

Stability of Polytopic Polynomial Matrices¹

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Abstract: This paper gives a necessary and sufficient condition for robust D -stability of Polytopic Polynomial Matrices.

Keywords: Robust Stability, Polynomial Matrices, Polytopic Polynomials, Edge Theorem, Kharitonov's Theorem.

1 Introduction

Motivated by the seminal theorem of Kharitonov on robust stability of interval polynomials[1, 2], a number of papers on robustness analysis of uncertain systems have been published in the past few years[3, 4, 5, 6, 7, 8, 9, 10]. Kharitonov's theorem states that the Hurwitz stability of a real (or complex) interval polynomial family can be guaranteed by the Hurwitz stability of four (or eight) prescribed critical vertex polynomials in this family. This result is significant since it reduces checking stability of infinitely many polynomials to checking stability of finitely many polynomials, and the number of critical vertex polynomials need to be checked is independent of the order of the polynomial family. An important extension of Kharitonov's theorem is the edge theorem discovered by Bartlett, Hollot and Huang[4]. The edge theorem states that the stability of a polytope of polynomials can be guaranteed by the stability of its one-dimensional exposed edge polynomials. The significance of the edge theorem is that it allows some (affine) dependency among polynomial coefficients, and applies to more general stability regions, e.g., unit circle, left sector, shifted half plane, hyperbola region, etc. When the dependency among polynomial coefficients is nonlinear, however, Ackermann shows that checking a subset of a polynomial family generally can not guarantee the stability of the entire family[11, 12, 13].

Parallel to this line of research, robust stability of uncertain matrices has also received considerable attention. Bialas 'proved' that for robust Hurwitz stability of an interval matrix, it suffices to check all vertices[14]. But Barmish and Hollot gave a counter-example to show that Bialas's claim is incorrect[15]. Kokame and Mori considered Hurwitz stability of an interval polynomial matrix, and by using some result in signal processing theory, established a necessary and sufficient condition for robust stability[16].

This paper studies robust D -stability of polytopic polynomial matrices, i.e., matrices with entries being polytopes of polynomials. We give a necessary and sufficient condition for robust D -stability of Polytopic Polynomial Matrices, namely, the stability of a subset of this family guarantees the stability of the entire family.

2 Preliminaries

Definition 1 An interval polynomial matrix $A = (p_{ij})_{n \times n}$ is a matrix whose entries p_{ij} are interval polynomials, i.e., $p_{ij} = q_{ij}^0 + q_{ij}^1 s + \dots + q_{ij}^m s^m$, $q_{ij}^k \in [\underline{q}_{ij}^k, \bar{q}_{ij}^k]$, $k = 0, \dots, m$, where k stands for superscript. A polytopic polynomial matrix $A = (p_{ij})_{n \times n}$ is a matrix whose entries p_{ij} are polytopic polynomials, i.e., $p_{ij} = \sum_{k=1}^m \lambda_{ij}^k p_{ij}^k$, $\lambda_{ij}^k \geq 0$, $\sum_{k=1}^m \lambda_{ij}^k = 1$, $i, j = 1, \dots, n$, where p_{ij}^k are fixed polynomials.

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Definition 2 Suppose $\Omega \subset \mathbf{R}^{n+1}$ is an m -dimensional polytope. Its supporting plane H is defined as an n -dimensional affine set, satisfying $\Omega \cap H \neq \emptyset$, and all points of Ω lie on the same side of H ; Its exposed set is defined as the intersection of Ω and its supporting plane H ; Its exposed edge set is defined as the one-dimensional exposed set.

An n -th order polynomial can be regarded as a point in its $(n + 1)$ -dimensional coefficient space.

Definition 3 Given an open region D in the complex plane, the polynomial matrix A is said to be D -stable, if all roots of $\det A = 0$ lie within D ; A polynomial matrix set \mathcal{A} is said to be D -stable, if every member in \mathcal{A} is D -stable. When D is taken as the open left half of the complex plane, D -stable is also called Hurwitz stable.

In fact, Edge Theorem holds for more general stability regions. For simplicity, we only consider simply-connected stability regions in this paper.

