PID Controller Design Using Bode's Integrals

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Abstract

The feedback relay test is usually used to identify one point on the Nyquist diagram of the plant model. It is shown that the derivatives of amplitude and phase of the plant model with respect to frequency at that point can be approximated by the Bode's integrals without any model of the plant. The precision of the approximation for typical industrial plant models is studied. The derivatives are used to design a PID controller for slope adjustment of the loop Nyquist diagram and improve the closed-loop performance. Simulation examples illustrate the effectiveness and the simplicity of the proposed method to design the PID controllers.

Keywords: PID controller, relay feedback test, Ziegler-Nichols method, auto-tuning

1 Introduction

The Ziegler-Nichols methods [8] are still extensively used for determining the parameters of PID controllers. The design is based on the measurement of the critical gain and critical frequency of the plant and using simple formulas to compute the controller parameters. In 1984, Åström and Hägglund [3] proposed an automatic tuning method based on a simple relay feedback test which gives, using the describing function analysis, the critical gain and the critical frequency of the system. This information can be used to compute a controller with the desired gain or phase margins. However, in order to obtain the desired phase margin, the closed-loop system should oscillate at the desired crossover frequency (the frequency at which the loop gain is equal to 1). This can be obtained using a relay with hysteresis [3] or with introducing an adjustable time delay in the closed-loop system [5]. The hysteresis or the time delay should slowly change up to obtain a limit cycle at crossover frequency. This experiment, compared to the standard one, is more time consuming. A closed-loop relay test scheme was proposed in [7] which identifies directly the crossover frequency. In this scheme the plant operates in closed loop with an existing controller and the output of the relay is connected to the reference of the closed-loop system. The advantage is that the noise effect is attenuated and the amplitude of the relay can be easily adjusted.

After identifying a point on the frequency response of the plant, the so-called modified Ziegler-Nichols method can be used to move this point to another position in the complex plane [4]. Two equations for phase and amplitude assignment are obtained which can be solved to find the parameters of a PI controller. For a PID controller, however, an additional equation should be introduced. In the modified Ziegler-Nichols method, the ratio between integral time T_i and derivative time T_d is chosen to be constant ($T_d = 0.25T_i$) in order to have a unique solution. In [4], it was proposed to adjust the slope of the Nyquist curve at the crossover frequency such that a minimum distance to the critical point can be assured in the high frequencies. In this way the robustness of the closed-loop system with respect to the unmodeled dynamics may be improved. However, this method requires inevitably the derivatives of the plant transfer function with respect to frequency which are not known a priori.

The main contribution of this paper is the use of Bode's integrals for slope adjustment in PID controller design. The Bode's integrals [1] show the relation between the phase and the amplitude of minimum phase stable systems. It will be shown how these integrals can be used to approximate the derivatives of the amplitude and the phase of a system with respect to frequency at a given frequency. It is interesting to notice that the approximation is made only with the knowledge of the amplitude and the phase of the system at the given frequency and the system static gain. The derivatives can be used in the modified Ziegler-Nichols method to adjust the slope of the Nyquist curve at the given frequency.

The paper is organized as follows: In Section 2 a formula is derived which gives the relation between T_i and T_d for obtaining the desired slope of the Nyquist curve of the loop transfer function at a given frequency. The Bode's integrals are used to approximate the derivatives of the plant transfer function in Section 3. Section 4 shows the precision of the approximations. A

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PID controller design method for phase margin and slope adjustment is proposed in Section 5. Section 6 presents the simulation results. Finally, Section 7 gives some concluding remarks.

2 Loop slope adjustment

The slope of the Nyquist curve at crossover frequency affects drastically the performance and robustness of the closed-loop system. In this section, a formula is derived which gives the relation between T_i and T_d for obtaining the desired slope of the Nyquist curve of the loop transfer function at a given frequency.

Consider the loop transfer function $L(j\omega) = G(j\omega)K(j\omega)$ where

$$K(j\omega) = K_p(1 + \frac{1}{j\omega T_i} + j\omega T_d) \quad (1)$$

is the PID controller. The slope of the Nyquist curve of the loop transfer function $L(j\omega)$ at ω_0 defined by ψ is equal to the phase of the derivative of $L(j\omega)$ at ω_0. The derivative of the loop transfer function with respect to ω is computed as follows:

$$\frac{dL(j\omega)}{d\omega} = G(j\omega)\frac{dK(j\omega)}{d\omega} + K(j\omega)\frac{dG(j\omega)}{d\omega} \quad (2)$$

