix} + \begin{pmatrix} \tilde{B}_1^1 \\ \tilde{B}_1^2 \end{pmatrix} \tilde{v}_1,$$
$$\tilde{z}_1 = (0 \quad \tilde{C}_1^2) \begin{pmatrix} \tilde{x}_1^1 \\ \tilde{x}_1^2 \end{pmatrix}, \tag{37}$$

where $\tilde{x}_1^1$ represent the zero dynamics. Note the structure of $\tilde{v}_1$ and $\tilde{z}_1$ that we used in constructing $\tilde{\Sigma}_1$:

$$\tilde{v}_1 = (z_{11}^{\mathrm{T}}, z_{12}^{\mathrm{T}}, z_2)^{\mathrm{T}}, \quad \tilde{z}_1 = z_{11}.$$

Using this decomposition explicitly, we get:

$$\begin{pmatrix} \dot{\tilde{x}}_1^1 \\ \dot{\tilde{x}}_1^2 \end{pmatrix} = \begin{pmatrix} \tilde{A}_1^1 & \tilde{A}_1^2 \\ 0 & \tilde{A}_1^3 \end{pmatrix} \begin{pmatrix} \tilde{x}_1^1 \\ \tilde{x}_1^2 \end{pmatrix} + \begin{pmatrix} \tilde{B}_1^{11} & \tilde{B}_1^{12} & \tilde{B}_1^{13} \\ \tilde{B}_1^{21} & \tilde{B}_1^{22} & \tilde{B}_1^{23} \end{pmatrix} \begin{pmatrix} z_{11} \\ z_{12} \\ z_2 \end{pmatrix},$$

$$z_{11} = (0 \quad \tilde{C}_1^2) \begin{pmatrix} \tilde{x}_1^1 \\ \tilde{x}_1^2 \end{pmatrix}.$$

We can eliminate the z_{11} from the state equation by substituting the output equation and we obtain the following system:

$$\bar{\Sigma}_1 : \begin{cases} \begin{pmatrix} \dot{\tilde{x}}_1^1 \\ \dot{\tilde{x}}_1^2 \end{pmatrix} = \begin{pmatrix} \tilde{A}_1^1 & \bar{A}_1^2 \\ 0 & \bar{A}_1^3 \end{pmatrix} \begin{pmatrix} \tilde{x}_1^1 \\ \tilde{x}_1^2 \end{pmatrix} + \begin{pmatrix} \bar{B}_1^1 \\ \bar{B}_1^2 \end{pmatrix} \bar{v}_1, \\ \tilde{z}_1 = (0 \quad \tilde{C}_1^2) \begin{pmatrix} \tilde{x}_1^1 \\ \tilde{x}_1^2 \end{pmatrix}, \end{cases} \tag{38}$$

where $\bar{A}_1^2 = \tilde{A}_1^2 + \tilde{B}_1^{11}\tilde{C}_1^2$, $\bar{A}_1^3 = \tilde{A}_1^3 + \tilde{B}_1^{21}\tilde{C}_1^2$, $\bar{B}_1^1 = (\tilde{B}_1^{12}, \tilde{B}_1^{13})$, $\bar{B}_1^2 = (\tilde{B}_1^{22}, \tilde{B}_1^{23})$, $\bar{v}_1 = (z_{12}^{\mathrm{T}}, z_2^{\mathrm{T}})^{\mathrm{T}}$, and $\tilde{z}_1 = z_{11}$.

Note that if we impose amplitude constraints on $\bar{v}_1$ and $\tilde{z}_1$, then we can always translate these back to constraints on the original output z provided the constraint set $\mathscr{V}$ for $\bar{v}_1$ decomposes as $\mathscr{V} = \mathscr{V}_1 \times \mathscr{V}_2$ compatible with the decomposition of $\bar{v}_1$ into z_{12} and z_2.

Theorem 30. *Consider the system Σ given by* (1) *with* (A, B) *stabilizable. Then the following two statements are equivalent*:

(1) *Semi-global constrained stabilization is possible for all constraint sets $\mathscr{S}$ and $\mathscr{T}$ satisfying Assumption* 1.
(2) *The constraints of system Σ are at most weakly non-minimum phase. Moreover, if we construct $\bar{\Sigma}_1$ of the form* (38), *then we have $\bar{A}_1^3 = \alpha I$ with $\alpha \leqslant 0$ and $\tilde{C}_1^2$ injective.*

For global constrained stabilization we can use similar arguments to obtain the following result:

Theorem 31. *Consider the system Σ given by* (1) *with* (A, B) *stabilizable. Then the following two statements are equivalent*:

(1) *Global constrained stabilization is possible for all constraint sets $\mathscr{S}$ and $\mathscr{T}$ satisfying Assumption* 1.
(2) *The constraints of system Σ are at most weakly non-minimum phase and of type one. Moreover, if we construct $\bar{\Sigma}_1$ of the form* (38) *then we have $\bar{A}_1^3 = \alpha I$ with $\alpha \leqslant 0$ and $\tilde{C}_1^2$ injective.*

The proofs of Theorems 30 and 31 are omitted. Interested readers can find the proofs from Saberi et al. (2001).

5. Conclusions

We focus here on a wide range of constrained stabilization problems, where the constraints exist on both the amplitude and rate of change of state as well as control variables in a very broad frame work. An important aspect that emerged from our study is the taxonomy of constraints. This taxonomy is based on the structural properties of the mapping from the input to the constraint output. The formulated stabilization problems in global, semi-global, as well as regional frame work are studied at length. Also, construction methods of appropriate stabilizing controllers are developed. We would like to point out that some of the concepts and ideas of constraint stabilization which have been presented in this paper can be extended to nonlinear systems with the proper structure such as normal form in more or less straightforward manner.

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