Definition 4 Given an interval polynomial set $\mathcal{F}(s) = \{\sum_{i=0}^m q_i s^i, \quad q_i \in [\underline{q}_i, \bar{q}_i]\}$, its Kharitonov vertex set is $K_{\mathcal{F}}^0 = \{f_k^1, f_k^2, f_k^3, f_k^4\}$, and $E_{\mathcal{F}}^0 = \{\lambda f_k^s + (1 - \lambda)f_k^t, \quad (s, t) \in \{(1, 2), (2, 4), (4, 3), (3, 1)\}, \lambda \in [0, 1]\}$ is called its Kharitonov exposed edge set, where

$$\begin{aligned} f_k^1 &= \underline{q}_0 + \underline{q}_1 s + \bar{q}_2 s^2 + \dots & f_k^2 &= \underline{q}_0 + \bar{q}_1 s + \bar{q}_2 s^2 + \dots \\ f_k^3 &= \bar{q}_0 + \underline{q}_1 s + \underline{q}_2 s^2 + \dots & f_k^4 &= \bar{q}_0 + \bar{q}_1 s + \underline{q}_2 s^2 + \dots \end{aligned}$$

Consider the polytopic polynomial sets

$$\mathcal{P}_{ij} = \left\{ \sum_{k=1}^m \lambda_{ij}^k p_{ij}^k : \lambda_{ij}^k \geq 0, \sum_{k=1}^m \lambda_{ij}^k = 1 \right\} \quad (i, j = 1, \dots, n) \quad (1)$$

p_{ij}^k are fixed polynomials, $k = 1, \dots, m$.

Their vertex sets are

$$K_{ij} = \{p_{ij}^k \mid k = 1, \dots, m\} \quad (i, j = 1, \dots, n)$$

and by definition, their exposed edge sets are contained in

$$E_{ij} = \{\lambda p_{ij}^s + (1 - \lambda)p_{ij}^t, \quad s, t = 1, \dots, m\} \quad (i, j = 1, \dots, n)$$

Let $\mathcal{A} = \{(p_{ij})_{n \times n} : p_{ij} \in \mathcal{P}_{ij}, \quad i, j = 1, \dots, n\}$, and let P_n^n be the set of all permutations of $1, 2, \dots, n$.

Definition 5 Define $\epsilon_{\mathcal{A}}$ as

$$\left\{ (p_{ij})_{n \times n} : \begin{array}{l} p_{sl_s} \in E_{sl_s}, (l_1, \dots, l_n) \in P_n^n, s = 1, \dots, n \\ p_{si_s} \in K_{si_s}, i_s = 1, \dots, l_s - 1, l_s + 1, \dots, n \end{array} \right\} \quad (2)$$

It is easy to see that, $\epsilon_{\mathcal{A}}$ is produced by taking only one entry from its exposed edge set in every row/column and all other entries from their vertex sets in \mathcal{A} .

3 Main Results

3.1 Polytopic Polynomial Matrices

Consider the polytopic polynomial matrix set

$$\mathcal{A} = \{(p_{ij})_{n \times n} : \text{where } p_{ij} \in \mathcal{P}_{ij}, \quad i, j = 1, \dots, n\} \quad (3)$$

Suppose $\forall A \in \mathcal{A}, \deg(\det A) = \text{const}$.

Theorem 1 \mathcal{A} is D -stable if and only if $\epsilon_{\mathcal{A}}$ is D -stable.

Proof: Necessity is obvious. To prove sufficiency, suppose $\epsilon_{\mathcal{A}}$ is D -stable. Let

$$A = \begin{pmatrix} p_{11} & \dots & p_{1n} \\ \dots & \dots & \dots \\ p_{n1} & \dots & p_{nn} \end{pmatrix}$$

By using Laplace formula on the first column, we have

$$\det A = p_{11}M_{11} + \dots + p_{n1}M_{n1}$$

Let

$$\mathcal{T} = \left\{ \begin{pmatrix} p_{11}^* & p_{12} & \dots & p_{1n} \\ \dots & \dots & \dots & \dots \\ p_{n1}^* & p_{n2} & \dots & p_{nn} \end{pmatrix}, \begin{array}{l} \text{and } p_{i1}^* \in \mathcal{P}_{i1}; i = 1, \dots, n \\ p_{ij} \text{ are entries of } A \\ j = 2, \dots, n \end{array} \right\}$$