Furthermore one has:

$$\ln G(j\omega) = \ln|G(j\omega)| + j\angle G(j\omega) \quad (3)$$

Differentiating this equation gives:

$$\begin{aligned}\frac{d\ln G(j\omega)}{d\omega} &= \frac{1}{G(j\omega)}\frac{dG(j\omega)}{d\omega} \\ &= \frac{d\ln|G(j\omega)|}{d\omega} + j\frac{d\angle G(j\omega)}{d\omega} \quad (4)\end{aligned}$$

On the other hand, the derivative of the controller with respect to ω is:

$$\frac{dK(j\omega)}{d\omega} = jK_p(T_d + \frac{1}{\omega^2 T_i}) \quad (5)$$

Substituting Eqs (1), (4) and (5) into Eq. (2), one obtains:

$$\begin{aligned}\frac{dL(j\omega)}{d\omega} &= K_pG(j\omega)\Big[j(T_d + \frac{1}{\omega^2 T_i}) \\ &+ \left(1 + j(T_d\omega - \frac{1}{\omega T_i})\right) \\ &\times \left(\frac{d\ln|G(j\omega)|}{d\omega} + j\frac{d\angle G(j\omega)}{d\omega}\right)\Big] \quad (6)\end{aligned}$$

Hence, the slope of the Nyquist curve at ω_0 is given by:

$$\psi = \angle\frac{dL(j\omega)}{d\omega}\Big|_{\omega_0} = \varphi_0 + \arctan \quad (7)$$
$$\frac{(T_dT_i\omega_0^2 + 1) + (T_dT_i\omega_0^2 - 1)s_a(\omega_0) + s_p(\omega_0)T_i\omega_0}{s_a(\omega_0)T_i\omega_0 - (T_dT_i\omega_0^2 - 1)s_p(\omega_0)}$$

where $\varphi_0 = \angle G(j\omega_0)$ and $s_a(\omega_0)$ and $s_p(\omega_0)$ are defined as follows:

$$s_a(\omega_0) = \omega_0 \frac{d\ln|G(j\omega)|}{d\omega}\Big|_{\omega_0} \quad (8)$$

$$s_p(\omega_0) = \omega_0 \frac{d\angle G(j\omega)}{d\omega}\Big|_{\omega_0} \quad (9)$$

It is desired to adjust the slope of the Nyquist curve of the loop transfer function $L(j\omega)$ to a specified value ψ. Then straightforward calculation gives:

$$\begin{aligned}T_d = &[s_a(\omega_0) - 1 + s_p(\omega_0)\tan(\psi - \varphi_0) \\ &-T_i\omega_0(s_p(\omega_0) - s_a(\omega_0)\tan(\psi - \varphi_0))] \\ &\times[\omega_0^2 T_i(1 + s_a(\omega_0) + s_p(\omega_0)\tan(\psi - \varphi_0))]^{-1}\end{aligned}$$

In the next part, $s_a(\omega_0)$ and $s_p(\omega_0)$ are directly approximated using the Bode's integrals.

3 Bode's integrals

The relations between the phase and the amplitude of a stable minimum-phase system have been investigated for the first time by Bode [1]. The results are based on Cauchy's residue theorem and have been extensively used in network analysis. Two integrals are presented in this section. The first one, which is well known in the control engineering field, shows the relation between the phase of the system at each frequency as a function of the derivative of its amplitude. But the second integral, to the best of the authors' knowledge, has been never used in the control design. The integral shows how the amplitude of the system at each frequency is related to the derivative of the phase and the static gain of the system.

3.1 Derivative of amplitude

Bode has shown in [1] that for a stable minimum-phase transfer function $G(j\omega)$, the phase of the system at ω_0 is given by:

$$\angle G(j\omega_0) = \frac{1}{\pi}\int_{-\infty}^{+\infty} \frac{d\ln|G(j\omega)|}{d\nu} \ln\coth\frac{|\nu|}{2} d\nu \quad (10)$$

where $\nu = \ln\frac{\omega}{\omega_0}$. Since $\ln\coth\frac{|\nu|}{2}$ decreases rapidly as ω deviates from ω_0, the integral depends mostly on $\frac{d\ln|G(j\omega)|}{d\nu}$ (the slope of the Bode plot) near ω_0. Therefore, assuming that the slope of the Bode plot is almost constant in the neighborhood of ω_0, $\angle G(j\omega_0)$ can be approximated by:

$$\begin{aligned}\angle G(j\omega_0) &\approx \frac{1}{\pi}\frac{d\ln|G(j\omega)|}{d\nu}\Big|_{\omega_0}\int_{-\infty}^{+\infty}\ln\coth\frac{|\nu|}{2}d\nu \\ &\approx \frac{\pi}{2}\frac{d\ln|G(j\omega)|}{d\nu}\Big|_{\omega_0} \quad (11)\end{aligned}$$

This property is often used in loop shaping where the slope of the amplitude Bode plot at crossover frequency is limited to -20dB/decade in order to obtain approximately a phase margin of 90°. Here the measured phase of the system at ω_0 is used to determine approximately the slope of the amplitude Bode plot (s_a):

$$\begin{aligned} s_a(\omega_0) &= \left.\frac{d\ln|G(j\omega)|}{d\nu}\right|_{\omega_0} = \omega_0 \left.\frac{d\ln|G(j\omega)|}{d\omega}\right|_{\omega_0} \\ &\approx \frac{2}{\pi}\angle G(j\omega_0) \end{aligned} \quad (12)$$

3.2 Derivative of phase

The second Bode's integral shows that the amplitude of a stable minimum-phase system can be determined uniquely from its phase and its static gain. More precisely, the logarithm of the system amplitude at ω_0 is given by [1]:

$$\begin{aligned} \ln|G(j\omega_0)| &= \ln|K_g| - \\ &\frac{\omega_0}{\pi}\int_{-\infty}^{+\infty}\frac{d(\angle G(j\omega)/\omega)}{d\nu}\ln\coth\frac{|\nu|}{2}d\nu \end{aligned}$$

where K_g is the static gain of the plant. In the same way, assuming that $\angle G(j\omega)/\omega$ is linear (in a logarithmic scale) in the neighborhood of ω_0, one has:

$$\begin{aligned} \ln|G(j\omega_0)| &\approx \ln|K_g| - \frac{\omega_0}{\pi}\left.\frac{d(\angle G(j\omega)/\omega)}{d\nu}\right|_{\omega_0}\frac{\pi^2}{2} \\ \ln|G(j\omega_0)| &\approx \ln|K_g| - \frac{\pi\omega_0^2}{2}\left[\frac{1}{\omega_0}\left.\frac{d\angle G(j\omega)}{d\omega}\right|_{\omega_0}\right. \\ &\qquad \left. - \frac{\angle G(j\omega_0)}{\omega_0^2}\right] \end{aligned} \quad (13)$$

which gives:

$$\begin{aligned} s_p(\omega_0) &= \omega_0 \left.\frac{d\angle G(j\omega)}{d\omega}\right|_{\omega_0} \\ &\approx \angle G(j\omega_0) + \frac{2}{\pi}[\ln|K_g| - \ln|G(j\omega_0)|] \end{aligned} \quad (14)$$

Note that for the systems containing an integrator, the static gain cannot be computed. For such systems, the static gain of the system without the integrator should be estimated and used in the above formula (note that the phase of the integrator is constant and its derivative is zero).

4 Precision of the estimates

The precision of the estimates of the derivatives of amplitude and phase depends on the system dynamics and on the frequency at which the experiments are performed. However, extensive simulations on the typical models of industrial plants have shown that the absolute normalized error of the estimates is within an acceptable range. In order to give an idea about the precision of the estimates, let us consider the following system:

$$G(s) = \frac{1}{(s+1)^n} \quad (15)$$

where n is a positive integer. The true values of $s_a(\omega)$ and $s_p(\omega)/\omega$ computed on the basis of the model for different frequencies, are compared with the estimated ones based on the Bode's integrals in Fig. 1 and Fig. 2, respectively. It can be observed that the maximum of the absolute normalized error does not exceed 0.1 for this system.

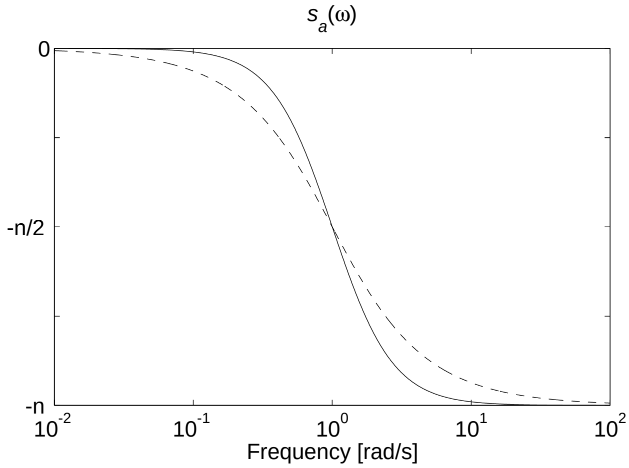

Figure 1: Comparison of true $s_a(\omega)$ and the estimated one based on Bode's integral (solid line: true values, dashed line: estimates)