It is easy to see that $A \in \mathcal{T}$ and $\forall T \in \mathcal{T}$

$$\det T = p_{11}^*M_{11} + \dots + p_{n1}^*M_{n1}$$

Apparently, $\det T$ is an affine function of $p_{11}^*, \dots, p_{n1}^*$. By Edge Theorem

$$\det T \text{ is } D\text{-stable} \Leftrightarrow \text{the edge set of } \det T \text{ is } D\text{-stable.}$$

The edge set of $\det T$ is

$$\left\{ E_{i1}M_{i1} + \sum_{i \neq j=1}^n K_{j1}M_{j1}, i = 1, \dots, n \right\}.$$

The corresponding matrix collection is

$$\mathcal{A}_1 = \left\{ \begin{pmatrix} q_{11} \\ \dots \\ q_{i1} & (p_{st})_{n \times (n-1)} \\ \dots \\ q_{n1} \end{pmatrix}, \begin{array}{l} q_{i1} \in E_{i1}; i \in \{1, \dots, n\} \\ q_{k1} \in K_{k1}; k = 2, \dots, n \\ p_{kt}, p_{it} \text{ are entries of } A \\ k = 1, \dots, i-1, i+1, \dots, n \end{array} \right\}$$

In this case, \mathcal{T} is D -stable $\Leftrightarrow \mathcal{A}_1$ is D -stable. Moreover, $\forall A_1 \in \mathcal{A}_1$, there exists $i \in \{1, \dots, n\}, q_{i1} \in E_{i1}, q_{k1} \in K_{k1}, k = 1, \dots, i-1, i+1, \dots, n$ such that

$$A_1 = \begin{pmatrix} q_{11} & p_{12} \\ \dots & \dots \\ q_{i1} & p_{i2} & (p_{st})_{n \times (n-2)} \\ \dots & \dots \\ q_{n1} & p_{n2} \end{pmatrix}$$

Again, by using Laplace formula on the second column, we have

$$\det A_1 = p_{12}M_{12} + \dots + p_{n2}M_{n2}$$

Set

$$\mathcal{B} = \left\{ \begin{pmatrix} q_{11} & p_{12}^* & p_{13} & \dots & p_{1n} \\ \dots & \dots & \dots & \dots & \dots \\ q_{n1} & p_{n2}^* & p_{n3} & \dots & p_{nn} \end{pmatrix}, \begin{array}{l} q_{ij} \text{ are entries of } A_1 \\ p_{ij}^* \text{ are entries of } A \\ p_{ij}^* \in \mathcal{P}_{ij} \\ i, j = 1, \dots, n \end{array} \right\}$$

Let $\mathcal{T}_1 = \bigcup_{A_1 \in \mathcal{A}_1} \mathcal{B}$, then, its edge set \mathcal{A}_2 is

$$\left\{ \begin{pmatrix} q_{11} & q_{12} \\ \dots & \dots \\ q_{i1} & \dots \\ \dots & \dots & (p_{st}) \\ \dots & q_{j2} \\ \dots & \dots \\ q_{n1} & q_{n2} \end{pmatrix}, \begin{array}{l} j \in \{1, \dots, n\}; l = 1, \dots, n \\ q_{lm} \in \begin{cases} K_{lm}; & m = 1, l \neq i \\ E_{lm}; & m = 2, l \neq j \\ & m = 1, l = i \\ & m = 2, l = j \end{cases} \\ p_{lt} \text{ are entries of } A \\ t = 3, \dots, n \end{array} \right\}$$

and $\mathcal{A}_1 \subset \mathcal{T}_1$. By definition and Edge Theorem

$$\mathcal{T}_1 \text{ is } D\text{-stable} \Leftrightarrow \mathcal{A}_2 \text{ is } D\text{-stable}.$$