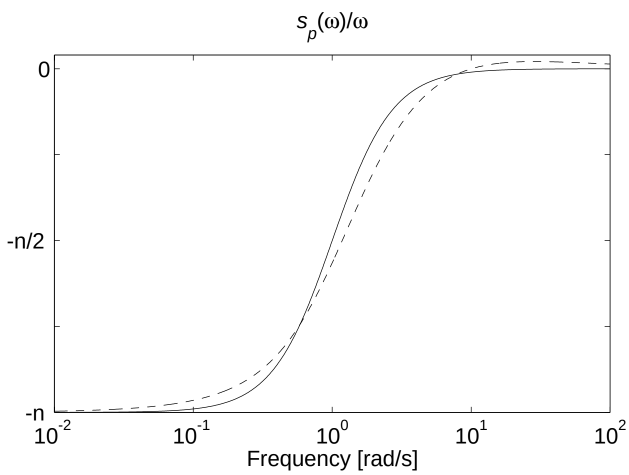

Figure 2: Comparison of true $s_p(\omega)/\omega$ and the estimated one based on Bode's integral (solid line: true values, dashed line: estimates)

Similar results can be obtained for the systems presented by several first-order models in cascade. For oscillatory systems, the estimation may be poor in certain frequencies, but in general the results remain satisfactory. For non-minimum-phase systems the Bode's integrals are no longer valid and the proposed formulas give incorrect results. However, it will be shown next that the pure time delay has no effect on the estimation of s_p and its effect on the estimation of s_a can be neglected if it is small with respect to the dominant time constant of the system.

4.1 Effect of pure time delay

Consider the following system with a pure time delay τ:

$$G_\tau(j\omega) = G(j\omega)e^{-j\tau\omega} \tag{16}$$

where $G(j\omega)$ is a stable minimum-phase system. Differentiating the amplitude and phase of $G_\tau(j\omega)$ with respect to ω gives:

$$\frac{d\ln|G_\tau(j\omega)|}{d\omega} = \frac{d\ln|G(j\omega)|}{d\omega} \tag{17}$$

$$\frac{d\angle G_\tau(j\omega)}{d\omega} = \frac{d\angle G(j\omega)}{d\omega} - \tau \tag{18}$$

Now using the Bode's integral from Eq. (11), the phase of $G_\tau(j\omega)$ at ω_0 is approximated by:

$$\begin{aligned}\angle G_\tau(j\omega_0) &= \angle G(j\omega_0) - \tau\omega_0 \\ &\approx \frac{\pi}{2}\left.\frac{d\ln|G(j\omega)|}{d\omega}\right|_{\omega_0} - \tau\omega_0\end{aligned} \tag{19}$$

Then $s_a(\omega_0)$ and $s_p(\omega_0)$ for a system including a pure time delay are computed as follows:

$$\begin{aligned}s_a(\omega_0) &= \omega_0\left.\frac{d\ln|G_\tau(j\omega)|}{d\omega}\right|_{\omega_0} \\ &\approx \frac{2}{\pi}(\angle G_\tau(j\omega_0) + \tau\omega_0)\end{aligned} \tag{20}$$

$$\begin{aligned}s_p(\omega_0) &= \omega_0\left.\frac{d\angle G_\tau(j\omega)}{d\omega}\right|_{\omega_0} \\ &= \omega_0\left.\frac{d\angle G(j\omega)}{d\omega}\right|_{\omega_0} - \tau\omega_0 \\ &\approx \angle G(j\omega_0) + \frac{2}{\pi}[\ln|K_g| - \ln|G(j\omega_0)|] - \tau\omega_0 \\ &\approx \angle G_\tau(j\omega_0) + \frac{2}{\pi}[\ln|K_g| - \ln|G_\tau(j\omega_0)|]\end{aligned} \tag{21}$$

The above relations show that the pure time delay should be known for the calculation of $s_a(\omega_0)$ but it has no effect on the calculation of $s_p(\omega_0)$. It should be remembered that τ represents the pure time delay of the system which is usually related to the mass transport delay and is often negligible or can be easily measured. This value should not be confounded with the time delay that is used to model a high-order system as a first- or second-order system with delay. For example, $G(s) = 1/(s+1)^5$ has no pure time delay whereas it can be approximated by a first-order model with a large time delay.