By repeating the process above, we have

$$\begin{aligned} & \mathcal{A}_2, \dots, \mathcal{A}_{n-1}, \mathcal{A}_n; \quad \mathcal{T}_2, \dots, \mathcal{T}_{n-1} \\ & \mathcal{T}_k \text{ is } D\text{-stable} \Leftrightarrow \mathcal{A}_{k+1} \text{ is } D\text{-stable} \\ & \mathcal{A}_k \text{ is } D\text{-stable} \Leftrightarrow \mathcal{T}_k \text{ is } D\text{-stable} \quad k = 2, \dots, n-1 \end{aligned}$$

where \mathcal{A}_k is the collection of

$$\left\{ \left(\begin{array}{ccc} q_{11} & \dots & q_{1k} \\ \dots & \dots & \dots \\ q_{i_1 1} & \dots & q_{i_1 k} \\ \dots & \dots & \dots \\ q_{i_k 1} & \dots & q_{i_k k} \\ \dots & \dots & \dots \\ q_{n1} & \dots & q_{nk} \end{array} \right)_{n \times n} \quad (p_{st}) \quad \left. \begin{array}{l} q_{lt} \in \begin{cases} E_{i_t t}, l = i_t \\ K_{i_t t}, l \neq i_t \end{cases} \\ i_1, \dots, i_k \in \{1, \dots, n\} \\ t = 1, \dots, k \\ l = 1, \dots, n \\ p_{ls} \text{ are entries of } A \\ s = k+1, \dots, n \end{array} \right\}$$

Thus, for each element of \mathcal{A}_k , its entries have the following characteristics: for the first k columns, all entries of each column belong to their vertex sets except that one entry belongs to its exposed edge set, and the entries of the remaining $n-k$ columns are the corresponding entries of A . Hence, $\forall A_n \in \mathcal{A}_n$, we have

$$\mathcal{A}_n = \left\{ \left(\begin{array}{ccc} q_{11} & \dots & q_{1n} \\ \dots & \dots & \dots \\ q_{i_1 1} & \dots & q_{i_1 n} \\ \dots & \dots & \dots \\ q_{i_n 1} & \dots & q_{i_n n} \\ \dots & \dots & \dots \\ q_{n1} & \dots & q_{nn} \end{array} \right) \quad \left. \begin{array}{l} q_{lt} \in \begin{cases} E_{i_t t}, l = i_t \\ K_{i_t t}, l \neq i_t \end{cases} \\ i_1, \dots, i_n \in \{1, \dots, n\} \\ t = 1, \dots, n \\ l = 1, \dots, n \end{array} \right\}$$

If $i_s = i_t$ for some pair i_s, i_t , without loss of generality, suppose $i_1 = i_2 = 1$, namely

$$A_n = \left(\begin{array}{cccc} q_{11} & q_{12} & \dots & q_{1n} \\ \dots & \dots & \dots & \dots \\ q_{n1} & q_{n2} & \dots & q_{nn} \end{array} \right), \quad q_{11} \in E_{11}, \quad q_{12} \in E_{12}$$

By using Laplace formula on the first row of A_n , we have

$$\det A_n = q_{11}M_{11} + q_{12}M_{12} + \sum_{i=3}^n q_{1i}M_{1i}$$

By Edge Theorem

$$A_n \text{ is } D\text{-stable} \Leftrightarrow q_{11}M_{11} + q_{12}^0M_{12} + \sum_{i=3}^n q_{1i}M_{1i} \text{ and}$$

$$q_{11}^0M_{11} + q_{12}M_{12} + \sum_{i=3}^n q_{1i}M_{1i} \text{ are } D\text{-stable}.$$

The corresponding matrices are

$$\begin{aligned} & \left(\begin{array}{cccc} q_{11}^0 & q_{12} & \dots & q_{1n} \\ \dots & \dots & \dots & \dots \\ q_{n1} & q_{n2} & \dots & q_{nn} \end{array} \right), q_{11}^0 \in K_{11}, \\ & \left(\begin{array}{cccc} q_{11} & q_{12}^0 & \dots & q_{1n} \\ \dots & \dots & \dots & \dots \\ q_{n1} & q_{n2} & \dots & q_{nn} \end{array} \right), q_{12}^0 \in K_{12}, \end{aligned}$$

which belong to

$$\left\{ \begin{pmatrix} q_{11} & \cdots & q_{1n} \\ \vdots & \vdots & \vdots \\ q_{i_1 1} & \cdots & q_{i_1 n} \\ \vdots & \vdots & \vdots \\ q_{i_n 1} & \cdots & q_{i_n n} \\ \vdots & \vdots & \vdots \\ q_{n1} & \cdots & q_{nn} \end{pmatrix}, \begin{array}{l} q_{i_s s} \in E_{i_s s}, (i_1, \dots, i_n) \in P_n^n \\ q_{ls} \in K_{ls}, l \neq i_s \\ l = 1, \dots, n \\ s = 1, \dots, n \end{array} \right\}.$$

So, \mathcal{A}_n is D -stable $\Leftrightarrow \epsilon_{\mathcal{A}}$ is D -stable. Thus

$$\begin{aligned} \epsilon_{\mathcal{A}} \text{ is } D\text{-stable} &\Leftrightarrow \mathcal{A}_n \text{ is } D\text{-stable} \Leftrightarrow \mathcal{T}_{n-1} \text{ is } D\text{-stable} \\ &\Rightarrow \mathcal{A}_{n-1} \text{ is } D\text{-stable} \dots \Leftrightarrow \mathcal{T}_i \text{ is } D\text{-stable} \\ &\Rightarrow \mathcal{A}_i \text{ is } D\text{-stable} \Leftrightarrow \mathcal{T} \text{ is } D\text{-stable} \Rightarrow A \text{ is } D\text{-stable}. \end{aligned}$$

That is to say, $\epsilon_{\mathcal{A}}$ is D -stable $\Rightarrow \forall A \in \mathcal{A}, A$ is D -stable, namely, \mathcal{A} is D -stable. This completes the proof.

Remark 1 When $m = 2$, i.e. $A = (p_{ij})_{n \times n}$, where

$$\begin{aligned} p_{ij} &= p_{ij}^0 + \lambda_{ij} p_{ij}^1 \quad \lambda_{ij} \in [0, 1] \\ p_{ij}^0, p_{ij}^1 &\text{ are fixed polynomials} \quad i, j = 1, \dots, n \end{aligned} \quad (4)$$

Let

$$\mathcal{B}_1 = \{(p_{ij})_{n \times n} \mid p_{ij} \text{ satisfies (4)}\} \quad (5)$$

In this case, the vertex sets of p_{ij} are $\{p_{ij}^0, p_{ij}^0 + p_{ij}^1\}$, and their exposed sets are exactly themselves, namely, $\{p_{ij}^0 + \lambda_{ij} p_{ij}^1\}$. The corresponding conclusion has more concise form, this is due to the simplification of the edge set of \mathcal{B}_1

$$\epsilon_{\mathcal{B}_1} = \left\{ (p_{ij})_{n \times n} \mid \begin{array}{l} (l_1, \dots, l_n) \in P_n^n, s = 1, \dots, n \\ \lambda_{si} \in \begin{cases} [0, 1] & i = l_s \\ \{0, 1\} & i \neq l_s \end{cases} \end{array} \right\}$$

3.2 Interval Polynomial Matrices

Consider the subset of $R^{n \times n}(s)$

$$\mathcal{B}_2 = \{(p_{ij})_{n \times n}, p_{ij} \text{ are interval polynomials}\} \quad (6)$$

where $R^{n \times n}(s)$ is the collection of $n \times n$ polynomial matrices. Assume $\epsilon_{\mathcal{B}_2}$ is *Hurwitz* stable.

Consider $\det A$, using *Laplace* formula on first column, we have

$$\det A = p_{11} M_{11} + \dots + p_{n1} M_{n1}.$$

Similar to the proof of theorem 1, and by resort to the Generalized Kharitonov Theorem, we have

Theorem 2 If $\forall A \in \mathcal{B}_2, \deg(\det A) = m$. Then \mathcal{B}_2 is *Hurwitz* stable $\Leftrightarrow \epsilon_{\mathcal{B}_2}$ is *Hurwitz* stable, where $\epsilon_{\mathcal{B}_2}$ is

$$\left\{ (p_{ij})_{n \times n} : \begin{array}{l} p_{il_i} \in E_{il_i}^0; i = 1, \dots, n \\ p_{si_s} \in K_{si_s}^0; l_s \neq i_s = 1, \dots, n; \\ (l_1, \dots, l_n) \in P_n^n; s = 1, \dots, n \end{array} \right\}$$

and $E_{il_i}^0, K_{si_s}^0$ are defined in Definition 4.

Remark 2 Theorems 1 and 2 can be viewed as a generalization of the Edge Theorem and Kharitonov Theorem to MIMO case. Stability test of the entire family is reduced to a critical low-dimensional subset. No extra lemma from signal processing is needed in our proof. Furthermore, our results can be easily extended to polynomial matrices with complex coefficients.

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