5 PID design

Suppose that the amplitude and the phase of a plant at crossover frequency ω_c are known. These values may be obtained using the existing controller and by the method proposed in [6]. Suppose also that the static gain of the process is measured. The objective is to improve the controller performance by adjusting the phase margin and the slope of the Nyquist curve at the crossover frequency. The modified Ziegler-Nichols method is used but the derivatives are approximated by the Bode's integrals, so no model for the system is required. To obtain a desired phase margin Φ_d at the crossover frequency ω_c we have the following equations to solve:

$$\angle G(j\omega_c) + \angle K(j\omega_c) = \Phi_d - \pi \tag{22}$$

$$|G(j\omega_c)K(j\omega_c)| = 1 \tag{23}$$

Solving these equations one obtains:

$$K_p = \frac{\cos(\Phi_d - \varphi_c - \pi)}{|G(j\omega_c)|} \tag{24}$$

$$T_d\omega_c - \frac{1}{T_i\omega_c} = \tan(\Phi_d - \varphi_c) \tag{25}$$

where φ_c is the phase of $G(j\omega_c)$. Now we exploit Eq. (10) in order to obtain the desired slope ψ at the crossover frequency. Combining Eqs (10) and (25), we obtain after straightforward calculations the parameters T_i and T_d as follows:

$$T_i = \frac{1}{\omega_c(T_d\omega_c - \tan(\Phi_d - \varphi_c))} \tag{26}$$

$$\begin{aligned}T_d = {} & \frac{1}{2\omega_c}[(s_a(\omega_c) - s_p(\omega_c)\tan(\Phi_d - \varphi_c)) \\ & \times \tan(\psi - \varphi_c) + (1 - s_a(\omega_c))\tan(\Phi_d - \varphi_c) \\ & - s_p(\omega_c)]\end{aligned} \tag{27}$$

The improved PID controller is now defined by (24), (26) and (27).

6 Simulation Results

The PID design method presented above will be illustrated via a simulation example. Consider the following model:

$$G(s) = \frac{1}{(s+1)^5} \tag{28}$$

The specifications are set at 0.4 rad/s for the crossover frequency and 50° for the phase margin. First, the control parameters are obtained using the modified Ziegler-Nichols method. The resulting PID controller is:

$$K(s) = 1.35\,(1 + \frac{1}{3.44s} + 0.86s) \tag{29}$$

This controller moves the point $G(0.4\,j)$ of the Nyquist curve to a point of $K(j\omega)G(j\omega)$ on the unit circle having a phase of 130°. This conforms to the closed-loop system with the specifications mentioned above.

In order to improve the closed-loop performance, let us calculate now a controller that gives to the closed-loop system the same crossover frequency and phase margin, but with the desired slope of the open-loop Nyquist curve at the crossover frequency of 65°. This reduces the current slope of the Nyquist curve by 25° and ensures a greater distance of the Nyquist curve from the critical point in high frequencies. The controller parameters are obtained from Eqs (24), (27) and (26), where $s_a(0.4\,j)$ and $s_p(0.4\,j)$ are approximated using Eqs (12) and (14).

$$K(s) = 1.35\,(1 + \frac{1}{2.81s} + 1.27s) \tag{30}$$

Although the approximation error for s_a and s_p leads to a resultant slope of 74° (about 13% error), a comparison of the closed-loop performance for the two controllers shown in Fig. 3 illustrates a significant improvement of the closed-loop performance. The overshoot is about the same but the settling time is 44 % smaller with the proposed method. The Nyquist diagrams (Fig. 4) further show that the proposed controller modifies the slope of the Nyquist curve and as a result improves the gain margin as well as the modulus margin (the minimum distance between $L(j\omega)$ and the critical point) of the system.

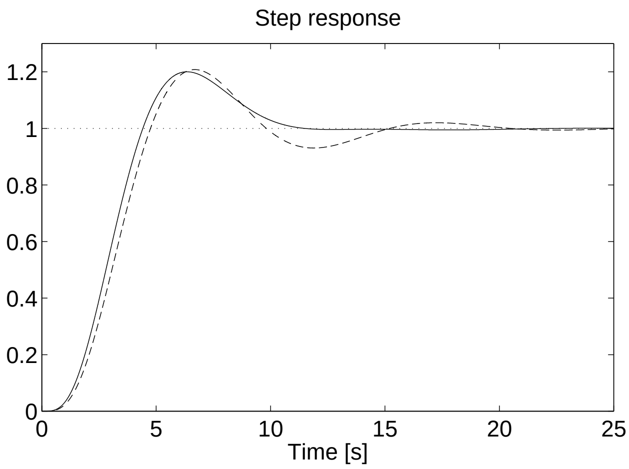

Figure 3: Step response of the closed-loop system (dashed line: modified Ziegler-Nichols, solid line: proposed)

Consider again the same plant model with a new initial PID-controller:

$$K(s) = 0.57\,(1 + \frac{1}{1.89s} + 1.89s) \tag{31}$$

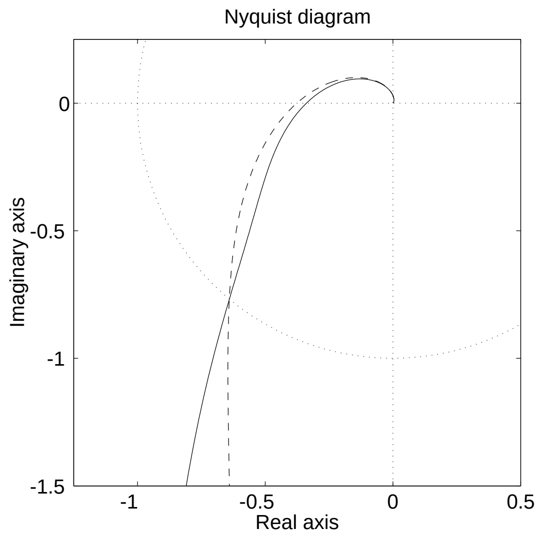

Figure 4: Nyquist diagram (dashed line: modified Ziegler-Nichols, solid line: proposed)

This controller has been proposed for the above mentioned plant in [2] to achieve a phase margin of 60° and a gain margin of 3. Suppose that only the static gain of the plant is a priori known. A closed-loop experiment as proposed in [6] measures for this system a phase margin of 52.8° and a crossover frequency of 0.243 rad/s (while the true values are 52.27° and 0.240 rad/s, respectively).

Let us define new specifications with the same crossover frequency in order to improve the closed-loop performance with the proposed method. The phase margin is set at 60° and the slope of the Nyquist curve in the crossover frequency at 80°. As the point $G(0.243j)$ is known from the closed-loop experiment mentioned above, Eqs (24), (27) and (26) can be used to compute the new controller

$$K(s) = 0.687\,(1 + \frac{1}{2.91s} + 0.421s) \tag{32}$$

A comparison of the closed-loop performance between the initial controller (dashed line) and the new controller (solid line) in Fig. 5 shows that a much smaller settling time and overshoot is achieved by the new controller.

7 Conclusions

The derivatives of phase and amplitude of minimum-phase and stable plant models with respect to the frequency have been approximated using the Bode's integrals. Only the value of the transfer function at the given frequency is used for the approximation and no

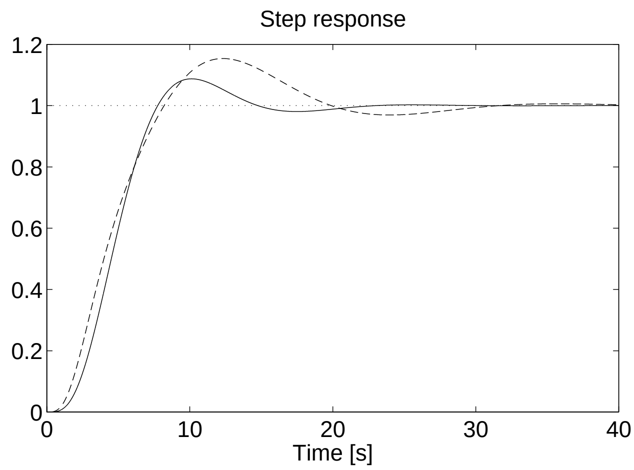

Figure 5: Step response of the closed-loop system (dashed line: initial, solid line: proposed method)

parametric model of the plant is required. The precision of the approximation for typical industrial plant models is adequate for PID controller tuning. The estimated derivatives of phase and amplitude of the plant can be used to adjust the slope of the Nyquist curve and improve the robustness of the closed-loop system with respect to the unmodeled dynamics. The proposed method requires a minimum information about the plant which can be obtained with a simple closed-loop relay test. Simulation examples have shown that the closed-loop performance is significantly related to the slope of the Nyquist curve and can be improved using the proposed method.

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